Homework 4

Please complete the problems on a separate sheet of paper with your name at the top. Make sure to show your work and/or provide an explanation for each problem. Please be clear in your work. Partial credit will be given when merited. The total credit is 8 points. The bonus problem worths 2 points.

Problem 1. Assume that X is a continuous random variable with the following *pdf*:

$$f(x) = \begin{cases} x+1 & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Derive the CDF of X. (0.5 point)

(b) Derive the mean of X. (0.5 point)

(c) Derive the variance of X. (0.5 point)

(d) Derive the 50th percentile of the distribution. (0.5 point)

Problem 2. Random variable X has the pdf $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.

(a) Derive the CDF of X. (0.5 point)

(b) Derive the moment generating function of X. (0.5 point)

(c) Derive the mean of X. (0.5 point)

(d) Derive the variance of X. (0.5 point)

(e) Find the 50th percentile of the distribution (0.5 point)

Problem 3 The cumulative distribution function of random variable X is

$$F(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 1 \\ 1. & x \ge 1 \end{cases}$$

Problem 1

a)

$$F(x) = \begin{cases} 0 & x < -1 \\ \int_{-1}^{x} (u+1)\,dx = \left(\frac{u^2}{2} + u\right)\Big|_{-1}^{x} = \underline{\frac{x^2}{2} + x + \frac{1}{2}}; & -1 < x < 0 \\ \int_{-1}^{0} (u+1)\,dx + \int_{0}^{x} (1-u)\,du = \frac{1}{2} + \left(u - \frac{u^2}{2}\right)\Big|_{0}^{x} & \\ \qquad = \underline{\frac{1}{2} + x - \frac{x^2}{2}} & 0 < x < 1 \\ & x > 1 \end{cases}$$

b) $\mu = E(X) = \int_{-1}^{0} x\underline{(x+1)}\,dx + \int_{0}^{1} x(1-x)\,dx$

$$= \int_{-1}^{0} (x + x^2)\,dx + \int_{0}^{1} (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{1}$$

$$= 0 - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - 0$$

$$= 0$$

c) $Var(x) = E(x^2) - \mu^2$

Since $\mu = 0$,

$$Var(X) = \int_{-1}^{0} x^2(x+1)\,dx + \int_{0}^{1} x^2(1-x)\,dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{-1}^{0} + \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{0}^{1}$$

$$= 0 - \left(\frac{1}{4} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

d) Since $F(0) = 0.5$, the 50th percentile is 0.5

Problem 2

a) $F(x) = \begin{cases} 0 & x < 0 \\ \int_0^x \lambda e^{-\lambda u} du = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$

b) $M(t) = E(e^{tX})$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty \lambda e^{(t-\lambda)x} dx$$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty = 0 - \frac{\lambda}{t-\lambda}$$

$$= \frac{\lambda}{\lambda - t} \qquad (t < \lambda)$$

c) $\mu = M'(0)$

$M'(t) = \frac{\lambda}{(\lambda-t)^2}$, Hence $\mu = \frac{1}{\lambda}$

d) $M''(t) = \frac{2\lambda}{(\lambda-t)^3}$

$M''(0) = \frac{2}{\lambda^2}$

Therefore $\sigma^2 = M''(0) - [M'(0)]^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

e) Solve $F(c) = 0.5$

$1 - e^{-\lambda c} = 0.5$

$-\lambda c = \log(0.5)$

$c = - \frac{0.693}{\lambda}$

(a) What is $P(|X| \leq 0.5)$? (0.5 point)

(b) What is the density (pdf) of the distribution? (0.5 point)

Problem 4 Let $X \sim N(\mu, \sigma^2)$. Find the following probabilities:

(a) $P(-2\sigma + \mu < X < 2\sigma + \mu)$. (0.25 point)

(b) $P(-\sigma + \mu < X < 3\sigma + \mu)$. (0.25 point)

(c) $P(1 < X < 3)$ (0.25 point)

(d) 50th percentile of the distribution (0.25 point)

Problem 5 The time until recharge for a battery in a laptop computer under common conditions is normally distributed with mean of 275 minutes and a standard deviation of 50 minutes.

(a) What is the probability that a battery lasts more than four hours? (0.25 point)

(b) What are the quartiles (the 25% and 75% values) of battery life? (0.5 point)

(c) **Given that a battery already lasts four hours, what is the probability that it lasts at least another two hours?** (0.25 point) Hint: denote B as a battery already lasts four hours $B = \{X > 240 minutes\}$ and A as a battery lasts at least 6 hours $A = \{X > 360 minutes\}$. The question is then: what is $P(A|B)$ (A given B)?

Problem 6. Random variable X has the density function (pdf) $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Find the pdf of $Y = \log(X)$. (1 point)

Bonus problem. Random variable X has the density function (pdf) $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Denote event A as $X < 3$.

(a) What is the probability of A? (0.5 point)

(b) What is the conditional probability of $X < x$ given that A happens. Here assume $x < 3$ (0.5 point)

(c) Denote the conditional probability in part (b) as $P(X < x|A)$. Find the derivative of $P(X < x|A)$, which is denoted by $f_{X|A}(x)$. Here, $f_{X|A}(x)$ is referred as a conditional density. (0.5 point)

(d) **Find the expected value (i.e. the mean)** of the conditional density in part (c). (0.5 point)

Problem 3

a) $p(|X| < 0.5) = p(-0.5 < X < 0.5)$

$= F(0.5) - F(-0.5)$

$= \left(\frac{0.5+1}{2}\right) - \left(\frac{-0.5+1}{2}\right)$

$= 0.5$

Both 0.5 and -0.5 are between -1 and 1

b) $f(x) = F'(x) = \begin{cases} \frac{1}{2} & -1 < X < 1 \\ 0 & \text{elsewhere} \end{cases}$

Problem 4

a) $p(-2\sigma + \mu < X < 2\sigma + \mu)$

$= P\left(-2 < \frac{X-\mu}{\sigma} < 2\right)$

$= \Phi_{0,1}(2) - \Phi_{0,1}(-2) \approx 0.95$

2) $P(-\sigma+\mu < X < 3\sigma+\mu)$

$= \Phi(3) - \Phi(-1)$

$= 0.9986 - 0.1586$

$= 0.840$ (It is okay to have approxiate answers)

3) $P(-1 < X < 3)$

$= P\left(\frac{-1-\mu}{\sigma} < X < \frac{3-\mu}{\sigma}\right)$

$= \Phi_{0,1}\left(\frac{3-\mu}{\sigma}\right) - \Phi_{0,1}\left(\frac{-1-\mu}{\sigma}\right)$

4) $F(c) = 0.5$

$c = \mu + \underbrace{Z_{0.5}}_{=0}\sigma = \mu$

Problem 5

Denote X as the time until recharge for a randomly selected battery.

a) $P(X > 4 \cdot 60) = P\left(\frac{X-275}{50} > \frac{240-275}{50}\right)$

$= 1 - \Phi_{0,1}(-0.7)$

$= 0.758$

b) Let C_1 be the 25% percentile and C_2 be the 75% pecentile.

$C_1 = \mu + Z_{0.25} \cdot 6 = 275 + (-0.674) \cdot 50$

$= 241.3$

$C_2 = 275 + 0.674 \cdot 50$

$= 308.7$

c) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(X > 360)}{P(X > 240)} = \frac{1 - \Phi_{0,1}(1.7)}{0.758}$

$= 0.059$

Problem 6

$$F_Y(y) = P(Y<y) = P(\log(x) < y)$$

$$= P(x < e^y)$$

$$= \int_0^{e^y} \lambda e^{-\lambda x} \, dx$$

$$= \left(-e^{-\lambda x}\right)\Big|_0^{e^y}$$

$$= 1 - e^{-\lambda e^y}$$

$$f_Y(y) = F_Y'(y) = \lambda e^y e^{-\lambda e^y}, \quad -\infty < y < \infty$$

Bonus Problem

(a) $$P(A) = P(X<3) = \int_0^3 \lambda e^{-\lambda x} \, dx$$

$$= 1 - \lambda e^{-3\lambda}$$

(b) $P(X<x \mid A)$

$$= P(X<x \mid X<3)$$

$$= \frac{P(\{X<x\} \cap \{X<3\})}{P(X<3)}$$ since $x<3$

$$= \frac{P(X<x)}{P(X<3)} = \frac{1-e^{-\lambda x}}{1-e^{-3\lambda}}$$

taking derivation of this with respect to x

c) $$f_{X|A}(x) = \frac{\lambda e^{-\lambda x}}{1-e^{-3\lambda}}, \quad 0<x<3$$

d) $$E(X|A) = \int_0^3 x \cdot \frac{\lambda e^{-\lambda x}}{1-e^{-3\lambda}}\, dx$$

$$= \frac{\lambda}{1-e^{-3\lambda}} \int_0^3 x e^{-\lambda x}\, dx$$

Integration by part

$$= \frac{\lambda}{1-e^{-3\lambda}} \left[\left(-\frac{1}{\lambda} \cdot x e^{-\lambda x}\right)\Big|_0^3 + \int_0^3 \frac{1}{\lambda} e^{-\lambda x}\, dx \right]$$

$$= \frac{\lambda}{1-e^{-3\lambda}} \left[-\frac{3}{\lambda} e^{-3\lambda} + \left(-\frac{1}{\lambda^2} e^{-\lambda x}\Big|_0^3\right) \right]$$

$$= \frac{\lambda}{1-e^{-3\lambda}}\left[-\frac{3}{\lambda}e^{-3\lambda} - \frac{1}{\lambda^2}e^{-3\lambda} + \frac{1}{\lambda^2}\right]$$

$$= \frac{1}{1-e^{-3\lambda}}\left[-3e^{-3\lambda} - \frac{1}{\lambda}e^{-3\lambda} + \frac{1}{\lambda}\right]$$