1) a) 25 Bernoulli trials total

b) Each queition has 0.25 proobability to be guessed correctly

c) All questions were guessed independently

$X \sim \text{Binomial}(n = 25,\ p = 0.25)$

$$f(x) = \binom{25}{x} 0.25^{x}\, 0.75^{25-x}$$

$x = 0, 1, \ldots 25$

2) $P(3 \le X \le 10)$

$= f(3) + f(4) + f(5) + \ldots f(10)$

$= 0.938$

3) $E(X) = \overset{n}{25} \overset{*}{\cdot} \overset{p}{0.25} = 6.25$

2') $P(X \geq 12) = 0.011$ (pass with at least D)

Problem 2 p: the probability of "success"

$$f(x) = (1-p)^{x-1} p \quad x = 1, 2, \cdots$$

Define X as the number of people needed to be tested to detect one person with the gene.

a) $P(X = 4) = 0.9^3 \times 0.1 = 0.0729$

b) $P(X \geq 4) = 1 - P(X < 4)$

$= 1 - [f(1) + f(2) + f(3)]$

$= 0.729$

c) $E(X) = \frac{1}{p} = 10 \quad (p = 0.1)$

Problem 3

a) X: the number of calls in one hours

$P(X \le 3)$, $X \sim Poisson(\lambda \cdot T)$, $\lambda \to 10$, $T \to 1$

$= f(0) + f(1) + f(2) + f(3)$

$$f_X(x) = \frac{e^{-\lambda T} (\lambda T)^x}{x!}$$

$x = 0, 1, 2, 3, \ldots$

$$= e^{-10} + e^{-10} \cdot 10 + \frac{e^{-10} \cdot 10^2}{2!} + \frac{e^{-10} \cdot 10^3}{3!}$$

$= 0.0103$

b)

Y: the number of calls in half an hour

$$Y \sim \text{Poisson}(\lambda T)$$

$\lambda \downarrow 10$, $T \downarrow 0.5$

$$P(Y \leq 3) = 0.265$$

Problem 4

X_1 : score of patient 1 ; X_2 : score of patient 2

Y: total score of 2 patients

y	2	3	4	5	6
$f(y)$	$P(X_1=1, X_2=1)$	$P(X_1=1, X_2=2) + P(X_1=2, X_1=1)$	$P(X_1=1, X_2=3) + P(X_1=3, X_2=1) + P(X_1=2)P(X_2=2)$	$P(X_1=1, X_2=4)$ $P(X_1=4, X_2=1)$ $P(X_1=2, X_2=3)$ $P(X_1=3, X_2=2)$	$\vdots$
independence	$P(X_1=1)P(X_2=1) = 0.05^2$	$2 \cdot 0.05 \cdot 0.1$	$2 \cdot 0.05 \cdot 0.2 + 0.1^2$	$2 \cdot 0.05 \cdot 0.25 + 2 \cdot 0.1 \cdot 0.2$	

Problem 5

Uniform distribution density

PDF: $f(x) = \frac{x}{b-a}$, $a < x < b$

CDF: $F(x) = \begin{cases} 0 & x < a \\ \boxed{\frac{x-a}{b-a}} & a < x < b \\ 1 & x > b \end{cases}$

$F'(x) = f(x)$

$F(x) = \int_{-\infty}^{x} f(u)du$

$E(X) = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

p-th percentile of the distribution

$F(c) = p/100$; $c = (b-a) \cdot \frac{p}{100} + a$

CDF

$p \cdot 100\%$ of all observations are below c.

or in other words, the probability for X falls below c is p.

a) $f(x) = 2$ for $49.75 < x < 50.25$

$$P(X > 50) = \int_{50}^{50.25} 2 \, dx$$

$$= 0.5$$

b) $$F(x) = \begin{cases} 0 & x < 49.75 \\ \dfrac{x - 49.75}{0.5} & 49.75 < x < 50.25 \\ 1 & x > 50.25 \end{cases}$$

c) $$c = (50.25 - 49.75) \cdot \frac{90}{100} + 49.75$$

$$= 0.45 + 49.75$$

$$= 50.2$$

d) $\mu = 50$, $\sigma^2 = \dfrac{0.5^2}{12}$

Problem 6

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

μ: distribution mean

σ^2: distribution variance

$X \sim N(\mu, \sigma^2)$

50th percentile (Median) is just μ

$$M(t) = e^{\mu t + \sigma^2 t^2/2}$$

1) Find $P(a < X < b)$ using the Normal table

2) Find p-th percentile using the Normal table

1) $P(a < X < b)$ let $Z = \frac{X-\mu}{\sigma}$

$= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$

$= P\left(Z < \frac{b-\mu}{\sigma}\right) - P\left(Z < \frac{a-\mu}{\sigma}\right)$

$= \Phi_{0,1}\left(\frac{b-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{a-\mu}{\sigma}\right)$

2) $P(X < c) = \frac{p}{100}$

$\rightarrow \quad c = z_p \cdot \sigma + \mu$

p-th percentile of standard Normal distribution

a) X: the weight of a running shoe

$X \sim N(12, 0.5^2)$

$P(X > 13) = 1 - P(X < 13)$

$= 1 - \Phi_{0,1}\left(\frac{13-12}{0.5}\right)$

$= 1 - \Phi_{0,1}(2)$

$= 1 - 0.97725$

$= 0.02275$

b) $P(\underbrace{\{X > 13\}}_{B} \mid \underbrace{\{X > 10\}}_{A})$ $\quad B \cap A = \{X > 13\}$

$\rightarrow P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(X > 13)}{P(X > 10)} \approx \frac{0.02275}{1}$

≈ 0.02275

c) 13 is the 99.9 percentile

$$13 = z_{99.9} \cdot \sigma + 12$$

$$13 = 3.09 \cdot \sigma + 12$$

$$\sigma = 0.3236$$

d) $13 = z_{99.9} \cdot 0.5 + \mu$

$$13 = 3.09 \cdot 0.5 + \mu$$

$$\mu = 13 - 1.545$$

$$= 11.455$$

Problem 7

$$\Phi_{0,1}'(x) = \phi_{0,1}(x)$$

$$X \sim N(\mu, \sigma^2)$$

$$\phi_{0,1}(x) = \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$

$$Y = e^X$$

1) Find out the CDF of Y : $F_Y(y)$

2) $f_Y(y) = F_Y'(y)$

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y)$$

$$= P(X \leq \log(y))$$

$$= \boxed{\Phi_{0,1}}\left(\frac{\log(y)-\mu}{\sigma}\right)$$

density of standard normal distribution

$$f_Y(y) = \phi_{0,1}\left(\boxed{\frac{\log(y)-\mu}{\sigma}}\right) \cdot \frac{1}{y\sigma}$$

Bernoulli distribution

Assumption: the random variable has two outcomes

Binomial distribution

Assumptions: 1) the random variable (X) is defined as the total number of success out of a <u>fixed</u> number N Bernoulli trials (each trial has exactly 2 outcomes

2) The Bernoulli trials are identical and independent

$X \sim$ Biomial(N, p) → the probability of success for each trial

3) Geometric distribution

Assumption 1): X is defined as the number of Bernoulli trials till the first success.

2) The Bernoulli trials are identical and independent

$X \sim \text{Geometric}(p)$: p is the probability of success for each trial

4) Poisson distribution

Assumption 1) X is defined as the number of events over a fixed number of units of time / length / space ...

2) ...

$X \sim \text{Poisson}(\lambda T)$

λ: the average number of events over 1 unit of time / space / length ...

T: the total number of units that the probability problem is associated with