Exercise set 1 for chapter 3

$m>1$

Problem 1. A discrete random variable X has a probability distribution $f(x_i)$ for x_i in the sample space where $X = \{x_1, x_2, \ldots, x_m\}$. Show the following:

(a) $P(\underbrace{1 < X \leq 5}_{A}) = P(\underbrace{1 < X \leq 2}_{B}) + P(\underbrace{2 < X \leq 5}_{C})$

$A = B \cup C \quad B \cap C = \phi$

$P(A) = P(B) + P(C)$

(b) $P(\underbrace{X > 1}_{A}) = 1 - P(\underbrace{X \leq 1}_{A^c})$

(c) $P(\underbrace{X \leq a}_{A}) \leq P(\underbrace{X \leq b}_{B})$ for $a < b$

$A \subset B \quad P(A) \leq P(B)$

(d) $F(a) \leq F(b)$ for $a < b$ where $F(\cdot)$ is the cumulative distribution function

$a < b$

$P(X \leq a) = P(X \leq b)$

$X = \{1, 2, 3\}$, $a = 1$, $b = 1.5$

$P(X \leq 1) = f(1)$

$P(X \leq 1.5) = f(1)$

(e) $F(x) \neq f(x)$

(f) $F(x_j) - F(x_{j-1}) = f(x_j)$

e) $\begin{cases} F(x_m) = 1 \\ f(x_m) < 1 \end{cases}$

$f(x_1)$

$\begin{cases} F(x) \neq 0 \text{ for } x_1 < X < x_2 \\ f(x) = 0 \end{cases}$

(f) $P(X \leq x_j) - P(X \leq x_{j-1}) = P(x_{j-1} < X \leq x_j)$

$= f(x_j)$

Problem 2. The random variable X has the following probability distribution:

(a) $P(X \leq 3 \text{ or } X > 6)$

(b) CDF $F(x)$

(c) $E(X)$

(d) $Var(X)$

Table 1: Probability distribution

	x_1	x_2	x_3	x_4	x_5
x	2	3	4	5	8
Probability	0.2	0.2	0.2	0.3	0.1
	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$

e) Second moment

(e) Second moment of the distribution

a) $P(\underbrace{X \le 3}_{A} \text{ or } \underbrace{X > 6}_{B})$

$= P(X \le 3) + P(X > 6)$

$= 0.4 + 0.1 = 0.5$

b) $F(x) = \begin{cases} 0 & x < 2 \\ 0.2 & 2 \le x < 3 \\ 0.4 & 3 \le x < 4 \\ 0.6 & 4 \le x < 5 \\ 0.9 & 5 \le x < 8 \\ 1 & x \ge 8 \end{cases}$

c) Mean) $\mu = x_1 f(x_1) + x_2 f(x_2) + \dots x_m f(x_m)$

$= 2 \cdot 0.2 + 3 \cdot 0.2 + 4 \cdot 0.2 + 5 \cdot 0.3 + 8 \cdot 0.1$

$= 4.1$

d) Variance

$\sigma^2 = x_1^2 f(x_1) + x_2^2 f(x_2) + \dots x_m^2 f(x_m) \boxed{-\mu^2}$

$= 2^2 \cdot 0.2 + 3^2 \cdot 0.2 + 4^2 \cdot 0.2 + 5^2 \cdot 0.3 + 8^2 \cdot 0.1 - 4.1^2$

$= 2.89$

Problem 3. A discrete random variable X has a probability distribution $f(x_i)$ for x_i in the sample space where $X = \{x_1, x_2, \dots, x_m\}$. Let μ denote distribution mean and σ^2 denote distribution variance. Show the following:

when the distribution is not uniform $\left(f(x_1) = f(x_2) = \dots f(x_m) = \frac{1}{m}\right)$

(a) $\mu \ne \frac{x_1 + x_2 + \dots x_m}{m}$ in general (you can give a counter example)

(b) $\sigma^2 = x_1^2 f(x_1) + x_2^2 f(x_2) + \dots + x_m^2 f(x_m) - \mu^2$ and hence showing $E(X-\mu)^2 = E(X^2) - \mu^2$

$\sigma^2 = (x_1 - \mu)^2 f(x_1) + (x_2 - \mu)^2 f(x_2) + \dots (x_m - \mu)^2 f(x_m)$

$= [x_1^2 - 2\mu x_1 + \mu^2] f(x_1) + [x_2^2 - 2\mu x_2 + \mu^2] f(x_2) + \dots [x_m^2 - 2\mu x_m + \mu^2] f(x_m)$

$= [x_1^2 f(x_1) + x_2^2 f(x_2) + \dots x_m^2 f(x_m)] - 2\mu \underbrace{[x_1 f(x_1) + \dots x_m f(x_m)]}_{= \mu}$

$+ \mu^2 \underbrace{[f(x_1) + f(x_2) + \dots f(x_m)]}_{1}$

$= x_1^2 f(x_1) + \dots x_m^2 f(x_m) - 2\mu^2 + \mu^2$

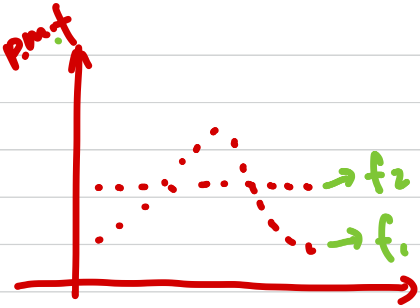
$$E(X-\mu)^2 = E(X^2) - \mu^2$$

$$\parallel$$

$$E\left[X^2 - 2X\mu + \mu^2\right] = E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$



$Var(f_2) > Var(f_1)$

$$\mu = E(X) = \sum_{i=1}^{m} x_i f(x_i)$$

$$\sigma^2 = E(X-\mu)^2 = \sum_{i=1}^{m} (x_i-\mu)^2 f(x_i)$$

e) Second moment

$$E(X^2) = \sum_{i=1}^{m} x_i^2 f(x_i)$$

$$E(g(x)) = \sum_{i=1}^{m} g(x_i) f(x_i)$$

$$E(X^2) = 19.7$$

Problem 4.

Table 2: Probability distribution

x	1	0
Probability	p	$1-p$

(a) The mean and variance of the distribution

(b) What is the maximum value of variance? At what p value?

$= 0.25$ $\quad p = 0.5$

a) $\mu = 1 \cdot p + 0 \cdot (1-p) = p$

$\sigma^2 = 1^2 \cdot p + 0^2 \cdot (1-p) - \mu^2 = \underline{p - p^2} = p(1-p) > 0$

b)