STAT 345 Fall 2018
Homework 3 - Discrete Random Variables   Name: ______________________

Please adhere to the homework rules as given in the Syllabus.

1. Frequency Distribution The total number of goals scored in a World Cup soccer match approximately follows the following distribution.

Goals Scored	0	1	2	3	4	5	6	7
Probability	0.08	0.2	0.26	0.2	0.14	0.07	0.03	0.02

a) Let X be the number of goals scored in a randomly selected World Cup soccer match. Write out the PMF for X and explain why it is a valid PMF.

b) Compute the mean and variance of X.

c) Find and sketch the CDF of X. Explain why it is a valid CDF.

2.

a) Recall that for a random variable X and constants a and b, $E(aX+b) = aE(X)+b$ (we proved this in class). Prove that

$$Var(aX+b) = a^2 Var(X)$$

b) Mike agrees to donate 50 dollars to charity, plus 25 dollars for every goal scored in a particular World Cup soccer game. Thus Mike's donation is a random variable $Y = 25X + 50$ where X is defined in problem 1. Find the expected value and the variance of Y using the results in part a.

3. **Probability Mass Function.** Let X be a discrete RV with the following PMF.

$$f(x) = \begin{cases} \frac{c}{2^x} & x = 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$$

a) Find the value of c that makes $f(x)$ a valid PMF.

b) Find $P(X > 2)$.

c) Find $E(X)$ and $Var(X)$.

d) Find $E(2^X)$.

4. Linearity of Expectation. The SAT is a multiple choice exam, with 5 possible answers for each question. To discourage test takers from guessing, they use the following point system. If you get a question right you get 1 point. If you get a question wrong you lose 0.25 points. (If you leave a question blank, you get/lose nothing, but that doesn't matter for this problem).

a) Assuming you are completely guessing, what is the probability of getting the question correct?

b) We can model this using a Bernoulli random variable, where $X = 1$ if you get the question right and $X = 0$ if you get the question wrong. The number of points earned on a single question is a random variable and can be expressed as

$$Y = X - 0.25(1 - X) = 1.25X - 0.25$$

Use linearity of expectation to show that $E(Y) = 0$ if you are guessing at random.

c) Assume now that there are k possible answers for each question. In terms of k, how many points should we take off for a wrong answer if we want to ensure $E(Y) = 0$ when a test-taker is guessing? *Hint: Let* $Y = X - L(1 - X)$ *where* L *is the number of points we remove. Find the expected value and solve for* L*.*

5. Get Rich Quick. Professor Halfbrain has come up with a betting strategy for roulette which he guarantees will make you rich. Bet 100 dollars on red (which has probability $\frac{18}{38}$). If you win, take the 100 dollar profit and quit. If you lose, bet 100 dollars on red two more times and then quit. Let X denote your earnings.

a) Find the probability mass function of X. *Hint: There are only* 3 *possible values that* X *can take. Find the two easier probabilities, and find the third by using the fact that the probabilities must sum to* 1.

b) What is $P(X > 0)$. What do you think of the Professors scheme?

c) Find $E(X)$. What do you think of the Professors shceme now?

6. *Challenge Problem.* **The Magic Money Box.** After cracking open your piggy bank, you find that you are down to your last 100 dollars. You decide that it is time to break open your... *Magic Money Box.* Each night, you may place some proportion $p \in [0, 1]$ of your money into the box. When you wake up in the morning, one of two things will happen with equal probability. Either your money will be tripled or it will have disappeared. Your money tomorrow is a random variable X which can be expressed as follows:

$$X = \begin{cases} 100(1-p), & \text{with probability } 1/2 \\ 100(1-p) + 3(100p), & \text{with probability } 1/2 \end{cases}$$

a) Find the expected value of your money tomorrow (i.e. $E(X)$) in terms of p. What value of $p \in [0, 1]$ maximizes your expected value?

b) Suppose you can use the Magic Money Box every night (as long as you still have money). Explain why the "optimal" strategy in part **a)** might not be the best idea.

c) The expression $U(p) = E(\log X)$ is called the log-utility function. Find the log-utility and use calculus to determine the value of $p \in [0, 1]$ that maximizes log-utility. It has been shown that this is the optimal strategy "in the long-run".