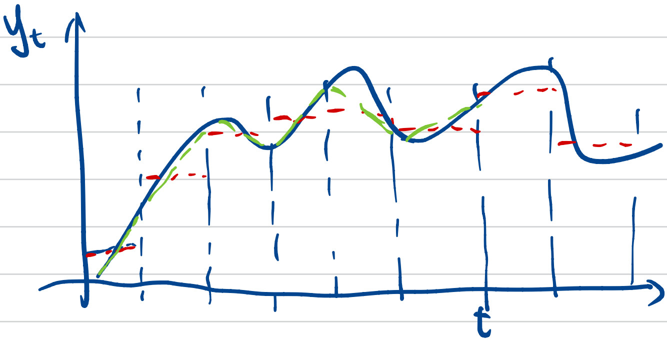
STAT481/581: Introduction to Time Series Analysis

Exponential smoothing

OTexts.org/fpp3/

Outline

Exponential smoothing



① Does the mean level change over the windows?

② Does the trend change?

③ Is there seasonal pattern?

ETS (Error, trend, seasonal)

Error	trend	seasonal
A	A	A
M	M	M
	Ad	

Simple exponential smoothing

ETS(A, N, N)

State space form: $y_t = \ell_{t-1} + \varepsilon_t$

$\ell_t = \ell_{t-1} + \alpha \varepsilon_t \quad 0 < \alpha < 1$

The change in the level ℓ_t is less than the change in observed y_t from ℓ_{t-1}

Component form

① $\hat{y}_{t+h|t} = E(y_{t+h} | \vec{\ell_t}, \vec{y_t}) = \ell_t$

② $\ell_t = E(\ell_t | \vec{\ell_{t-1}}, \vec{y_t}) = \alpha y_t + (1-\alpha)\ell_{t-1}$

Let $t = T$, $h = 1$

$$\hat{y}_{T+1|T} = \ell_T$$

$$\begin{aligned}\ell_T &= \alpha y_T + (1-\alpha)\ell_{T-1} \\ &= \alpha y_T + (1-\alpha)[\alpha y_{T-1} + (1-\alpha)\ell_{T-2}] \\ &= \alpha y_T + \alpha(1-\alpha) y_{T-1} + (1-\alpha)^2 \ell_{T-2} \\ &\vdots \\ &= \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \dots \\ &\quad \alpha(1-\alpha)^{T-1} y_1 + (1-\alpha)^T \ell_0\end{aligned}$$

State space form:

$$y_t = l_{t-1} + \varepsilon_t$$

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

Result 1: $\varepsilon_{t+1} \perp (y_1, y_2, \cdots y_t)$

Result 2: $\varepsilon_{t+1} \perp (l_1, l_2, \cdots l_t)$

$$y_t = l_{t-1} + \varepsilon_t$$

$$= l_{t-2} + \alpha \varepsilon_{t-1} + \varepsilon_t$$

$$= l_{t-3} + \alpha \varepsilon_{t-2} + \alpha \varepsilon_{t-1} + \varepsilon_t$$

$$\vdots$$

$$= l_0 + \alpha(\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_{t-1}) + \varepsilon_t$$

Since $\varepsilon_{t+1} \perp (\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t)$,

$y_t \perp \varepsilon_{t+1}$

Forecast error: $v_{t+h} = y_{t+h} - \hat{y}_{t+h|t} = \ell_t$

$$= y_{t+h} - \ell_t$$

$$= \ell_{t+h-1} + \varepsilon_{t+h} - \ell_t$$

$$= \ell_{t+h-2} + \alpha\varepsilon_{t+h-1} + \varepsilon_{t+h} - \ell_t$$

$$= \ell_{t+h-3} + \alpha\varepsilon_{t+h-2} + \alpha\varepsilon_{t+h-1} + \varepsilon_{t+h} - \ell_t$$

$$\vdots$$

$$= \ell_t + \alpha\varepsilon_{t+1} + \alpha\varepsilon_{t+2} + \cdots \alpha\varepsilon_{t+h-1} + \varepsilon_{t+h} - \ell_t$$

$$= \alpha(\varepsilon_{t+1} + \varepsilon_{t+2} + \cdots \varepsilon_{t+h-1}) + \varepsilon_{t+h}$$

h-1 terms

$$Var(v_t) = \alpha^2\sigma^2 + \alpha^2\sigma^2 + \cdots \alpha^2\sigma^2 + \sigma^2 = \sigma^2(1+h^2\alpha^2)$$

If $X_1, X_2, \ldots X_p$ are mutually independent,

$$Var(a_1X_1 + a_2X_2 + \ldots a_pX_p) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cdots + a_p^2 Var(X_p)$$

Outline

Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”: α, β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

Big idea: control the rate of change (smoothing)

α controls the flexibility of the **level**

- If $\alpha = 0$, the level never updates (mean)
- If $\alpha = 1$, the level updates completely (naive)

β controls the flexibility of the **trend**

- If $\beta = 0$, the trend is linear (regression trend)
- If $\beta = 1$, the trend updates every observation

Big idea: control the rate of change (smoothing)

γ controls the flexibility of the **seasonality**

- If $\gamma = 0$, the seasonality is fixed (seasonal means)
- If $\gamma = 1$, the seasonality updates completely (seasonal naive)

A model for levels, trends, and seasonalities

We want a model that captures the level (ℓ_t), trend (b_t) and seasonality (s_t).

How do we combine these elements?

Additively?

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

Multiplicatively?

$$y_t = \ell_{t-1} b_{t-1} s_{t-m}(1 + \varepsilon_t)$$

Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

ETS models

General notation E T S : **E**xponen**T**ial **S**moothing

$\nearrow$ $\uparrow$ $\nwarrow$

Error **T**rend **S**eason

Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

Seasonality: None (`"N"`), additive (`"A"`) or multiplicative (`"M"`)

Outline

Simple methods

Time series $y_1, y_2, \dots, y_T$.

Random walk forecasts

$$\hat{y}_{T+h|T} = y_T$$

Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

- Want something in between these methods.
- Most recent data should have more weight.

Simple Exponential Smoothing

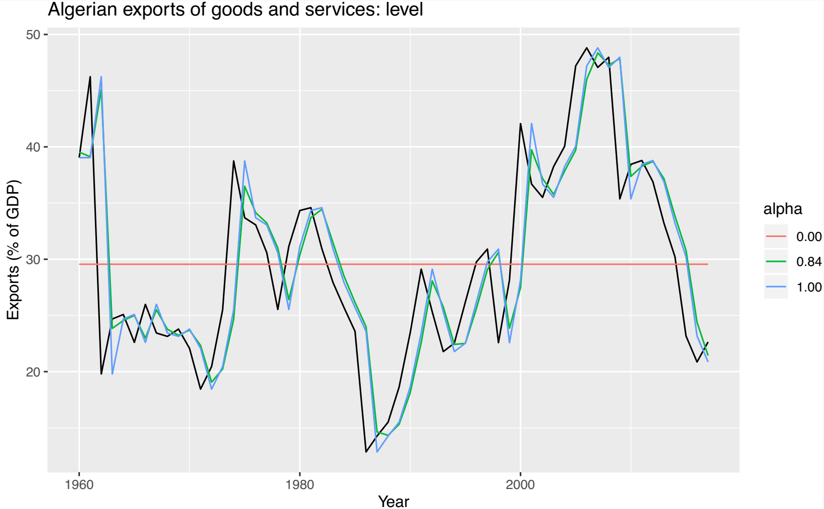
Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \cdots$$

where $0 \le \alpha \le 1$.

	Weights assigned to observations for:			
Observation	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2	0.4	0.6	0.8
y_{T-1}	0.16	0.24	0.24	0.16
y_{T-2}	0.128	0.144	0.096	0.032
y_{T-3}	0.1024	0.0864	0.0384	0.0064
y_{T-4}	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
y_{T-5}	$(0.2)(0.8)^5$	$(0.4)(0.6)^5$	$(0.6)(0.4)^5$	$(0.8)(0.2)^5$

Simple Exponential Smoothing



Simple Exponential Smoothing

Methods

- Algorithms that return point forecasts.

Models

- Generate same point forecasts but can also generate forecast distributions.
- A stochastic (or random) data generating process that can generate an entire forecast distribution.
- Allow for "proper" model selection.

ETS(A,N,N)

State space form:

Measurement equation	$y_t = \ell_{t-1} + \varepsilon_t$
State equation	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- "innovations" or "single source of error" because equations have the same error process, ε_t.
- Measurement equation: relationship between observations and states.
- Transition/state equation(s): evolution of the state(s) through time.

ETS(A,N,N)

ETS(A,N,N)

Component form

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t$
Smoothing equation	$\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}$

- ℓ_t is the level (or the smoothed value) of the series at time t.
- $\hat{y}_{t+1|t} = \alpha y_t + (1-\alpha)\hat{y}_{t|t-1}$
  Iterate to get exponentially weighted moving average form.

Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j} + (1-\alpha)^T \ell_0$$

Optimising smoothing parameters

- Need to choose best values for α and ℓ_0.
- Similarly to regression, choose optimal parameters by minimising SSE:

$$\text{SSE} = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2.$$

- Unlike regression there is no closed form solution — use numerical optimization.

ETS(A,N,N): Specifying the model

```
ETS(y ~ error("A") + trend("N") + season("N"))
```

By default, an optimal value for α and ℓ_0 is used.

α can be chosen manually in `trend()`.

```
trend("N", alpha = 0.5)
trend("N", alpha_range = c(0.2, 0.8))
```

Algeria economy data

```r
algeria_economy <- tsibbledata::global_economy %>%
  filter(Country == "Algeria")
fit <- algeria_economy %>%
  model(ETS(Exports ~ error("A") + trend("N")
              +season("N"),opt_crit = "mse"))
tidy(fit)$term
```

```
## [1] "alpha" "l"
```

```r
tidy(fit)$estimate
```

```
## [1]  0.8398 39.5401
```

Algeria economy data

- For Algerian Exports example:
  - $\hat{\alpha} = 0.8398$
  - $\hat{\ell}_0 = 39.54$
- Here l_0

Example: Algerian Exports

```
report(fit)
```

```
## Series: Exports
## Model: ETS(A,N,N)
##   Smoothing parameters:
##     alpha = 0.8398
##
##   Initial states:
##      l
##  39.54
##
##   sigma^2:  35.63
##
##   AIC  AICc   BIC
## 446.7 447.2 452.9
```

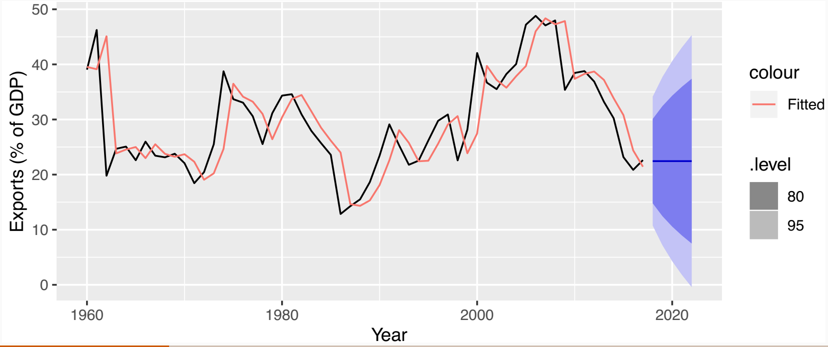
Example: Algerian Exports

```r
components(fit) %>%
  left_join(fitted(fit), by = c("Country", ".model", "Year"))
```

```
## # A tsibble: 59 x 7 [1Y]
## # Key:       Country, .model [1]
##    Country .model  Year Exports level remainder .fitted
##    <fct>   <chr>  <dbl>   <dbl> <dbl>     <dbl>   <dbl>
##  1 Algeria ANN     1959    NA    39.5    NA        NA
##  2 Algeria ANN     1960    39.0  39.1    -0.496    39.5
##  3 Algeria ANN     1961    46.2  45.1     7.12     39.1
##  4 Algeria ANN     1962    19.8  23.8   -25.3      45.1
##  5 Algeria ANN     1963    24.7  24.6     0.841    23.8
##  6 Algeria ANN     1964    25.1  25.0     0.534    24.6
##  7 Algeria ANN     1965    22.6  23.0    -2.39     25.0
##  8 Algeria ANN     1966    26.0  25.5     3.00     23.0
##  9 Algeria ANN     1967    23.4  23.8    -2.07     25.5
## 10 Algeria ANN     1968    23.1  23.2    -0.630    23.8
## # ... with 49 more rows
```

Simple Exponential Smoothing is the initial state.

```
fc <- fit %>% forecast(h = 5)
fc %>%autoplot(algeria_economy) + geom_line(aes(y = .fitted,
colour = "Fitted"), data = augment(fit)) +
  ylab("Exports (% of GDP)") + xlab("Year")
```



ETS(M,N,N)

SES with multiplicative errors.

$y_{t+1} = \ell_t (1+ \varepsilon_{t+1})$

Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

$\ell_{t+1} = \ell_t (1+\alpha \varepsilon_{t+1})$

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

ETS(M,N,N)

$$\hat{\ell}_t = E(\ell_t \mid \vec{y}_t, \vec{\ell}_{t-1}) = E[\ell_{t-1}(1+\alpha\varepsilon_t) \mid \vec{y}_t, \vec{\ell}_{t-1})$$

$$= E\left\{ \ell_{t-1}\left[1+\alpha\left(\frac{y_t}{\ell_{t-1}}-1\right)\right] \,\middle|\, \vec{y}_t, \vec{\ell}_{t-1} \right\}$$

$$= \ell_{t-1}\left[1+\alpha\left(\frac{y_t}{\ell_{t-1}}-1\right)\right]$$

$$= \ell_{t-1} + \alpha(y_t - \ell_{t-1})$$

$$= \alpha y_t + (1-\alpha)\ell_{t-1}$$

$$E(y_{t+1} \mid \vec{y}_t, \vec{\ell}_t)$$

If $X \perp (Y, Z)$

$E(X \mid Y, Z) = E(X)$

$$= E[\ell_t(1+\varepsilon_{t+1}) \mid \vec{y}_t, \vec{\ell}_t]$$

$$= \ell_t E[1+\varepsilon_{t+1} \mid \vec{y}_t, \vec{\ell}_t] = \ell_t$$

$$E(y_{t+2} \mid \vec{y}_t, \vec{\ell}_t)$$

If $X \perp Y$

$E(XY \mid Z) = E(X \mid Z) \cdot E(Y \mid Z)$

$$= E(\ell_{t+1}(1+\varepsilon_{t+2}) \mid \vec{y}_t, \vec{\ell}_t)$$

$$= E[\ell_{t+1} \mid \vec{y}_t, \vec{\ell}_t]$$

$$= E[\ell_t(1+\alpha\varepsilon_{t+1}) \mid \vec{y}_t, \vec{\ell}_t]$$

$$= \ell_t \cdot \underbrace{E[1+\alpha\varepsilon_{t+1} \mid \vec{y}_t, \vec{\ell}_t]}_{=1}$$

$$= \ell_t$$

Forecast error of ETS(M, N, N)

$$v_{t+h} = y_{t+h} - \hat{y}_{t+h|t}$$

$$= y_{t+h} - \ell_t$$

$$= \ell_t\left[(1+\alpha\varepsilon_{t+1})(1+\alpha\varepsilon_{t+2})\cdots(1+\alpha\varepsilon_{t+h-1})(1+\varepsilon_{t+h})\right] - \ell_t$$

$$y_{t+h} = \ell_{t+h-1}(1+\varepsilon_{t+h})$$

$$= \ell_{t+h-2}(1+\alpha\varepsilon_{t+h-1})(1+\varepsilon_{t+h})$$

$$= \ell_{t+h-3}(1+\alpha\varepsilon_{t+h-2})(1+\alpha\varepsilon_{t+h-1})\cdot(1+\varepsilon_{t+h})$$

$$\vdots$$

$$= \ell_t(1+\alpha\varepsilon_{t+1})(1+\alpha\varepsilon_{t+2})\cdots(1+\alpha\varepsilon_{t+h-1})(1+\varepsilon_{t+h})$$

$(1+\varepsilon$:

Outline

ETS(A,A,N)

Holt's linear method with additive errors.

- b_t: slope
- State space form:

$$
\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \alpha\beta^*\varepsilon_t
\end{aligned}
$$

$\ell_{t+1} = \ell_t + b_t + \alpha \varepsilon_{t+1}$

- For simplicity, set $\beta = \alpha\beta^*$.

Holt's linear trend

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

- Two smoothing parameters α and β^* $(0 \le \alpha, \beta^* \le 1)$.

Holt's linear trend

- ℓ_t level: weighted average between y_t and one-step ahead forecast for time t, $(\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1})$.
- b_t slope: weighted average of $(\ell_t - \ell_{t-1})$ and b_{t-1}, current and previous estimate of slope.
- Choose $\alpha, \beta^*, \ell_0, b_0$ to minimise SSE.

$$SSE = \sum_{t=1}^{T} (y_t - \hat{y}_{t|t-1})^2$$

Holt's linear trend

$$\ell_t \triangleq E(\ell_t \mid \vec{y}_t, \vec{\ell}_{t-1}, \vec{b}_{t-1}) = E(\ell_{t-1} + b_{t-1} + \alpha \varepsilon_t \mid \vec{y}_t, \vec{\ell}_{t-1}, \vec{b}_{t-1})$$

$$b_t \triangleq E(b_t \mid \vec{y}_t, \vec{\ell}_t, \vec{b}_{t-1}) = E(b_{t-1} + \alpha\beta^* \varepsilon_t \mid \vec{y}_t, \vec{\ell}_t, \vec{b}_{t-1})$$

$$\hat{y}_{t+h|t} = E(y_{t+h} \mid \vec{y}_t, \vec{\ell}_t, \vec{b}_t)$$

Since $\varepsilon_t = y_t - \ell_{t-1} - b_{t-1}$,

$$= \ell_{t-1} + b_{t-1} + \alpha E(y_t - \ell_{t-1} - b_{t-1} \mid \vec{y}_t, \vec{\ell}_{t-1}, \vec{b}_{t-1})$$

$$= \ell_{t-1} + b_{t-1} + \alpha (y_t - \ell_{t-1} - b_{t-1})$$

$$= \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

Since $\varepsilon_t = (\ell_t - \ell_{t-1} - b_{t-1})/\alpha$

$$b_t = E(b_{t-1} + \alpha\beta^* \varepsilon_t \mid \vec{y_t}, \vec{\ell_t}, \vec{b_{t-1}})$$

$$= b_{t-1} + \alpha\beta^* (\ell_t - \ell_{t-1} - b_{t-1})/\alpha$$

$$= (1-\beta^*) b_{t-1} + \beta^* (\ell_t - \ell_{t-1})$$

$$\hat{y}_{t+1|t} = E(y_{t+1} \mid \vec{y_t}, \vec{\ell_t}, \vec{b_t})$$

$$= E(\ell_t + b_t + \varepsilon_{t+1} \mid \vec{y_t}, \vec{\ell_t}, \vec{b_t})$$

$$= \ell_t + b_t$$

$$\hat{y}_{t+2|t} = E(y_{t+2} \mid \vec{y_t}, \vec{\ell_t}, \vec{b_t})$$

$$= E(\ell_{t+1} + b_{t+1} + \varepsilon_{t+2} \mid \vec{y_t}, \vec{\ell_t}, \vec{b_t})$$

$$= E(\ell_t + b_t + \alpha\varepsilon_{t+1} + b_t + \alpha\beta^*\varepsilon_{t+1} + \varepsilon_{t+2} \mid \vec{y_t}, \vec{\ell_t}, \vec{b_t})$$

$$= \ell_t + b_t + b_t = \ell_t + 2b_t$$

Exponential smoothing: trend/slope

ETS(A,A,N): Specifying the model

```
ETS(y ~ error("A") + trend("A") + season("N"))
```

By default, optimal values for β and b_0 are used.

β can be chosen manually in `trend()`.

```
trend("A", beta = 0.004)
trend("A", beta_range = c(0, 0.1))
```

AUS pupulation data

```
aus_economy <- global_economy %>%filter(Code == "AUS") %>%
  mutate(Pop = Population / 1e6)
fit <- aus_economy %>%
  model(AAN = ETS(Pop ~ error("A") + trend("A") +
                    season("N")))
fc <- fit %>% forecast(h = 10)
```

ETS(A,A,N)

AUS pupulation data

```
report(fit)
```

```
## Series: Pop
## Model: ETS(A,A,N)
##   Smoothing parameters:
##     alpha = 0.9999
##     beta  = 0.3266
##
##   Initial states:
##      l      b
##   10.05 0.2225
##
##   sigma^2:  0.0041
##
##     AIC   AICc    BIC
## -76.99 -75.83 -66.68
```

→ The level is close to a random walk behavior

AUS pupulation data

```r
components(fit) %>%
  left_join(fitted(fit), by = c("Country", ".model", "Year"))
```

```
## # A tsibble: 59 x 8 [1Y]
## # Key:       Country, .model [1]
##    Country .model  Year   Pop level slope remainder
##    <fct>   <chr>  <dbl> <dbl> <dbl> <dbl>     <dbl>
##  1 Austra~ AAN     1959  NA    10.1 0.222 NA
##  2 Austra~ AAN     1960  10.3  10.3 0.222 -0.000145
##  3 Austra~ AAN     1961  10.5  10.5 0.217 -0.0159
##  4 Austra~ AAN     1962  10.7  10.7 0.231  0.0418
##  5 Austra~ AAN     1963  11.0  11.0 0.223 -0.0229
##  6 Austra~ AAN     1964  11.2  11.2 0.221 -0.00641
##  7 Austra~ AAN     1965  11.4  11.4 0.221 -0.000314
##  8 Austra~ AAN     1966  11.7  11.7 0.235  0.0418
##  9 Austra~ AAN     1967  11.8  11.8 0.206 -0.0869
## 10 Austra~ AAN     1968  12.0  12.0 0.208  0.00350
## # ... with 49 more rows, and 1 more variable:
```

AUS pupulation data

```
fc
```

```
## # A fable: 10 x 5 [1Y]
## # Key:     Country, .model [1]
##    Country   .model  Year   Pop .distribution
##    <fct>     <chr>  <dbl> <dbl> <dist>
##  1 Australia AAN     2018  25.0 N(25, 0.0041)
##  2 Australia AAN     2019  25.3 N(25, 0.0114)
##  3 Australia AAN     2020  25.7 N(26, 0.0227)
##  4 Australia AAN     2021  26.1 N(26, 0.0389)
##  5 Australia AAN     2022  26.4 N(26, 0.0609)
##  6 Australia AAN     2023  26.8 N(27, 0.0895)
##  7 Australia AAN     2024  27.2 N(27, 0.1257)
##  8 Australia AAN     2025  27.6 N(28, 0.1704)
##  9 Australia AAN     2026  27.9 N(28, 0.2243)
## 10 Australia AAN     2027  28.3 N(28, 0.2885)
```

T=59

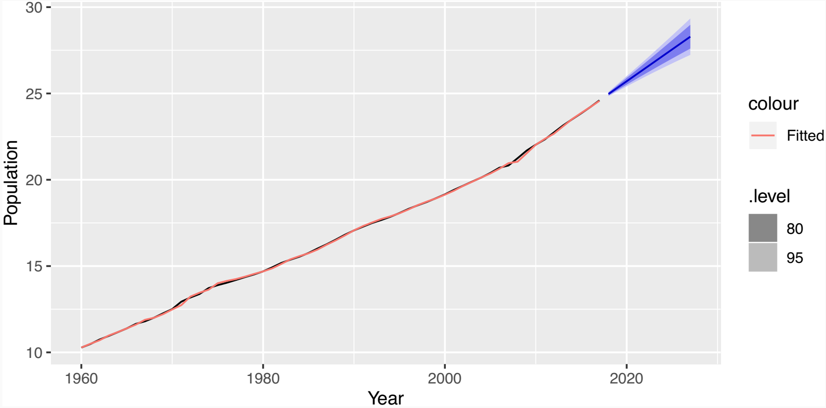
$\hat{y}_{T+1|T}$

$\hat{y}_{T+2|T}$

$\hat{y}_{T+10|T}$

AUS pupulation data

```
fit %>%forecast(h = 10) %>%autoplot(aus_economy) +
  geom_line(aes(y=.fitted,colour="Fitted"),data =
augment(fit))+ylab("Population") + xlab("Year")
```



ETS(M,A,N)

Holt's linear method with multiplicative errors.

- Following a similar approach as above, the innovations state space model underlying Holt's linear method with multiplicative errors is specified as

$$
\begin{aligned}
y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t
\end{aligned}
$$

  where again $\beta = \alpha\beta^*$ and $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(M,A,N) component form

Damped trend method

Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t$$
$$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$$
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}.$$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \to \infty$, $\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1-\phi)$.
- Short-run forecasts trended, long-run forecasts constant.

$b_t = \phi b_{t-1} + \alpha\beta^* \varepsilon_t$

Your turn

- Write down the state space model for ETS(A,Ad,N)
- Derive the forecast equation and the smoothing equations.

Your turn

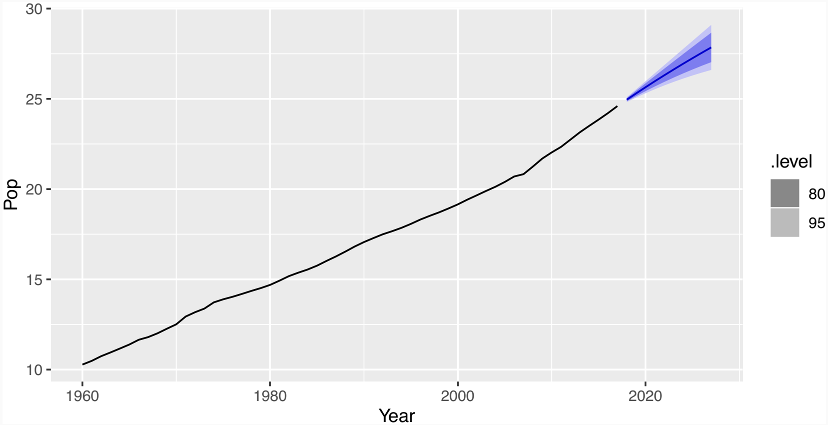
$$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \qquad \text{ETS}(A, A_d, N)$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$$

$$b_t = \phi b_{t-1} + \alpha \beta^* \varepsilon_t$$

Example: Australian population

```
aus_economy %>%
  model(holt = ETS(Pop ~ error("A") + trend("Ad") + season("N"))) %>%
  forecast(h = 10) %>%
  autoplot(aus_economy)
```



Example: Australian population

```
fit <- aus_economy %>%
  filter(Year <= 2010) %>%
  model(
    ses = ETS(Pop ~ error("A") + trend("N") + season("N")),
    holt = ETS(Pop ~ error("A") + trend("A") + season("N")),
    damped = ETS(Pop ~ error("A") + trend("Ad") + season("N"))
  )
```

```
tidy(fit)
accuracy(fit)
```

Example: Australian population

term	SES	Linear trend	Damped trend
α	1.00	1.00	1.00
β^*		0.30	0.40
ϕ			0.98
ℓ_0	10.28	10.05	10.04
b_0		0.22	0.25
Training RMSE	0.24	0.06	0.07
Test RMSE	1.63	0.15	0.21
Test MASE	6.18	0.55	0.75
Test MAPE	6.09	0.55	0.74
Test MAE	1.45	0.13	0.18

Your turn

`fma::eggs` contains the price of a dozen eggs in the United States from 1900–1993

1. Use simple exponential smoothing (SES) and Holt's method (with and without damping) to forecast "future" data.
   [Hint: use h=100 so you can clearly see the differences between the options when plotting the forecasts.]
2. Which method gives the best training RMSE?
3. Are these RMSE values comparable?
4. Do the residuals from the best fitting method

Outline

Simple exponential smoothing — ETS(A, N, N) (→ Error, → trend, → season)

$$y_t = l_{t-1} + \varepsilon_t \quad ; \quad Var(y_t | l_{t-1}) = \sigma_\varepsilon^2$$

$$l_t = l_{t-1} + \alpha \varepsilon_t$$

ETS(M, N, N)

$$y_t = l_{t-1}(1+\varepsilon_t)$$

$$Var(y_t | l_{t-1}) = l_{t-1}^2 \, Var(1+\varepsilon_t) = l_{t-1}^2 \sigma_\varepsilon^2$$

$$l_t = l_{t-1}(1+\alpha\varepsilon_t)$$

Models with trend

Holt's linear model ETS (A, A, N)

$$y_t = l_{t-1} + b_{t-1} + \varepsilon_t$$

$$l_t = l_{t-1} + b_{t-1} + \alpha \varepsilon_t$$

$$b_t = b_{t-1} + \beta \varepsilon_t \quad (\beta = \alpha\beta^*)$$

ETS(M, A, N)

$$y_t = (l_{t-1} + b_{t-1})(1+\varepsilon_t)$$

$$l_t = (l_{t-1} + b_{t-1})(1+\alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta(l_{t-1}+b_{t-1})\varepsilon_t$$

Damped trend model with additive errors ETS(A, A_d, N)

$$y_t = l_{t-1} + \phi b_{t-1} + \varepsilon_t$$

$$l_t = l_{t-1} + \phi b_{t-1} + \alpha \varepsilon_t$$

$$b_t = \phi b_{t-1} + \beta \varepsilon_t$$

ETS(M, A_d, N)

$$y_t = (l_{t-1} + \phi b_{t-1})(1+\varepsilon_t)$$

$$l_t = (l_{t-1} + \phi b_{t-1})(1+\alpha\varepsilon_t)$$

$$b_t = \phi b_{t-1} + \beta(l_{t-1} + \phi b_{t-1})\varepsilon_t$$

ETS(A,A,A)

- Holt and Winters extended Holt's method to capture seasonality.
- Holt-Winters additive method in state space form:

$$\begin{aligned}\text{Observation equation} \quad y_t &= \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\ \text{State equations} \quad \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} + \beta\varepsilon_t \\ s_t &= s_{t-m} + \gamma\varepsilon_t\end{aligned}$$

- k is integer part of $(h-1)/m$.
- s_t: seasonal state.

Holt-Winters additive method component form

Component form

(4) $\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$

(1) $\ell_t = \alpha(y_t - s_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$

(2) $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$

(3) $s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1-\gamma)s_{t-m}$

- $k =$ integer part of $(h-1)/m$. Ensures estimates from the final year are used for forecasting.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and $m =$ period of seasonality (e.g. $m = 4$ for quarterly data).

Holt-Winters additive method component form

① $\ell_t \triangleq E(\ell_t \mid \vec{y}_t, \vec{\ell}_{t-1}, \vec{b}_{t-1}, \vec{S}_{t-m})$

$= E(\ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \mid \square)$

$= \ell_{t-1} + b_{t-1} + \alpha E(y_t - \ell_{t-1} - b_{t-1} - S_{t-m} \mid \square)$

$= \alpha(y_t - S_{t-m}) + (1-\alpha)(\ell_{t-1} + b_{t-1})$

From state-space form

$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$

$y_t = \ell_{t-1} + b_{t-1} + S_{t-m} + \varepsilon_t$

$b_t = b_{t-1} + \beta\varepsilon_t$

② $b_t \triangleq E(b_t \mid \vec{y}_t, \vec{\ell}_t, \vec{b}_{t-1}, \vec{S}_{t-m})$

$= E(b_{t-1} + \beta\varepsilon_t \mid \square) = b_{t-1} + \beta \cdot E(y_t - \ell_{t-1} - b_{t-1} - S_{t-m} \mid \square)$

or $= b_{t-1} + \beta \cdot E\left(\frac{\ell_t - \ell_{t-1} - b_{t-1}}{\alpha} \mid \square\right)$ ✓

$\beta = \beta^* \cdot \alpha$

$= (1-\beta^*) b_{t-1} + \beta^*(\ell_t - \ell_{t-1})$

③

$$s_t = s_{t-m} + \gamma \varepsilon_t$$

$$\hat{s}_t = E(s_t \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_{t-m}})$$

$$= E[s_{t-m} + \gamma (y_t - l_{t-1} - b_{t-1} - s_{t-m}) \mid \square]$$

$$= (1-\gamma) s_{t-m} + \gamma (y_t - l_{t-1} - b_{t-1})$$

④ $E(y_{t+h} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$.

$$E(y_{t+1} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t}) = E(l_t + b_t + s_{t+1-m} + \varepsilon_{t+1} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$$

$$= l_t + b_t + s_{t+1-m} \quad \text{since } t+1-m \le t$$

(m is at least 2)

$$E(y_{t+2} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t}) = E(l_{t+1} + b_{t+1} + s_{t+2-m} + \varepsilon_{t+2} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$$

$$= E(l_{t+1} + b_{t+1} + s_{t+2-m} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$$

since $\varepsilon_{t+2} \perp (\vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$

$$= E(l_t + b_t + \alpha \varepsilon_{t+1} + b_t + \beta \varepsilon_{t+1} + s_{t+2-m} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})$$

Note I did not substitute s_{t+2-m} by $s_{t+2-2m} + \varepsilon_{t+2-m}$ since s_{t+2-m} is already in given in the condition vector $\vec{s_t}$ $(t+2-m \le t)$.

Hence $E(y_{t+2} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t}) = l_t + 2b_t + s_{t+2-m}$

In general

$$E(y_{t+h} \mid \vec{y_t}, \vec{l_t}, \vec{s_t}, \vec{b_t})$$

$$= E(l_{t+h-1} + b_{t+h-1} + s_{t+h-m} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t}) \quad (E(\varepsilon_{t+h} \mid \square) = 0)$$

$$= \underbrace{E(l_{t+h-1} + b_{t+h-1} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})}_{\triangleq A} + \underbrace{E(s_{t+h-m} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t})}_{\triangleq B}$$

For A, repeat subsituting l and b by $\begin{cases} l_t = l_{t-1} + b_{t-1} + \alpha\varepsilon_t \\ b_t = b_{t-1} + \beta\varepsilon_t \end{cases}$ till the subscripts of l and b are t but the subsript of ε is $t+1$. This takes (h-1) more steps. Hence

$$E(l_{t+h-2} + 2b_{t+h-2} \mid \vec{y_t}, \vec{l_t}, \vec{b_t}, \vec{s_t}) = l_t + hb_t$$

For B, repeat subsituting s by $s_t = s_{t-m} + \gamma\varepsilon_t$ till the subscript of s is less (or equal) than t but the subscript of ε is greater than t.

$$E(S_{t+h-m} \mid \vec{y_t}, \vec{\ell_t}, \vec{s_t}, \vec{b_t})$$

$$= E(S_{t+h-2m} + \gamma \varepsilon_{t+h-m} \mid \vec{y_t}, \vec{\ell_t}, \vec{s_t}, \vec{b_t})$$

Is $t+h-2m \leq t$ and $t+h-m > t$? If so, stop the substitution. $B = S_{t+h-2m}$

If not, continue the substitution of the S term.

Even if we continue with the substitution, the final form of $B = S_{t+h-(k+1)m}$ where

$t+h-(k+1)m \leq t$ (subscript for S) and $t+h-km > t \triangleq t+h-km \geq t+1$ (subscript for the error)

Note

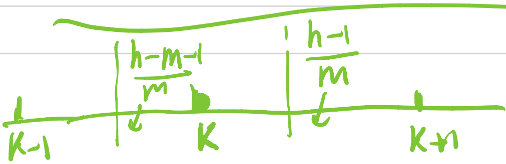
$t+h-(k+1)m \leq t$ $\quad$ $t+h-km \geq t+1$

$k \geq \frac{h}{m} - 1 = \frac{h-m}{m}$ $\quad$ $k \leq \frac{h-1}{m}$

Therefore k is the integer part of $\frac{h-1}{m}$ (why?)

For example $h=2$, $m=2$, then $k=0$, $B = S_{t+2-2} = S_t$

$h=4$, $m=2$, then $k=1$, $B = S_{t+4-2\cdot 2} = S_t$

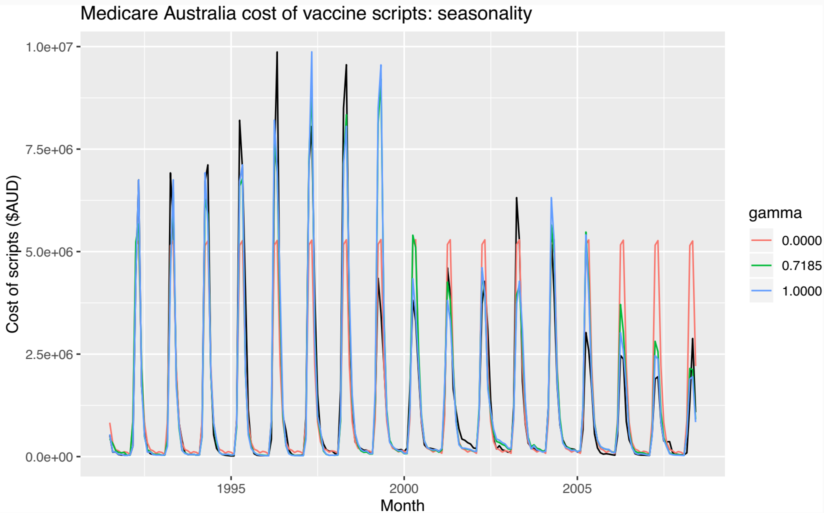


k has to be an integer while between $\frac{h-1}{m}$ and $\frac{h-m-1}{m}$

Holt-Winters additive method

- Seasonal component is usually expressed as $s_t = \gamma^*(y_t - \ell_t) + (1 - \gamma^*)s_{t-m}$.
- Substitute in for ℓ_t: $s_t = \gamma^*(1-\alpha)(y_t - \ell_{t-1} - b_{t-1}) + [1 - \gamma^*(1-\alpha)]s_{t-m}$
- We set $\gamma = \gamma^*(1 - \alpha)$.
- The usual parameter restriction is $0 \leq \gamma^* \leq 1$, which translates to $0 \leq \gamma \leq (1 - \alpha)$.

Exponential smoothing: seasonality



Your turn

$$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$$

$$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$$

$$s_t = s_{t-m} + \gamma\varepsilon_t$$

- Write down the state space model and component form for ETS(A,N,A).

Your turn

ETS(M,A,M)

- Holt-Winters multiplicative method with multiplicative errors for when seasonal variations are changing proportional to the level of the series.

$$
\begin{aligned}
\text{Observation equation} \quad y_t &= (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t) \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1}(1 + \beta\varepsilon_t) \\
s_t &= s_{t-m}(1 + \gamma\varepsilon_t)
\end{aligned}
$$

$b_{t-1} + \beta(\ell_{t-1} + b_{t-1})\varepsilon_t$

- k is integer part of $(h-1)/m$.

Holt-Winters multiplicative method

Component form

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$$

$$\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1-\alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1-\gamma)s_{t-m}$$

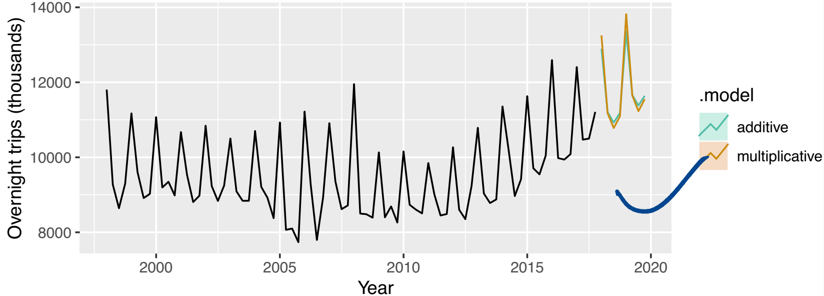
ETS(M, A, M)

- k is integer part of $(h-1)/m$.
- With additive method s_t is in absolute terms: within each year $\sum_i s_i \approx 0$.
- With multiplicative method s_t is in relative terms: within each year $\sum_i s_i \approx m$.

Holt-Winters multiplicative method component form

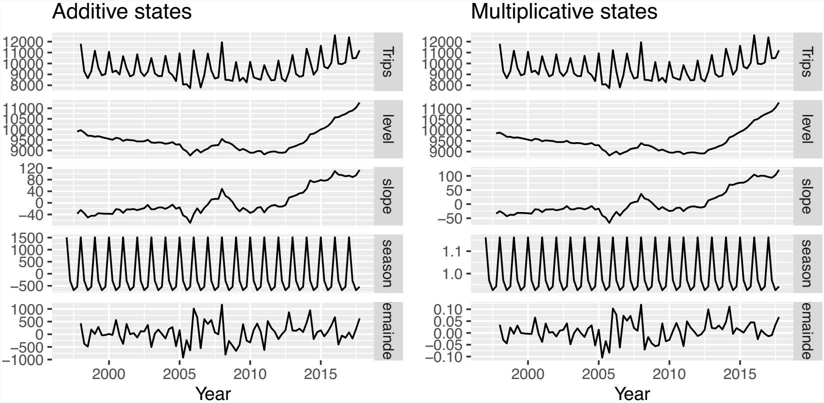
Example: Australian holiday tourism

```
aus_holidays <- tourism %>% filter(Purpose == "Holiday") %>%
  summarise(Trips = sum(Trips))
fit <- aus_holidays %>%
  model(
    additive = ETS(Trips ~ error("A") + trend("A") + season("A")),
    multiplicative = ETS(Trips ~ error("M") + trend("A") + season("M"))
  )
fc <- fit %>% forecast()
```



Estimated components

```
components(fit)
```



Holt-Winters damped method

ETS(M, Ad, M)

Often the single most accurate forecasting method for seasonal data:

$$\hat{y}_{t+h|t} = [\ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t]s_{t+h-m(k+1)}$$
$$\ell_t = \alpha(y_t/s_{t-m}) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1})$$
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}$$
$$s_t = \gamma\frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1-\gamma)s_{t-m}$$

Your turn

Apply Holt-Winters' multiplicative method to the Gas data from `aus_production`.

1. Why is multiplicative seasonality necessary here?
2. Experiment with making the trend damped.
3. Check that the residuals from the best method look like white noise.

Outline

Exponential smoothing methods

Trend Component		Seasonal Component N (None)	Seasonal Component A (Additive)	Seasonal Component M (Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A_d	(Additive damped)	(A_d,N)	(A_d,A)	(A_d,M)

(N,N): Simple exponential smoothing   ETS(A, N, N)

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

ETS models

Additive Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	A,N,M
A	(Additive)	A,A,N	A,A,A	A,A,M
A_d	(Additive damped)	A,A_d,N	A,A_d,A	A,A_d,M

Multiplicative Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Additive error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$	$y_t = \ell_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/\ell_{t-1}$
A	$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$ $b_t = b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + b_{t-1})$
A_d	$y_t = \ell_{t-1} + \phi b_{t-1} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$	$y_t = \ell_{t-1} + \phi b_{t-1} + s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t$ $b_t = \phi b_{t-1} + \beta\varepsilon_t$ $s_t = s_{t-m} + \gamma\varepsilon_t$	$y_t = (\ell_{t-1} + \phi b_{t-1})s_{t-m} + \varepsilon_t$ $\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t/s_{t-m}$ $b_t = \phi b_{t-1} + \beta\varepsilon_t/s_{t-m}$ $s_t = s_{t-m} + \gamma\varepsilon_t/(\ell_{t-1} + \phi b_{t-1})$

ETS(A, N, A)

Multiplicative error models

Trend	Seasonal N	Seasonal A	Seasonal M
N	$y_t = \ell_{t-1}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$	$y_t = (\ell_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\alpha(\ell_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+s_{t-m})\varepsilon_t$	$y_t = \ell_{t-1}s_{t-m}(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}(1+\alpha\varepsilon_t)$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A	$y_t = (\ell_{t-1}+b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+b_{t-1}+\alpha(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = b_{t-1}+\beta(\ell_{t-1}+b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$
A$_d$	$y_t = (\ell_{t-1}+\phi b_{t-1})(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1}+s_{t-m})(1+\varepsilon_t)$ $\ell_t = \ell_{t-1}+\phi b_{t-1}+\alpha(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$ $s_t = s_{t-m}+\gamma(\ell_{t-1}+\phi b_{t-1}+s_{t-m})\varepsilon_t$	$y_t = (\ell_{t-1}+\phi b_{t-1})s_{t-m}(1+\varepsilon_t)$ $\ell_t = (\ell_{t-1}+\phi b_{t-1})(1+\alpha\varepsilon_t)$ $b_t = \phi b_{t-1}+\beta(\ell_{t-1}+\phi b_{t-1})\varepsilon_t$ $s_t = s_{t-m}(1+\gamma\varepsilon_t)$

Estimating ETS models

$SSE = \sum_{t=1}^{T} (y_t - y_{t|t-1})^2$

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, b_0, $s_0, s_{-1}, \ldots, s_{-m+1}$ are estimated by maximising the "likelihood" $=$ the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

Likelihood

1) $y_1, y_2, \ldots, y_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\mathcal{L}(\mu, \sigma^2 \mid \vec{y}) = \prod_{i=1}^{n} \phi(y_i ; \mu, \sigma^2)$$

↳ Normal distribution density

2) $y_i \sim N(x_i'\beta, \sigma^2)$

$$\mathcal{L}(\beta, \sigma^2 \mid \vec{y}) = \prod_{i=1}^{n} \phi(y_i ; x_i'\beta, \sigma^2)$$

3) ETS(A, N, N), what is $\mathcal{L}$?

ETS(A, A, N), what is $\mathcal{L}$?

Variance of
Forecast error

Innovations state space models

Let $\boldsymbol{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$ and $\varepsilon_t \overset{\text{iid}}{\sim} \text{N}(0, \sigma^2)$.

$$
\begin{aligned}
y_t &= \underbrace{h(\boldsymbol{x}_{t-1})}_{\mu_t} + \underbrace{k(\boldsymbol{x}_{t-1})\varepsilon_t}_{e_t} \\
\boldsymbol{x}_t &= f(\boldsymbol{x}_{t-1}) + g(\boldsymbol{x}_{t-1})\varepsilon_t
\end{aligned}
$$

ETS(A,A,N)

$$y_t = h(\ell_{t-1}, b_{t-1}) + \varepsilon_t$$

$$\begin{pmatrix} \ell_t \\ b_t \end{pmatrix} = \begin{pmatrix} \ell_{t-1} + b_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \varepsilon_t$$

Additive errors

$$k(x) = 1. \qquad y_t = \mu_t + \varepsilon_t.$$

Multiplicative errors

$$k(\boldsymbol{x}_{t-1}) = \mu_t. \qquad y_t = \mu_t(1 + \varepsilon_t).$$

$\varepsilon_t = (y_t - \mu_t)/\mu_t$ is relative error.

Innovations state space models

Estimation

$$L^*(\boldsymbol{\theta}, \boldsymbol{x}_0) = n \log\left(\sum_{t=1}^{n} \varepsilon_t^2 / k^2(\boldsymbol{x}_{t-1}) \right) + 2 \sum_{t=1}^{n} \log |k(\boldsymbol{x}_{t-1})|$$
$$= -2 \log(\text{Likelihood}) + \text{constant}$$

- Estimate parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \phi)$ and initial states $\boldsymbol{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$ by minimizing L^*.

Parameter restrictions

Usual region

- Traditional restrictions in the methods
  $0 < \alpha, \beta^*, \gamma^*, \phi < 1$
  (equations interpreted as weighted averages).
- In models we set $\beta = \alpha\beta^*$ and $\gamma = (1 - \alpha)\gamma^*$.
- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

Parameter restrictions

Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts.
- Usually (but not always) less restrictive than the *traditional* region.
- For example for ETS(A,N,N):
  traditional $0 < \alpha < 1$ — *admissible* is $0 < \alpha < 2$.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.
- These are: ETS(A,N,M), ETS(A,A,M), ETS(A,A_d,M).
- Models with multiplicative errors are useful for strictly positive data, but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.

Exponential smoothing models

Additive Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	~~A,N,M~~
A	(Additive)	A,A,N	A,A,A	~~A,A,M~~
A_d	(Additive damped)	A,A_d,N	A,A_d,A	~~A,A_d,M~~

Multiplicative Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Example: Australian holiday tourism

```
fit <- aus_holidays %>% model(ETS(Trips))
report(fit)
```

```
## Series: Trips
## Model: ETS(M,N,M)
##   Smoothing parameters:
##     alpha = 0.3578
##     gamma = 0.0009686
##
##   Initial states:
##     l     s1     s2     s3     s4
##   9667 0.943 0.9268 0.9684 1.162
##
##   sigma^2:  0.0022
##
##  AIC AICc  BIC
## 1331 1333 1348
```

Example: Australian holiday tourism

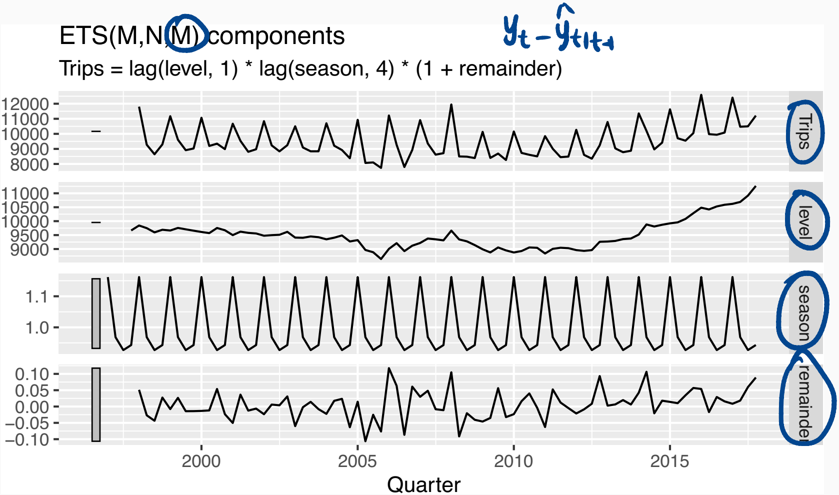
Model selected: ETS(M,N,M)

$$y_t = \ell_{t-1}s_{t-m}(1 + \varepsilon_t)$$
$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$
$$s_t = s_{t-m}(1 + \gamma\varepsilon_t).$$

$\hat{\alpha} = 0.3578$, and $\hat{\gamma} = 0.000969$.

Example: Australian holiday tourism

```
components(fit)
```



Residuals

Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

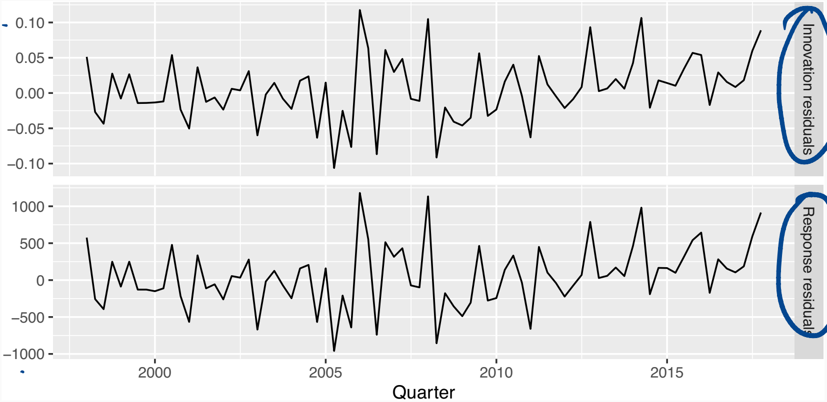
$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$

$y_t = \mu_t (1 + \varepsilon_t)$

$\varepsilon_t = \frac{y_t}{\mu_t} - 1 = \frac{y_t - \mu_t}{\mu_t}$

Example: Australian holiday tourism

```r
residuals(fit)
residuals(fit, type = "response")
```



Outline

Forecasting with ETS models

Point forecasts: iterate the equations for $t = T+1, T+2, \dots, T+h$ and set all $\varepsilon_t = 0$ for $t > T$.

- Not the same as $\text{E}(y_{t+h}|\boldsymbol{x}_t)$ unless trend and seasonality are both additive.
- Point forecasts for ETS(A,*,*) are identical to ETS(M,*,*) if the parameters are the same.

Example: ETS(A,A,N)

$$
\begin{aligned}
y_{T+1} &= \ell_T + b_T + \varepsilon_{T+1} \\
\hat{y}_{T+1|T} &= \ell_T + b_T \\
y_{T+2} &= \ell_{T+1} + b_{T+1} + \varepsilon_{T+2} \\
&= (\ell_T + b_T + \alpha\varepsilon_{T+1}) + (b_T + \beta\varepsilon_{T+1}) + \varepsilon_{T+2} \\
\hat{y}_{T+2|T} &= \ell_T + 2b_T
\end{aligned}
$$

etc.

Example: ETS(M,A,N)

$$
\begin{aligned}
y_{T+1} &= (\ell_T + b_T)(1 + \varepsilon_{T+1}) \\
\hat{y}_{T+1|T} &= \ell_T + b_T. \\
y_{T+2} &= (\ell_{T+1} + b_{T+1})(1 + \varepsilon_{T+2}) \\
&= \{(\ell_T + b_T)(1 + \alpha\varepsilon_{T+1}) + [b_T + \beta(\ell_T + b_T)\varepsilon_{T+1}]\} (
\\
\hat{y}_{T+2|T} &= \ell_T + 2b_T
\end{aligned}
$$

etc.

Forecasting with ETS models

Prediction intervals: can only generated using the models.

$y_{T+1} = \ell_T + \varepsilon_{T+1}$

$y_{T+2} = \ell_{T+1} + \varepsilon_{T+2}$; $\ell_{T+1} = \ell_T + \alpha\varepsilon_{T+1}$

- The prediction intervals will differ between models with additive and multiplicative errors.
- Exact formulae for some models.
- More general to simulate future sample paths, conditional on the last estimate of the states, ℓ_T and to obtain prediction intervals from the percentiles of these simulated future paths.

① Simulate 100 r.v from $N(0, \sigma^2)$. Denote them as $\varepsilon^{(i)}_{T+1}$, $i = 1, 2, \ldots 100$

② Let $\ell^{(i)}_{T+1} = \ell_T + \alpha\varepsilon^{(i)}_{T+1}$; ③ $y^{(i)}_{T+2} = \ell^{(i)}_{T+1} + \varepsilon^{(i)}_{T+2}$

Prediction intervals

PI for most ETS models: $\hat{y}_{T+h|T} \pm c\sigma_h$, where c depends on coverage probability and σ_h is forecast standard deviation.

(A,N,N) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1)\Big]$

(A,A,N) $\quad \sigma_h = \sigma^2\Big[1 + (h-1)\big\{\alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1)\big\}\Big]$

(A,A$_d$,N) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2}\left\{2\alpha(1-\phi) + \beta\phi\right\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\right\}\Big]$

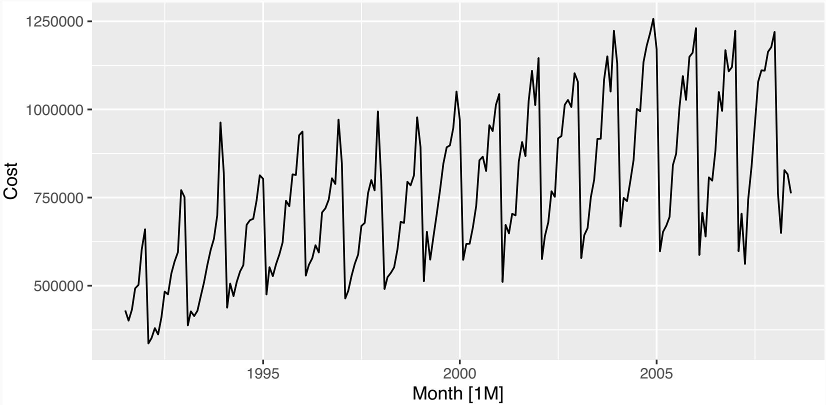
(A,N,A) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1) + \gamma k(2\alpha+\gamma)\Big]$

(A,A,A) $\quad \sigma_h = \sigma^2\Big[1 + (h-1)\big\{\alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1)\big\} + \gamma k\big\{2\alpha + \gamma + \beta m(k$

(A,A$_d$,A) $\quad \sigma_h = \sigma^2\Big[1 + \alpha^2(h-1) + \frac{\beta\phi h}{(1-\phi)^2}\left\{2\alpha(1-\phi) + \beta\phi\right\} - \frac{\beta\phi(1-\phi^h)}{(1-\phi)^2(1-\phi^2)}\left\{2\alpha(1-\phi^2) + \beta\phi(1+2\phi-\phi^h)\right\} + \gamma k(2\alpha+\gamma) + \frac{2\beta\gamma\phi}{(1-\phi)(1-\phi^m)}\left\{k(1-\phi^m) - \phi^m(1-\phi^{mk})\right\}\Big]$

Example: Corticosteroid drug sales

```
h02 <- tsibbledata::PBS %>%
  filter(ATC2 == "H02") %>%
  summarise(Cost = sum(Cost))
h02 %>%
  autoplot(Cost)
```



Example: Corticosteroid drug sales

```r
h02 %>% model(ETS(Cost)) %>% report
```

```
## Series: Cost
## Model: ETS(M,Ad,M)
##   Smoothing parameters:
##     alpha = 0.3071
##     beta  = 0.0001007
##     gamma = 0.0001007
##     phi   = 0.9775
##
##   Initial states:
##         l    b     s1    s2     s3     s4     s5    s6
##   417269 8206 0.8717 0.826 0.7563 0.7733 0.6872 1.284
##      s7   s8    s9   s10   s11    s12
##   1.325 1.18 1.164 1.105 1.048 0.9806
##
##   sigma^2:  0.0046
##
##   AIC AICc  BIC
## 5515 5519 5575
```

ETS(M,A,M)

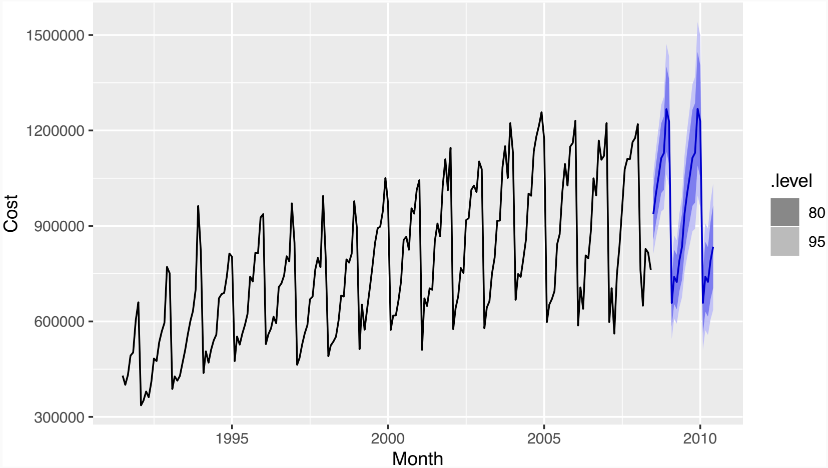
Example: Corticosteroid drug sales

```
h02 %>% model(ETS(Cost ~ error("A") + trend("A") + season("A"))) %>% report
```

```
## Series: Cost
## Model: ETS(A,A,A)
##   Smoothing parameters:
##     alpha = 0.1702
##     beta  = 0.006311
##     gamma = 0.4546
##
##   Initial states:
##        l     b     s1       s2       s3       s4       s5
##   409706 9097 -99075 -136602 -191496 -174531 -241437
##       s6      s7      s8      s9    s10    s11     s12
##   210644 244644 145368 130570 84458 39132 -11674
##
##   sigma^2:  3.499e+09
##
##   AIC AICc  BIC
## 5585 5589 5642
```

Example: Corticosteroid drug sales

```
h02 %>% model(ETS(Cost)) %>% forecast() %>% autoplot(h02)
```



Example: Corticosteroid drug sales

```
h02 %>%
  model(
    auto = ETS(Cost),
    AAA = ETS(Cost ~ error("A") + trend("A") + season("A"))
  ) %>%
  accuracy()
```

Model	ME	MAE	RMSE	MAPE	MASE
auto	2461	38649	51102	4.989	0.6376
AAA	-5780	43378	56784	6.048	0.7156

Your turn

- Use `ETS()` on some of these series:

  `tourism`, *`gafa_stock`*, *`pelt`*

- Does it always give good forecasts?
- Find an example where it does not work well. Can you figure out why?

Likelihood for ETS(A,N,N)

$y_t = \ell_{t-1} + \varepsilon_t$

$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

① y_t is dependent with $y_{t-1}, y_{t-2}, \dots y_1$

$\underline{Cov(y_t, y_{t-1})} = cov(\ell_{t-1}+\varepsilon_t, \ell_{t-2}+\varepsilon_{t-1})$

$= cov(\ell_{t-2}+\varepsilon_t+\alpha\varepsilon_{t-1}, \ell_{t-2}+\varepsilon_{t-1})$

$= cov(\ell_{t-2}, \ell_{t-2}) + cov(\ell_{t-2}, \varepsilon_{t-1}) + cov(\varepsilon_t, \ell_{t-2})$
$+ cov(\varepsilon_t, \varepsilon_{t-1}) + \alpha\, cov(\varepsilon_{t-1}, \ell_{t-2}) + \alpha^2 cov(\varepsilon_{t-1}, \varepsilon_{t-1})$

$= Var(\ell_{t-2}) + 0 + 0 + 0 + 0 + \alpha^2\sigma^2 \neq 0$

Therefore y_t and y_{t-1} are dependent

② $L(\alpha, \ell_0 | \mathcal{D}) = f(y_1, y_2, \dots y_T)$ ($\mathcal{D} \to (y_1, y_2, \dots y_T)$)

③ $f(y_1, y_2, \dots y_T)$

$= f(y_1) \cdot f(y_2|y_1) \cdot f(y_3|y_2, y_1) \cdots f(y_T|y_{T-1}, y_{T-2}, \dots y_1)$

$f(y_1) \cdot \frac{f(y_1, y_2)}{f(y_1)} \cdot \frac{f(y_1, y_2, y_3)}{f(y_1, y_2)} \cdots \frac{f(y_1, \dots y_T)}{f(y_1, \dots y_{T-1})}$

A and B are independent if and only if

$P(A|B) = P(A)$ ✓

$P(B|A) = P(B)$ ✓

$P(A \cap B) = P(A)P(B)$

X and Y are ind. if only if

$f(x|y) = f(x)$ ✓

$f(y|x) = f(y)$ ✓

$f(y, x) = f(x)f(y)$

If $cov(X, Y) \neq 0$

$\Rightarrow$ X and Y are dependent

$$y_1 = \ell_0 + \varepsilon_1 \; ; \; \varepsilon_1 \sim N(0, \sigma^2)$$

$$f(y_1) = \phi(y_1; \ell_0, \sigma^2)$$

$$y_2 = \ell_1 + \varepsilon_2 \qquad y_2 | \ell_1 \sim N(\ell_1, \sigma^2) \qquad \ell_1 = \ell_0 + \alpha \varepsilon_1$$

$$\ell_2 = \ell_1 + \alpha \varepsilon_2 \qquad \ell_1 = \alpha y_1 + (1-\alpha)\ell_0 \qquad y_1 = \ell_0 + \varepsilon_1$$

$$= \varepsilon_1 = y_1 - \ell_0$$

$$f(y_2 | y_1) = \phi(y_2; \alpha y_1 + (1-\alpha)\ell_0, \sigma^2) \qquad \ell_1 = \ell_0 + \alpha(y_1 - \ell_0)$$

$$= \alpha y_1 + (1-\alpha)\ell_0$$

$$f(y_3 | y_2, y_1) = \phi(y_3; \alpha^2 y_1 + \alpha y_2 + \alpha(1-\alpha)\ell_0, \sigma^2)$$

$$y_3 = \ell_2 + \varepsilon_3$$

$$\ell_2 = \alpha^2 y_1 + \alpha(1-\alpha)\ell_0 + \alpha y_2$$

$$L = \phi(y_1; \ell_0, \sigma^2)\, \phi(y_2; \alpha y_1 + (1-\alpha)\ell_0, \sigma^2) \cdot \phi(y_3; \alpha^2 y_1 + \alpha y_2 + \alpha(1-\alpha)\ell_0, \sigma^2)$$

$$\ldots \; \phi(y_T; \alpha y_{T-1} + \alpha^2 y_{T-2} + \ldots (1-\alpha)^T \ell_0, \sigma^2)$$

$\hat{y}_{t|t-1} = l_{t-1} + b_{t-1}$

$l_{t-1} =$

$b_{t-1} =$

For general ETS models

$L(\theta | \mathcal{D}) = f(y_1, y_2, \ldots y_T)$ (θ → all parameters)

$= f(y_1)\, f(y_2|y_1) \cdots f(y_T | y_1, \ldots y_{T-1})$

$f(y_t | y_1, \ldots y_{t-1}) = \phi(y_t; \hat{y}_{t|t-1}, \hat{\sigma}^2_{t|t-1})$

For ETS (A,N,N)

$\hat{y}_{t|t-1} = \alpha y_{t-1} + \alpha^2 y_{t-2} + \ldots (1-\alpha)^T l_0$

$\hat{y}_{t|t-1} = l_{t-1}$

$l_t = \alpha y_t + (1-\alpha) l_{t-1}$

$l_{t-1} = \alpha y_{t-1} + (1-\alpha) l_{t-2}$

$= \alpha y_{t-1} + (1-\alpha)(\alpha y_{t-2} + (1-\alpha) l_{t-3})$

$\vdots$

$= h(y_{t-1}, y_{t-2}, \ldots y_1, l_0)$

ETS(A, N, N)

$$\left.\begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + \alpha\varepsilon_t \end{aligned}\right\} \Rightarrow \begin{aligned} y_t &= \ell_{t-1} + \varepsilon_t \\ &= \ell_{t-2} + \alpha\varepsilon_{t-1} + \varepsilon_t \\ &= \ell_{t-3} + \alpha\varepsilon_{t-2} + \alpha\varepsilon_{t-1} + \varepsilon_t \\ &= \cdots \\ &= \ell_0 + \alpha(\varepsilon_1 + \varepsilon_2 + \cdots \varepsilon_{t-1}) + \varepsilon_t \end{aligned}$$

Hence marginally $y_t \sim N\left(\ell_0, (t-1)\alpha^2\sigma^2 + \sigma^2\right), t = 1, \cdots T$

For $i < j$, $\text{cov}(y_i, y_j) = \text{cov}\left(\ell_0 + \alpha(\varepsilon_1 + \cdots \varepsilon_{i-1}) + \varepsilon_i, \ \ell_0 + \alpha(\varepsilon_1 + \cdots \varepsilon_{j-1}) + \varepsilon_j\right)$

$$= (i-1)\alpha^2\sigma^2 + \alpha\sigma^2$$

Hence
$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \\ y_T \end{pmatrix} \sim N\left(\begin{pmatrix} \ell_0 \\ \ell_0 \\ \vdots \\ \\ \ell_0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \alpha\sigma^2 & \cdots & (T-1)\alpha^2\sigma^2 + \alpha\sigma^2 \\ \alpha\sigma^2 & \alpha^2\sigma^2 + \sigma^2 & & \\ & & \ddots & \\ (T-1)\alpha^2\sigma^2 + \alpha\sigma^2 & & & (T-1)\alpha^2\sigma^2 + \sigma^2 \end{pmatrix}\right)$$

The likelihood is then the multinormal density.

Denote L^*

$$= f(l_1, y_1, l_2, y_2, \dots l_T, y_T)$$

$$= f(l_1, y_1)\, f(l_2, y_2 | l_1, y_1)\, f(l_3, y_3 | l_1, y_1, l_2, y_2) \cdots$$

$$f(l_T, y_T | \square)$$

$$\left.\begin{aligned} y_t &= l_{t-1} + \varepsilon_t \\ l_t &= l_{t-1} + \alpha \varepsilon_t \end{aligned}\right\} \Rightarrow \left(\begin{matrix} y_t \\ l_t \end{matrix} \middle| l_{t-1} \right) \sim N\left(\begin{pmatrix} l_{t-1} \\ l_{t-1} \end{pmatrix}, \begin{pmatrix} \sigma^2 & \alpha\sigma^2 \\ \alpha\sigma^2 & \alpha^2\sigma^2 \end{pmatrix} \right)$$

Hence

$$L^* = \phi\left(\begin{pmatrix} l_0 \\ l_0 \end{pmatrix}, \Sigma \right) \cdot \phi\left(\begin{pmatrix} l_1 \\ l_1 \end{pmatrix}, \Sigma \right) \cdots \phi\left(\begin{pmatrix} l_{T-1} \\ l_{T-1} \end{pmatrix}, \Sigma \right)$$

The marginal likelihood of $y_1, \dots y_T$ is

$$L = \int L^*(l_1, l_2, \dots l_{T-1})\, dl_1\, dl_2 \cdots dl_{T-1}$$

ETS (A, A, N)

$$\left.\begin{aligned} y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\ \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\ b_t &= b_{t-1} + \gamma\varepsilon_t \end{aligned}\right\} \Rightarrow$$

$$\ell_{t-1} = \ell_{t-2} + b_{t-2} + \alpha\varepsilon_{t-1}$$
$$b_{t-1} = b_{t-2} + \gamma\varepsilon_{t-1}$$

① $y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$

$= \ell_{t-2} + b_{t-2} + \alpha\varepsilon_{t-1} + b_{t-2} + \gamma\varepsilon_{t-1} + \varepsilon_t$

$= \ell_{t-2} + 2b_{t-2} + (\alpha+\gamma)\varepsilon_{t-1} + \varepsilon_t$

$= \ell_{t-3} + b_{t-3} + \alpha\varepsilon_{t-2} + 2(b_{t-3} + \gamma\varepsilon_{t-2}) + (\alpha+\gamma)\varepsilon_{t-1} + \varepsilon_t$

$= \ell_{t-3} + 3b_{t-3} + (\alpha+2\gamma)\varepsilon_{t-2} + (\alpha+\gamma)\varepsilon_{t-1} + \varepsilon_t$

$=$

$\vdots$

$= \ell_0 + tb_0 + [\alpha + (t-1)\gamma]\varepsilon_1 + [\alpha + (t-2)\gamma]\varepsilon_2 + \cdots$
$(\alpha+\gamma)\varepsilon_{t-1} + \varepsilon_t$

From ①, $y_t \sim N(\ell_0 + tb_0,\ [\alpha+(t-1)\gamma]^2\sigma^2 + [\alpha+(t-2)\gamma]^2\sigma^2 + \cdots$
$(\alpha+\gamma)^2\sigma^2 + \sigma^2)$

For $i<j$ $\quad cov(y_i, y_j) = \sum_{k=1}^{i-1} [\alpha + (k-1)\gamma]^2\sigma^2 + [\alpha + (i-1)\gamma]\sigma^2$

$$L^* = f(y_1, l_1, b_1 \mid l_0, b_0)\, f(y_2, l_2, b_2 \mid l_1, b_1, y_1, b_0, l_0) \cdots$$

$$f(y_T, l_T, b_T \mid l_0, b_0, l_1, b_1, y_1, \cdots l_{T-1}, b_{T-1}, y_{T-1})$$

$$f(y_t, l_t, b_t \mid l_{t-1}, b_{t-1}) = \phi\left(\begin{pmatrix} l_{t-1} + b_{t-1} \\ l_{t-1} + b_{t-1} \\ b_{t-1} \end{pmatrix}, \Sigma \right)$$

where $\Sigma = \begin{pmatrix} \sigma^2 & \alpha\sigma^2 & \beta\sigma^2 \\ \alpha\sigma^2 & \alpha\sigma^2 & \alpha\beta\sigma^2 \\ \beta\sigma^2 & \alpha\beta\sigma^2 & \beta\sigma^2 \end{pmatrix}$

Marginal likelihood for $(y_1, y_2, \cdots y_T)$

$$L = \int L^*(l_1, b_1, l_2, b_2, \cdots l_{T-1}, b_{T-1} \mid y_1, y_2, \cdots y_T)\, dl_1\, db_1 \cdots dl_{T-1}, b_{T-1}$$