HW 3 solutions

Problem 1:

a) $p(2 \le X < 4) = f(2) + f(3) = \dfrac{2\cdot2+1}{25} + \dfrac{2\cdot3+1}{25} = \dfrac{12}{25} = 0.48$

b) $$F(x) = \begin{cases} 0 & x<0 \\ \frac{1}{25} & 0 \le x < 1 \\ \frac{4}{25} & 1 \le x < 2 \\ \frac{9}{25} & 2 \le x < 3 \\ \frac{16}{25} & 3 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

c) μ or $E(X) = 0 \cdot \frac{1}{25} + 1 \cdot \frac{3}{25} + 2 \cdot \frac{5}{25} + 3 \cdot \frac{7}{25} + 4 \cdot \frac{9}{25}$

$= \frac{70}{25} = 2.8$

d) σ^2 or $Var(X) = E(X-2.8)^2 = (0-2.8)^2 \cdot \frac{1}{25} + (1-2.8)^2 \cdot \frac{3}{25} + (2-2.8)^2 \cdot \frac{5}{28}$

$+ (3-2.8)^2 \cdot \frac{7}{25} + (4-2.8)^2 \cdot \frac{9}{25}$

$= 1.36$

or $Var(X) = E(X^2) - \mu^2 = 0^2 \cdot \frac{1}{25} + 1^2 \cdot \frac{3}{25} + 2^2 \cdot \frac{5}{28} + 3^2 \cdot \frac{7}{25} + 4^2 \cdot \frac{9}{25}$

$- 2.8^2$

$= 9.2 - 2.8^2 = 1.36$

Problem 2

a) X follows a Binomial distribution with

$n = 5, \; p = 0.05$

$p(x) = \binom{5}{x} 0.05^{x} \, 0.95^{5-x}$, $x = 0, 1, 2, 3, 4, 5$

If this is missing, you only receive 0.1 point for this part.

b) $\mu = n \cdot p = 0.25$

c) $\sigma^2 = n \cdot p \cdot (1-p) = 0.2375$

d) $p(\text{less than 2 defective fuses}) = p(X<2)$

$= f(0) + f(1)$

$\simeq 0.774 + 0.204$

$= 0.978$

e) $p(\text{greater or equal to 2 defective fuses}) = p(X \geq 2)$

$= 1 - p(X<2)$

$= 1 - 0.978$

$= 0.012$

problem 3.

a) $P(X \le 45) = F(45) = 1$

b) $P(35 \le X \le 55) = P(X \le 55) - P(X < 35)$

$= F(55) - P(X \le 35)$ → Since the CDF is constant before and after 35, i.e. CDF is constant over 25 and 44.999

$= 1 - 0.7$

$= 0.3$

c) Since CDF has jumps at -15, 25, and 45

$$f(x) = \begin{cases} 0.15, & x = -15 \\ 0.70 - 0.15 = 0.55, & x = 25 \\ 1 - 0.7 = 0.3, & x = 45 \end{cases}$$

i.e.

x	-15	25	45
$f(x)$	0.15	0.55	0.3

Problem 4

a) X follows a Binomial distribution with $n = 125$,

$p = 0.95$ (the probability of showing up for each passenger)

$$f(x) = \binom{125}{x} 0.95^{x} \cdot 0.05^{125-x}, \quad \underline{\underline{x = 0, 1, 2, \cdots 125}}$$

If you miss this, you ~~only get~~ miss 0.25 credit.

b) $P(X \le 115) = 1 - P(X > 115)$

$= 1 - f(116) - f(117) - f(118) - f(119) - f(120)$

$- f(121) - f(122) - f(123) - f(124) - f(125)$

$= 0.097$

c) $E(X) = n \cdot p = 125 \cdot 0.95 = 118.75$

Problem 5.

a) X follows a Geometric distribution with $p = 0.03$

$f(x) = 0.97^{x-1} \cdot 0.03, \quad \underline{\underline{x = 1, 2, 3, \cdots}}$

b) $P(X=10) = f(10) = 0.97^{9} \cdot 0.03 = 0.0228$

c) P(It requires more than 5 calls to connect)

$= P(X > 5) = 1 - P(X \leq 5)$

$= 1 - f(1) - f(2) - f(3) - f(4) - f(5)$

$= 0.8329$

Problem 6

(a) $$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda T}(\lambda T)^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{(e^t \cdot \lambda T)^x}{x!} \cdot e^{-\lambda T}$$

$$= e^{e^t \cdot \lambda T} \cdot e^{-\lambda T}$$

$$= e^{\lambda T(e^t - 1)}$$

(b) $$M'(t) = e^{\lambda T(e^t-1)} \cdot \lambda T \cdot e^t$$

$M'(0) = \lambda T$; Hence $\mu = \lambda T$

(c) $$M''(t) = e^{\lambda T(e^t-1)} \cdot (\lambda T)^2 \cdot e^t + e^{\lambda T(e^t-1)} \cdot \lambda T e^t$$

$M''(0) = (\lambda T)^2 + \lambda T$

Hence $\sigma^2 = M''(0) - (M'(0))^2 = \lambda T$

Problem 7

a) Assume X has sample space $\{x_1, x_2, \dots x_n\}$ and probability mass function $f(x)$. ← This is valid since we assumed X is a discrete random variable

$$E(Y) = E(X+1) = \sum_{i=1}^{n} (x_i+1)\cdot f(x_i)$$

$$= \sum_{i=1}^{n} x_i f(x_i) + \sum_{i=1}^{n} 1 \cdot f(x_i)$$

$$= E(X) + 1$$

$$= \mu + 1$$

(b) Following the same assumption in part (a),

$$Var(Y) = Var(X+1)$$

$$= E[(X+1)^2] - [E(X+1)]^2$$

$$= \sum_{i=1}^{n} (x_i+1)^2 f(x_i) - (\mu+1)^2$$

$$= \sum_{i=1}^{n} x_i^2 f(x_i) + 2\sum_{i=1}^{n} x_i f(x_i) + \sum_{i=1}^{n} f(x_i) - \mu^2 - 2\mu - 1$$

$$= \underbrace{\sum_{i=1}^{n} x_i^2 f(x_i)}_{=} + 2\mu + 1 - \mu^2 - 2\mu - 1$$

$$= \sigma^2 - \mu^2 + \mu^2 = \sigma^2$$

Hence $Var(Y)$ is the same as $Var(X)$

— Second way to show it is on the back —

$$Var(Y) = Var(X+1)$$

$$= E[(X+1) - E(X+1)]^2$$

$$= E[X+1 - E(X) - 1]^2$$

$$= E(X - E(X)]^2$$

$$= Var(X)$$