STAT 345 Spring 2020
Midterm

Name: ______________________________

1. (15 points) Three boys and four girls are about to sit on a bench.

a) How many different ways can these friends sit on the bench?

$$|S| = 7! = 5040$$

b) How many ways can they sit on the bench if the three boys insist on sitting next to each other?

* 5 ways to choose where the group of boys sit.
* $3! = 6$ ways to order the boys.
* $4! = 24$ ways to order the girls.

$$|A| = 5 \times 6 \times 24 = 720$$

c) If each person chooses a seat completely at random, what is the probability that the three boys will sit next to each other?

$$P(A) = \frac{|A|}{|S|} = \frac{720}{5040} = 0.143$$

2. (15 points) During the Monopoly promotion at McDonald's, certain items that are purchased have a sticker. Each sticker has a 1 in 5 chance of winning a prize. Alf buys 14 hash browns (each hash brown has one sticker). Let X be the number of prizes that Alf wins.

a) What is the expected value and standard deviation of X?

$X \sim Binom(14, 0.2)$ so

$$E(X) = np = 14(0.2) = 2.8$$

$$Var(X) = np(1-p) = 14(0.2)(0.8) = 2.24$$

$$SD(X) = \sqrt{Var(X)} = \sqrt{2.24} = 1.25$$

b) What is the probability that he wins *exactly* 5 prizes?

$$P(X=5) = p(5) = \binom{14}{5} 0.2^5 (1-0.2)^{14-5} = 0.086$$

b) What is the probability that he wins **at most** 10 prizes? *Note: You don't have to compute the final answer, but show your steps and take it as far as you can before doing any calculations.*

$$\begin{aligned} P(X \leq 10) &= 1 - P(X > 10) \\ &= 1 - (p(11) + p(12) + p(13) + p(14)) \\ &= 1 - \left(\binom{14}{11} 0.2^{11} 0.8^3 + \binom{14}{12} 0.2^{12} 0.8^2 + \binom{14}{13} 0.2^{13} 0.8^1 + \binom{14}{14} 0.2^{14} 0.8^0 \right) \\ &= 0.9999959 \qquad \text{(this step not required for midterm)} \end{aligned}$$

3. (25 points) Let X be a continuous RV with the following PDF.

$$f(X) = \begin{cases} 1.5\sqrt{x}, & 0 < x < 1. \\ 0, & otherwise \end{cases}$$

a) Find the expected value and variance of X.

$$E(X) = \int_0^1 x1.5\sqrt{x}dx = 1.5\int_0^1 x^{1.5}dx = \frac{1.5}{2.5} = 0.6$$
$$E(X^2) = \int_0^1 x^2 1.5\sqrt{x}dx = 1.5\int_0^1 x^{2.5}dx = \frac{1.5}{3.5} = 0.429$$
$$Var(X) = E(X^2) - E(X)^2 = 0.429 - 0.6^2 = 0.069$$

b) Find the CDF of X. *Hint: Don't forget to define the CDF for all values of X.*

$$\int_{-\infty}^{x} f(t)dt = 1.5\int_0^x t^{0.5}dt = \frac{1.5}{1.5}t^{1.5}|_0^x = x^{1.5}$$

So the CDF is

$$F(x) = \begin{cases} 0, & x < 0 \\ x^{1.5}, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

c) What is the probability that X is **equal to** 0.3?

Since X is a continuous RV, we have

$$P(X = 0.3) = 0$$

d) What is the probability that X is greater than 0.3?

$$P(X > 0.3) = 1 - P(X \leq 0.3) = 1 - F(0.3) = 1 - 0.3^{1.5} = 0.836$$

e) What is the probability that X is between 0.3 and 0.7?

$$P(0.3 \leq X \leq 0.7) = F(0.7) - F(0.3) = 0.7^{1.5} - 0.3^{1.5} = 0.421$$

4. (20 points) The number of cups of coffee that Frank drinks per day is a Poisson random variable. Assume that 78% of the time, Frank has at least one cup of coffee.

a) Rounded to one decimal place, how many cups of coffee does Frank drink per day on average (i.e. what is λ)? *Hint: The answer is not* 0.78 *or* 0.22.

$$P(X > 0) = 0.78$$

or equivalently,

$$P(X = 0) = 0.22$$

so we have

$$\frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = 0.22$$

solving for λ we get

$$\lambda = -\ln(0.22) = 1.51$$

b) What is the probability that Frank drinks at least three cups of coffee in a day. *Hint: Use the Poisson pmf. Do NOT use a normal approximation here.*

$$\begin{aligned} P(X \geq 3) &= 1 - P(X < 3) \\ &= 1 - (p(0) + p(1) + p(2)) \\ &= 1 - \frac{e^{-1.51}1.51^0}{0!} - \frac{e^{-1.51}1.51^1}{1!} - \frac{e^{-1.51}1.51^2}{2!} \\ &= 1 - 0.221 - 0.334 - 0.252 \\ &= 0.194 \end{aligned}$$

c) Assuming independence, let Y be the number of cups of coffee that Frank drinks over the course of two weeks (14 days). What is the distribution, mean and variance of Y?

Since $14(1.51) = 21.14$, we have that $Y \sim Pois(21.14)$ so $E(Y) = 21.14$ and $Var(Y) = 21.14$.

d) Use a Normal approximation to estimate the probability that Frank drinks at least 20 cups of coffee over a two week period.

$$P(Y \geq 20) \approx 1 - \Phi\left(\frac{20 - 0.5 - 21.14}{\sqrt{21.14}}\right) = 1 - \Phi(-0.36) = 1 - 0.3594 = 0.6406$$

5. (20 points) **Spam detection**. An analysis of emails has shown that

- A non-spam email contains the phrase "Be your own boss!" just 0.01% of the time.
- A spam email contains the phrase "Be your own boss!" 12.50% of the time.

Assume that 45% of emails are spam. If you receive an email and it contains the phrase "Be your own boss!" what is the probability that it is spam?

Let S be the event that an email is spam and let B be the event that an email contains the phrase "Be your own boss!". We are given

$$P(S) = 0.45 \qquad \textit{and} \qquad P(S^c) = 0.55$$

$$P(B|S) = 0.125 \qquad \textit{and} \qquad P(B|S^c) = 0.0001$$

Using Law of total probability, we can find the probability of B as

$$P(B) = P(B|S)P(S) + P(B|S^c)P(S^c) = 0.125(0.45) + 0.0001(0.55) = 0.0563$$

Now using Bayes rule we get

$$P(S|B) = \frac{P(B|S)P(S)}{P(B)} = \frac{0.125(0.55)}{0.0563} = 0.999$$

Side note: This is why spam filters on your email are so succesful.

6. (10 points) Timothy and Jimothy are fighting over who has to take out the trash. They decide to flip a coin but, unfortunately, they have only an *unfair coin* which lands heads with probability 0.7. They agree to use the following strategy. They will flip the coin two times and define

$$X = \begin{cases} 1, & \text{if first flip is heads} \\ 0, & \text{if first flip is tails} \end{cases} \qquad Y = \begin{cases} 1, & \text{if second flip is heads} \\ 0, & \text{if second flip is tails} \end{cases}$$

If X and Y are the same, then they will re-flip the coin two more times. If X and Y are different, then Timothy takes out the trash if $Y = 1$ and Jimothy takes out the trash if $Y = 0$.

a) Show that this procedure is "fair" in the sense that both Timothy and Jimothy have equal probability of taking out the trash. *Hint: Find* $P(Y = 1|X \neq Y)$.

First, we want to find $P(X \neq Y)$. There are two (disjoint) ways that this can happen, so

$$P(X \neq Y) = P(X = 1 \cap Y = 0) + P(X = 0 \cap Y = 1) = (0.7)(0.3) + (0.3)(0.7) = 0.21$$

Now we find $P(Y = 1|X \neq Y)$ which can be interpreted as *the probability that Timothy takes out the trash, given that somebody takes out the trash.*

$$\begin{aligned} P(Y = 1|X \neq Y) &= \frac{P(X \neq Y|Y = 1)P(Y = 1)}{P(X \neq Y)} && \text{(Bayes rule)} \\ &= \frac{P(X = 0)P(Y = 1)}{P(X \neq Y)} && \text{(using logic)} \\ &= \frac{(0.3)(0.7)}{0.42} \\ &= 1/2 \end{aligned}$$

Since Timothy has a 1/2 chance of taking out the trash, the procedure is fair. (:

b) **Challenge:** Let V be the total number of times a coin will be flipped before somebody takes out the trash. What is the mean and variance of V?

Let U be the number of "trials" which are needed before somebody takes out the trash, i.e. until $X \neq Y$. Thus the "success" probability in this case is 0.42 so $U \sim Geom(0.42)$. There are 2 flips per trial so $V = 2U$.

$$E(V) = E(2U) = 2E(U) = 2\frac{1}{0.42} = 4.76$$

$$Var(V) = Var(2U) = 2^2 Var(U) = 4\frac{1 - 0.42}{0.42^2} = 13.15$$