STAT481/581: Introduction to Time Series Analysis

Ch3. The forecasters' toolbox

OTexts.org/fpp3/

Outline

Outline

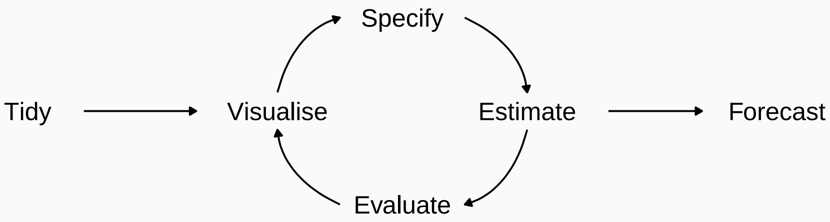
1. A tidy forecasting workflow
2. Some simple forecasting methods
3. The workflow in action
4. Transformations
5. Distributional forecasts

A tidy forecasting workflow

The process of producing forecasts can be split up into a few fundamental steps.

1. Preparing data
2. Data visualisation
3. Specifying a model
4. Model estimation
5. Accuracy & performance evaluation
6. Producing forecasts

A tidy forecasting workflow



Outline

Some simple forecasting methods

How would you forecast these series?

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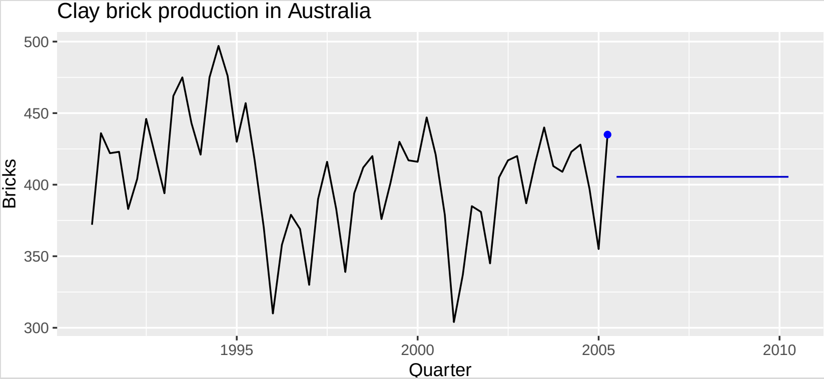
Some simple forecasting methods

`MEAN(y)`: Average method

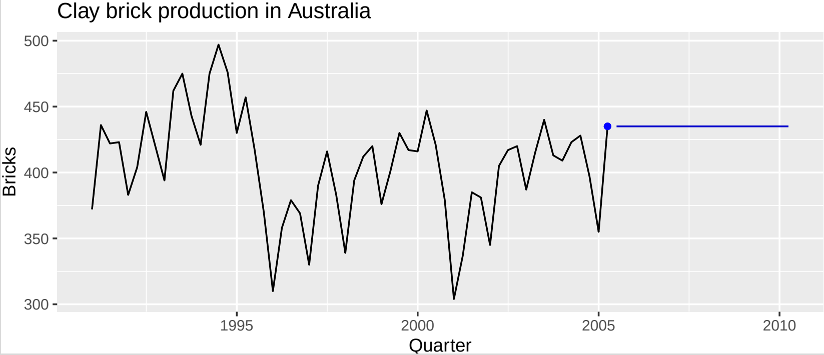
- Forecast of all future values is equal to mean of historical data $\{y_1, \dots, y_T\}$.
- Forecasts: $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \cdots + y_T)/T$



Some simple forecasting methods

`NAIVE(y)`: Naïve method

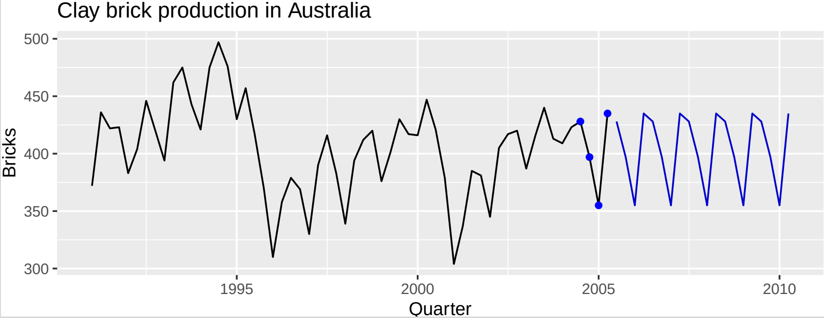
- Forecasts equal to last observed value.
- Forecasts: $\hat{y}_{T+h|T} = y_T$.



Some simple forecasting methods

`SNAIVE(y ~ lag(m))`: Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts: $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$, where $m =$ seasonal period and k is the integer part of $(h-1)/m$.



Some simple forecasting methods

`RW(y ~ drift())`: Drift method

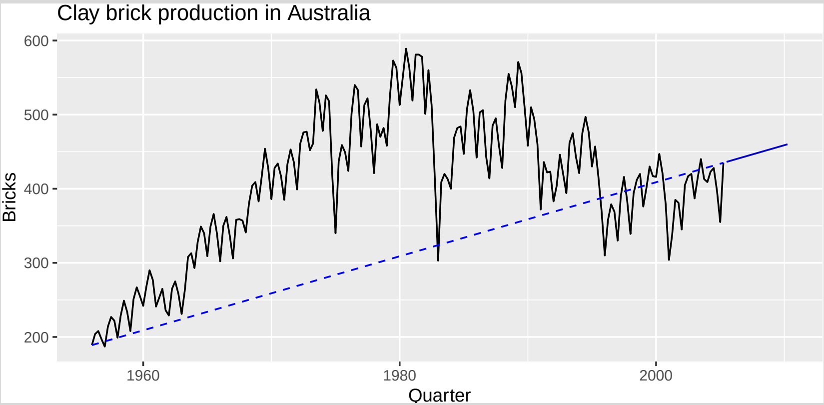
- Forecasts equal to last value plus average change.
- Forecasts:

$$\hat{y}_{T+h|T} = y_T + \frac{h}{T-1}\sum_{t=2}^{T}(y_t - y_{t-1})$$
$$= y_T + \frac{h}{T-1}(y_T - y_1).$$

- Equivalent to extrapolating a line drawn between first and last observations.

Some simple forecasting methods

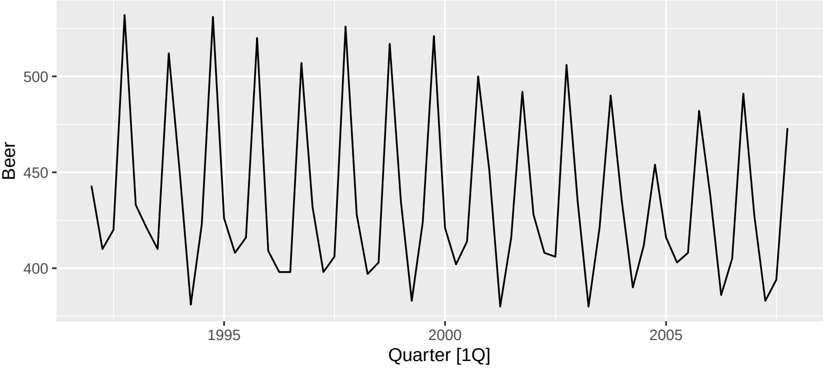
Drift method



Outline

Data preparation and visualisation

```r
# Set training data from 1992 to 2007
train <- aus_production %>%
  filter(between(year(Quarter), 1992, 2007))
train %>% autoplot(Beer)
```



Model estimation

The `model()` function trains models to data.

```
# Fit the models
beer_fit <- train %>%
  model(
    Mean = MEAN(Beer),
    Naïve = NAIVE(Beer),
    Seasonal naïve = SNAIVE(Beer), #or SNAIVE(Beer~lag(4))
    # or SNAIVE(Beer~lag("year"))
    Drift = RW(Beer ~ drift())
  )
```

Model estimation

```
beer_fit
```

```
## # A mable: 1 x 4
##   Mean    Naïve   Seasonal naïve Drift
##   <model> <model> <model>          <model>
## 1 <MEAN>  <NAIVE> <SNAIVE>         <RW w/ drift>
```

A `mable` is a model table, each cell corresponds to a fitted model.

Producing forecasts

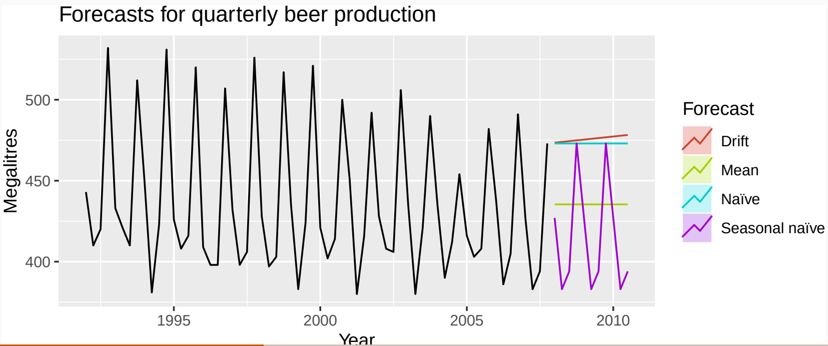
```
beer_fc <- beer_fit %>%
  forecast(h = 11)
```

```
## # A fable: 44 x 4 [1Q]
## # Key:     .model [4]
##   .model Quarter  Beer .distribution
##   <chr>    <qtr> <dbl> <dist>
## 1 Mean   2008 Q1  435. N(435, 1964)
## 2 Mean   2008 Q2  435. N(435, 1964)
## 3 Mean   2008 Q3  435. N(435, 1964)
## 4 Mean   2008 Q4  435. N(435, 1964)
## # ... with 40 more rows
```

A `fable` is a forecast table with point forecasts and distributions.

Visualising forecasts

```r
beer_fc %>%
  autoplot(train, level = NULL) +
  #level=NULL means no prediction interval
  ggtitle("Forecasts for quarterly beer production") +
  xlab("Year") + ylab("Megalitres") +
  guides(colour=guide_legend(title="Forecast"))
```



Facebook closing stock price

```r
# Extract training data
fb_stock <- gafa_stock %>%
  group_by(Symbol) %>%
  mutate(trading_day = row_number()) %>%
  update_tsibble(index=trading_day, regular=TRUE) %>%
  filter(Symbol == "FB",
         between(Date, ymd("2018-01-01"), ymd("2018-09-01")))
# Specify, estimate and forecast
fb_stock %>%
  model(
    Mean = MEAN(Close),
    Naïve = NAIVE(Close),
    Drift = RW(Close ~ drift())
  ) %>%
  forecast(h=42) %>%
  autoplot(fb_stock, level = NULL) +
  ggtitle("Facebook closing stock price (daily ending Sep 2018)") +
  xlab("Day") + ylab("") +
  guides(colour=guide_legend(title="Forecast"))
```

Facebook closing stock price

Your turn

- Produce forecasts from the appropriate method for Amazon closing price (`gafa_stock`) and Australian takeaway food turnover (`aus_retail`).
- Plot the results using `autoplot()`.

Outline

Variance stabilization

If the data show different variation at different levels of the series, then a transformation can be useful.

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Denote original observations as $y_1, \dots, y_n$ and transformed observations as $w_1, \dots, w_n$.

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Mathematical transformations for stabilizing variation

Square root	$w_t = \sqrt{y_t}$	$\downarrow$
Cube root	$w_t = \sqrt[3]{y_t}$	Increasing
Logarithm	$w_t = \log(y_t)$	strength

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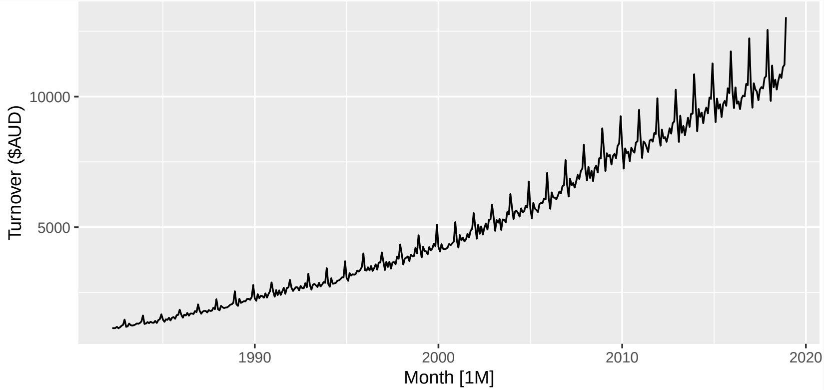
Logarithm	$w_t = \log(y_t)$	strength

Logarithms, in particular, are useful because they are more interpretable: changes in a log value are **relative (percent) changes on the original scale**.

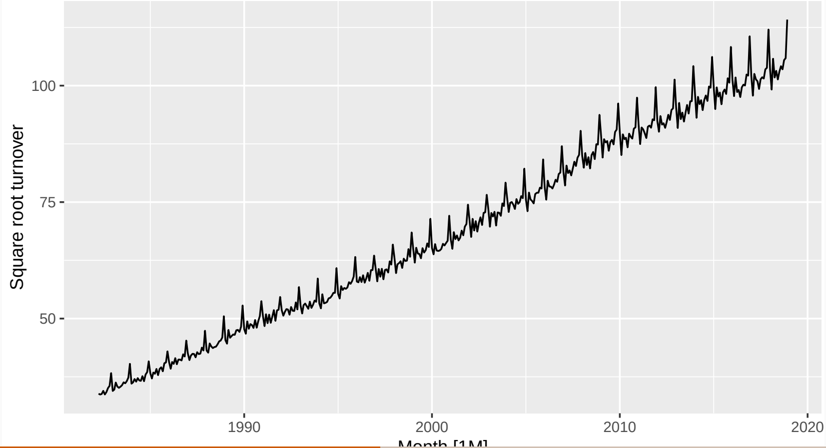
Variance stabilization

```r
food <- aus_retail %>%
  filter(Industry == "Food retailing") %>%
  summarise(Turnover = sum(Turnover))
```



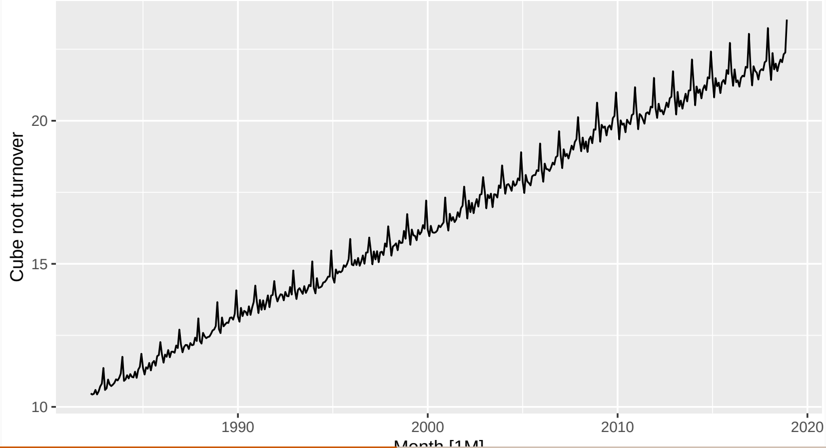
Variance stabilization

```r
food %>% autoplot(sqrt(Turnover)) +
  labs(y = "Square root turnover")
```



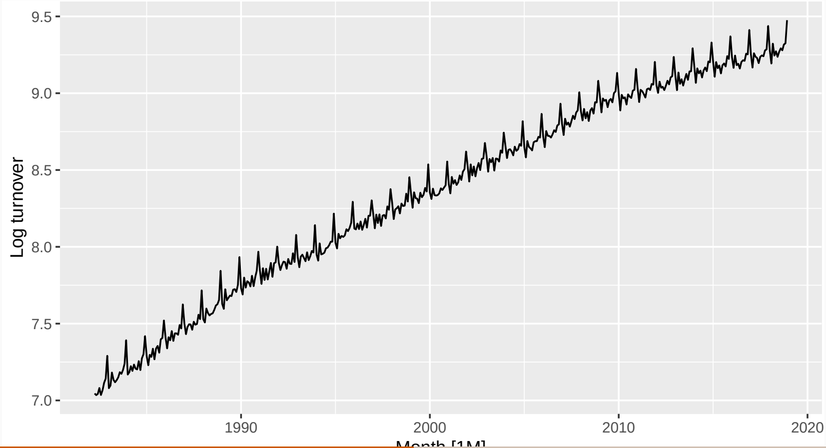
Variance stabilization

```
food %>% autoplot(Turnover^(1/3)) +
  labs(y = "Cube root turnover")
```



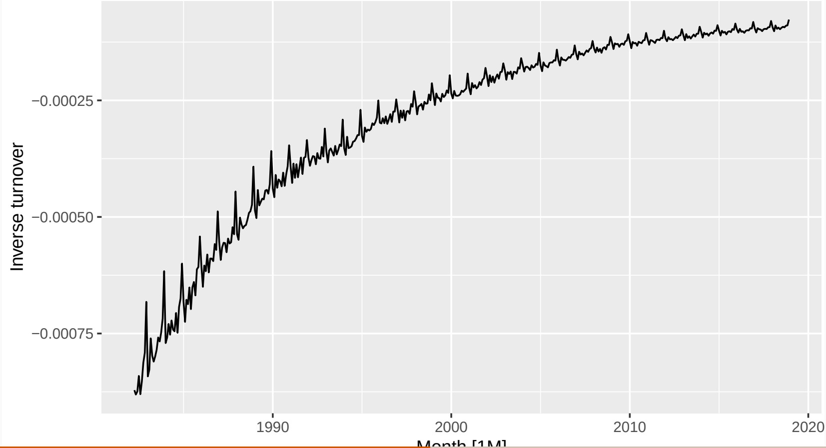
Variance stabilization

```r
food %>% autoplot(log(Turnover)) +
  labs(y = "Log turnover")
```



Variance stabilization

```r
food %>% autoplot(-1/Turnover) +
  labs(y = "Inverse turnover")
```



Box-Cox transformations

Each of these transformations is close to a member of the family of **Box-Cox transformations**:

$$w_t = \begin{cases} \log(y_t), & \lambda = 0; \\ (y_t^\lambda - 1)/\lambda, & \lambda \neq 0. \end{cases}$$

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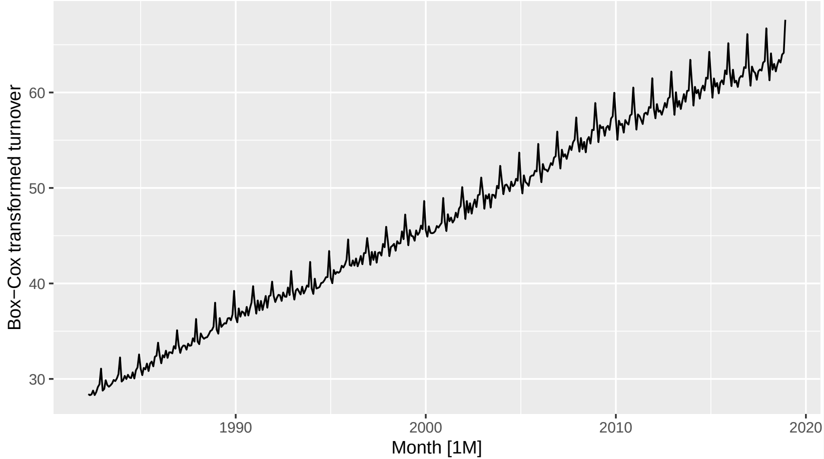
$$w_t = \begin{cases} \log(y_t), & \lambda = 0; \\ (y_t^\lambda - 1)/\lambda, & \lambda \neq 0. \end{cases}$$

- $\lambda = 1$: (No substantive transformation)
- $\lambda = \frac{1}{2}$: (Square root plus linear transformation)
- $\lambda = 0$: (Natural logarithm)
- $\lambda = -1$: (Inverse plus 1)

Box-Cox transformations

```
food %>% autoplot(box_cox(Turnover, 1/3)) +
  labs(y = "Box-Cox transformed turnover")
```



Box-Cox transformations

- y_t^λ for λ close to zero behaves like logs.
- If some $y_t = 0$, then must have $\lambda > 0$
- if some $y_t < 0$, no power transformation is possible unless all y_t adjusted by **adding a constant to all values**.
- Simple values of λ are easier to explain.
- Results are relatively insensitive to λ.
- Often no transformation ($\lambda = 1$) needed.
- Transformation can have very large effect on PI.
- Choosing $\lambda = 0$ is a simple way to force forecasts to be positive

Box-Cox transformations

```r
food %>%
  features(Turnover, features = guerrero)
```

```
## # A tibble: 1 x 1
##   lambda_guerrero
##             <dbl>
## 1          0.0524
```

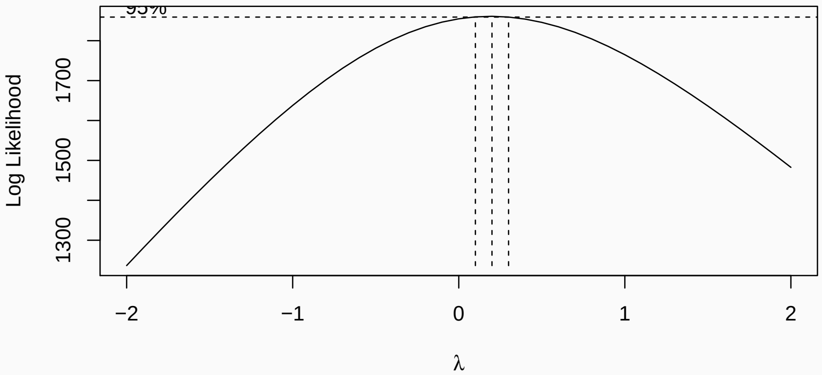
```r
# it uses the BoxCoxLambda function in the forecast package
# use the guerrero function for an automated approach
# Guerrero, V.M. (1993) Time-series analysis supported by
# power transformations.
# Journal of Forecasting, 12, 37--48.
```

Box-Cox transformations

- The `guerrero` approach attempts to balance the seasonal fluctuations and random variation across the series.
- Always check the results.
- A low value of λ can give extremely large prediction intervals.

Box-Cox transformations

```
#use library(TSA)
lambda.fit <- BoxCox.ar(food$Turnover)
```



Box-Cox transformations

`BoxCox.ar` is to estimate the power transformation so that the transformed time series is approximately a Gaussian AR process (no seasonal/cyclic pattern or trend is allowed). Hence the food turner over data is not appropriate here.

```
lambda.fit$mle
```

```
## [1] 0.2
```

```
lambda.fit$ci
```

```
## [1] 0.1 0.3
```

Back-transformation

We must reverse the transformation (or *back-transform*) to obtain forecasts on the original scale. The reverse Box-Cox transformations are given by

$$y_t = \begin{cases} \exp(w_t), & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda}, & \lambda \neq 0. \end{cases}$$

Modelling with transformations

Transformations used in the left of the formula will be automatically back-transformed. To model log-transformed food retailing turnover, you could use:

```r
fit <- food %>%
  model(SNAIVE(log(Turnover) ~ lag("year")))
```

```
## # A mable: 1 x 1
##   SNAIVE(log(Turnover) ~ lag("year"))
##   <model>
## 1 <SNAIVE>
```

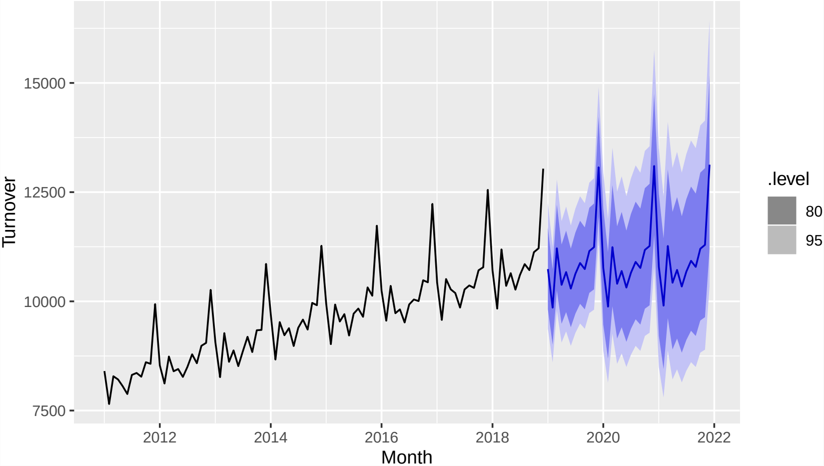
Forecasting with transformations

```r
fc <- fit %>%
  forecast(h = "3 years")
```

```
## # A fable: 36 x 4 [1M]
## # Key:     .model [1]
##   .model                            Month Turnover .distribution
##   <chr>                             <mth>    <dbl> <dist>
## 1 "SNAIVE(log(Turnover) ~  2019 Jan   10738. t(N(9.3, 0.004~
## 2 "SNAIVE(log(Turnover) ~  2019 Feb    9856. t(N(9.2, 0.004~
## 3 "SNAIVE(log(Turnover) ~  2019 Mar   11214. t(N(9.3, 0.004~
## 4 "SNAIVE(log(Turnover) ~  2019 Apr   10378. t(N(9.2, 0.004~
## 5 "SNAIVE(log(Turnover) ~  2019 May   10670. t(N(9.3, 0.004~
## 6 "SNAIVE(log(Turnover) ~  2019 Jun   10292. t(N(9.2, 0.004~
## # ... with 30 more rows
```

Forecasting with transformations

```r
fc %>% autoplot(filter(food,year(Month)>2010))
```



Your turn

Find a transformation that works for the Australian gas production (`aus_production`).

Bias adjustment

- Back-transformed median forecasts for w_{T+h} are median forecasts for y_{T+h}.
- Back-transformed PI have the correct coverage.
- A forecast $\hat{y}_{T+h|T}$ is (usually) the mean of the conditional distribution $y_{T+h} \mid y_1, \dots, y_T$.

Bias adjustment

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Back-transformed means

Let X be have mean μ and variance σ^2.

Let $f(x)$ be back-transformation function, and $Y = f(X)$.

Bias adjustment

Taylor series expansion about μ:

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2 f''(\mu).$$

Bias adjustment

Taylor series expansion about μ:

$$f(X) = f(\mu) + (X-\mu)f'(\mu) + \frac{1}{2}(X-\mu)^2 f''(\mu).$$

$$\mathrm{E}[Y] = \mathrm{E}[f(X)] = f(\mu) + \tfrac{1}{2}\sigma^2 f''(\mu)$$

Bias adjustment

Box-Cox back-transformation:

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f(x) = \begin{cases} e^x & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} e^x & \lambda = 0; \\ (1-\lambda)(\lambda x + 1)^{1/\lambda - 2} & \lambda \neq 0. \end{cases}$$

Bias adjustment

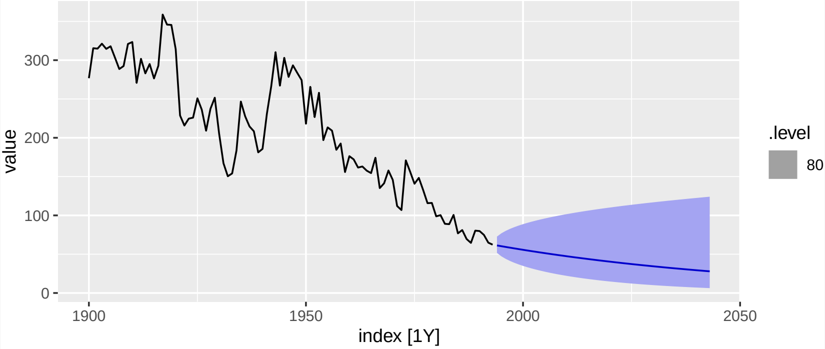
$$\mathrm{E}[Y] = \begin{cases} e^{\mu}\left[1 + \frac{\sigma^2}{2}\right] & \lambda = 0; \\ (\lambda\mu + 1)^{1/\lambda}\left[1 + \frac{\sigma^2(1-\lambda)}{2(\lambda\mu+1)^2}\right] & \lambda \neq 0. \end{cases}$$

Bias adjustment

```
eggs <- as_tsibble(fma::eggs)
fit <- eggs %>% model(RW(log(value) ~ drift()))
fc <- fit %>% forecast(h=50, bias_adjust = TRUE)
fc_biased <- fit %>% forecast(h=50, bias_adjust = FALSE)
```

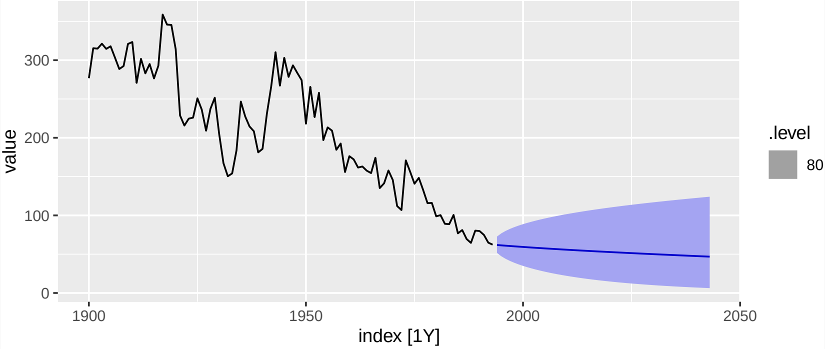
Bias adjustment

```
# the point forecasts are medians
eggs %>% autoplot(value) +
  autolayer(fc_biased, series="Simple back transformation", level=80)
```



Bias adjustment

```r
# the point forecasts are bias adjusted means
eggs %>% autoplot(value) +
  autolayer(fc, series="Simple back transformation", level=80)
```



Outline

Forecast distributions

- Most time series models produce normally distributed forecasts.
- The forecast distribution describes the probability of observing any future value.

Forecast distributions

Assuming residuals are normal, uncorrelated, sd $= \hat{\sigma}$:

Mean:	$\hat{y}_{T+h\|T} \sim N(\bar{y}, (1 + 1/T)\hat{\sigma}^2)$
Naïve:	$\hat{y}_{T+h\|T} \sim N(y_T, h\hat{\sigma}^2)$
Seasonal naïve:	$\hat{y}_{T+h\|T} \sim N(y_{T+h-m(k+1)}, (k + 1)\hat{\sigma}^2)$
Drift:	$\hat{y}_{T+h\|T} \sim N(y_T + \frac{h}{T-1}(y_T - y_1), h\frac{T+h}{T}$

where k is the integer part of $(h - 1)/m$.

Note that when $h = 1$ and T is large, these all give the same approximate forecast variance: $\hat{\sigma}^2$.

Prediction intervals

- A prediction interval gives a region within which we expect y_{T+h} to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

where $\hat{\sigma}_h$ is the st dev of the h-step distribution.

- When $h = 1$, $\hat{\sigma}_h$ can be estimated from the residuals.

Prediction intervals

```r
fit <- fb_stock %>% model(NAIVE(Close))
forecast(fit)
```

```
## # A fable: 2 x 5 [1]
## # Key:     Symbol, .model [1]
##   Symbol .model    trading_day Close .distribution
##   <chr>  <chr>           <dbl> <dbl> <dist>
## 1 FB     NAIVE(Cl~        3693  176. N(176, 21)
## 2 FB     NAIVE(Cl~        3694  176. N(176, 42)
```

Prediction intervals

```
res_sd <- sqrt(mean(augment(fit)$.resid^2, na.rm = TRUE))
last(fb_stock$Close) + 1.96 * res_sd * c(-1,1)
```

```
## [1] 166.7196 184.7404
```

Prediction intervals

```r
forecast(fit, h = 1) %>%
  transmute(interval = hilo(.distribution, level = 95))
```

```
## # A tsibble: 1 x 4 [1]
## # Key:       Symbol, .model [1]
##   Symbol .model       trading_day                 interval
##   <chr>  <chr>              <dbl>                   <hilo>
## 1 FB     NAIVE(Close)        3693 [166.7198, 184.7402]95
```

```r
# transmute: add a new variable to the tsibble object by forecast
# hilo: use the forecast object and compute its 95% PI
# .distribution is a column of the forecast object
```

Prediction intervals

- Point forecasts are often useless without a measure of uncertainty (such as prediction intervals).
- Prediction intervals require a stochastic model (with random errors, etc).
- Multi-step forecasts for time series require a more sophisticated approach (with PI getting wider as the forecast horizon increases).

Prediction intervals

- Computed automatically from the forecast distribution.
- Use `level` argument to control coverage.
- Check residual assumptions before believing them (we will see this next class).
- Usually too narrow due to unaccounted uncertainty.