STAT 345 Spring 2018
Challenge Homework - Moment Generating Functions Name: ______________________

Please adhere to the homework rules as given in the Syllabus.

The moment generating function of a Random Variable X is defined as,

$$M_X(t) = E(e^{Xt}) = \begin{cases} \sum_{x \in \mathcal{X}} e^{xt} f(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{xt} f(x) dx, & \text{if } X \text{ is continuous} \end{cases}$$

On a previous homework, we showed that the moments $E(X^k)$ can be found by taking the k^{th} derivative of $M_X(t)$ and evaluating at 0. This is a useful fact, but it is actually not the main reason that MGFs are so useful. The MGF of a distribution is *unique*. Combining this with properties we will show in this homework, we can easily show relationships between random variables that are otherwise difficult to show.

1 The MGF of $Y = aX + b$

Suppose that X is a random variable with MGF $M_X(t)$, and a and b are constants. Finally, let $Y = aX + b$.

a) Using properties of the exponential function, show that the MGF of Y is $M_Y(t) = e^{bt} M_X(at)$.

b) Assume that $X \sim Exp(\lambda)$. Find the MGF of X.

c) Assume that $X \sim Exp(\lambda)$ and $Y = aX + b$. Find the MGF of Y. If $b = 0$, what is the distribution of Y?

2 The MGF of $Z = X_1 + X_2$

Suppose that X_1 and X_2 are *independent* random variables with MGFs $M_1(t)$ and $M_2(t)$ respectively. Finally, let $Z = X_1 + X_2$.

d) Using properties of the exponential function and independence, show that $M_Z(t) = M_1(t)M_2(t)$.

e) Find the MGF of $X \sim Poiss(\lambda)$.

f) Assume that $X_1 \sim Poiss(\lambda)$ and $X_2 \sim Poiss(\delta)$ and $Z = X_1 + X_2$. Find the MGF of Z, and give the distribution of Z.

We can obviously extend this easily. If $X_1, X_2, \cdots X_n$ are independent and $Z = \sum_{i=1}^{n} X_i$, then

$$M_Z(t) = \prod_{i=1}^{n} M_i(t)$$

and if the X_i are iid, then $M_Z(t) = M_X(t)^n$.
g) Determine the MGF of $Z \sim Binom(n, p)$ by using the sum of Bernoulli random variables.