STAT 345 Fall 2018
Homework 6 - Continuous Probability Distributions

Name: ______________________

Please adhere to the homework rules as given in the Syllabus.

1. **Standard Normal.** Let Z be a Standard Normal RV.

a) Find $P(Z < 1.55)$ and $P(Z > 1.55)$

b) Find $P(Z < -0.64)$ and $P(Z > 0.64)$

c) Find $P(-0.64 < Z < 1.55)$.

d) Find $P(|Z| < k)$ for $k = 1, 2, 3$. *Hint: the event* $|Z| < k$ *can be written in a more familiar way.*

2. Let X be the height of a randomly selected UNM male student. Assume that X is normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 3$.

a) Find the probability that a randomly selected UNM male student is taller than legendary rapper Lonzo Ball (6 ft 6 in).

b) Find the probability that a randomly selected UNM male student is between 5 feet and 6 feet tall.

c) If exactly 25% of UNM males are taller than Timothy, how tall is Timothy?

3. **The Log-normal Distribution.** In atmospheric science, the Log-normal distribution is often used to characterize particle size distributions. In one study, the distribution of silicone nanoparticle size (in nm) was found to be approximately Log-normal with Log-mean $\theta = 3.91$ and Log-sd $\omega = 0.47$. Suppose the desired range of particle sizes is $(40nm, 110nm)$. What percentage of silicone nanoparticles do you expect to fall within this range?

4. Normal Approximations.

a) Suppose that the number of cars that drive past Lomas on the I-25 between 5pm and 6pm on a Wednesday can be modeled as a Poisson random variable with mean $\lambda = 2100$. Use the Normal approximation to the Poisson distribution to determine the probability that less than 2000 cars drive past Lomas on the I-25 between 5pm and 6pm this wednesday.

b) 37% of cars in Albuquerque are white. Last wednesday, 2200 cars drove past Lomas on the I-25 between 5pm and 6pm. Use the Normal approximation to the Binomial distribution to determine the probability that more than 800 of these cars were white.

5. Skewness. Let X be a random variable with mean μ and standard deviation σ. Then the Skew of X can be defined mathematically as follows.

$$Sk(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{E(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

The Exponential Distribution is *Skewed Right.* Sometimes this is called *Positive Skew.* Show that for $X \sim Exp(\lambda)$ we have $Sk(X) = 2$ (positive) regardless of λ. *Hint: Find* $E(X^3)$ *via Integration by parts, tabular integration or the gamma function.*

6. Lack of Memory. The Exponential has a special property called *Lack of Memory* (the geometric distribution and Professor Halfbrain also have this property). Assuming that $s < t$, this property is defined mathematically as follows:

$$P(X > t | X > s) = P(X > t - s)$$

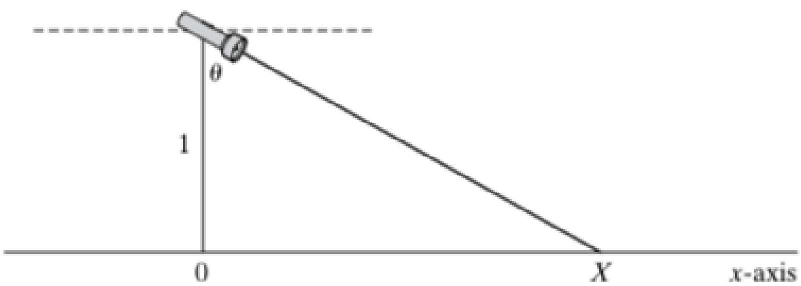
a) Prove that this property holds for $X \sim Exp(\lambda)$. *Hint: Find the LHS using the definition of conditional probability. Find the RHS using the CDF. Show that they give the same answer.*

b) Suppose that X is a positive random variable that represents the lifetime of some item or person (Example: Lifetime of a battery or lifetime of a person with some disease.) The Hazard rate (sometimes called the failure rate) is the function of time

$$h(t) = \frac{f(t)}{1 - F(t)}$$

which describes the instantaneous probability of death (or failure) at time t. Different distributions lead to different forms of the hazard function. Show that the Hazard rate for an exponential random variable is constant, and explain why this makes sense according to the lack of memory property.

7. **Challenge Problem** *The Cauchy Distribution.* Suppose that a flashlight is spun around its center, which is located 1 unit away from the x-axis. (See Figure below). Consider the point X at which the beam intersects the x-axis when the flashlight has stopped spinning. (If the beam is not touching the x-axis at all, repeat the experiment.) In this problem, we will show that X has a Cauchy distribution (equivalent to a t-distribution with only 1 degree of freedom).



a) The angle θ (shown in picture) is also a random variable, and it is reasonable to assume that $\theta \sim Unif(-\pi/2, \pi/2)$. Write down the CDF of θ.

b) Determine (using Trig) the relationship between X and θ. Use this relationship, and your answer in a), to find $F(x)$, the CDF of X. What is the median of X? Does this make sense?

c) Find $f(x)$, the PDF of X.

d) An interesting feature of the Cauchy distribution, is that the expected value is *undefined* (even though the median is perfectly reasonable). Show that this is true by finding

$$I(a,b) = \int_a^b x f(x) dx$$

and showing that

$$\lim_{a \to -\infty} \lim_{b \to \infty} I(a,b)$$

does not equal

$$\lim_{b \to \infty} \lim_{a \to -\infty} I(a,b)$$