Problem 1

a) Likelihood is $L(\lambda) = \prod_{i=1}^{n} \left(\frac{1}{\lambda} e^{-\frac{x_i}{\lambda}}\right)$

$$= \left(\frac{1}{\lambda}\right)^n e^{-\frac{1}{\lambda}\sum_{i=1}^{n} x_i}$$

The Log of the likelihood is

$$\log(L(\lambda)) = -n\log(\lambda) - \frac{1}{\lambda}\sum_{i=1}^{n} x_i$$

The Maximum likelihood estimator maximizes $\log(L(\lambda))$. To obtain it, we solve $\frac{d\log L(\lambda)}{d\lambda} = 0$ and check if $\left.\frac{d^2\log L(\lambda)}{d^2\lambda}\right|_{\lambda = \hat{\lambda}_{MLE}} < 0$.

$\frac{d\log L(\lambda)}{d\lambda} = -\frac{n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} x_i$; setting it to zero we solve $\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n}$

$$\frac{d^2\log L(\lambda)}{d\lambda^2} = \frac{n}{\lambda^2} - \frac{2}{\lambda^3}\sum_{i=1}^{n} x_i = \frac{1}{\lambda^2}\left(n - \frac{2}{\lambda}\sum_{i=1}^{n} x_i\right)$$

$$\left.\frac{d^2\log L(\lambda)}{d\lambda^2}\right|_{\hat{\lambda}_{MLE}} = \frac{1}{\hat{\lambda}_{MLE}^2}\left(n - \frac{2n}{\sum_{i=1}^{n} x_i} \cdot \sum_{i=1}^{n} x_i\right) = \frac{1}{\hat{\lambda}_{MLE}^2} \cdot (-n) < 0$$

Hence $\frac{\sum_{i=1}^{n} x_i}{n}$ is indeed the maximum likelihood estimator

b) $E(\hat{\lambda}_{MLE}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$E(X_i) = M'(0)$

$M(t) = \int_0^\infty e^{tx} \cdot \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx$

$= \frac{1}{\lambda}\int_0^\infty e^{x(t-\frac{1}{\lambda})} dx$ for $t < \frac{1}{\lambda}$

$= \frac{1}{\lambda} \cdot \frac{1}{t-\frac{1}{\lambda}} e^{x(t-\frac{1}{\lambda})} \Big|_{x=0}^{\infty}$

$= -\frac{1}{\lambda} \cdot \frac{1}{t-\frac{1}{\lambda}} = \frac{-1}{\lambda t - 1} = \frac{1}{1-\lambda t}$

$M'(t) = -\frac{1}{(1-\lambda t)^2} \cdot (-\lambda) = \frac{\lambda}{(1-\lambda t)^2}$

Hence $M'(0) = \lambda$

Hence $E(X_i) = \lambda$ and $E(\hat{\lambda}_{MLE}) = \lambda$. Therefore $\hat{\lambda}_{MLE}$ is unbiased.

c) $MSE = E(\hat{\lambda}_{MLE} - \lambda)^2 = Var(\hat{\lambda}_{MLE}) = Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)$

$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$

$$Var(X_i) = M''(0) - M'(0)^2$$

$$M''(t) = \left(\frac{\lambda}{(1-\lambda t)^2}\right)' = -2\lambda \cdot \frac{1}{(1-\lambda t)^3} \cdot (-\lambda)$$

$$= \frac{2\lambda^2}{(1-\lambda t)^3}$$

Hence $M''(0) = 2\lambda^2$, $Var(X_i) = 2\lambda^2 - \lambda^2 = \lambda^2$

Hence $MSE = Var(\hat{\lambda}_{MLE}) = \frac{1}{n^2}(\overbrace{\lambda^2 + \lambda^2 + \cdots \lambda^2}^{n \text{ of them}})$

$$= \frac{\lambda^2}{n}$$

Problem 2.

a) $\left[\bar{x} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}, \ \bar{x} + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right]$

$Z_{\alpha/2}$: $100(1-\frac{\alpha}{2})$ percentile of standard Normal distribution

$\bar{x}$: sample mean

σ: standard deviation of the population

n: sample size

b) The 95% CI is $\left[\bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \ \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right]$

$$= \left[98 - 1.96 \cdot \frac{2}{\sqrt{9}}, \ 98 + 1.96 \cdot \frac{2}{\sqrt{9}}\right]$$

$= [96.69, 99.31]$

Interpretation: with 95% confidence, the mean breaking strength of the yarn is between 96.69 and 99.31 psi.

c) The 99% CI is $\left[\bar{x} - 2.58 \cdot \frac{\sigma}{\sqrt{n}},\ \bar{x} + 2.58 \cdot \frac{\sigma}{\sqrt{n}}\right]$

$= [96.28, 99.72]$

d) $$n \geq \left(\frac{Z_{\frac{\alpha}{2}} \cdot \sigma}{E}\right)^2 = \left(\frac{Z_{0.025} \cdot 2}{1}\right)^2 = (1.96 \cdot 2)^2 = 15.37$$

Hence the smallest sample size is 16.

Problem 3.

a) CI formula for proportion is

$$\left[\hat{p} - Z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + Z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$\hat{p}$: sample proportion

$Z_{\alpha/2}$: $100(1-\frac{\alpha}{2})$ percentile of standard Normal distribution

n: sample size

b) 95% CI is

$$\left[0.823 - 1.96\sqrt{\frac{0.823 \cdot 0.177}{1000}}\ ,\ 0.823 + 1.96\sqrt{\frac{0.823 \cdot 0.177}{1000}}\right]$$

$$= [0.499, 0.547]$$

c) p from this sample is 0.823

$$n \geq \left(\frac{z_{\frac{\alpha}{2}}}{E}\right)^2 p(1-p) = \left(\frac{1.96}{0.03}\right)^2 \cdot 0.823 \cdot 0.177$$

$$= 621.8$$

Hence the smallest sample size is 622

d) using $p = 0.5$

$$n \geq \left(\frac{1.96}{0.03}\right)^2 \cdot 0.5 \cdot 0.5 = 1067.11$$

Hence the smallest sample size is 1068.