Mathematics from a Historical Perspective
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Contents

1 Thales of Miletus and Pythagoras of Samos

Thales (622–547 B.C.) traveled to Egypt, where a learned geometry.
Pythagoras, 505–500 B.C.

A list of geometric propositions:

1. If two straight lines intersect, the opposite angles are equal.
2. If two parallel lines are intersected by a straight line, then corresponding angles are equal. This can be used to show:
3. The sum of the three angles of a triangle add up to 180 degrees.
4. In an isosceles triangle the angles at the base are equal to each other.
5. Thales Theorem: An angle inscribed in a semi–circle is a right angle.
6. Two triangles are similar if they have the same angles. The lengths (a, b, c and A, B, C) of the sides of two similar triangles are proportional, i.e., $a/b = A/B$ etc.

   This result can be used to give a nice proof of the Theorem of Pythagoras. See Burton, p. 159.
7. The Theorem of Pythagoras
   If a, b, c are the lengths of the sides of a rectangular triangle then

$$a^2 + b^2 = c^2 \ .$$

2 Euclidean Geometry and Beyond

Euclid lived in Alexandria, Egypt, 323–285 B.C.

1. Constructions with straight edge and compass
   Examples: Perpendiculars and parallels

2. The sum of angles in a triangle

3. The theorem of Thales of Miletus
   Thales of Miletus lived in an area which is now in Turkey; 624–545 BC.

4. The inscribed angle theorem

5. The theorem of Pythagoras
   If a, b, c are the lengths of the sides of a rectangular triangle then

$$c^2 = a^2 + b^2 \ .$$

6. Law of cosines
   Let a, b, c denote the lengths of the sides of a triangle. Then

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

   where γ is the angle opposite to the side of length c.

7. $\sqrt{2}$ is irrational
   Zeno's paradox

8. Archimedes, 287–212 BC
   Archimedes lived in Syracuse, Sicily, an Italian island.

   He studied the volumes and surface areas of spheres.

9. Construction of a regular pentagon
   Which lengths can be constructed?

10. Construction of a regular N–gone
    Carl Friedrich Gauss, 1777–1855, German.

    A regular N–gone can be constructed with straight edge and compass if and only if

$$N = 2^k \, F_1 F_2 \cdots F_n$$

    where $k \in \{0, 1, 2, \ldots\}$ and $F_1, F_2, \ldots, F_n$ are distinct Fermat primes.

3 Introduction to Number Theory

1. Pythagorean triples
   Three positive integers a, b, c form a Pythagorean triple if
   $$a^2 + b^2 = c^2 \ .$$
   Euclid knew that the formulas
   $$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2$$
   give a Pythagorean triple if u and v are positive integers and $u > v$. Are there other Pythagorean triples?

2. Triangular numbers (Burton, p. 95–101)
   For $n \in \mathbb{N}$ the triangular number t_n is
   $$t_n = 1 + 2 + \ldots + n = \frac{1}{2}\, n(n+1) \ .$$
   One can notice geometrically that
   $$t_n + t_{n+1} = (n+1)^2 \quad \text{for all} \quad n \in \mathbb{N} \ .$$
   A formal proof can be given by induction.

   Another observation:
   $$1 + 3 + 5 + \ldots + (2n+1) = n^2 \ .$$
   Again, a formal proof can be given by induction.

   Sum of cubes:
   $$1^3 + 2^3 + 3^3 + \ldots + n^3 = t_n^2 \ .$$
   Proof by induction:

   The equation holds for $n = 1$. Set
   $$Q_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$$
   and assume that for some fixed n we have $Q_n = t_n^2$. Then we have

$$
\begin{aligned}
Q_{n+1} &= Q_n + (n+1)^3 \\
&= \frac{1}{4}\, n^2(n+1)^2 + (n+1)^3 \\
&= \frac{1}{4}\,(n+1)^2\Big(n^2 + 4(n+1)\Big) \\
&= \frac{1}{4}\,(n+1)^2\Big(n^2 + 4n + 4\Big) \\
&= \frac{1}{4}\,(n+1)^2(n+2)^2 \\
&= t_{n+1}^2
\end{aligned}
$$

This completes the induction.

4 Perfect Numbers: From Euclid to Euler

Leonard Euler, Swiss, 1707–1783

Let $d, n \in \mathbb{N}$, i.e., d and n are positive integers. The number d is called a divisor of n if $n = dm$ for some positive integer m. A divisor d of n is called a *proper divisor* of n if $1 \leq d < n$. A number $n \in \mathbb{N}$ is called *perfect* if the proper divisors d of n add up to n, i.e.,

$$\sum_{1 \leq d < n,\, d|n} d = n \ .$$

Example 1: The number $n = 6$ has the proper divisors $1, 2, 3$. Since

$$1 + 2 + 3 = 6$$

the number 6 is perfect.

Example 2: The number $n = 28$ has the proper divisors $1, 2, 4, 7, 14$. Since

$$1 + 2 + 4 + 7 + 14 = 28$$

the number 28 is perfect.

According to Burton, p. 505, Euclid knew the following:

Theorem 4.1 *(Euclid) Let $k \in \mathbb{N}, k \geq 2$. If the number*

$$P := 2^k - 1$$

is prime then the number

$$n = 2^{k-1} P = 2^{k-1} \Big(2^k - 1 \Big) \tag{4.1}$$

is perfect.

Note that for $k = 2$ the number n in (4.1) is $n = 2 \cdot (4 - 1) = 6$ and for $k = 3$ the number is $n = 4 \cdot (8 - 1) = 28$.

Proof: First recall the geometric sum formula

$$1 + a + a^2 + \ldots + a^j = \frac{a^{j+1} - 1}{a - 1} \quad \text{for} \quad a \neq 1 \ . \tag{4.2}$$

The proper divisors of $n = 2^{k-1} P$ are

$$1, 2, 4, \ldots, 2^{k-1} \quad \text{and} \quad P, 2P, 4P, \ldots, 2^{k-2} P \ .$$

Using the geometric sum formula for $a = 2$ one obtains that the proper divisors of n add up to

$$\Big(2^k - 1 \Big) + P \Big(2^{k-1} - 1 \Big) =: S \ .$$

Since $P = 2^k - 1$ one obtains that

$$S = P + P\Big(2^{k-1} - 1\Big) = 2^{k-1} P \ .$$

Since $P = 2^k - 1$ this proves that $S = n$. The number n agrees with the sum of its proper divisors. $\diamond$

Euclid's theorem leads to the question: For which $k \in \mathbb{N}$ is $P = 2^k - 1$ a prime number? The following lemma says that $P = 2^k - 1$ is not prime unless k is prime

Lemma 4.1 *Let $m \in \mathbb{N}$ be composite. Then $2^m - 1$ is not prime.*

Proof: Let $m = sk$ where $s, k \in \mathbb{N}, s \geq 2, k \geq 2$. Then we have

$$2^m = (2^s)^k = a^k \quad \text{with} \quad a := 2^s \geq 4 \ .$$

Using the geometric sum formula (4.2) one obtains that

$$2^m - 1 = a^k - 1 = (a-1)\Big(1 + a + \ldots + a^{k-1}\Big) =: AB \ .$$

Here $A = a - 1 \geq 3$ and $B = 1 + a + \ldots + a^{k-1} \geq 5$. This proves that $2^m - 1$ is composite if m is composite. $\diamond$

Euler proved that Euclid's formula (4.1) gives *all even perfect numbers.*

Theorem 4.2 *(Euler) If $n \in \mathbb{N}$ is an even perfect number then the formula*

$$n = 2^{k-1}\Big(2^k - 1\Big)$$

holds for some prime number k where $P := 2^k - 1$ is prime.

To prove Euler's Theorem we will use the **divisor function**:

$$\sigma(n) = \sum_{1 \leq d \leq n,\, d|n} d$$

defined for $n \in \mathbb{N}$. Note that n is a perfect number if and only if $\sigma(n) = 2n$.

If $m, n \in \mathbb{N}$ then the greatest common divisor of m and n is the largest integer which divides both m and n. The greatest common divisor is often denote by $gcd(m, n)$.

Example: Let $m = p$ and $n = q$ denote two distinct primes. We have $gcd(p, q) = 1$. Also,

$$\sigma(p) = 1 + p, \quad \sigma(q) = 1 + q \ ,$$

and

$$\sigma(pq) = 1 + p + q + pg = (1+p)(1+q) = \sigma(p)\sigma(q) \ .$$

The following theorem generalizes this result:

Theorem 4.3 *Let $m, n \in \mathbb{N}$ and assume that $gcd(m,n) = 1$. Then we have*

$$\sigma(m)\sigma(n) = \sigma(mn) .$$

Proof: Let $d_1, \ldots, d_k$ denote all divisors of m and let $q_1, \ldots, q_l$ denote all divisors of n. Then we have

$$\sigma(m) = \sum_{i=1}^{k} d_i \quad \sigma(n) = \sum_{j=1}^{l} q_j$$

and

$$\sigma(m)\sigma(n) = \sum_{i=1}^{k}\sum_{j=1}^{l} d_i q_j .$$

Here the products $d_i q_j$ are all the distinct divisors of mn. The claim follows. $\diamond$

Proof of Euler's Theorem: Assume that n is an even perfect number. Write n in the form

$$n = 2^{k-1}x$$

where $x \in \mathbb{N}$ is odd. We will prove that $x = 2^k - 1$.

In the formula $n = 2^{k-1}x$ the integer k is greater than or equal to 2 since, by assumption, the number n is even. Since n is perfect we have

$$2n = \sigma(n) = \sigma(2^{k-1}x) = \sigma\Big(2^{k-1}\Big)\sigma(x) .$$

Using the geometic sum formula one obtains that

$$\sigma\Big(2^{k-1}\Big) = 2^k - 1 .$$

Therefore,

$$2^k x = 2n = \Big(2^k - 1\Big)\sigma(x) . \tag{4.3}$$

The number $2^k - 1$ is odd. The above equation implies that $2^k - 1$ divides x. We set

$$y := \frac{x}{2^k - 1}$$

and note that $y \in \mathbb{N}$. If we can show that $y = 1$ then we obtain that $x = 2^k - 1$ and $n = 2^{k-1}x = 2^{k-1}\Big(2^k - 1\Big)$. This is the claim of Euler's Theorem. Thus, it remains to prove that $y = 1$.

From (4.3) we have

$$2^k \frac{x}{2^k - 1} = \sigma(x) ,$$

thus

$$2^k y = \sigma(x) \ . \tag{4.4}$$

Also, the equation

$$x = y\Big(2^k - 1\Big)$$

follows from the definition of y.

The number x has at least the divisors y and x. (Since $k \geq 2$ we have $x > y$, thus $x \neq y$.)

It follows that

$$\sigma(x) = x + y + S$$

where $S = 0$ or S is the sum of the divisors of x different from y and x if such divisors exist.

Since $x = y\Big(2^k - 1\Big)$ one obtains that

$$\sigma(x) = x + y + S = y\Big(2^k - 1\Big) + y + S = 2^k y + S \ .$$

But we have $2^k y = \sigma(x)$; see (4.4). Therefore, $S = 0$. This implies that $\sigma(x) = x + y$. The only divisors of x are x and y. Clearly, x has the divisor 1; it follows that $y = 1$. As noted above, this gives us Euler's result that $n = 2^{k-1}\Big(2^k - 1\Big)$. $\diamond$

We summarize the theorems of Euclid and Euler:

Theorem 4.4 *An even number n is perfect if and only if n has the form*

$$n = 2^k\Big(2^k - 1\Big)$$

for some $k \in \mathbb{N}$, where $2^k - 1$ is prime.

Recall Lemma 4.1: For $2^k - 1$ to be prime, it is necessary, but not sufficient, that k is prime.

5 Mersenne Primes and Fermat Primes

5.1 Mersenne Primes

Marin Mersenne, French, 1588–1648. He was a polymath and a Catholic priest, also interested in music theory.

The numbers

$$M_k = 2^k - 1, \quad k = 2, 3, \ldots$$

are called Mersenne numbers. If M_k is prime, then M_k is called a Mersenne prime. The Mersenne primes are important because an even number n is perfect if and only if

$$n = 2^k\, M_k = 2^k \Big(2^k - 1\Big)$$

where $M_k = 2^k - 1$ is prime. This is the Euclid–Euler Theorem.

We have shown in Lemma 4.1 that M_k is not prime unless k is prime. It can be checked that M_k is prime for

$$k = 2, 3, 5, 7, 13, 17, 19, 31 \ .$$

For example,

$$M_2 = 3, M_3 = 7, M_5 = 31, M_7 = 127, M_{13} = 8191, M_{17} = 131071 \ .$$

However,

$$M_{11} = 2^{11} - 1 = 2047 = 23 \cdot 89$$

is composite.

As of 2023, 51 Mersenne primes are known. It is not known if there are infinitely many primes k for which M_k is prime. It is also not known if there are infinitely many primes k for which M_k is not prime.

According to Wikipedia: M_k is prime only for 43 prime numbers k for the first $2 * 10^6$ primes k.

5.2 Fermat Primes

A prime number of the form

$$F_j = 2^{(2^j)} + 1 \quad \text{where} \quad j = 0, 1, 2, \ldots$$

is called a Fermat prime. We have:

j	2^j	$F_j = 2^{2^j} + 1$
0	1	3
1	2	5
2	4	17
3	8	257
4	16	65537

The numbers $F_j = 2^{2^j} + 1$ are prime for $j = 0, 1, 2, 3, 4$. However, in 1732 Euler showed that F_5 is not prime,

$$F_5 = 2^{32} + 1 = 641 \cdot 6700417 \ .$$

It is not known if there exists any Fermat prime F_j with $j > 4$.

Fermat primes are of interest since they are connected with constructability of an N–gone: A regular N–gone can be constructed with straight edge and compass if and only if N has the form

$$N = 2^k \, F_1 F_2 \cdots F_n$$

where $k \in \{0, 1, 2, \ldots\}$ and the numbers $F_1, F_2, \ldots, F_n$ are distinct Fermat primes.

6 Ellipses: Orbits of Planets

1. **Two Definitions of an Ellipse**
   Planes which intersect a circular cone in a closed curve intersect the cone in a circle or an ellipse.

   Another definition of an ellipse is as follows: Let F_1 and F_2 denote two distinct points in a plane E. The points F_1 and F_2 will be the foci of the ellipse. If P is a point in the plane E then let $d_j(P)$ denote the distance between P and F_j for $j = 1, 2$. Let $a > 0$ be a length, larger than the distance between F_1 and F_2. The ellipse with foci F_1 and F_2 and major semi–axis a consists of all points $P \in E$ with

   $$d_1(P) + d_2(P) = 2a \ .$$

   A nice proof that the two definitions of an ellipse agree with each other was given by

   Pierre Dandelin, Belgium, 1794– 1847.

2. **An Ellipse in Cartesian Coordinates**
   Let $a > 0$ and let $0 < \varepsilon < 1$. Let

   $$F_1 = (\varepsilon a, 0), \quad F_2 = (-\varepsilon a, 0)$$

   denote the foci of the ellipse with major semi–axis a. The parameter ε is called the eccentricity of the ellipse. The minor semi–axis of the ellipse is

   $$b = a\sqrt{1 - \varepsilon^2} \ .$$

   One can show that in Cartesian coordinates the ellipse is given by the equation

   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ . \qquad (6.1)$$

   Here $P = (x, y)$ is the general point of the ellipse.

3. **An Ellipse in Polar Coordinates**
   As above, let $a > 0$ and $0 < \varepsilon < 1$. Let

   $$F_1 = (\varepsilon a, 0), \quad F_2 = (-\varepsilon a, 0) \ .$$

   In Cartesian coordinates the ellipse with foci F_1, F_2, with major semi–axis a and minor semi–axis $b = a\sqrt{1 - \varepsilon^2}$ is given by (6.1).

   Take the point $F_1 = (\varepsilon a, 0)$ as the origin of polar coordinates (r, θ), where the angle $\theta = 0$ corresponds to the straight line from the point $F_1 = (\varepsilon a, 0)$ to the point $(a, 0)$.

   One can show: The ellipse given by (6.1) has the equation

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta} \tag{6.2}$$

in the polar coordinates (r, θ).

Note that the origin $(x, y) = (0, 0)$ of the Cartesian coordinates (x, y) used in (6.1) is different from the origin $F_1 = (\varepsilon a, 0)$ of the polar coordinates (r, θ) used in (6.2).

4. **Newton's Derivation of Kepler's Laws**