Partial Differential Equations for Engineering
Math 312, Spring 2021

Jens Lorenz

May 7, 2021

Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 The Heat Equation; Initial and Boundary Conditions

1.1 Derivation of the 3D Heat Equation

Read Haberman Sect. 1.2, 1.5

We assume that the specific heat c, the material density ρ, and the thermal conductivity K_0 are constants.

Let $\mathbf{x} = (x, y, z)$ denote the space variable and let t denote the time variable. The temperature function is

$$u(\mathbf{x}, t) = u(x, y, z, t)$$

and

$$e(\mathbf{x}, t) = c\rho\, u(\mathbf{x}, t)$$

is the thermal energy. By Fourier's law of heat conduction, the vector function

$$\mathbf{q}(\mathbf{x}, t) = -K_0 \nabla u(\mathbf{x}, t) \tag{1.1}$$

is the heat flux. Here

$$\nabla u(\mathbf{x}, t) = (u_x, u_y, u_z)(\mathbf{x}, t)$$

denotes the gradient of the temperature function $u(\mathbf{x}, t)$. Note that $\nabla u(\mathbf{x}, t)$ points in the direction of increasing temperature and $\mathbf{q}(\mathbf{x}, t)$ points in the opposite direction.

Let B denote a rectangular box with sides of length $\Delta x, \Delta y, \Delta z$. With $\mathbf{n} = \mathbf{n}(\mathbf{x})$ we denote the unit outward normal at the point $\mathbf{x}$ on the surface S of B. We will use the temperature function $u(\mathbf{x}, t)$ to compute the heat energy H which flows into B through the surface S in the time interval from t to $t + \Delta t$.

The box B has six rectangular surfaces. Let us consider the sides S_1 and S_2 which are perpendicular to $\mathbf{i}$. On S_1 we have $\mathbf{n} = -\mathbf{i}$, on S_2 we have $\mathbf{n} = \mathbf{i}$.

Let

$$\mathbf{x} = (x, y, z) \quad \text{and} \quad \mathbf{x} + \Delta x\, \mathbf{i} = (x + \Delta x, y, z)$$

denote two corners of the box B. To leading order, the heat energy entering the box through the surface S_1 in the time interval $[t, t + \Delta t]$ is

$$H_1 = \mathbf{i} \cdot \mathbf{q}(x, y, z, t)\, \Delta y \Delta z \Delta t\ .$$

(Note that the vector $\mathbf{i}$ points inside the box B at the surface S_1.)

The corresponding energy through the surface S_2 is

$$H_2 = -\mathbf{i} \cdot \mathbf{q}(x + \Delta x, y, z, t)\, \Delta y \Delta z \Delta t\ .$$

Using Fourier's law (1.1) one obtains that

$$H_1 + H_2 = K_0 \Big(u_x(x + \Delta x, y, z) - u_x(x, y, z) \Big)\, \Delta y \Delta z \Delta t\ .$$

Here, to leading order,

$$u_x(x+\Delta x,y,z)-u_x(x,y,z)=u_{xx}(x,y,z,t)\,\Delta x\ .$$

This yields that

$$H_1+H_2=K_0u_{xx}(x,y,z,t)\,\Delta V\Delta t$$

where

$$\Delta V=\Delta x\Delta y\Delta z$$

is the volume of the box B. Adding the heat flux through the other four sides of B one obtains that

$$H=K_0\nabla^2u(x,y,x,t)\Delta V\Delta t$$

is the heat energy entering the box B through its surface in the time interval from t to $t+\Delta t$. Here

$$\nabla^2u(x,y,x,t)=(u_{xx}+u_{yy}+u_{zz})(x,y,z,t)$$

denotes the Laplacian of the function $u(x,y,z,t)$.

The heat flux through the surface of B changes the temperature in B. Since $\rho\,\Delta V$ is the mass in B one obtains that

$$c\rho\,\Delta V\Big(u(x,y,z,t+\Delta t)-u(x,y,z,t)\Big)\sim H=K_0\nabla^2u(x,y,x,t)\,\Delta V\Delta t\ .$$

In the limit $\Delta V\to 0,\Delta t\to 0$ one obtains the heat equation,

$$c\rho\,u_t(x,y,z,t)=K_0\nabla^2u(x,y,z,t)\ .$$

The equation is also written as

$$u_t=k(u_{xx}+u_{yy}+u_{zz})\quad\text{with}\quad k=K_0/(c\rho)$$

or

$$u_t=k\,\Delta u$$

where

$$\Delta=\nabla^2$$

is another notation for the Laplace operator.

1.2 Boundary Conditions

Read Haberman Sect. 1.3

Let $D \subset \mathbb{R}^3$ denote a domain with surface $\mathcal{S}$. Let $\mathbf{n}(\mathbf{x})$ denote the unit outward normal at $\mathbf{x} \in \mathcal{S}$.

The heat equation

$$u_t(x, y, z, t) = k \nabla^2 u(x, y, z, t) \quad \text{for} \quad \mathbf{x} \in D, \quad t \geq 0 \ , \tag{1.2}$$

with initial condition

$$u(\mathbf{x}, 0) = F(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in D \ ,$$

has to be supplemented by boundary conditions. The following three boundary conditions are often used in applications.

1. Dirichlet Boundary Conditions

$$u(\mathbf{x}, t) = G(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}, \quad t \geq 0 \ . \tag{1.3}$$

Here $G(\mathbf{x}, t)$ is a given function, the temperature at the boundary surface $\mathcal{S}$ of the domain D.

2. Neumann Boundary Conditions

$$-K_0 \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}, t) = H(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}, \quad t \geq 0 \ . \tag{1.4}$$

The outward component of the heat flux vector $-K_0 \nabla u(\mathbf{x}, t)$ is a given function $H(\mathbf{x}, t)$ at the boundary surface $\mathcal{S}$. If one chooses $H(\mathbf{x}, t) \equiv 0$ then one obtains the homogeneous Neumann boundary condition: No heat flux occurs through the boundary surface $\mathcal{S}$.

3. Newton's Law of Cooling

$$-K_0 \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}, t) = C\Big(u(\mathbf{x}, t) - G(\mathbf{x}, t)\Big) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}, \quad t \geq 0 \ . \tag{1.5}$$

Here the constant $C > 0$ is the heat transfer coefficient, which is assumed to be known. The temperature difference $u(\mathbf{x}, t) - G(\mathbf{x}, t)$ is proportional to the heat flux through the boundary surface. If $G(\mathbf{x}, t) < u(\mathbf{x}, t)$ then heat energy leaves the domain D at the point $\mathbf{x} \in \mathcal{S}$; cooling occurs inside D.

1.3 A 1D Initial–Boundary Value Problem for the Heat Equation

Read Haberman Sect. 2.3

Consider the 1D heat equation

$$u_t(x, t) = k u_{xx}(x, t) \quad \text{for} \quad 0 \leq x \leq L \quad \text{and} \quad t \geq 0 \tag{1.6}$$

with initial condition

$$u(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq L \tag{1.7}$$

and homogeneous Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for} \quad t \ge 0 \; . \tag{1.8}$$

In the PDE (1.6), k is a positive constant assumed to be known.

We first ignore the initial condition and determine solutions of the PDE in separated variables. The boundary conditions will be enforced soon.

Let

$$u(x,t) = \phi(x) G(t) \quad \text{for} \quad 0 \le x \le L \quad \text{and} \quad t \ge 0 \; .$$

The PDE requires that

$$\phi(x) G'(t) = k \phi''(x) G(t) \; ,$$

thus

$$\frac{G'(t)}{G(t)} = k \, \frac{\phi''(x)}{\phi(x)} \; .$$

One obtains that $G'(t)/G(t)$ and $\phi''(x)/\phi(x)$ must be constant:

$$\begin{aligned} \frac{G'(t)}{G(t)} &= -k\lambda \\ \frac{\phi''(x)}{\phi(x)} &= -\lambda \end{aligned}$$

The ODE $G'(t) = -k\lambda \, G(t)$ has the solution

$$G(t) = G(0) e^{-k\lambda t} \; .$$

One expects the solution $u(x,t)$ to decay to zero as $t \to \infty$. Clearly, the function $G(t) = G(0)e^{-k\lambda t}$ decays to zero as $t \to \infty$ if $\lambda > 0$.

The ODE for $\phi(x)$ reads

$$\phi''(x) + \lambda \phi(x) = 0 \; .$$

If $\lambda > 0$ then the general solution is

$$\phi(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$$

where c_1 and c_2 are free constants.

If we impose the boundary conditions

$$\phi(0) = \phi(L) = 0$$

we obtain that $c_1 = 0$. Also, $c_2 = 0$ unless

$$\sin(\sqrt{\lambda}\, L) = 0 \; . \tag{1.9}$$

The above condition requires that

$$\sqrt{\lambda}\,L = n\pi$$

for some $n \in \mathbb{N}$, i.e,

$$\lambda = \lambda_n = (n\pi/L)^2 \quad \text{for} \quad n = 1, 2, 3, \dots$$

We have obtained the following solutions of the heat equation $u_t = k u_{xx}$ satisfying the boundary conditions (1.8):

$$u_n(x,t) = B_n \sin(n\pi x/L) e^{-k(n\pi/L)^2\, t}$$

for $n = 1, 2, 3, \dots$ Here $B_n \in \mathbb{R}$ is an arbitrary constant. Clearly, any finite sum of the form

$$u(x,t) = \sum_{n=1}^{M} B_n \sin(n\pi x/L) e^{-k(n\pi/L)^2\, t}$$

also satisfies the heat equation and the homogeneous Dirichlet boundary conditions (1.8). Here $M \in \mathbb{N}$ is arbitrary.

We can choose $M = \infty$ if the series

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) e^{-k(n\pi/L)^2\, t}$$

converges to a smooth function $u(x,t)$ which can be differentiated term by term. The initial condition (1.7) requires that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \quad \text{for} \quad 0 \le x \le L\ .$$

One can show the following: If $f(x)$ is a continuously differentiable functions satisfying the boundary conditions

$$f(0) = f(L) = 0$$

then the series representation

$$f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \quad \text{for} \quad 0 \le x \le L$$

holds with

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L)\, dx\ .$$

Also, the function $u(x,t)$ defined by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) e^{-k(n\pi/L)^2\, t}$$

is a C^∞ function for $0 \le x \le L$ and $t > 0$. This function $u(x,t)$ is the unique solution of the PDE (1.6) with initial condition (1.7) and boundary condition (1.8).

1.4 More on Eigenvalues and Eigenfunctions

Consider the BVP

$$-\phi''(x) = \lambda\phi(x) \quad \text{for} \quad 0 \le x \le L, \quad \phi(0) = \phi(L) = 0 \ . \tag{1.10}$$

Claim: If $\lambda \le 0$ then $\phi \equiv 0$ is the only solution.

Proof: a) If $\lambda = 0$ then the general solution of the ODE is $\phi(x) = c_1 + c_2 x$. The boundary conditions imply that $c_1 = c_2 = 0$.

b) Let $\lambda = -\omega^2 < 0$ where$\omega > 0$. The general solution of the ODE $\phi'' = \omega^2\phi$ is

$$\phi(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} \ .$$

The boundary conditions require

$$c_1 + c_2 = 0, \quad c_1 e^{\omega L} + c_2 e^{-\omega L} = 0 \ .$$

In matrix form:

$$\begin{pmatrix} 1 & 1 \\ e^{\omega L} & e^{-\omega L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ .$$

The determinant of the matrix is

$$e^{-\omega L} - e^{\omega L} \neq 0$$

and $c_1 = c_2 = 0$ follows.

Exercise: Consider the BVP (1.10) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Prove the $\phi \equiv 0$ is the only solution.

Orthogonality of Eigenfunctions: Let $C[a,b]$ denote the vector space of all continuous functions $\phi : [a,b] \to \mathbb{C}$. For $\phi, \psi \in C[a,b]$ one defines the L_2–inner product by

$$(\phi, \psi) = (\phi, \psi)_{L_2} = \int_a^b \bar{\phi}(x)\psi(x)\,dx \ .$$

(Here $\bar{\phi}(x)$ is the complex conjugate of $\phi(x)$.) The corresponding L_2–norm is

$$\|\phi\| = \sqrt{\int_a^b |\phi(x)|^2\,dx} \ .$$

In the following, let $[a,b] = [0,L]$. We have obtained that the eigenvalue problem (1.10) has the eigenvalues

$$\lambda_n = (n\pi/L)^2, \quad n = 1, 2, 3, \ldots$$

with eigenfunctions

$$\phi_n(x) = \sin(n\pi x/L) \ .$$

Lemma 1.1 *The following orthogonality relation holds*

$$(\phi_m, \phi_n) = \frac{L}{2}\,\delta_{mn} \quad \textit{for} \quad m, n \in \mathbb{N} \ .$$

Here δ_{mn} *is the Kronecker symbol*

$$\delta_{mn} = \begin{cases} 0 & \textit{if} \quad m \neq n \\ 1 & \textit{if} \quad m = n \end{cases} \ .$$

Proof: a) First let $m = n$. We have

$$\begin{aligned} (\phi_n, \phi_n) &= \|\phi_n\|^2 \\ &= \int_0^L \sin^2(n\pi x/L)\, dx \quad (\text{substitute } \xi = \pi x/L) \\ &= \frac{L}{\pi}\int_0^\pi \sin^2(n\xi)\, d\xi \end{aligned}$$

Since $\sin^2(n\xi) + \cos^2(n\xi) = 1$ and since

$$\int_0^\pi \sin^2(n\xi)\, d\xi = \int_0^\pi \cos^2(n\xi)\, d\xi$$

one obtains that $\|\phi_n\|^2 = \frac{L}{2}$.

b) Let $m \neq n$. We have

$$\begin{aligned} (\phi_m, \phi_n) &= \int_0^L \sin(m\pi x/L)\sin(n\pi x/L)\, dx \\ &= \frac{L}{\pi}\int_0^\pi \sin(m\xi)\sin(n\xi)\, d\xi \end{aligned}$$

Using Euler's identity

$$e^{i\alpha} = \cos\alpha + i\sin\alpha, \quad e^{-i\alpha} = \cos\alpha - i\sin\alpha$$

one can express the sine functions by the exponential function and obtain that $(\phi_m, \phi_n) = 0$ for $m \neq n$.

c) Another proof for $m \neq n$ uses that ϕ_m and ϕ_n are eigenfunctions to different eigenvalues,

$$-\phi_m'' = \lambda_m \phi_m, \quad -\phi_n'' = \lambda_n \phi_n \ .$$

Using integration by parts and the boundary conditions on obtains

$$\begin{aligned}
\lambda_m(\phi_m,\phi_n) &= (\lambda_m\phi_m,\phi_n) \\
&= -\int_0^L \phi_m''\phi_n\,dx \\
&= -\int_0^L \phi_m\phi_n''\,dx \\
&= \lambda_n(\phi_m,\phi_n)
\end{aligned}$$

Since $\lambda_m \neq \lambda_n$ one obtains that $(\phi_m,\phi_n)=0$. $\diamond$

1.5 Examples of the 1D Heat Equation

Example 1: Consider the IBVP

$$\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \;, \\
u(x,0) &= \sin x \quad \text{for} \quad 0 \le x \le \pi \;, \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \;.
\end{aligned}$$

Separation of variables

$$u(x,t) = \phi(x)G(t)$$

leads to the eigenvalue problem

$$-\phi''(x) = \lambda\phi(x) \quad \text{for} \quad 0 \le x \le L, \quad \phi(0)=\phi(L)=0$$

and the time–function

$$G(t) = G(0)e^{-\lambda t} \;.$$

The eigenvalue problem has the eigenvalues $\lambda_n = n^2$ and eigenfunctions

$$\phi_n(x) = \sin(nx) \quad \text{for} \quad n = 1,2,\ldots$$

Any function of the form

$$u(x,t) = \sum_{n=1}^{M} B_n \sin(nx) e^{-n^2 t}$$

satisfies the heat equation and the boundary conditions. Given the initial condition $u(x,0)=\sin x$, one obtains the solution

$$u(x,t) = e^{-t}\sin x \;.$$

Example 2: Consider the IBVP

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= \sin^3 x \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ .
\end{aligned}
$$

We will write the initial function $f(x) = \sin^3 x$ as a linear combination of the eigenfunctions $\phi_n(x) = \sin(nx)$.

Claim:

$$
\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin(3x)
$$

Derivation using Euler's identity: From

$$
\sin x = \frac{1}{2i} \left(e^{ix} - e^{-ix}\right)
$$

obtain that

$$
\begin{aligned}
\sin^3 x &= -\frac{1}{8i}\Big(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}\Big) \\
&= -\frac{1}{4} \cdot \frac{1}{2i} \Big(e^{3ix} - e^{-3ix}\Big) + \frac{3}{4} \cdot \frac{1}{2i} \Big(e^{ix} - e^{-ix}\Big) \\
&= -\frac{1}{4} \sin(3x) + \frac{3}{4} \sin x
\end{aligned}
$$

Obtain the following solution of the IBVP:

$$
u(x,t) = \frac{3}{4} (\sin x) e^{-t} - \frac{1}{4} \sin(3x) e^{-9t} \ .
$$

Example 3: Consider the IBVP

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= x(\pi - x) \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ .
\end{aligned}
$$

We will write the initial function $f(x) = x(\pi - x)$ as a series

$$
x(\pi - x) = \sum_{n=1}^{\infty} B_n \sin(nx) \ .
$$

Multiply the above equation by $\sin(jx)$, integrate over $0 \le x \le \pi$, and use Lemma 1.1 to obtain that

$$
\int_0^\pi x(1-x) \sin(jx) \, dx = \frac{\pi}{2} B_j \ .
$$

We now use that the eigenfunction

$$\phi_j(x) = \sin(jx)$$

satisfies $-\phi_j'' = j^2 \phi_j$. Using integration by parts and using that $f''(x) = -2$ holds for the function $f(x) = x(1-x)$ one obtains that

$$\begin{aligned} B_j &= \frac{2}{\pi} \int_0^\pi f(x) \sin(jx)\, dx \\ &= -\frac{2}{\pi j^2} \int_0^\pi f(x) \phi_j''(x)\, dx \\ &= -\frac{2}{\pi j^2} \int_0^\pi f''(x) \phi_j(x)\, dx \\ &= \frac{4}{\pi j^2} \int_0^\pi \sin(jx)\, dx \\ &= -\frac{4}{\pi j^3} \cos(jx)\Big|_0^\pi \end{aligned}$$

Here

$$\cos(j\pi) = \begin{cases} -1 & \text{if} \quad j \text{ is odd} \\ 1 & \text{if} \quad j \text{ is even} \end{cases} .$$

Therefore,

$$\cos(jx)\Big|_0^\pi = \begin{cases} -2 & \text{if} \quad j \text{ is odd} \\ 0 & \text{if} \quad j \text{ is even} \end{cases} .$$

Obtain

$$B_j = \begin{cases} \frac{8}{\pi j^3} & \text{if} \quad j \text{ is odd} \\ 0 & \text{if} \quad j \text{ is even} \end{cases} .$$

One obtains the sine–series representation

$$x(1-x) = \frac{8}{\pi} \sum_{j=0}^\infty \frac{\sin((2j+1)x)}{(2j+1)^3} .$$

The solution of the IBVP is

$$u(x,t) = \frac{8}{\pi} \sum_{j=0}^\infty \frac{\sin((2j+1)x)}{(2j+1)^3}\, e^{-(2j+1)^2 t} .$$

Example 4: Consider the IBVP

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 , \\ u(x,0) &= 1 \quad \text{for} \quad 0 \le x \le \pi , \\ u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 . \end{aligned}$$

We will write the initial function $f(x) = 1$ as a series

$$1 = \sum_{n=1}^{\infty} B_n \sin(nx) \ .$$

We have

$$\begin{aligned} B_n &= \frac{2}{\pi} \int_0^\pi \sin(nx)\, dx \\ &= \frac{2}{\pi} \cdot \frac{1}{n} \Big(-\cos(nx) \Big) \Big|_0^\pi \end{aligned}$$

Thus $B_n = \frac{4}{\pi n}$ if n is odd and $B_n = 0$ if n is even. Obtain

$$\begin{aligned} u(x,t) &= \frac{4}{\pi} \sum_{n=1, n\ odd}^{\infty} \frac{1}{n} e^{-n^2 t} \sin(nx) \\ &= \frac{4}{\pi} \Big(e^{-t} \sin x + \frac{1}{3}\, e^{-9t} \sin(3x) + \ldots \Big) \end{aligned}$$

1.6 Energy Estimate and Uniqueness of Solutions

Consider the IBVP

$$\begin{aligned} u_t(x,t) &= k u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le L, \quad t \ge 0 \ , \\ u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le L \ , \\ u(0,t) &= u(L,0) = 0 \quad \text{for} \quad t \ge 0 \ . \end{aligned}$$

Here k is a given positive constant.

If $u(x,t)$ is a solution then

$$E(t) = \frac{1}{2} \int_0^L u^2(x,t)\, dx = \frac{1}{2} \, \|u(\cdot,t)\|^2$$

is often called the energy at time t, but this is typically not the heat energy. One obtains through integration by parts

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_0^L u(x,t) u_t(x,t)\, dx \\ &= k \int_0^L u(x,t) u_{xx}(x,t)\, dx \\ &= -k \int_0^L (u_x)^2(x,t)\, dx \\ &\le 0 \end{aligned}$$

Therefore, $E(t) \le E(0)$ for $t \ge 0$.

Assume that $u(x,t)$ and $v(x,t)$ are two solutions of the IBVP. Then the function

$$w(x,t) = u(x,t) - v(x,t)$$

solves the IBVP with initial data

$$w(x,0) = 0 \quad \text{for} \quad 0 \le x \le L \ .$$

The energy estimate implies that

$$\int_0^L w^2(x,t)\,dx \le 0 \quad \text{for} \quad t \ge 0 \ ,$$

thus $w \equiv 0, u \equiv v$.

The argument proves that the IBVP cannot have more than one solution.

1.7 The Inhomogeneous Heat Equation

First recall that the homogeneous IVP

$$u'(t) = Lu(t), \quad u(0) = u_0 \ ,$$

has the solution

$$u_{hom}(t) = e^{Lt}u_0$$

and the inhomogeneous IVP

$$u'(t) = Lu(t) + q(t), \quad u(0) = u_0 \ ,$$

has the solution

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}q(s)\,ds \ .$$

The term

$$u_{inh}(t) = \int_0^t e^{L(t-s)}q(s)\,ds$$

satisfies

$$\begin{aligned} u'_{inh}(t) &= q(t) + L\int_0^t e^{L(t-s)}q(s)\,ds \\ &= q(t) + Lu_{inh}(t) \end{aligned}$$

and $u_{inh}(0) = 0$.

The ODE example shows: If one can solve a homogeneous IVP, then one can also solve a corresponding inhomogeneous IVP. The idea works for many PDEs.

Consider the homogeneous heat equation with homogeneous boundary conditions

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\ u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le \pi \ , \\ u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ . \end{aligned}$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2 t}$$

with

$$B_n = \frac{2}{\pi} \int_0^\pi f(y) \sin(ny) \, dy \ .$$

Therefore,

$$\begin{aligned} u(x,t) &= \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin(nx) \int_0^\pi \sin(ny) f(y) \, dy \\ &= \int_0^\pi G(x,y,t) f(y) \, dy \end{aligned}$$

with

$$G(x,y,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin(nx) \sin(ny) \ .$$

One can show that inhomogeneous heat equation (with homogeneous boundary conditions)

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + q(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\ u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le \pi \ , \\ u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ . \end{aligned}$$

has the solution

$$u(x,t) = \int_0^\pi G(x,y,t) f(y) \, dy + \int_0^t \int_0^\pi G(x,y,t-s) q(y,s) \, dy ds \ .$$

This generalizes the ODE formula

$$u(t) = e^{Lt} u_0 + \int_0^t e^{L(t-s)} q(s) \, ds$$

to the inhomogeneous PDE.

Example 5: Consider the IBVP

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t)+\sin(2x) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= 0 \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ .
\end{aligned}
$$

In the following, we will exchange summations and integrations. This can be justified, but we will leave the justification for later.

Set

$$
Int(x,t-s) = \int_0^\pi G(x,y,t-s)\sin(2y)\,dy \ .
$$

Then the solution of the IBVP is

$$
u(x,t) = \int_0^t Int(x,t-s)\,ds \ .
$$

We have

$$
Int = \frac{2}{\pi}\sum_{n=1}^\infty \int_0^\pi \phi_n(x)\phi_n(y)\phi_2(y)\,dy e^{-n^2(t-s)} \ .
$$

Here

$$
\sum_{n=1}^\infty \int_0^\pi \phi_n(y)\phi_2(y)\,dy = \frac{\pi}{2} \ .
$$

Therefore,

$$
Int = \phi_2(x)e^{-4(t-s)} = \sin(2x)e^{-4(t-s)}
$$

and

$$
\begin{aligned}
u(x,t) &= \sin(2x)\int_0^t e^{-4(t-s)}\,ds \\
&= \sin(2x)e^{-4t}\int_0^t e^{4s}\,ds \\
&= \sin(2x)e^{-4t}\frac{1}{4}\,(e^{4t}-1) \\
&= \frac{1}{4}\,\sin(2x)(1-e^{-4t})
\end{aligned}
$$

It is easy to check that the function $u(x,t)$ satisfies the homogeneous initial and boundary conditions. Also,

$$
\begin{aligned}
u_t(x,t) &= \sin(2x)e^{-4t} \\
u_{xx}(x,t) &= -\sin(2x)(1-e^{-4t}) \\
&= \sin(2x)(e^{-4t}-1)
\end{aligned}
$$

Thus

$$u_{xx}(x,t) + \sin(2x) = \sin(2x)e^{-4t} = u_t(x,t) \ .$$

We have checked that the function $u(x,t)$ given above solves the IBVP.

Next consider a problem with **inhomogeneous boundary data:**

$$\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) + q(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= \alpha(t) \quad \text{for} \quad t \ge 0 \\
u(\pi,t) &= \beta(t) \quad \text{for} \quad t \ge 0 \ .
\end{aligned}$$

The function

$$p(x,t) = \alpha(t) + \frac{1}{\pi}\,(\beta(t) - \alpha(t))x$$

satisfies the boundary conditions

$$p(0,t) = \alpha(t) \quad \text{and} \quad p(\pi,t) = \beta(t) \ .$$

If one defines a new unknown function $v(x,t)$ by

$$v(x,t) = u(x,t) - p(x,t)$$

then one obtains an inhomogeneous heat equation for $v(x,t)$ with homogeneous boundary conditions.

1.8 The Heat Equation: Relation to Eigenvalue Problems

Example 6:

Consider the heat equation with initial and boundary conditions:

$$\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le 1, \quad t \ge 0 \ , && (1.11)\\
u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le 1 \ , && (1.12)\\
u(0,t) &= 0 \quad \text{for} \quad t \ge 0 && (1.13)\\
u(1,t) + u_x(1,t) &= 0 \quad \text{for} \quad t \ge 0 \ . && (1.14)
\end{aligned}$$

Recall Newton's law of cooling:

$$-K_0\mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x},t) = C\Big(u(\mathbf{x},t) - g(\mathbf{x},t)\Big) \quad \text{for} \quad \mathbf{x} \in \mathcal{S}, \quad t \ge 0 \ . \qquad (1.15)$$

If we have $\mathbf{n} = \mathbf{i}, g(x,t) \equiv 0$ then Newton's law becomes

$$-Ku_x = Cu, \quad (C/K)u + u_x = 0 \ .$$

If $C/K = 1$ we obtain the boundary condition (1.14).

Using separation of variables, we first obtain solutions of the heat equation satisfying the homogeneous boundary conditions. Let

$$u(x,t) = \phi(x)G(t)$$

and obtain

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{G(t)} = -\lambda \ ,$$

thus

$$G(t) = G(0)e^{-\lambda t} \ .$$

Here λ is the separation constant. If $\lambda = p^2, p > 0$, then the ϕ–equation becomes

$$\phi''(x) + p^2\phi(x) = 0$$

with general solution

$$\phi(x) = a\cos(px) + b\sin(px) \ .$$

The boundary condition $u(0,t) = 0$ yields that $a = 0$. For $b = 1$ one obtains that

$$\phi(x) = \sin(px), \quad \phi'(x) = p\cos(px) \ .$$

The boundary condition (1.14) requires that

$$\sin p + p\cos p = 0 \ ,$$

thus

$$-p = \tan p \ .$$

Sketching the graphs of the functions $\tan p$ and $-p$ for $p > 0$ it is easy to see that the equation $\tan p = -p$ has a sequence of solution p_j with

$$(j - \frac{1}{2})\pi < p_j < j\pi \quad \text{for} \quad j = 1, 2, \ldots$$

and

$$p_j \sim (j - \frac{1}{2})\pi$$

for j large.

We will show below that the eigenvalue problem

$$-\phi''(x) = \lambda\phi(x), \quad \phi(0) = 0 = \phi(1) + \phi'(1)$$

has *only* the eigenvalues $\lambda_j = p_j^2$ where

$$\tan p_j = -p_j, \quad j = 1, 2, \ldots$$

The corresponding eigenfunctions are

$$\phi_j(x) = \sin(p_j x) \ .$$

This suggests to write the solution of the problem (1.11)-(1.14) in the form

$$u(x,t) = \sum_{j=1}^{\infty} b_j \sin(p_j x) e^{-p_j^2 t}$$

where the coefficients b_j must be determined by the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} b_j \sin(p_j x) = f(x) \quad \text{for} \quad 0 \le x \le 1 \ .$$

Notation: Let $C[0,1]$ denote the vector space of all functions $\phi : [0,1] \to \mathbb{C}$ which are continuous. If $\phi, \psi \in C[0,1]$ one defines their inner product by

$$(\phi, \psi) = \int_0^1 \bar{\phi}(x) \psi(x) \, dx \ .$$

Lemma 1.2 *Let*

$$\phi_j(x) = \sin(p_j x), \quad j = 1, 2, \ldots$$

where

$$-\phi_j''(x) = p_j^2 \phi_j(x), \quad \phi_j(0) = 0 = \phi_j(1) + \phi_j'(1) \ .$$

For $j \neq k$ we have orthogonality:

$$(\phi_j, \phi_k) = \delta_{jk} \quad \textit{if} \quad j \neq k \ .$$

Proof: We use integration by parts.

$$\begin{aligned}
-p_j^2 \int_0^1 \phi_j \phi_k \, dx &= \int_0^1 \phi_j'' \phi_k \, dx \\
&= \phi_j' \phi_k |_0^1 - \int_0^1 \phi_j' \phi_k' \, dx \\
&= \phi_j' \phi_k |_0^1 - \phi_j \phi_k' |_0^1 + \int_0^1 \phi_j \phi_k'' \, dx \\
&= \phi_j' \phi_k |_0^1 - \phi_j \phi_k' |_0^1 - p_k^2 \int_0^1 \phi_j \phi_k \, dx
\end{aligned}$$

Here the boundary term BT is zero at $x = 0$, thus

$$BT = \phi_j'(1) \phi_k(1) - \phi_j(1) \phi_k'(1) \ .$$

Using the boundary conditions

$$\phi_j(1) + \phi_j'(1) = 0 = \phi_k(1) + \phi_k'(1)$$

it is easy to check that the boundary term is zero, $BT = 0$. One obtains the equation

$$-p_j^2(\phi_j, \phi_k) = -p_k^2(\phi_j, \phi_k) \ .$$

Since $p_j^2 \neq p_k^2$ for $j \neq k$ the orthogonality follows. $\diamond$

If one assumes the expansion

$$\sum_{j=1}^\infty b_j \sin(p_j x) = f(x) \quad \text{for} \quad 0 \le x \le 1$$

then one obtains that

$$\int_0^1 f(x) \sin(p_k x) \, dx = b_k \int_0^1 \sin^2(p_k x) \, dx \ .$$

From this equation one can compute b_k.

Assume that the coefficients b_j are computed in this way. One can prove: If $f(x)$ is a continuous function then the formula

$$u(x,t) = \sum_{j=1}^\infty b_j \sin(p_j x) e^{-p_j^2 t}$$

defines a solution of the heat equation for $0 \le x \le 1, t > 0$. The boundary conditions are satisfied for $t > 0$. Also,

$$\int_0^1 (u(x,t) - f(x))^2 \, dx \to 0 \quad \text{as} \quad t \to 0+ \ .$$

Consider the **eigenvalue problem**

$$-\phi''(x) = \lambda \phi(x), \quad \phi(0) = 0 = \phi(1) + \phi'(1) \ . \tag{1.16}$$

We have determined all positive eigenvalues,

$$\lambda_j = p_j^2 \quad \text{where} \quad p_j > 0 \quad \text{and} \quad \tan p_j = -p_j, \quad j = 1, 2, \ldots$$

We will show that there are no other eigenvalues.

Consider $\lambda = 0$. If $\lambda = 0$ then $\phi(x) = a + bx$. One easily checks that the boundary conditions imply that $a = b = 0$.

Let $\lambda = -p^2 < 0, p > 0$. The general solution of the ODE $\phi'' = p^2 \phi$ is

$$\phi(x) = a e^{px} + b e^{-px} \ .$$

The boundary condition $\phi(0) = 0$ yields that

$$a + b = 0 \ .$$

Since

$$\begin{aligned}\phi(1) &= ae^p + be^{-p}\\ \phi'(1) &= ape^p - bpe^{-p}\end{aligned}$$

one obtains that the boundary condition at $x = 1$ requires that

$$ae^p(1+p) + be^{-p}(1-p) = 0 \ .$$

In matrix form, one obtains the condition

$$\begin{pmatrix} 1 & 1 \\ e^p(1+p) & e^{-p}(1-p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies $a = b = 0$ unless the determinant of the matrix is zero. The determinant is zero if and only if

$$e^{-p}(1-p) = e^p(1+p) \ ,$$

i.e.,

$$e^{2p} = \frac{1-p}{1+p} \ .$$

It is easy to check that this equation has no positive solution p. Therefore, the eigenvalue problem (1.16) has no negative eigenvalue $\lambda = -p^2$.

Suppose that the eigenvalue problem (1.16) has an eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We use integration by parts, as in the proof of Lemma 1.2. The boundary terms drop out.

Suppose that $-\phi'' = \lambda\phi$ and ϕ satisfies the boundary conditions. We have

$$\begin{aligned}\lambda\|\phi\|^2 &= (\phi, \lambda\phi)\\ &= (\phi, -\phi'')\\ &= (-\phi'', \phi)\\ &= (\lambda\phi, \phi)\\ &= \bar\lambda\|\phi\|^2\end{aligned}$$

Therefore, if ϕ is not identical zero, then $\lambda = \bar\lambda$, i.e., λ is real.

Example 7: Consider the heat equation with initial and boundary conditions:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le 1, \quad t \ge 0 \ , && (1.17)\\ u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le 1 \ , && (1.18)\\ u(0,t) &= 0 \quad \text{for} \quad t \ge 0 && (1.19)\\ \beta u(1,t) + u_x(1,t) &= 0 \quad \text{for} \quad t \ge 0 \ . && (1.20)\end{aligned}$$

Here β is a real parameter. We want to show that the sign of β is important. If the boundary condition (1.20) models Newton's law of cooling (with $g(t) \equiv 0$) then $\beta \geq 0$.

The eigenvalue problem obtained by separation of variables is

$$-\phi''(x) = \lambda\phi(x), \quad \phi(0) = 0 = \beta\phi(1) + \phi'(1) \ . \tag{1.21}$$

If $\lambda = p^2, p > 0$, then

$$\phi(x) = a\cos(px) + b\sin(px)$$

and the boundary condition $\phi(0) = 0$ implies that $a = 0$. For $b = 1$ we have

$$\phi(1) = \sin p, \quad \phi'(1) = p\cos p \ ,$$

and the boundary condition $0 = \beta\phi(1) + \phi'(1)$ requires that

$$\beta \sin p + p\cos p = 0 \ .$$

For $\beta = 0$ one obtains the equation $\cos p = 0$. This leads to the eigenvalues

$$\lambda_j = p_j^2 \quad \text{with} \quad p_j = (j - \frac{1}{2})\pi, \quad j = 1, 2, \ldots$$

Let $\beta \neq 0$. Obtain the equation

$$\tan p = -\frac{1}{\beta}\, p \ .$$

Sketching the functions $\tan p$ and $-p/\beta$ for $p > 0$ one obtains that there exists a sequence of positive solutions p_j with

$$0 < p_1 < p_2 < p_3 < \ldots \quad \text{and} \quad p_j \to \infty \ .$$

To summarize, for every coefficient $\beta \in \mathbb{R}$ the eigenvalue problem (1.21) has a sequence of positive eigenvalues $\lambda_j = p_j^2$ with $\lambda_j \to \infty$.

Can there be **other real eigenvalues?** Consider $\lambda = 0$, thus $\phi(x) = a + bx$. The boundary condition $\phi(0) = 0$ implies $a = 0$. Then the boundary condition $\beta\phi(1) + \phi'(1) = 0$ requires that

$$0 = \beta b + b = b(\beta + 1) \ .$$

One obtains that $\lambda = 0$ is an eigenvalues of (1.21) if and only if $\beta = -1$. We will show that there exists a negative eigenvalue if $\beta < -1$.

Assume that $\lambda = -p^2 < 0$ is an eigenvalue, $p > 0$. The general solution of the equation $\phi''(x) = p^2\phi(x)$ is

$$\phi(x) = ae^{px} + be^{-px} \ .$$

The boundary conditions require that

$$\begin{pmatrix} 1 & 1 \\ e^p(\beta + p) & e^{-p}(\beta - p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The number $\lambda = -p^2$ is an eigenvalue if and only if the determinant of the matrix is zero. This holds if and only if

$$e^p(\beta + p) = e^{-p}(\beta - p) \ .$$

It's easy to check that $p = -\beta, p > 0$, is no solution.

Assuming $\beta + p \neq 0$ one obtains

$$e^{2p} = \frac{\beta - p}{\beta + p} \ . \tag{1.22}$$

If $\beta \geq 0$ there is no solution $p > 0$. Let $\beta < 0$. The function

$$g(p) = \frac{\beta - p}{\beta + p}$$

satisfies

$$g'(p) = \frac{-2\beta}{(\beta + p)^2}$$

thus

$$g(0) = 1, \quad g'(0) = -\frac{2}{\beta} = \frac{2}{|\beta|} \ .$$

Since the function $h(p) = e^{2p}$ satisfies

$$h(0) = 1, \quad h'(0) = 2 \ ,$$

one obtains that the equation (1.22) has a solution p_0 with

$$0 < p_0 < |\beta|$$

if $\beta < -1$. Thus, for $\beta < -1$ the eigenvalue problem (1.21) has the negative eigenvalue

$$\lambda_0 = -p_0^2 \ .$$

In the solution formula

$$u(x,t) = \sum_{j=0}^{\infty} b_j \phi_j(x) e^{-\lambda_j t}$$

for the heat equation problem the term

$$b_0 \phi_0(x) e^{p_0^2 t}$$

grows exponentially. This shows that $\beta < -1$ is not physically reasonable.

1.9 Series Expansion

Let $L_2(0,\pi)$ denote the space of all functions $f:(0,\pi)\to\mathbb{R}$ with

$$\int_0^\pi f^2(x)\,dx < \infty \ .$$

For $f,g\in L_2(0,\pi)$ let

$$(f,g)=\int_0^\pi f(x)g(x)\,dx \quad \text{and} \quad \|f\|=\sqrt{(f,f)} \ .$$

Recall that the functions

$$\phi_n(x)=\sin(nx), \quad n=1,2,\ldots$$

satisfy

$$(\phi_m,\phi_n)=\frac{\pi}{2}\,\delta_{mn} \ .$$

Let $f\in L_2(0,\pi)$. Let's assume that

$$f(x)=\sum_{n=1}^\infty B_n\phi_n(x) \tag{1.23}$$

and proceed formally to obtain

$$B_n=\frac{2}{\pi}\,(\phi_n,f) \ .$$

In the following, let B_n be defined by this equation.

One can prove:

$$\|f(x)-\sum_{n=1}^N B_n\phi_n(x)\|\to 0 \quad \text{as} \quad n\to\infty \ .$$

In this sense, the equation (1.23) holds for all $f\in L_2(0,\pi)$.

Consider the IBVP

$$\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0\le x\le\pi, \quad t\ge 0 \ ,\\
u(x,0) &= f(x) \quad \text{for} \quad 0\le x\le\pi \ ,\\
u(0,t) &= u(\pi,0)=0 \quad \text{for} \quad t\ge 0 \ .
\end{aligned}$$

where $f\in L_2(0,\pi)$. Set

$$u(x,t)=\sum_{n=1}^\infty B_n\sin(nx)e^{-n^2t} \ . \tag{1.24}$$

One can prove that the series defines a function

$$u\in C^\infty\Big([0,\pi]\times(0,\infty)\Big) \ .$$

The series can be differentiated arbitrarily often term by term for $t > 0$. Therefore, $u(x,t)$ satisfies

$$u_t(x,t) = u_{xx}(x,t)$$

for $0 \le x \le \pi$ and $t > 0$. Also,

$$u(0,t) = u(\pi,0) = 0 \quad \text{for} \quad t > 0 \ .$$

Furthermore, one can prove that

$$\|f(x) - u(x,t)\| \to 0 \quad \text{as} \quad t \to 0+ \ .$$

In this sense, the C^∞–function $u(x,t)$ defined by (1.24) for $t > 0$ satisfies the initial condition $u(x,0) = f(x)$.

1.10 Well-Posed and Ill–Posed Problems

We want to show by example that the forward heat equation is well–posed whereas the backward heat equation is ill–posed.

Consider the IBVP for the forward heat equation

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\ u(x,0) &= f(x) \quad \text{for} \quad 0 \le x \le \pi \ , \\ u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ , \end{aligned}$$

where $f \in L_2(0,\pi)$. The problem has the unique solution

$$u(x,t) = \int -0^\pi G(x,y,t) f(y)\, dy$$

where

$$G(x,y,t) = \frac{2}{\pi} \sum_{n=1}^\infty e^{-n^2 t} \sin(nx) \sin(ny) \ .$$

Also, the energy estimate yields that

$$\|u(\cdot,t)\| \le \|f\| \quad \text{for} \quad t \ge 0 \ .$$

If $\tilde{u}(x,t)$ denotes the solution with perturbed initial data

$$\tilde{u}(x,0) = f(x) + \varepsilon g(x)$$

then

$$\|\tilde{u}(\cdot,t) - u(\cdot,t)\| \le \varepsilon \|g\| \ .$$

Thus, if the data get perturbed a little, then the solutions get perturbed a little. The solution depends continuously on the data.

Consider the IBVP for the backward heat equation

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= \sin(nx) \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ .
\end{aligned}
$$

The function

$$u(x,t) = \sin(nx)e^{n^2 t}$$

is a solution. We have

$$\|u(x,0)\| = \|\sin(nx)\| = \sqrt{\pi/2}$$

and

$$\|u(x,t)\| = \|\sin(nx)\|e^{n^2 t} = \sqrt{\pi/2}e^{n^2 t} \ .$$

For $t > 0$ one cannot estimate $\|u(x,t\|$ by $\|u(x,0)\|$

If

$$f_n(x) = \frac{1}{n} \sin(nx)$$

then

$$u_n(x,t) = \frac{1}{n} \sin(nx)e^{n^2 t} \ .$$

We have

$$\|u_n(x,0)\| \to 0 \quad \text{as} \quad n \to \infty \ ,$$

but

$$\|u_n(x,t)\| \to \infty \quad \text{as} \quad n \to \infty \quad \text{for} \quad t > 0 \ .$$

The solution does not depend continuously on the data. The IBVP for the backward heat equation is ill–posed.

2 Review: Second–Order ODEs with Constant Coefficients

2.1 General Solution

Consider the differential operator L defined by

$$Ly(x) = ay''(x) + by'(x) + cy(x)$$

where a, b, c are real or complex constants and $a \neq 0$. We want to determine the general solution of the homogeneous equation

$$Ly = 0 \ .$$

Using the ansatz

$$y(x) = e^{px} \ ,$$

where p is a parameter, one obtains the characteristic equation

$$ap^2 + bp + c = 0 \ .$$

The solutions are

$$p_{1,2} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \ .$$

Case 1: $b^2 - 4ac \neq 0$
In this case, $p_1 \neq p_2$ and the general solution of the equation $Ly = 0$ is

$$y(x) = c_1 e^{p_1 x} + c_2 e^{p_2 x} \ .$$

Here c_1 and c_2 are arbitrary constants.
Case 2: $b^2 - 4ac = 0$
In this case, $p_1 = p_2 = p = -\frac{b}{2a}$. The general solution of $Ly = 0$ is

$$y(x) = c_1 e^{px} + c_2 x e^{px} \ .$$

Here c_1 and c_2 are arbitrary constants.

Remarks: If $p = \alpha + i\beta$ with real α, β, then

$$e^{px} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) \ .$$

Thus, the real part of p determines the exponential behavior of the function e^{px} and the imaginary part of p determines the wavelength of the oscillatory behavior of e^{px}. Note that the wavelength of the function $\cos \beta x$ is $L = 2\pi/|\beta|$. One calls β the wave number of the oscillation $\cos \beta x$. Roughly speaking, $|\beta|$ is the number of waves of the oscillation if x varies from 0 to 2π.

If the independent variable is time, we write

$$e^{pt} = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) \ .$$

In this case, the oscillation $\cos \beta t$ has period $T = 2\pi/|\beta|$ and one calls β the frequency of the oscillation.

2.2 Initial Value Problems

Consider the equation

$$y''(t) = \lambda y(t) \ ,$$

where λ is a real constant. We want to determine the solution satisfying the initial conditions

$$y(0) = b_0, \quad y'(0) = b_1 \ .$$

Case 1: $\lambda = \omega^2 > 0, \omega > 0$. The general solution is

$$y(t) = c_1 e^{\omega t} + c_2 e^{-\omega t} \ .$$

The initial conditions require

$$y(0) = c_1 + c_2 = b_0, \quad y'(0) = c_1\omega - c_2\omega = b_1 \ .$$

These are two conditions for the constants c_1 and c_2. We write these in matrix form as

$$\begin{pmatrix} 1 & 1 \\ \omega & -\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \ .$$

Note that the determinant of the matrix is

$$detA = -2\omega \neq 0 \ .$$

One obtains that

$$c_1 = \frac{1}{2}\, b_0 + \frac{1}{2\omega}\, b_1, \quad c_2 = \frac{1}{2}\, b_0 - \frac{1}{2\omega}\, b_1 \ .$$

Thus, the solution of the initial–value problem is

$$y(t) = b_0 \frac{e^{\omega t} + e^{-\omega t}}{2} + \frac{b_1}{\omega} \frac{e^{\omega t} - e^{-\omega t}}{2} \ .$$

Recall the definitions

$$\cosh \tau = \frac{1}{2}(e^\tau + e^{-\tau}), \quad \sinh \tau = \frac{1}{2}(e^\tau - e^{-\tau}) \ .$$

Using these notations,

$$y(t) = b_0 \cosh(\omega t) + \frac{b_1}{\omega} \sinh(\omega t) \ .$$

Case 2: $\lambda = -\omega^2 < 0, \omega > 0$. The general solution is

$$y(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \ .$$

The initial conditions require

$$y(0) = c_1 + c_2 = b_0, \quad y'(0) = c_1 i\omega - c_2 i\omega = b_1 \ .$$

These are two conditions for the constants c_1 and c_2. We write these in matrix form as

$$\begin{pmatrix} 1 & 1 \\ i\omega & -i\omega \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \ .$$

Note that the determinant of the matrix is

$$det A = -2i\omega \neq 0 \ .$$

One obtains that

$$c_1 = \frac{1}{2} b_0 + \frac{1}{2i\omega} b_1, \quad c_2 = \frac{1}{2} b_0 - \frac{1}{2i\omega} b_1 \ .$$

Thus, the solution of the initial–value problem is

$$y(t) = b_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2} + \frac{b_1}{\omega} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \ .$$

Here

$$\frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos \omega t, \quad \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin \omega t \ .$$

One obtains:

$$y(t) = b_0 \cos(\omega t) + \frac{b_1}{\omega} \sin(\omega t) \ .$$

Case 3: $\lambda = 0$. In this case, the general solution is

$$y(t) = c_1 + c_2 t \ .$$

The solution of the initial value problem is

$$y(t) = b_0 + b_1 t \ .$$

2.3 Boundary Value Problems

Consider the equation

$$y''(x) = \lambda y(x), \quad 0 \leq x \leq \pi \ ,$$

where λ is a real constant. We want to determine the solution satisfying the boundary conditions

$$y(0) = b_0, \quad y(\pi) = b_1 \ .$$

Case 1: $\lambda = \omega^2 > 0, \omega > 0$. The general solution is

$$y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} \ .$$

The boundary conditions require

$$y(0) = c_1 + c_2 = b_0, \quad y(\pi) = c_1 e^{\omega\pi} + c_2 e^{-\omega\pi} = b_1 \ .$$

These are two conditions for the constants c_1 and c_2. We write these in matrix form as

$$\begin{pmatrix} 1 & 1 \\ e^{\omega\pi} & e^{-\omega\pi} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \ .$$

We see that the determinant of the matrix is

$$detA = e^{-\omega\pi} - e^{\omega\pi} \neq 0 \ .$$

Therefore, the boundary value problem has a unique solution if $\lambda > 0$.

Case 2: $\lambda = -\omega^2 < 0, \omega > 0$. The general solution is

$$y(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \ .$$

The boundary conditions require

$$y(0) = c_1 + c_2 = b_0, \quad y(\pi) = c_1 e^{i\omega\pi} + c_2 e^{-i\omega\pi} = b_1 \ .$$

These are two conditions for the constants c_1 and c_2. We write these in matrix form as

$$\begin{pmatrix} 1 & 1 \\ e^{i\omega\pi} & e^{-i\omega\pi} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \ .$$

We see that the determinant of the matrix is

$$detA = e^{-i\omega\pi} - e^{i\omega\pi} = -2i\sin(\omega\pi) \ .$$

We see that the boundary value problem has a unique solution if and only if the positive number ω is not an integer.

In case that $\omega = n \in \{1, 2, \ldots\}$ we have $detA = 0$. Then, for most boundary data b_0, b_1, the above matrix equation has no solution and the boundary value problem has no solution. If it happens that the matrix equation is solvable, then the solution of the equation is not unique and the solution of the boundary value problem is not unique.

Case 3: $\lambda = 0$. In this case, the general solution is

$$y(t) = c_1 + c_2 t \ .$$

One obtains that the boundary value problem is uniquely solvable.

2.4 An Eigenvalue Problem

Consider the boundary value problem

$$
\begin{aligned}
-\phi''(x) &= \lambda\phi(x) \quad \text{for} \quad 0 \le x \le 1 && (2.1)\\
\alpha\phi(0) + \beta\phi'(0) &= 0 && (2.2)\\
\gamma\phi(1) + \delta\phi'(1) &= 0 && (2.3)
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and

$$(\alpha, \beta) \neq (0,0) \neq (\gamma, \delta) \ .$$

The number $\lambda \in$ is called an eigenvalue if the boundary value problem has a solution $\phi(x)$ which is not identically zero.

We will show that all eigenvalues λ are real and that eigenfunctions to different eigenvalues are orthogonal.

Lemma 2.1 *Assume that the functions $\phi, \psi \in C^2[0,1]$ satisfy the boundary conditions. Then*

$$\int_0^1 \phi\psi''\,dx = \int_0^1 \phi''\psi\,dx \ .$$

Proof: Using integration by parts we have

$$
\begin{aligned}
\int_0^1 \phi\psi''\,dx &= \phi\psi'|_0^1 - \int_0^1 \phi'\psi'\,dx\\
&= \phi\psi'|_0^1 - \phi'\psi|_0^1 + \int_0^1 \phi''\psi\,dx
\end{aligned}
$$

Consider the boundary term at $x = 1$:

$$BT_1 = \phi(1)\psi'(1) - \phi'(1)\psi(1) \ .$$

If $\delta = 0$ in (2.3) then $\phi(1) = \psi(1) = 0$ and $BT_1 = 0$ follows. If $\delta \neq 0$ then

$$\phi'(1) = -\gamma\phi(1)/\delta, \quad \psi'(1) = -\gamma\psi(1)/\delta \ .$$

Again, it follows that $BT_1 = 0$. With the same arguments, $BT_0 = 0$. This proves the lemma. $\diamond$

Lemma 2.2 *All eigenvalues of the eigenvalue problem (2.1)–(2.3) are real.*

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue and let $\phi : [0,1] \to \mathbb{C}, \phi \in C^2$, denote the eigenfunction. We have

$$
\begin{aligned}
-\lambda \int_0^1 \bar{\phi}\phi \, dx &= \int_0^1 \bar{\phi}\phi'' \, dx \\
&= \int_0^1 \bar{\phi}'' \phi \, dx \\
&= -\bar{\lambda} \int_0^1 \bar{\phi}\phi \, dx
\end{aligned}
$$

Since

$$
\int_0^1 \bar{\phi}(x)\phi(x) \, dx = \int_0^1 |\phi(x)|^2 \, dx > 0
$$

it follows that $\lambda = \bar{\lambda}$, thus λ is real. $\diamond$

Lemma 2.3 *All eigenfunctions of the eigenvalue problem (2.1)–(2.3) to different eigenvalues are orthogonal.*

Proof: Let ϕ_1 and ϕ_2 denote eigenfunctions to the eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$.

We have

$$
\begin{aligned}
\lambda_1 \int_0^1 \phi_1 \phi_2 \, dx &= \int_0^1 \lambda_1 \phi_1 \phi_2 \, dx \\
&= -\int_0^1 \phi_1'' \phi_2 \, dx \\
&= -\int_0^1 \phi_1 \phi_2'' \, dx \\
&= \lambda_2 \int_0^1 \phi_1 \phi_2 \, dx
\end{aligned}
$$

Since $\lambda_1 \neq \lambda_2$ it follows that $\int_0^1 \phi_1 \phi_2 \, dx = 0$. $\diamond$

Lemma 2.4 *Consider the eigenvalue problem*

$$
\begin{aligned}
-\phi''(x) &= \lambda \phi(x) \quad \textit{for} \quad 0 \le x \le 1 && (2.4) \\
\alpha \phi(0) + \phi'(0) &= 0 && (2.5) \\
\gamma \phi(1) + \phi'(1) &= 0 && (2.6)
\end{aligned}
$$

1) If $\alpha \le 0 \le \gamma$ then all eigenvalues λ are non–negative.

2) If $\alpha \le 0 \le \gamma$ and $(\alpha, \gamma) \neq (0,0)$ then all eigenvalues λ are strictly positive.

Proof: Let $\phi(x)$ denote an eigenfunction to the eigenvalue λ. Since $\lambda \in \mathbb{R}$ we may assume that $\phi(x)$ is real valued.

We have

$$
\begin{aligned}
-\lambda \int_0^1 \phi^2 \, dx &= \int_0^1 \phi \phi'' \, dx \\
&= \phi \phi' |_0^1 - \int_0^1 (\phi')^2 \, dx
\end{aligned}
$$

The boundary term is

$$
\begin{aligned}
\phi \phi' |_0^1 &= \phi(1)\phi'(1) - \phi(0)\phi'(0) \\
&= -\gamma \phi^2(1) + \alpha \phi^2(0) \\
&\leq 0
\end{aligned}
$$

since $\alpha \leq 0 \leq \gamma$. Therefore,

$$
-\lambda \int_0^1 \phi^2 \, dx \leq - \int_0^1 (\phi')^2 \, dx \leq 0 \ .
$$

This implies that $\lambda \geq 0$. Also, $\lambda > 0$ unless $\phi' \equiv 0$. The case $\phi' \equiv 0$ only occurs if

$$
\phi(x) = const = c \neq 0 \ .
$$

The function $\phi(x) \equiv c$ satisfies the boundary conditions only of $\alpha = \gamma = 0$. $\diamond$

3 Second–Order Cauchy–Euler Equations

3.1 General Solution

An equation of the form

$$Ly(x) \equiv ax^2y''(x) + bxy'(x) + cy(x) = 0, \quad x > 0 ,$$

where a, b, c are real or complex constants and $a \neq 0$, is called a Cauchy–Euler equation. To determine the general solution one uses the ansatz

$$y(x) = x^r$$

where r is a parameter. One obtains the characteristic equation

$$ar(r-1) + br + c = 0 .$$

If this quadratic equation has two distinct solutions r_1 and r_2, then the general solution of $Ly = 0$ is

$$y(x) = c_1x^{r_1} + c_2x^{r_2} .$$

If $r_1 = r_2 = r$, then the general solution of the equation $Ly = 0$ is

$$y(x) = c_1x^r + c_2x^r \ln x .$$

Example 1: Ly(x)= $x^2y''(x) - 2y(x) = 0$
The characteristics equation is

$$r(r-1) - 2 = 0$$

with solutions

$$r_1 = 2, \quad r_2 = -1 .$$

The general solution of $Ly = 0$ is

$$y(x) = c_1x^2 + \frac{c_2}{x} .$$

Example 2: $x^2y'' + xy' + y = 0$
The characteristics equation is

$$r(r-1) + r + 1 = 0$$

with solutions

$$r_1 = i, \quad r_2 = -i .$$

The general solution is

$$y(x) = c_1x^i + c_2x^{-i} .$$

Since the equation $Ly = 0$ has only real coefficients, one wants to obtain a general solution in real form.

We have $x = e^{\ln x}$ for $x > 0$, thus

$$\begin{aligned} x^i &= e^{i \ln x} \\ &= \cos(\ln x) + i \sin(\ln x) \end{aligned}$$

and

$$\begin{aligned} x^{-i} &= e^{-i \ln x} \\ &= \cos(\ln x) - i \sin(\ln x) \end{aligned}$$

Since the functions x^i and x^{-i} are solutions of $Ly = 0$ one obtains that

$$\cos(\ln x) = \frac{1}{2}\left(x^i + x^{-i}\right) \quad \text{and} \quad \sin(\ln x) = \frac{1}{2i}\left(x^i - x^{-i}\right)$$

are also solutions of the equation $Ly = 0$.

Thus, we can write the general solution of $Ly = 0$ as

$$y(x) = d_1 \cos(\ln x) + d_2 \sin(\ln x)$$

where d_1 and d_2 are arbitrary constants. Note that the solution oscillates rapidly as $x \to 0$.

4 A Boundary Value Problem for the Laplace Equation in a Rectangle

In Section 1.5 we considered the heat equation with homogeneous Dirichlet boundary conditions and an initial condition $u(x,0)=f(x)$.

In Example 2 we had

$$f(x)=\sin^3 x=\frac{3}{4}\,\sin x-\frac{1}{4}\,\sin(3x) \tag{4.1}$$

and in Example 3 we had

$$f(x)=x(1-x)=\frac{8}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{(2j+1)^3} \tag{4.2}$$

We used separation of variables to solve the heat equation.

In this section we consider the Laplace equation

$$\Delta u(x,y)=(u_{xx}+u_{yy})(x,y)=0$$

in the square

$$D=(0,\pi)\times(0,\pi)$$

with boundary conditions

$$\begin{aligned} u(0,y) &= u(\pi,y)=0 \quad \text{for} \quad 0\le y\le\pi && (4.3)\\ u(x,0) &= 0 \quad \text{for} \quad 0\le x\le\pi && (4.4)\\ u(x,\pi) &= f(x) \quad \text{for} \quad 0\le x\le\pi && (4.5) \end{aligned}$$

First ignore the boundary conditions and determine solutions of $\Delta u=0$ in separated variables,

$$u(x,y)=X(x)Y(y)\ .$$

One obtains the equations

$$X''(x)=\lambda X(x) \quad \text{and} \quad Y''(y)=-\lambda Y(y)$$

where λ is the separation constant. If we impose the homogeneous Dirichlet conditions

$$X(0)=X(\pi)=0$$

then the function $u(x,y)=X(x)Y(y)$ satisfies the boundary conditions (4.3).

The BVP

$$X''(x)=\lambda X(x), \quad X(0)=X(\pi)=0\ ,$$

has the eigenvalues $\lambda=\lambda_n=-n^2$ for $n=1,2,\ldots$ and the eigenfunctions

$$X_n(x) = \sin(nx) \ .$$

For $\lambda = -n^2$ the Y–equation

$$Y''(y) = -\lambda Y(y)$$

has the general solution

$$Y_n(y) = ae^{ny} + be^{-ny} \ .$$

The functions

$$u_n(x,y) = c \, \sin(nx) \, Y_n(y)$$

satisfy $\Delta u_n = 0$ and satisfy the two homogeneous boundary conditions (4.3). The boundary condition $u(x,0) = 0$ applied to $u_n(x,y)$ leads to the requirement $Y_n(0) = 0$, thus $a = -b$. One obtains that the functions

$$u_n(x,y) = c_n \sin(nx)(e^{ny} - e^{-ny}), \quad n = 1, 2, \ldots$$

satisfy the PDE and the three homogeneous boundary conditions (4.3), (4.4). It remains to satisfy the inhomogeneous boundary condition

$$u(x,\pi) = f(x) \quad \text{for} \quad 0 \le x \le \pi \ .$$

Example 1: Let $f(x) = \sin x$. We consider

$$u_1(x,y) = c_1 \sin x (e^y - e^{-y})$$

with

$$u_1(x,\pi) = c_1 \sin x (e^\pi - e^{-\pi}) \ .$$

Requiring that $u_1(x,\pi) = f(x) = \sin x$ leads to

$$c_1 = \frac{1}{e^\pi - e^{-\pi}} \ .$$

The solution of $\Delta u = 0$ with the boundary conditions (4.3), (4.4), (4.5) and $f(x) = \sin x$ is

$$u(x,y) = \sin x \, \frac{e^y - e^{-y}}{e^\pi - e^{-\pi}} \ .$$

Example 2: Let

$$f(x) = \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin(3x) \ .$$

We consider

$$u(x,y) = c_1 \sin x (e^y - e^{-y}) + c_3 \sin(3x) \, (e^{3y} - e^{-3y})$$

with

$$u(x,\pi) = c_1 \sin x (e^{\pi} - e^{-\pi}) + c_3 \sin(3x)\,(e^{3\pi} - e^{-3\pi})\ .$$

The boundary condition

$$u(x,\pi) = f(x) = \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

leads to

$$c_1 = \frac{3}{4}\,\frac{1}{e^{\pi} - e^{-\pi}} \quad \text{and} \quad c_3 = -\frac{1}{4}\,\frac{1}{e^{3\pi} - e^{-3\pi}}\ .$$

The solution is

$$u(x,y) = \frac{3}{4}\sin x \frac{e^{y} - e^{-y}}{e^{\pi} - e^{-\pi}} - \frac{1}{4}\sin(3x)\,\frac{e^{3y} - e^{-3y}}{e^{3\pi} - e^{-3\pi}}\ .$$

Example 3: Let

$$f(x) = x(1-x) = \frac{8}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{(2j+1)^3}$$

and consider

$$u(x,y) = \frac{8}{\pi}\sum_{j=0}^{\infty} c_j (2j+1)^{-3}\,\sin((2j+1)x)\left(e^{(2j+1)y} - e^{-(2j+1)y}\right)\ .$$

To obtain that

$$u(x,\pi) = f(x)$$

we choose

$$c_j = \frac{1}{e^{(2j+1)\pi} - e^{-(2j+1)\pi}}\ .$$

The solution is

$$u(x,y) = \frac{8}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{(2j+1)^3}\cdot\frac{e^{(2j+1)y} - e^{-(2j+1)y}}{e^{(2j+1)\pi} - e^{-(2j+1)\pi}}\ . \tag{4.6}$$

Remarks: Consider the solution $u(x,y)$ of Example 3 at the corner point

$$P = (x,y) = (0,\pi)\ .$$

We have

$$u(x,\pi) = f(x) = x(1-x) \quad \text{and} \quad u_{xx}(x,\pi) = f''(x) = -2\ .$$

Also, $u(0,y) = 0$ for $0 \le y \le \pi$, thus

$$u_{yy}(0,\pi) = 0 \ .$$

This shows that $u(x,y)$ does not satisfy Laplace's equation at the corner point $P = (0,\pi)$. One can prove that the solution (4.6) is a C^∞–function in the open square $D = (0,\pi) \times (0,\pi)$. But $u(x,y)$ is not a C^2–function in the closed square $\bar{D} = [0,\pi] \times [0,\pi]$ since $\Delta u(0,\pi) = -2 \neq 0$.

In general, if one considers Laplace's equation in an open region D with Dirichlet boundary conditions, the solution $u(x,y)$ will satisfy

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

at every point (x,y) in the open region D, but not necessarily for points (x,y) on the boundary of D.

5 The Laplace Operator in Polar Coordinates

Let

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan(y/x) \tag{5.1}$$

denote polar coordinates in the (x, y)–plane. If $u(x, y)$ and $\tilde{u}(r, \phi)$ are functions satisfying

$$u(x, y) = \tilde{u}\Big(\sqrt{x^2 + y^2}, \arctan(y/x)\Big) \tag{5.2}$$

then $u(x, y)$ and $\tilde{u}(r, \phi)$ represent the same function in the plane.

Strictly speaking, one has to be careful how to choose the arctan–branch. One should choose

$$\phi = \arctan(y/x)$$

so that

$$x = r \cos\phi \quad \text{and} \quad y = r \sin\phi \quad \text{if} \quad r = \sqrt{x^2 + y^2}\ .$$

Assume that $u(x, y)$ and $\tilde{u}(r, \phi)$ are smooth functions satisfying (5.2). We want to express the Laplacian

$$\Delta u = u_{xx} + u_{yy}$$

in terms of $\tilde{u}$. In the following, we will assume $r > 0$.

Let $r(x, y)$ and $\phi(x, y)$ denote the functions given by (5.1).

We have

$$\begin{aligned}
r_x &= \frac{x}{r} \\
r_{xx} &= \frac{1}{r} - \frac{x}{r^2}\, r_x = \frac{1}{r} - \frac{x^2}{r^3} \\
r_y &= \frac{y}{r} \\
r_{yy} &= = \frac{1}{r} - \frac{y^2}{r^3} \\
\phi_x &= \frac{1}{1 + y^2/x^2}\,(-y/x^2) = -\frac{y}{r^2} \\
\phi_{xx} &= 2\,\frac{y}{r^3}\, r_x = \frac{2xy}{r^4} \\
\phi_y &= \frac{1}{1 + y^2/x^2} \cdot \frac{1}{x} = \frac{x}{r^2} \\
\phi_{yy} &= -2\,\frac{x}{r^3}\, r_y = -\frac{2xy}{r^4}
\end{aligned}$$

The following equations are easy consequences:

$$\begin{aligned}
(r_x)^2 + (r_y)^2 &= 1 \\
(\phi_x)^2 + (\phi_y)^2 &= \frac{1}{r^2} \\
r_x\phi_x + r_y\phi_y &= 0 \\
r_{xx} + r_{yy} &= \frac{1}{r} \\
\phi_{xx} + \phi_{yy} &= 0
\end{aligned}$$

From (5.2) it follows that

$$\begin{aligned}
u_x &= \tilde{u}_r r_x + \tilde{u}_\phi \phi_x \\
u_{xx} &= \tilde{u}_{rr}(r_x)^2 + 2\tilde{u}_{r\phi} r_x \phi_x + \tilde{u}_r r_{xx} + \tilde{u}_\phi \phi_{xx} + \tilde{u}_{\phi\phi}(\phi_x)^2
\end{aligned}$$

and a similar equations holds for u_{yy}. Just replace x by y. Therefore,

$$\begin{aligned}
u_{xx} + u_{yy} &= \tilde{u}_{rr}\Big((r_x)^2 + r_y)^2\Big) + \tilde{u}_{\phi\phi}\Big((\phi_x)^2 + (\phi_y)^2\Big) \\
&\quad +2\tilde{u}_{r\phi}(r_x\phi_x + r_y\phi_y) + \tilde{u}_r(r_{xx} + r_{yy}) + \tilde{u}_\phi(\phi_{xx} + \phi_{yy}) \\
&= \tilde{u}_{rr} + \frac{1}{r}\,\tilde{u}_r + \frac{1}{r^2}\,\tilde{u}_{\phi\phi}
\end{aligned}$$

The equation

$$\Delta u = u_{xx} + u_{yy} = \tilde{u}_{rr} + \frac{1}{r}\,\tilde{u}_r + \frac{1}{r^2}\,\tilde{u}_{\phi\phi}$$

expresses the Lapacian in polar coordinates. It is common to drop the tilde notation and write

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\phi\phi}\ . \tag{5.3}$$

Example: Let

$$u(x,y) = \frac{1}{x^2 + y^2} = (x^2 + y^2)^{-1}\ .$$

We have

$$\begin{aligned}
\tilde{u}(r,\phi) &= r^{-2} \\
\tilde{u}_r(r,\phi) &= -2r^{-3} \\
\tilde{u}_{rr}(r,\phi) &= 6r^{-4}
\end{aligned}$$

Using the formula

$$\Delta \tilde{u} = \tilde{u}_{rr} + \frac{1}{r}\,\tilde{u}_r + \frac{1}{r^2}\,\tilde{u}_{\phi\phi}$$

one obtains

$$\Delta \tilde{u} = \frac{6}{r^4} - \frac{2}{r^4} = \frac{4}{r^4} \ .$$

Let us check this by computing

$$\Delta u = u_{xx} + u_{yy} \ .$$

We have

$$\begin{aligned} u_x &= -2x(x^2+y^2)^{-2} \\ u_{xx} &= -2(x^2+y^2)^{-2} + 8x^2(x^2+y^2)^{-3} \\ u_y &= -2y(x^2+y^2)^{-2} \\ u_{yy} &= -2(x^2+y^2)^{-2} + 8y^2(x^2+y^2)^{-3} \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta u &= -4(x^2+y^2)^{-2} + 8\,\frac{x^2+y^2}{(x^2+y^2)^3} \\ &= 4(x^2+y^2)^{-2} \end{aligned}$$

The two results agree. For this example, the computation of the Laplacian is simpler if one uses $\tilde{u} = 1/r^2$.

6 The Laplace Equation on the Unit Disk

6.1 A Dirichlet Problem

Let

$$D = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\}$$

denote the open unit disk with boundary curve

$$\mathcal{S} = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\} \ .$$

The set $\bar{D} = D \cup \mathcal{S}$ is the closed unit disk.

Let $f : \mathcal{S} \to \mathbb{R}$ denote a continuous function. We want to determine a function $u : \bar{D} \to \mathbb{R}$ satisfying $u \in C(\bar{D}), u \in C^2(D)$ and

$$\Delta u = 0 \quad \text{in} \quad D, \quad u = f \quad \text{on} \quad \mathcal{S} \ . \tag{6.1}$$

This is the Dirichlet problem for the Laplace equation in the unit disk.

In polar coordinates, the equations (6.1) become

$$\begin{aligned} u_{rr} + \frac{1}{r}\, u_r + \frac{1}{r^2}\, u_{\phi\phi} &= 0 \quad \text{for} \quad 0 \le r < 1 \quad \text{and} \quad \phi \in \mathbb{R} \ , \\ u(1,\phi) &= f(\phi) \quad \text{for} \quad \phi \in \mathbb{R} \ . \end{aligned}$$

Here the functions $f(\phi)$ and $u(r,\phi)$ must be 2π–periodic in ϕ.

We first ignore the boundary condition and determine solutions of the PDE in separated variables,

$$u(r,\phi) = R(r)\Phi(\phi) \ .$$

The PDE becomes

$$R''(r)\Phi(\phi) + \frac{1}{r}\, R'(r)\Phi(\phi) + \frac{1}{r^2}\, R(r)\Phi''(\phi) = 0 \ ,$$

thus

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Phi''(\phi)}{\Phi(\phi)} = \lambda \ .$$

Here λ is the separation constant. Since $\Phi(\phi)$ has to be 2π–periodic one obtains that

$$\lambda = \lambda_n = n^2 \quad \text{for} \quad n \in \mathbb{Z} \ .$$

The equation

$$\Phi'' + n^2 \Phi = 0$$

has the 2π–periodic solutions

$$\Phi_n(\phi) = e^{in\phi} \quad \text{for} \quad n \in \mathbb{Z} \ .$$

The corresponding equation for $R(r)$ is the Cauchy–Euler equation

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0 \ . \tag{6.2}$$

The ansatz $R(r) = r^p$ leads to the quadratic equation

$$p(p-1) + p - n^2 = 0 \ ,$$

thus $p^2 = n^2, p = \pm n$. For $n \in \mathbb{Z}, n \neq 0$, the general solution of the equation (6.2) is

$$R(r) = ar^n + br^{-n} \ .$$

For $n = 0$ the general solution is

$$R(r) = a + b \ln r \ .$$

For $n = 1, 2, \ldots$ the functions r^{-n} are singular at $r = 0$; also, the function $\ln r$ is singular at $r = 0$. Since we want to determine solutions $u_n(r, \phi)$ which are smooth in D, these singular solutions are not of interest.

One obtains the following smooth solutions of the equation $\Delta u = 0$ in D in separated variables:

$$u_0(r, \phi) = c_0 \quad \text{and} \quad u_n(r, \phi) = r^n \left(a_n e^{in\phi} + b_n e^{-in\phi} \right) \quad \text{for} \quad n = 1, 2, \ldots$$

This suggests to try to solve the Dirichlet problem (6.1) by a function of the form

$$u(r, \phi) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} \, e^{in\phi} \ . \tag{6.3}$$

The coefficients c_n must be determined by the boundary condition $u(1, \phi) = f(\phi)$.

We first proceed formally, ignoring questions of convergence. The boundary condition $u(1, \phi) = f(\phi)$ requires that

$$f(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi} \ .$$

Since

$$\int_{-\pi}^{\pi} e^{in\alpha} e^{-ij\alpha} \, d\alpha = 2\pi \delta_{nj}$$

one obtains that

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) e^{-in\alpha} \, d\alpha \ .$$

Using these coefficients c_n in (6.3) and exchanging integration and summation one obtains

$$u(r,\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{n=-\infty}^{\infty} r^{|n|} \, e^{in(\phi-\alpha)} \Big) f(\alpha) d\alpha \; . \tag{6.4}$$

The function

$$P(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \, e^{in\theta} \tag{6.5}$$

is called the Poisson kernel for Laplace's equation in the unit disk.

6.2 Properties of the Poisson Kernel

We will now become more careful and consider issues of convergence. Does the series (6.5) converge? We will prove that it converges for $0 \le r < 1$, but the convergence for $r = 1$ is doubtful. For $r = 1$ the series (6.5) certainly does not converge absolutely.

We now derive another formula for the Poisson kernel for $0 \le r < 1$. Set

$$w = re^{i\theta}, \quad \bar{w} = re^{-i\theta} \quad \text{for} \quad 0 \le r < 1 \; .$$

Using the geometric sum formula we have

$$\begin{aligned} 2\pi P(r,\theta) &= \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \bar{w}^n \\ &= \frac{1}{1-w} + \frac{\bar{w}}{1-\bar{w}} \\ &= \frac{1-\bar{w} + (1-w)\bar{w}}{(1-w)(1-\bar{w})} \\ &= \frac{1-|w|^2}{|1-w|^2} \\ &= \frac{1-r^2}{1-2r\cos\theta + r^2} \end{aligned}$$

In the last equation we have used that

$$1 - w = 1 - r(\cos\theta + i \sin\theta) \; ,$$

thus

$$|1-w|^2 = (1 - r\cos\theta)^2 + r^2 \sin^2\theta \; .$$

This proves that

$$P(r,\theta) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2} \quad \text{for} \quad 0 \le r < 1, \quad \phi \in \mathbb{R} \; . \tag{6.6}$$

We now derive some properties of the Poisson kernel, which will be used later to get a better understanding of the Dirichlet problem (6.1).

First note that

$$\begin{aligned} 1-2r\cos\theta+r^2 &= (1-r)^2+2r(1-\cos\theta) \\ &\geq (1-r)^2>0 \end{aligned}$$

for $0 \leq r < 1$. Thus, the denominator in (6.6) is positive for $0 \leq r < 1$.

For later use we note the following: Let

$$\frac{1}{2} \leq r < 1 \quad \text{and} \quad 0 < \delta \leq |\theta| \leq \pi \ .$$

Then

$$1-2r\cos\theta+r^2 \geq 2r(1-\cos\theta) \geq c_\delta > 0$$

where

$$c_\delta = 1-\cos\delta > 0 \ .$$

One obtains that

$$0 < P(r,\theta) \leq \frac{1}{2\pi}\,\frac{1-r^2}{c_\delta} \tag{6.7}$$

for $\frac{1}{2} \leq r < 1, 0 < \delta \leq |\theta| \leq \pi$. The estimate (6.7) implies that

$$P(r,\theta) \to 0 \quad \text{as} \quad r \to 1- \quad \text{if} \quad 0 < |\theta| \leq \pi \ . \tag{6.8}$$

On the other hand, for $\theta = 0$ we have

$$2\pi P(r,0) = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r} \ ,$$

thus

$$P(r,\theta) \to \infty \quad \text{as} \quad r \to 1- \quad \text{if} \quad \theta = 0 \ . \tag{6.9}$$

Recall formula (6.5) for the Poisson kernel $P(r,\theta)$, which is valid for $0 \leq r < 1, \theta \in \mathbb{R}$. Since

$$\int_{-\pi}^{\pi} e^{in\theta}\,d\theta = 2\pi\,\delta_{n0}$$

one obtains that

$$\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1 \quad \text{for} \quad 0 \leq r < 1 \ . \tag{6.10}$$

Summary: For $0 \leq r < 1, \theta \in \mathbb{R}$ the Poisson kernel is given by the formulas

$$
\begin{aligned}
P(r,\theta) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{in\theta} && (6.11)\\
&= \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\theta + r^2} && (6.12)
\end{aligned}
$$

The following holds:

$$
\begin{aligned}
\int_{-\pi}^{\pi} P(r,\theta)\, d\theta = 1 \qquad & \text{for} \qquad 0 \le r < 1 && (6.13)\\
0 < P(r,\theta) < \infty \qquad & \text{for} \qquad 0 \le r < 1, \quad \theta \in \mathbb{R} && (6.14)\\
P(r,\theta) \to 0 \quad \text{as} \quad r \to 1- \qquad & \text{if} \qquad 0 < |\theta| \le \pi && (6.15)\\
P(r,\theta) \to \infty \quad \text{as} \quad r \to 1- \qquad & \text{if} \qquad \theta = 0\ . && (6.16)
\end{aligned}
$$

A more refined form of (6.15) is

$$
0 < P(r,\theta) \le \frac{1}{2\pi} \frac{1-r^2}{c_\delta} \quad \text{for} \quad \frac{1}{2} \le r < 1, \quad 0 < \delta \le |\theta| \le \pi \tag{6.17}
$$

where $c_\delta = 1 - \cos\delta > 0$.

Remarks about Dirac's Delta Function: Paul Dirac (1902-1984), an English theoretical physicist, introduced the Delta function, $\delta(x)$. He formally used a function satisfying

$$
\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ \infty & \text{for} \quad x = 0 \end{cases}
$$

and

$$
\int_{-\infty}^{\infty} \delta(x)\, dx = 1\ .
$$

He used the Delta function to make a calculation like

$$
\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0)\ .
$$

However, an ordinary function $\delta(x)$ with these properties does not exist. Laurent Schwartz (1915-2002), a French mathematician, made the mathematics precise by introducing distribution theory. In distribution theory, the δ–distribution is the assignment $f \to f(0)$ where f is a function in the Schwartz space.

6.3 Solution of the Dirichlet Problem

Recall the definitions of the open unit disk D and its boundary $\mathcal{S}$. Let $f : \mathcal{S} \to \mathbb{R}$ be a continuous function. We claim that function

$$
u(r,\phi) = \begin{cases} \int_{-\pi}^{\pi} P(r,\phi-\alpha) f(\alpha)\, d\alpha & \text{for} \quad 0 \le r < 1 \\ f(\phi) & \text{for} \quad r = 1 \end{cases} \tag{6.18}
$$

solves the Dirichlet problem for Laplace's equation in D with boundary data f.

Recall that the functions $r^{|n|}\,e^{in\theta}$ satisfy Laplace's equation for $n = 0, 1, 2\ldots$ Since the series in (6.11) converges absolutely for $0 \le r < 1$ one can show that $u(r,\phi)$ is a $C^\infty-$ function in D satisfying Laplace's equation in D. Trivially, the function $u(r,\phi)$ given in (6.18) satisfies the boundary condition.

The main difficulty is to prove that $u(r,\phi)$ is continuous at every boundary point $(r_0,\phi_0) = (1,\phi_0)$. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ so that

$$|f(\phi) - f(\phi_0)| \le \varepsilon \quad \text{for} \quad |\phi - \phi_0| \le 2\delta \ .$$

In the following, we assume that $|\phi - \phi_0| \le \delta$ and $\frac{1}{2} \le r < 1$. We have

$$\begin{aligned} u(r,\phi) &= \int_{-\pi}^{\pi} P(r,\phi-\alpha) f(\alpha), d\alpha \quad (\text{substitute } \phi - \alpha = \theta) \\ &= -\int_{\pi+\phi}^{-\pi+\phi} P(r,\theta) f(\phi-\theta)\,d\theta \\ &= \int_{-\pi+\phi}^{\pi+\phi} P(r,\theta) f(\phi-\theta)\,d\theta \\ &= \int_{-\pi}^{\pi} P(r,\theta) f(\phi-\theta)\,d\theta \end{aligned}$$

In the last equation we have used periodicity.

Using (6.13) we obtain

$$\begin{aligned} |u(r,\phi) - f(\phi_0)| &\le \int_{-\pi}^{\pi} P(r,\theta)|f(\phi-\theta) - f(\phi_0)|\,d\theta \\ &\le \int_{-\delta}^{\delta} P(r,\theta)|f(\phi-\theta) - f(\phi_0)|\,d\theta + 2|f|_\infty \int_{\delta\le|\theta|\le\pi} P(r,\theta)\,d\theta \end{aligned}$$

If $|\phi - \phi_0| \le \delta$ and $|\theta| \le \delta$ then

$$|\phi - \theta - \phi_0| \le 2\delta, \quad \text{thus} \quad |f(\phi-\theta) - f(\phi_0)| \le \varepsilon \ .$$

This yields that

$$|u(r,\phi) - f(\phi_0)| \le \varepsilon + 2|f|_\infty \int_{\delta\le|\theta|\le\pi} P(r,\theta)\,d\theta \ .$$

Using the estimate (6.17) obtain that

$$\int_{\delta\le|\theta|\le\pi} P(r,\theta)\,d\theta \le \frac{1-r^2}{c_\delta} \quad \text{where} \quad c_\delta = 1 - \cos\delta > 0 \ .$$

Therefore, there exists $\tilde{\delta} > 0$ so that

$$|u(r,\phi) - f(\phi_0)| \le 2\varepsilon \quad \text{if} \quad |\phi - \phi_0| \le \delta \quad \text{and} \quad 1 - \tilde{\delta} \le r < 1 \ .$$

This proves continuity of the function $u(r,\phi)$ at the point (r,ϕ_0).

6.4 Dirac's Delta Function

Recall the formula for Poisson's kernel:

$$P(r,\theta)=\frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos\theta+r^2}\quad \text{for}\quad 0\le r<1,\quad |\theta|\le\pi\ .$$

We have

$$\begin{array}{rcll}
\int_{-\pi}^{\pi} P(r,\theta)\,d\theta = 1 & & \text{for} & 0\le r<1\\
P(r,\theta)\to 0 \quad \text{as} & r\to 1- & \text{if} & 0<|\theta|\le\pi\\
P(r,\theta)\to \infty \quad \text{as} & r\to 1- & \text{if} & \theta=0\ .
\end{array}$$

This may suggest to write

$$P(1,\theta)=\delta_0(\theta)\quad \text{for}\quad |\theta|\le\pi$$

where

$$\delta_0(\theta)=\left\{\begin{array}{ccc} 0 & \text{for} & 0<|\theta|\le\pi\\ \infty & \text{for} & \theta=0\end{array}\right.$$

and

$$\int_{-\pi}^{\pi}\delta_0(\theta)\,d\theta=1\ .$$

Recall: If $f\in C_{2\pi}$ then the solution of $\Delta u=0$ in D is

$$u(r,\phi)=\int_{-\pi}^{\pi}P(r,\theta)f(\theta+\phi)\,d\theta\quad \text{for}\quad 0\le r<1\ .$$

Also,

$$u(1,\phi)=f(\phi)\ .$$

If we formally take the limit $r\to 1-$ in the equation for $u(r,\phi)$ then we obtain

$$\begin{aligned}
f(\phi) &= u(1,\phi)\\
&= \int_{-\pi}^{\pi}P(1,\theta)f(\theta+\phi)\,d\theta\\
&= \int_{-\pi}^{\pi}\delta_0(\theta)f(\theta+\phi)\,d\theta\ .
\end{aligned}$$

Thus, formally, the Dirac Delta function δ_0 has the property

$$\int_{-\pi}^{\pi}\delta_0(\theta)f(\theta+\phi)\,d\theta=f(\phi)\ .$$

Just taking $\phi=0$ one obtains

$$\int_{-\pi}^{\pi} \delta_0(\theta) f(\theta)\, d\theta = f(0)\ .$$

The trouble is: A function $\delta_0(\theta)$ with the above properties does not exist. Dirac's formal introduction of the delta–function lead to the mathematical theory of distributions. The French mathematician Laurent Schwartz (1915–2002) played a major role.

7 The Heat Equation on the Line

7.1 The 1D Heat Kernel

The function

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \quad x \in \mathbb{R}, \quad t > 0 \; ,$$

is called the heat kernel in one space dimension.

We have

$$\begin{aligned}
G_x(x,t) &= G(x,t)(-x/2t) \\
G_{xx}(x,t) &= G(x,t)(-1/2t) + G(x,t)(x^2/4t^2) \\
G_t(x,t) &= -\frac{1}{2}\, t^{-3/2}\, \frac{1}{\sqrt{4\pi}}\, e^{-x^2/4t} + G(x,t)(x^2/4t^2) \\
&= (-1/2t)G(x,t) + G(x,t)(x^2/4t^2)
\end{aligned}$$

This shows that

$$G_t(x,t) = G_{xx}(x,t) \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0 \; .$$

Thus, $G(x,t)$ satisfies the heat equation in the open upper half–plane.

It is easy to check that

$$\begin{array}{cccc}
G(x,t) \to 0 & \text{as} \quad t \to 0+ & \text{for} & x \neq 0 \\
G(x,t) \to \infty & \text{as} \quad t \to 0+ & \text{for} & x = 0
\end{array}$$

Integration of a Gaussian

Lemma 7.1

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Proof: Set

$$J = \int_{-\infty}^{\infty} e^{-x^2}\, dx \; .$$

Then we have

$$\begin{aligned}
J^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr d\phi \\
&= \pi \int_0^{\infty} 2re^{-r^2} dr \\
&= \pi \int_0^{\infty} e^{-q} dq \\
&= \pi
\end{aligned}$$

This proves the lemma. $\diamond$

Lemma 7.2

$$\int_{-\infty}^{\infty} G(x,t)\,dx = 1 \quad \textit{for} \quad t > 0 \ .$$

Proof: We use the substitution $q = x/\sqrt{4t}$ and the previous lemma:

$$\begin{aligned}
\int_{-\infty}^{\infty} G(x,t)\,dx &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-x^2/4t}\,dx \\
&= \frac{1}{\sqrt{4\pi t}} \sqrt{4t} \int_{-\infty}^{\infty} e^{-q^2}\,dq \\
&= 1
\end{aligned}$$

$\diamond$

7.2 Solution of the Heat Equation

Consider the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0 \ .$$

with initial condition

$$u(x,t) = f(x) \quad \text{for} \quad x \in \mathbb{R} \ .$$

We will assume that $f : \mathbb{R} \to \mathbb{R}$ is a given continuous, bounded function.

We claim that the function

$$u(x,t) = \begin{cases} \int_{-\infty}^{\infty} G(x-y,t) f(y)\,dy & \text{for} \quad t > 0 \\ f(x) & \text{for} \quad t = 0 \end{cases} \tag{7.1}$$

solves the heat equation. Since $G(x,t)$ is a C^∞–functions for $t > 0$ which satisfies the heat equation for $t > 0$ one can show that the function $u(x,t)$ defined above is a C^∞–function for $t > 0$ which satisfies the heat equation for $t > 0$.

It remains to prove that $u(x,t)$ is continuous at every point $(x,t) = (x_0, 0)$.

We will use the following lemma.

Lemma 7.3 *Let $\delta > 0$. Then*

$$\int_{|x| \ge \delta} G(x,t)\,dx \to 0 \quad \textit{as} \quad t \to 0+ \ .$$

Proof: Using the substitution $q = x/\sqrt{4t}$ we obtain for $t > 0$:

$$
\begin{aligned}
\int_{|x|\geq\delta} G(x,t)\,dx &= 2\int_\delta^\infty G(x,t)\,dx \\
&= \frac{2}{\sqrt{4\pi t}}\int_\delta^\infty e^{-x^2/4t}\,dx \\
&= \frac{2}{\sqrt{4\pi t}}\sqrt{4t}\int_{\delta/\sqrt{4t}}^\infty e^{-q^2}\,dq
\end{aligned}
$$

The claim follows. $\diamond$

Fix any $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$ be given. There exists $\delta > 0$ so that

$$
|f(x) - f(x_0)| \leq \varepsilon \quad \text{for} \quad |x - x_0| \leq 2\delta \ .
$$

In the following we will assume that $|x - x_0| \leq \delta$. We have for $t > 0$:

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^\infty G(x-y,t)f(y)\,dy \\
&= \int_{-\infty}^\infty G(y-x),t)f(y)\,dy \quad \text{(substitute } q = y - x) \\
&= \int_{-\infty}^\infty G(q,t)f(q+x)\,dq
\end{aligned}
$$

Using Lemma 7.2 we obtain that

$$
\begin{aligned}
|u(x,t) - f(x_0)| &\leq \int_{-\infty}^\infty G(q,t)|f(q+x) - f(x_0)|\,dq \\
&\leq \int_{|q|\leq\delta} G(q,t)|f(q+x) - f(x_0)|\,dq + 2M\int_{|q|\leq\delta} G(q,t)\,dq
\end{aligned}
$$

where $M > 0$ is a constant with

$$
|f(x)| \leq M \quad \text{for all} \quad x \in \mathbb{R} \ .
$$

Since $|q| \leq \delta$ and $|x - x_0| \leq \delta$ yields that $|q + x - x_0| \leq 2\delta$ we obtain that

$$
|f(q+x) - f(x_0)| \leq \varepsilon \ .
$$

Therefore,

$$
|u(x,t) - f(x_0)| \leq \varepsilon + 2M\int_{|q|\geq\delta} G(q,t)\,dq \ .
$$

Using Lemma 7.3 it follows that there exists $\tilde{\delta} > 0$ so that

$$
|u(x,t) - f(x_0)| \leq 2\varepsilon
$$

for $|x - x_0| \leq \delta$ and $0 < t \leq \tilde{\delta}$. This proves that $u(x,t)$ is continuous at the point $(x,t) = (x_0, 0)$.

7.3 Derivation of the Heat Kernel

Consider the heat equation

$$u_t(x,t) = u_{xx}(x,t), \quad x \in \mathbb{R}, \quad t \geq 0 \; .$$

Try to obtain solutions in separated variables,

$$u(x,t) = X(x)T(t) \; .$$

One obtains

$$X(x)T'(t) = X''(x)T(t) \; ,$$

thus

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda = -k^2 \; .$$

Since one expects $T(t)$ to decay as $t \to \infty$ it is reasonable to choose a non–positive separation constant, $-\lambda = -k^2 \leq 0$. One obtains

$$T(t) = T(0)e^{-k^2 t} \; .$$

The equation

$$X''(x) + k^2 X(x) = 0$$

has the solution

$$e^{ikx}, \quad k \in \mathbb{R} \; .$$

The corresponding solution of the heat equation is

$$u_k(x,t) = e^{ikx}e^{-k^2 t} \; .$$

If the initial condition

$$u(x,0) = f(x) \, , \quad x \in \mathbb{R} \; ,$$

is given, one should try to write $f(x)$ as a combination of the functions e^{ikx}. This leads to the Fourier representation of $f(x)$.

Fourier Transformation: Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous functions of moderate decrease, i.e,

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for} \quad x \in \mathbb{R}$$

for some constant A. The function

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \quad \text{for} \quad k \in \mathbb{R} \tag{7.2}$$

is called the Fourier transform of f. One can prove that $\hat{f}(k)$ is continuous. If $\hat{f}(k)$ is also of moderate decrease, the Fourier inversion formula holds:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\hat{f}(k)\,dk\quad\text{for}\quad x\in\mathbb{R}\;. \tag{7.3}$$

See [Stein, Sharkarchi: Fourier Analysis].

Since the solution of the heat equation with initial condition

$$u(x,0)=e^{ikx},\quad x\in\mathbb{R}\;,$$

has the solution

$$u_k(x,t)=e^{ikx}e^{-k^2t}$$

it is plausible that the solution $u(x,t)$ with initial condition

$$u(x,0)=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\hat{f}(k)\,dk\quad\text{for}\quad x\in\mathbb{R}\;,$$

is

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}\hat{f}(k)e^{-k^2t}\,dk\;.$$

We now proceed formally and use the formula (7.2) for $\hat{f}(k)$. (We use the variable y instead of x.) One obtains

$$\begin{aligned}u(x,t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ik(x-y)}e^{-k^2t}f(y)\,dydk\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ik(x-y)}e^{-k^2t}\,dkf(y)dy\end{aligned}$$

where we have exchanged the order of integration.

We will now prove that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}e^{-k^2t}\,dk=\sqrt{\frac{1}{4\pi t}}\,e^{-x^2/4t}=G(x,t)\quad\text{for}\quad x\in\mathbb{R},\quad t>0\;.$$

This will be the derivation of the heat kernel via Fourier transformation.

Set

$$g(x)=\int_{-\infty}^{\infty}e^{ikx}e^{-k^2t}\,dk\quad\text{for}\quad t>0\;.$$

Then we have

$$\begin{aligned}g(0)&=\int_{-\infty}^{\infty}e^{-k^2t}\,dk\quad(\text{substitute }q=k\sqrt{t})\\&=\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}e^{-q^2}\,dq\\&=\sqrt{\frac{\pi}{t}}\end{aligned}$$

and

$$
\begin{aligned}
g'(x) &= i\int_{-\infty}^{\infty} e^{ikx}\, k\, e^{-k^2 t}\, dk \\
&= \frac{-i}{2t}\int_{-\infty}^{\infty} e^{ikx}\,(-2kt)\, e^{-k^2 t}\, dk \\
&= \frac{-i}{2t}\int_{-\infty}^{\infty} e^{ikx}\,\frac{d}{dk}\, e^{-k^2 t}\, dk \\
&= \frac{i}{2t}\, ix \int_{-\infty}^{\infty} e^{ikx} e^{-k^2 t}\, dk \\
&= -\frac{x}{2t}\, g(x)
\end{aligned}
$$

The ODE initial value problem

$$
g'(x) = -\frac{x}{2t}\, g(x), \quad g(0) = \sqrt{\frac{\pi}{t}}\ ,
$$

has the unique solution

$$
g(x) = \sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}\ .
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} e^{-k^2 t}\, dk &= \frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\, e^{-x^2/4t} \\
&= \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t} \\
&= G(x,t)
\end{aligned}
$$

7.4 The Heat Equation with Extra Terms

Recall the heat kernel

$$
G(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \quad x \in \mathbb{R}, \quad t > 0\ ,
$$

and recall that the solution of the initial value problem

$$
u_t(x,t) = u_{xx}(x,t), \quad u(x,0) = f(x)\ ,
$$

is

$$
u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy\ .
$$

Here we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, bounded function.

Example 1: Consider the IVP

$$u_t(x,t) = u_{xx}(x,t) - bu(x,t), \quad u(x,0) = f(x) \ , \tag{7.4}$$

where $b \in \mathbb{R}$ is a constant. Let

$$u(x,t) = v(x,t)e^{-bt}$$

where $v(x,t)$ is a new unknown. Assuming that $u_t = u_{xx} - bu$ it is easy to check that $v_t = v_{xx}$.

Details: We have

$$u_t = v_t e^{-bt} - bve^{-bt} = v_t e^{-bt} - bu$$

and

$$u_{xx} = v_{xx}e^{-bt} \ .$$

Therefore, $u_t = u_{xx} - bu$ yields that

$$v_t e^{-bt} - bu = v_{xx}e^{-bt} - bu \ ,$$

and $v_t = v_{xx}$ follows.

Since $f(x) = u(x,0) = v(x,0)$ one obtains that

$$v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy,$$

thus

$$u(x,t) = e^{-bt}\int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy \ .$$

It is easy to check that this formula gives a solution of the IVP (7.4).

Example 2: Consider the IVP

$$u_t(x,t) + au_x(x,t) = u_{xx}(x,t), \quad u(x,0) = f(x) \ , \tag{7.5}$$

where $a \in \mathbb{R}$ is a constant. Assume that $u(x,t)$ is a solution and set

$$v(x,t) := u(x+at,t) \ .$$

We have

$$\begin{aligned} v_t(x,t) &= u_t(x+at,t) + au_x(x+at,t) \\ &= u_{xx}(x+at,t) \\ &= v_{xx}(x,t) \end{aligned}$$

thus

$$u(x+at,t) = v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy \ .$$

Setting $x + at = \xi, x = \xi - at$, one obtains that

$$u(\xi, t) = \int_{-\infty}^{\infty} G(\xi - at - y, t) f(y)\, dy \ .$$

It is easy to check that this formula gives a solution of the IVP (7.5).

Example 3: The IVP

$$u_t(x,t) + au_x(x,t) + bu(x,t) = u_{xx}(x,t), \quad u(x,0) = f(x) \ , \tag{7.6}$$

has the solution

$$u(x,t) = e^{-bt} \int_{-\infty}^{\infty} G(x - at - y, t) f(y)\, dy \ .$$

8 First Order PDEs

Problem 1: Consider the IVP

$$u_t(x,t) + au_x(x,t) = 0, \quad u(x,0) = f(x) \ , \tag{8.1}$$

where $f(x)$ is a smooth function and $a \in \mathbb{R}$ is a constant. The function

$$u(x,t) = f(x - at)$$

solves the IVP since $u(x,0) = f(x)$ and

$$u_t(x,t) = -af'(x-at) \quad \text{and} \quad u_x(x,t) = f'(x-at) \ .$$

Let us show that there is no other solution:

Let $x_0 \in \mathbb{R}$ be fixed and consider the line given by

$$(x(t),t) = (x_0 + at, t), \quad t \geq 0 \ .$$

Assume that $u(x,t)$ solves (8.1) and set

$$h(t) = u(x_0 + at, t), \quad t \geq 0 \ .$$

We have $h(0) = u(x_0, 0) = f(x_0)$ and

$$h'(t) = au_x(x_0 + at, t) + u_t(x_0 + at, t) = 0 \ ,$$

thus $h(t) \equiv f(x_0)$. This proves that $u(x_0 + at, t) = f(x_0)$. Setting

$$x = x_0 + at, \quad x_0 = x - at \ ,$$

one obtains that

$$u(x,t) = f(x - at) \ .$$

The line

$$(x(t),t) = (x_0 + at, t), \quad t \geq 0 \ ,$$

is called the characteristic in the (x,t)–plane starting at the point $(x_0, 0)$. The solution $u(x,t)$ of the IVP carries the value $f(x_0)$ on this line. The function $f(x)$ is carried with velocity a into the future. If $a > 0$ the characteristics move to the right with speed a; if $a < 0$ they move to the left.

Problem 2: Adding of a Forcing.
Consider the IVP

$$u_t(x,t) + au_x(x,t) = F(x,t), \quad u(x,0) = f(x) \ , \tag{8.2}$$

where $f(x)$ is a smooth function, $a \in \mathbb{R}$ is a constant, and $F(x,t)$ is a continuous function.

Assume that $u(x,t)$ solves the IVP. As in Problem 1, we fix $x_0 \in \mathbb{R}$ and consider the characteristic line

$$(x(t), t) = (x_0 + at, t), \quad t \geq 0 \ .$$

Set

$$h(t) = u(x_0 + at, t), \quad t \geq 0 \ .$$

We have $h(0) = f(x_0)$ and

$$h'(t) = u_t(x_0 + at, t) + au_x(x_0 + at) = F(x_0 + at, t) \ ,$$

thus

$$u(x_0 + at) = h(t) = f(x_0) + \int_0^t F(x_0 + as, s)\, ds \ .$$

Setting

$$x = x_0 + at, \quad x_0 = a - at \ ,$$

one obtains that

$$u(x, t) = f(x - at) + \int_0^t F(x - at + as, s)\, ds \ .$$

It is easy to check that the formula gives a solution of the IVP (8.2).

Example 1: The IVP

$$u_t + au_x = 1, \quad u(x, 0) = \sin x \ ,$$

has the solution

$$u(x, t) = \sin(x - at) + t \ .$$

Problem 3: Addition of a Zero–Order Term

Consider the IVP

$$u_t(x, t) + au_x(x, t) = bu(x, t) + F(x, t), \quad u(x, 0) = f(x) \ , \tag{8.3}$$

where $f(x)$ is a smooth function, a and b are real constants, and $F(x, t)$ is a continuous function.

Assume that $u(x, t)$ is a solution. As in the previous examples, we fix $x_0 \in \mathbb{R}$ and consider $u(x, t)$ along the characteristic $(x_0 + at, t)$. Set

$$h(t) = u(x_0 + at), \quad t \geq 0 \ .$$

We have $h(0) = f(x_0)$ and

$$\begin{aligned} h'(t) &= u_t(x_0 + at, t) + au_x(x_0 + at, t) \\ &= bu(x_0 + at, t) + F(x_0 + at, t) \\ &= bh(t) + g(t) \end{aligned}$$

with

$$g(t) = F(x_0 + at, t) \ .$$

The IVP

$$h'(t) = bh(t) + g(t), \quad h(0) = f(x_0) \ ,$$

has the solution

$$h(t) = f(x_0)e^{bt} + \int_0^t e^{b(t-s)} g(s) \, ds \ .$$

This leads to the solution formula

$$u(x,t) = e^{bt} f(x - at) + \int_0^t e^{b(t-s)} F(x - at + as, s) \, ds \ .$$

Problem 4: Variable Signal Speed
Consider the IVP

$$u_t(x,t) + a(x,t)u_x(x,t) = 0, \quad u(x,0) = f(x) \ , \tag{8.4}$$

where $f(x)$ and $a(x,t)$ are smooth functions. Fix $x_0 \in \mathbb{R}$ and consider a line in the (x,t)–plane parameterized by t:

$$(\xi(t), t), \quad t \geq 0 \ ,$$

with $\xi(0) = x_0$. Let $u(x,t)$ solve the IVP (8.4) and set

$$h(t) = u(\xi(t), t) \ .$$

One obtains that $h(0) = u(x_0, 0) = f(x_0)$ and

$$h'(t) = u_t(\xi(t), t) + \xi'(t) u_x(\xi(t), t) \ .$$

This suggests how to choose $\xi(t)$: Assume that

$$\xi'(t) = a(\xi(t), t), \quad \xi(0) = x_0 \ .$$

Then obtain that $h'(t) \equiv 0$, thus

$$u(\xi(t), t) = h(t) \equiv f(x_0) \ .$$

The line

$$(\xi(t), t), \quad t \geq 0 \ ,$$

where

$$\xi'(t) = a(\xi(t), t), \quad \xi(0) = x_0 \ ,$$

is the characteristic in the (x,t)–plane starting at the point $(x_0, 0)$. The solution $u(x,t)$ of the IVP (8.4) is constant along any characteristic.

Example 2: Consider the IVP

$$u_t + xu_x = 0, \quad u(x,0) = \sin x \ .$$

The characteristic IVP is

$$\xi'(t) = \xi(t), \quad \xi(0) = x_0 \ .$$

One obtains that

$$\xi(t) = x_0 e^t \ .$$

Given $x \in \mathbb{R}$ and $t \geq 0$ determine $x_0 \in \mathbb{R}$ so that

$$x_0 e^t = x \ .$$

Clearly, $x_0 = xe^{-t}$. This yields that

$$u(x,t) = \sin(e^{-t}x) \ .$$

Check: We have $u(x,0) = \sin x$ and

$$u_t(x,t) = -e^{-t}x\cos(e^{-t}x), \quad u_x(x,t) = e^{-t}\cos(e^{-t}x) \ .$$

It follows that

$$u_t(x,t) + xu_x(x,t) = 0 \ .$$

Problem 5: Shock Formation
Consider an IVP for the **Inviscid Burgers' Equation**:

$$u_t(x,t) + u(x,t)u_x(x,t) = 0, \quad u(x,0) = \sin x \ . \tag{8.5}$$

Assume that $u(x,t)$ is a solution for $t \geq 0$. As above, fix $x_0 \in \mathbb{R}$ and let

$$\xi'(t) = u(\xi(t),t), \quad \xi(0) = x_0 \ .$$

If one sets

$$h(t) = u(\xi(t),t)$$

then $h(0) = \sin x_0$ and

$$\begin{aligned} h'(t) &= u_t(\xi(t),t) + \xi'(t)u_x(\xi(t),t) \\ &= u_t(\xi(t),t) + u(\xi(t),t)u_x(\xi(t),t) \\ &= 0 \end{aligned}$$

It follows that

$$h(t) \equiv \sin x_0 \equiv u(\xi(t),t) \ .$$

Therefore,

$$\xi(t) = x_0 + t \sin x_0 \ .$$

Along any straight line

$$(x_0 + t \sin x_0, t)$$

the solution (if it exists) carries the value $\sin x_0$.

First, take $x_0 = \pi/2$, thus $\sin x_0 = 1$. Along the line

$$\left(\frac{\pi}{2} + t, t\right)$$

the solution carries the value

$$u(\pi/2 + t, t) = \sin(\pi/2) = 1 \ .$$

Second, take $x_0 = \pi$, thus $\sin x_0 = 0$. Along the line

$$(\pi, t)$$

the solution carries the value

$$u(\pi, t) = \sin(\pi) = 0 \ .$$

However, the lines

$$\left(\frac{\pi}{2} + t, t\right) \quad \text{and} \quad (\pi, t)$$

intersect at $t = \pi/2$. This leads to a contradiction since $u(x,t)$ carries the value 1 on the first line and the value 0 on the second line.

For the IVP (8.5) one can show that a smooth solution exists for $0 \le t < 1$, but at time $t = 1$ a shock starts to form.

The inviscid Burgers' equation is a simple model for shock formation. For the viscous Burgers' equation $u_t + uu_x = \nu u_{xx}$ with $\nu > 0$ the viscosity term νu_{xx} smoothes the shocks. One can show that a smooth solution exists for $0 \le t < \infty$.

Derivation of a Solution for $0 \le t < 1$: Consider the IVP (8.5). Fix $0 \le t < 1$ and consider the function

$$\phi(x_0) = x_0 + t \sin x_0, \quad x_0 \in \mathbb{R} \ .$$

The function increases strictly and

$$\phi(x_0) \to \infty \quad \text{for} \quad x_0 \to \infty, \quad \phi(x_0) \to -\infty \quad \text{for} \quad x_0 \to -\infty \ .$$

Therefore, for every $x \in \mathbb{R}$ there exists a unique $x_0 \in \mathbb{R}$ with

$$x = x_0 + t \sin x_0 \ . \tag{8.6}$$

Set

$$u(x,t) = u(x_0 + t\sin x_0, t) = \sin x_0 \ .$$

The function $u(x,t)$ is defined for $x \in \mathbb{R}$ and $0 \le t < 1$. It satisfies $u(x_0, 0) = \sin x_0$ for all $x_0 \in \mathbb{R}$. Set

$$h(t) = u(x_0 + t\sin x_0, t) \quad \text{for} \quad 0 \le t < 1 \ .$$

Then $h(t) \equiv \sin x_0$ and

$$\begin{aligned} 0 &= h'(t) \\ &= u_t(x_0 + t\sin x_0, t) + \sin x_0 \, u_x(x_0 + t\sin x_0, t) \\ &= u_t(x_0 + t\sin x_0, t) + u(x_0 + t\sin x_0, t)\, u_x(x_0 + t\sin x_0, t) \end{aligned}$$

This proves that $u_t + uu_x = 0$ for $x \in \mathbb{R}, 0 \le t < 1$.

Remark: One can use the implicit function theorem so show that the solution $x_0 = x_0(x,t)$ of (8.6) depends smoothly on (x,t). Therefore, $u(x,t)$ is a smooth function defined for $x \in \mathbb{R}, 0 \le t < 1$.

It is not difficult to generalize the above example. Consider Burgers' equation with initial condition

$$u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R}$$

where $f \in C^1(\mathbb{R})$. Assume that for some $T > 0$ the following holds: For $0 \le t < T$ and for all $x \in \mathbb{R}$ the equation

$$x_0 + tf(x_0) = x$$

has a unique solution $x_0 = x_0(x,t)$. Then the function

$$u(x,t) = f(x_0(x,t))$$

defined for $x \in \mathbb{R}$ and $0 \le t < T$ satisfies Burgers' equation for $0 \le t < T$ and satisfies $u(x,0) = f(x)$.

9 The 1D Wave Equation

9.1 Derivation of the Equation

Imagine a homogeneous string placed along the x–axis in the interval $0 \le x \le L$. Assume it is set to vibrate vertically and the function $u(x,t)$ describes the vertical displacement as a function of position x and time t.

Let $N \in \mathbb{N}$ denote a large integer; let $h = L/N$ denote a step–size, and let $x_n = nh, n = 0, 1, \ldots, N$.

If $\rho > 0$ is the density of the string (assumed to be constant), then the mass in the interval $[x_n - h/2, x_n + h/2]$ is ρh. By Newton's law,

$$\rho h \, u_{tt}(x_n, t) = \text{force} \ .$$

The force is linked to the position of the neighbors,

$$y_{n+1} = u(x_{n+1}, t) \quad \text{and} \quad y_{n-1} = u(x_{n-1}, t) \ ,$$

and the tension of the string. In a simple model one assumes that the force coming from the right is $\tau(y_{n+1} - y_n)/h$ and the force coming from the left is $-\tau(y_n - y_{n-1})/h$. Here $\tau > 0$ is the coefficient of tension of the string. (The forces are assumed to be accurate to order $\mathcal{O}(h^2)$.)

One obtains

$$u_{tt}(x_n, t) \sim \frac{\tau}{\rho} \cdot \frac{1}{h} \Big(\frac{y_{n+1} - y_n}{h} - \frac{y_n - y_{n-1}}{h} \Big) \ .$$

As $h \to 0$ one obtains the equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \ , \tag{9.1}$$

where $c^2 = \tau/\rho$.

The equation $u_{tt} = c^2 u_{xx}$ is the 1D wave equation. With similar arguments one can derive the 2D and 3D wave equations,

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad \text{and} \quad u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \ .$$

9.2 d'Alembert's Formula

The 1D wave equation can be written as

$$\Big(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \Big) u(x,t) = 0$$

where

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \Big(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \Big) \Big(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \Big) \ .$$

Therefore, if the function $u(x,t)$ has the form

$$u(x,t) = f(x - ct) + g(x + ct) \tag{9.2}$$

with smooth functions f and g, then $u(x,t)$ solves the wave equation.

The term $f(x-ct)$ describes the motion of the function $f(x)$ to the right at speed c (assuming $c>0$) since the value $f(x_0)$ is carried along the line

$$(x(t),t)=(x_0+ct,t)\ .$$

The term $g(x+ct)$ describes motion to the left.

Consider the Cauchy Problem for the 1D wave equation,

$$\begin{aligned}
u_{tt}(x,t) &= c^2u_{xx}(x,t) \quad \text{for} \quad x\in\mathbb{R},\ t\geq 0\ , && (9.3)\\
u(x,0) &= F(x) \quad \text{for} \quad x\in\mathbb{R}\ , && (9.4)\\
u_t(x,0) &= G(x) \quad \text{for} \quad x\in\mathbb{R}\ , && (9.5)
\end{aligned}$$

where $F(x)$ and $G(x)$ are given smooth functions.

Assuming the form (9.2) for $u(x,t)$ one obtains the following equations for $f(x)$ and $g(x)$:

$$\begin{aligned}
f(x)+g(x) &= F(x)\\
-cf'(x)+cg'(x) &= G(x)
\end{aligned}$$

The second equation holds if

$$-cf(x)+cg(x)=\int_0^x G(s)\,ds =: H(x)\ .$$

In matrix form,

$$\begin{pmatrix} 1 & 1\\ -c & c\end{pmatrix}\begin{pmatrix} f(x)\\ g(x)\end{pmatrix}=\begin{pmatrix} F(x)\\ G(x)\end{pmatrix}\ ,$$

thus

$$\frac{1}{2c}\begin{pmatrix} c & -1\\ c & 1\end{pmatrix}\begin{pmatrix} F(x)\\ G(x)\end{pmatrix}=\begin{pmatrix} f(x)\\ g(x)\end{pmatrix}\ .$$

One obtains that

$$\begin{aligned}
f(x) &= \frac{1}{2}F(x)-\frac{1}{2c}H(x)\\
g(x) &= \frac{1}{2}F(x)+\frac{1}{2c}H(x)
\end{aligned}$$

Obtain the solution formula

$$\begin{aligned}
u(x,t) &= f(x-ct)+g(x+ct)\\
&= \frac{1}{2}\Big(F(x-ct)+F(x+ct)\Big)+\frac{1}{2c}\Big(H(x+ct)-H(x-ct)\Big)\\
&= \frac{1}{2}\Big(F(x-ct)+F(x+ct)\Big)+\frac{1}{2c}\int_{x-ct}^{x+ct}G(s)\,ds
\end{aligned}$$

The equation

$$u(x,t) = \frac{1}{2}\Big(F(x-ct) + F(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds \tag{9.6}$$

is called d'Alembert's formula for the solution of the wave equation.

It is one of the rare cases of an explicit and rather simple general solution of a PDE.

One can use the formula to discuss the domain of dependence and the domain of influence for the 1D wave equation. The discussion shows that disturbances travel at speed c in positive and negative direction.

For F–data (prescribing u at time zero), c is the sharp speed of propagation. For G–data (prescribing u_t at time zero), c is the maximal speed of propagation.

9.3 The 1D Wave Equation in a Strip

See Haberman, Set. 4.4.

Consider the wave equation $u_{tt} = c^2 u_{xx}$ with initial and boundary conditions:

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0\ , \\ u(x,0) &= F(x) \quad \text{for} \quad 0 \le x \le \pi\ , \\ u_t(x,0) &= G(x) \quad \text{for} \quad 0 \le x \le \pi\ , \\ u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0\ . \end{aligned}$$

We proceed as for the heat equation in a strip. The ansatz

$$u(x,t) = \phi(x)h(t)$$

leads to

$$\frac{h''(t)}{h(t)} = c^2\,\frac{\phi''(x)}{\phi(x)} = -c^2\lambda$$

where we expect $\lambda > 0$. The eigenvalue problem

$$-\phi''(x) = \lambda\phi(x), \quad \phi(0) = \phi(\pi) = 0\ ,$$

has the eigenvalues and eigenfunctions

$$\lambda_n = n^2, \quad \phi_n(x) = \sin(nx) \quad \text{for} \quad n = 1, 2, \ldots$$

The corresponding equation for $h(t)$ is

$$h''(t) + c^2 n^2 h(t) = 0$$

with solutions

$$h_n(t) = a_n \cos(nct) + b_n \sin(nct)\ .$$

The functions

$$u_n(x,t) = \sin(nx)\, h_n(t)$$

solve the wave equation and satisfy the boundary conditions

$$u_n(0,t) = u_n(\pi,0) = 0 \quad \text{for} \quad t \geq 0 \ .$$

Let

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(nct)\sin(nx) + \sum_{n=1}^{\infty} b_n \sin(nct)\sin(nx) \ . \tag{9.7}$$

The coefficients a_n and b_n must be determined by the initial conditions.

The equation

$$u(x,0) = F(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

requires that

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(ny)F(y)\, dy \ .$$

The equation

$$u_t(x,0) = G(x) = \sum_{n=1}^{\infty} b_n\, nc\, \sin(nx)$$

requires that

$$b_n = \frac{2}{\pi} \frac{1}{nc} \int_0^{\pi} \sin(ny)G(y)\, dy \ .$$

Next, we will use trigonometric identities to show that the solution $u(x,t)$ can be written in the form

$$u(x,t) = R(x-ct) + S(x+ct) \ .$$

The term $R(x-ct)$ describes the motion of the profile $R(x)$ to the right at speed c. The term $S(x+ct)$ describes the motion of the profile $S(x)$ to the left at speed c.

Recall that

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{aligned}$$

thus

$$\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big) \ .$$

For

$$\alpha = nx, \quad \beta = nct$$

obtain

$$\sin(nx)\sin(nct) = \frac{1}{2}\Big(\cos(n(x-ct)) - \cos(n(x+ct))\Big)\,.$$

Similarly,

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha\\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \sin\beta\cos\alpha\end{aligned}$$

thus

$$\sin\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha-\beta) + \sin(\alpha+\beta)\Big)\,.$$

For

$$\alpha = nx, \quad \beta = nct$$

obtain

$$\sin(nx)\cos(nct) = \frac{1}{2}\Big(\sin(n(x-ct)) + \sin(n(x+ct))\Big)\,.$$

The function $u(x,y)$ given by (9.7) is a sum of functions

$$\begin{aligned}u_n(x,t) &= \sin(nx)\Big(a_n\cos(nct) + b_n\sin(nct)\Big)\\ &= \frac{a_n}{2}\Big(\sin(n(x-ct)) + \sin(n(x+ct))\Big) + \frac{b_n}{2}\Big(\cos(n(x-ct)) - \cos(n(x+ct)\Big)\\ &= \frac{1}{2}\Big(a_n\sin(n(x-ct)) + b_n\cos(n(x-ct))\Big) + \frac{1}{2}\Big(a_n\sin(n(x+ct)) - b_n\cos(n(x+ct))\Big)\end{aligned}$$

Therefore,

$$u(x,t) = R(x-ct) + S(x+ct)$$

with

$$\begin{aligned}R(x) &= \frac{1}{2}\sum_{n=1}^{\infty}\Big(a_n\sin(nx) + b_n\cos(nx)\Big)\\ S(x) &= \frac{1}{2}\sum_{n=1}^{\infty}\Big(a_n\sin(nx) - b_n\cos(nx)\Big)\end{aligned}$$

Example: Consider the IBVP for the 1D wave equation in a strip:

$$\begin{aligned}
u_{tt}(x,t) &= c^2 u_{xx}(x,t) \quad \text{for} \quad 0 \le x \le \pi, \quad t \ge 0 \ , \\
u(x,0) &= \sin(x) \quad \text{for} \quad 0 \le x \le \pi \ , \\
u_t(x,0) &= \sin(3x) \quad \text{for} \quad 0 \le x \le \pi \ , \\
u(0,t) &= u(\pi,0) = 0 \quad \text{for} \quad t \ge 0 \ .
\end{aligned}$$

We have

$$\sin x = F(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

for $a_1 = 1$ and $a_n = 0$ for $n \ge 2$. Also,

$$\sin(3x) = G(x) = \sum_{n=1}^{\infty} b_n \, nc \sin(nx)$$

for $b_3 = 1/(3c)$ and $b_n = 0$ for $n \ne 3$.

The general process described above gives the solution

$$u(x,t) = \sin x \, \cos(ct) + \frac{1}{3c} \sin(3x) \sin(3ct) \ .$$

If one wants to write the solution in the form

$$u(x,t) = R(x - ct) + S(x + ct)$$

then one uses that

$$\sin x \, \cos(ct) = \frac{1}{2} \Big(\sin(x - ct) + \sin(x + ct) \Big)$$

and

$$\sin(3x) \sin(3ct) = \frac{1}{2} \Big(\cos(3(x - ct)) - \cos(3(x + ct)) \Big) \ .$$

Obtain

$$\begin{aligned}
R(x - ct) &= \frac{1}{2} \sin(x - ct) + \frac{1}{6c} \cos(3(x - ct)) \\
S(x + ct) &= \frac{1}{2} \sin(x + ct) - \frac{1}{6c} \cos(3(x + ct))
\end{aligned}$$

thus

$$\begin{aligned}
R(x) &= \frac{1}{2} \sin(x) + \frac{1}{6c} \cos(3x) \\
S(x) &= \frac{1}{2} \sin(x) - \frac{1}{6c} \cos(3x)
\end{aligned}$$

9.4 Terminology for Sinusoidal Waves

Let

$$\begin{aligned}
u(x,t) &= A\sin\Big(2\pi(\frac{x}{\lambda} - ft)\Big) \\
&= A\sin\Big(\frac{2\pi}{\lambda}(x - ct)\Big) \quad \text{with} \quad c = \lambda f \\
&= A\sin\Big(kx - \omega t\Big) \quad \text{with} \quad k = 2\pi/\lambda \quad \text{and} \quad \omega = 2\pi f
\end{aligned}$$

$$\begin{aligned}
A &: \text{ amplitude} \\
\lambda &: \text{ wave length} \\
f &: \text{ frequency} \\
c = \lambda f &: \text{ wave speed} \\
k = 2\pi/\lambda &: \text{ angular wave number} \\
\omega = 2\pi f &: \text{ angular frequency}
\end{aligned}$$

10 The Wave Equation in a Disk

10.1 Separation of Variables in Polar Coordinates

Consider the 2D wave equation in polar coordinates,

$$u_{tt} = c^2 \Delta u = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} \right) . \tag{10.1}$$

We consider the equation in the unit disk, i.e., for $0 \le r \le 1$ and $\phi \in \mathbb{R}$ where $\phi \to u(r, \phi, t)$ is 2π–periodic. The boundary and initial conditions are

$$\begin{aligned} u(1, \phi, t) &= 0 \quad \text{for} \quad \phi \in \mathbb{R}, \quad t \ge 0 \ , && (10.2) \\ u(r, \phi, 0) &= \alpha(r, \phi) \quad \text{for} \quad \phi \in \mathbb{R}, \quad 0 \le r \le 1 \ , && (10.3) \\ u_t(r, \phi, 0) &= \beta(r, \phi) \quad \text{for} \quad \phi \in \mathbb{R}, \quad 0 \le r \le 1 \ . && (10.4) \end{aligned}$$

Here $\alpha(r, \phi)$ and $\beta(r, \phi)$ are given smooth functions which are 2π–periodic in ϕ.

The ansatz

$$u(r, \phi, t) = v(r, \phi) T(t)$$

leads to

$$\frac{T''(t)}{T(t)} = \frac{c^2 \Delta v(r, \phi)}{v(r, \phi)} = -\omega^2, \quad \omega \ge 0 \ , \tag{10.5}$$

thus

$$T_\omega(t) = \alpha_\omega \cos(\omega t) + \beta_\omega \sin(\omega t) \quad \text{for} \quad \omega > 0 \tag{10.6}$$

and

$$T_0(t) = \alpha_0 + \beta_0 t \quad \text{for} \quad \omega = 0 \ . \tag{10.7}$$

For the function $v(r, \phi)$ one obtains the equation

$$\Delta v(r, \phi) + k^2 v(r, \phi) = 0 \tag{10.8}$$

with $k = \omega / c \ge 0$. Equation (10.8) is called a Helmholtz equation.

Equation (10.8) leads us to an eigenvalue problem since the solutions of interest must be non–trivial. They must satisfy the boundary condition

$$v(1, \phi) = 0 \quad \text{for} \quad \phi \in \mathbb{R}$$

and, of course, $v(r, \phi)$ must be 2π–periodic in ϕ. In polar coordinates, the Helmholtz equation reads

$$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\phi\phi} + k^2 v = 0 \ . \tag{10.9}$$

Using the ansatz

$$v(r, \phi) = V(r) \Phi(\phi)$$

one obtains

$$\frac{r^2 V''(r) + rV'(r)}{V(r)} + \frac{\Phi''(\phi)}{\Phi(\phi)} + r^2 k^2 = 0 \ .$$

Since $\Phi''(\phi)/\Phi(\phi)$ must be constant and $\Phi(\phi)$ must be 2π–periodic, obtain that

$$\Phi''(\phi) + m^2 \Phi(\pi) = 0 \quad \text{where} \quad m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \ .$$

Therefore,

$$\Phi_m(\phi) = a_m \cos(m\phi) + b_m \sin(m\phi) \quad \text{for} \quad m = 1, 2, \ldots$$

and

$$\Phi_0(\phi) = a_0 \ .$$

For the function $V(r)$ obtain the equation

$$r^2 V''(r) + rV'(r) + (r^2 k^2 - m^2)V(r) = 0 \tag{10.10}$$

and the boundary condition $V(1) = 0$.

For $k = 0$ obtain the Cauchy–Euler equation

$$r^2 V''(r) + rV'(r) - m^2 V(r) = 0 \ . \tag{10.11}$$

For $m \geq 1$ the general solution is

$$V(r) = ar^m + br^{-m} \ .$$

Since r^{-m} is singular at $r = 0$ and we want to have a solution $u(r, \phi, t)$ which is smooth at $r = 0$, the term br^{-m} can be ignored. Then the boundary condition $V(1) = 0$ yields that $V \equiv 0$. For $m = 0$ the general solution of (10.11) is

$$V(r) = a + b \ln r \ .$$

Again, the singular term $b \ln r$ can be ignored and the boundary condition $V(1) = 0$ yields that $V \equiv 0$.

Therefore, we will consider the equation (10.10) only for $k > 0$ and $m \in \mathbb{N}_0$.

10.2 Introduction of Bessel Functions of the First Kind

To study the equation (10.10) with the boundary condition $V(1) = 0$ we introduce the variable $z = rk$ and let

$$V(r) = f(z) = f(rk)$$

where $f(z)$ is a new unknown function. Obtain

$$\begin{aligned} V'(r) &= kf'(rk) = kf'(z) \\ V''(r) &= k^2 f''(rk) = k^2 f''(z) \end{aligned}$$

The equation (10.10) becomes

$$z^2 f''(z) + zf'(z) + (z^2 - m^2)f(z) = 0 \tag{10.12}$$

and the boundary condition $V(1) = 0$ becomes $f(k) = 0$.

Equation (10.12) is called **Bessel's differential equation** of index m.

Regular Solutions of Bessel's Equation: To study the solutions of (10.12) for $z \sim 0$ let us first neglect the term $z^2 f(z)$ and consider the Cauchy–Euler equation

$$z^2 y''(z) + zy'(z) - m^2 y(z) = 0 \ .$$

For $m \geq 1$ the general solution is

$$y(z) = az^m + bz^{-m}$$

and for $m = 0$ the general solution is

$$y(z) = a + b \log z \ .$$

The functions z^{-m} and $\log z$ are singular at $z = 0$ whereas z^m and $z^0 = 1$ are well–behaved. To obtain a solution of (10.12) which is regular at $z = 0$ use the ansatz

$$f(z) = z^m \sum_{n=0}^{\infty} a_n z^n$$

and derive the power series coefficients a_n. We have

$$\begin{aligned}
f(z) &= z^m \sum_{n=0}^{\infty} a_n z^n \\
f'(z) &= mz^{m-1} \sum_{n=0}^{\infty} a_n z^n + z^m \sum_{n=0}^{\infty} n a_n z^{n-1} \\
f''(z) &= m(m-1)z^{m-2} \sum_{n=0}^{\infty} a_n z^n + 2mz^{m-1} \sum_{n=0}^{\infty} n a_n z^{n-1} + z^m \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} \\
zf'(z) &= z^m \sum_{n=0}^{\infty} a_n z^n (m+n) \\
z^2 f''(z) &= z^m \sum_{n=0}^{\infty} a_n z^n \Big(m(m-1) + 2mn + n(n-1) \Big)
\end{aligned}$$

Therefore,

$$Lf(z) = z^2 f''(z) + zf'(z) + (z^2 - m^2)f(z) = z^m \sum_{n=0}^{\infty} a_n z^n A_n + z^m \sum_{n=0}^{\infty} a_n z^{n+2}$$

with

$$A_n = m(m-1) + 2mn + n(n-1) + m + n - m^2 = n(n+2m) \ .$$

This yields that

$$\begin{aligned} Lf(z) &= z^m\Big(\sum_{n=0}^{\infty} a_n A_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n\Big) \\ &= z^m\Big(a_0 A_0 + a_1 A_1 z + \sum_{n=2}^{\infty} (a_n A_n + a_{n-2}) z^n\Big) \end{aligned}$$

One obtains that $Lf(z) \equiv 0$ if and only if

$$\begin{aligned} a_0 A_0 &= 0 \\ a_1 A_1 &= 0 \\ a_n A_n &= -a_{n-2} \quad \text{for} \quad n \geq 2 \end{aligned}$$

Since $A_0 = 0$ the coefficient a_0 is free. By convention,

$$a_0 = \frac{1}{2^m m!} \ .$$

Since $A_1 \neq 0$ obtain that $a_1 = 0$ and then $a_n = 0$ for all odd n. This yields that

$$f(z) = z^m \sum_{j=0}^{\infty} a_{2j} z^{2j} \quad \text{where} \quad a_0 = \frac{1}{2^m m!}$$

and

$$a_{2j} = -\frac{a_{2j-2}}{A_{2j}} \quad \text{with} \quad A_{2j} = 2j(2j+2m) = 4j(j+m) \ .$$

By induction, it follows that

$$a_{2j} = \frac{(-1)^j}{j!(j+m)!} \cdot \frac{1}{2^{m+2j}} \ . \tag{10.13}$$

Induction Argument: Formula (10.13) holds for $j = 0$. Suppose it holds for $j-1$:

$$a_{2j-2} = \frac{(-1)^{j-1}}{(j-1)!(j-1+m)!} \cdot \frac{1}{2^{m+2j-2}} \ .$$

Then

$$a_{2j} = -a_{2j-2}/A_{2j} = \frac{(-1)^j}{j!(j+m)!} \cdot \frac{1}{2^{m+2j}} \ .$$

We have derived the formula

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$$

for the Bessel function of the first kind on index m.

Since $j!$ grows much faster then $|z|^{2j}$ for $j \to \infty$ it is not difficult to prove that the power series converges absolutely for all $z \in \mathbb{C}$. The function $J_m(z)$ is entire.

For our application, it is important to understand the behavior of $J_m(x)$ for $0 \le x < \infty$. For $x \sim 0$ we have

$$J_m(x) \sim \frac{1}{m!} \left(\frac{x}{2}\right)^m .$$

Consider Bessel's differential equation for $0 < x < \infty$:

$$f''(x) + \frac{1}{x} f'(x) + \left(1 - \frac{m^2}{x^2}\right) f(x) = 0 .$$

If x is large it makes sense to neglect the term $-m^2/x^2$ and to replace the term $1/x$ by a small $\varepsilon > 0$. This leads to the approximate equation

$$y''(x) + \varepsilon y'(x) + y(x) = 0 .$$

The even simpler equation

$$y''(x) + y(x) = 0$$

has the oscillatory solutions

$$y(x) = a \cos x + b \sin x .$$

For small $\varepsilon > 0$, the term $\varepsilon y'(x)$ adds a damping. These somewhat vague considerations suggest that the Bessel function $J_m(x)$ decays to zero as $x \to \infty$ and oscillates, where 2π is the approximate period. One can prove that $J_m(x)$ has a sequence of positive zeros x_{mn},

$$0 < x_{m1} < x_{m2} < x_{m3} < \dots$$

with $x_{mn} \to \infty$ as $n \to \infty$. For large x the following approximation holds

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - m\frac{\pi}{2}\right) .$$

(Haberman, formula (7.8.3))

10.3 Application to the Wave Equation

Recall the equation (10.10):

$$r^2 V''(r) + r V'(r) + (r^2 k^2 - m^2) V(r) = 0 \tag{10.14}$$

and the boundary condition $V(1) = 0$. The function $V(r) = J_m(kr)$ satisfies the differential equation and the boundary condition $0 = v(1) = J_m(k)$ requires that the parameter k is a zero of $J_m(x)$, thus

$$k = x_{mn} \quad \text{for some } n \in \mathbb{N} \ .$$

One obtains that the functions

$$V_{mn}(r) = J_m(rx_{mn}) \ , \quad r \geq 0 \ ,$$

satisfy the differential equation (10.14) and the boundary condition $V_{mn}(1) = 0$ for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

The functions

$$v_{mn,c}(r,\phi) = V_{mn}(r)\cos(m\phi) \quad \text{and} \quad v_{mn,s}(r,\phi) = V_{mn}(r)\sin(m\phi)$$

satisfy the differential equation (10.9),

$$v_{rr} + \frac{1}{r}\, v_r + \frac{1}{r^2}\, v_{\phi\phi} + k^2 v = 0 \tag{10.15}$$

for $k = x_{mn}$. Recall that $k = \omega/c$, thus $\omega = \omega_{mn} = x_{mn}c$.

For simplicity, assume the the initial function $\beta(r,\phi)$ in the initial condition (10.4) is zero, thus

$$u_t(r,\phi,0) = 0 \ .$$

Then the terms $\sin(x_{mn}ct)$ will not be present in the solution $u(r,\phi,t)$ and one obtains the following series representation of the solution of the wave equation (10.1) with boundary condition (10.2):

$$\begin{aligned} u(r,\phi,t) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}V_{mn}(r)\cos(m\phi)\cos(x_{mn}ct) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}V_{mn}(r)\sin(m\phi)\cos(x_{mn}ct) \end{aligned}$$

Here

$$V_{mn}(r) = J_m(rx_{mn}) \ .$$

The coefficients A_{mn} and B_{mn} must be determined by the expansion of the initial function $u(r,\phi,0) = \alpha(r,\phi)$:

$$\alpha(r,\phi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}V_{mn}(r)\cos(m\phi) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}V_{mn}(r)\sin(m\phi) \tag{10.16}$$

10.4 Expansion in Terms of Bessel Functions

Lemma 10.1 *Fix $m \in \mathbb{N}_0$ and let $j, n \in \mathbb{N}$. Set*

$$\begin{aligned} f_j(r) &= V_{mj}(r) = J_m(r x_{mj}) \\ f_n(r) &= V_{mn}(r) = J_m(r x_{mn}) \end{aligned}$$

Then

$$\int_0^1 r f_j(r) f_n(r)\, dr = \delta_{jn} \, \frac{1}{2} \Big(J_m'(x_{mj}) \Big)^2 \, . \tag{10.17}$$

Proof: The functions $f_j(r)$ and $f_n(r)$ solve the differential equation (10.14) with $k = x_{mj}$ and $k = x_{mn}$, respectively. Therefore, for $r > 0$:

$$(r f_j')' + (x_{mj}^2 r - m^2/r) f_j = 0 \tag{10.18}$$

$$(r f_n')' + (x_{mn}^2 r - m^2/r) f_n = 0 \tag{10.19}$$

Multiply (10.18) by f_n and multiply (10.19) by f_j. Then integrate to obtain:

$$\begin{aligned} \int_0^1 (r f_j')' f_n \, dr + \int_0^1 (x_{mj}^2 r - m^2/r) f_j f_n \, dr &= 0 \\ \int_0^1 (r f_n')' f_j \, dr + \int_0^1 (x_{mn}^2 r - m^2/r) f_j f_n \, dr &= 0 \end{aligned}$$

By subtraction obtain:

$$\int_0^1 (r f_j')' f_n \, dr - \int_0^1 (r f_n')' f_j \, dr + \Big(x_{mj}^2 - x_{mn}^2 \Big) \int_0^1 r f_j f_n \, dr = 0 \, .$$

Here

$$x_{mj}^2 - x_{mn}^2 \neq 0$$

if $j \neq n$. Using that $f_j(1) = f_n(1) = 0$ one obtains through integration by parts that

$$\int_0^1 (r f_j')' f_n \, dr = \int_0^1 (r f_n')' f_j \, dr \, .$$

It follows that

$$\int_0^1 r f_j f_n \, dr = 0 \quad \text{for} \quad j \neq n \, .$$

If $n = j$ multiply (10.18) by $r f_j'$ and integrate to obtain that

$$\int_0^1 r f_j'(r f_j')'\,dr + x_{mj}^2 \int_0^1 r^2 f_j f_j'\,dr - m^2 \int_0^1 f_j f_j'\,dr = 0\ . \tag{10.20}$$

Here

$$m^2 \int_0^1 f_j f_j'\,dr = 0\ .$$

(Use integration by parts if $m \neq 0$.) With $y = r f_j'$ the first integral in (10.20) is

$$\begin{aligned}
\int_0^1 y y'\,dr &= \frac{1}{2}\int_0^1 (y^2)'\,dr \\
&= \frac{1}{2}\,y^2(1) \\
&= \frac{1}{2}\,(f_j'(1))^2 \\
&= \frac{1}{2}\,x_{mj}^2\Big(J_m'(x_{mj})\Big)^2
\end{aligned}$$

Also,

$$\int_0^1 r^2 f_j f_j'\,dr = \int_0^1 (-2r f_j f_j - r^2 f_j' f_j)\,dr\ ,$$

thus

$$\int_0^1 r^2 f_j f_j'\,dr = -\int_0^1 r f_j^2\,dr\ .$$

The equation

$$\int_0^1 r f_j^2\,dr = \frac{1}{2}\Big(J_m'(x_{mj})\Big)^2$$

follows and the lemma is proved. $\diamond$

Consider two functions

$$V_{m_1 j}(r)\cos(m_1\phi) \quad \text{and} \quad V_{m_2 n}(r)\cos(m_2\phi)$$

which occur in the expansion of the initial value $u(r,\phi,0) = \alpha(r,\phi)$ given above. Consider the integral of the product of the two functions over the disk

$$0 \leq r \leq 1, \quad 0 \leq \phi \leq 2\pi\ .$$

We claim that the integral is zero unless $m_1 = m_2$ and $j = n$.

Recall that the element of area in polar coordinates is

$$r\,dr d\phi\ .$$

The integral of the product is a product of integrals,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 V_{m_1 j}(r)\cos(m_1\phi)\,V_{m_2 n}(r)\cos(m_2\phi)\,rdrd\phi \\
&= \int_0^{2\pi}\cos(m_1\phi)\cos(m_2\phi)\,d\phi\cdot\int_0^1 rV_{m_1 j}(r)V_{m_2 n}(r)\,dr\ .
\end{aligned}$$

If $m_1 \neq m_2$ then the integral of the product of the cosines is zero. If $m_1 = m_2$ but $j \neq n$ then integral of the product of the Bessel functions is zero by the previous lemma.

The same orthogonality holds if $\cos(m\phi)$ is replaced by $\sin(m\phi)$. Therefore, the expansion functions occurring in (10.16) are all orthogonal to each other. One can show that any smooth function $\alpha(r,\phi)$ can be written in the series form (10.16).

10.5 The Γ–Function

For $s > 0$ one defines

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt\ .$$

Clearly,

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1\ .$$

The functional equation:

$$\begin{aligned}
\Gamma(s+1) &= \int_0^\infty t^s e^{-t}\,dt \\
&= -\int_0^\infty t^s e^{-t}|_0^\infty + s\int_0^\infty t^{s-1}e^{-t}\,dt \\
&= s\Gamma(s)
\end{aligned}$$

It follows that

$$\Gamma(n+1) = n! \quad \text{for} \quad n = 0,1,2,\dots$$

For $s = \frac{1}{2}$ one obtains, using the substitution

$$t^{1/2} = \tau, \quad 2d\tau = t^{-1/2}dt\ ,$$

$$\begin{aligned}
\Gamma(\frac{1}{2}) &= \int_0^\infty t^{-1/2}e^{-t}\,dt \\
&= 2\int_0^\infty e^{-\tau^2}\,d\tau \\
&= \sqrt{\pi}
\end{aligned}$$

Let $j \in \{0, 1, 2, \ldots\}$. We have

$$\begin{array}{rcl} \Gamma(\frac{1}{2}) & = & \sqrt{\pi} \\ \Gamma(1+\frac{1}{2}) & = & \frac{1}{2}\sqrt{\pi} \\ \Gamma(2+\frac{1}{2}) & = & \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \\ \Gamma(3+\frac{1}{2}) & = & \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \\ \Gamma(j+1+\frac{1}{2}) & = & \frac{1\cdot 3\ldots(2j+1)}{2^{j+1}}\sqrt{\pi} \end{array}$$

Here

$$1\cdot 3\cdot 5\ldots(2j+1) = \frac{(2j+1)!}{2^j j!} \ .$$

This yields

$$\Gamma\Big(j+1+\frac{1}{2}\Big) = \frac{(2j+1)!}{2^{2j+1}\, j!}\sqrt{\pi} \ .$$

For $m \geq 0, m \in \mathbb{R}$ we have

$$J_m(x) = \Big(\frac{x}{2}\Big)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1+m)} \Big(\frac{x}{2}\Big)^{2j}$$

This series representation shows the following:

Lemma 10.2 *For $m = \frac{1}{2}$ we have*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x, \quad x > 0 \ .$$

Proof: We have

$$\begin{array}{rcl} J_{1/2}(x) & = & \Big(\frac{x}{2}\Big)^{1/2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+1+\frac{1}{2})} \Big(\frac{x}{2}\Big)^{2j} \\ & = & \Big(\frac{2}{\pi x}\Big)^{1/2} \cdot \frac{x}{2} \cdot \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^{2j+1}}{(2j+1)!} \Big(\frac{x}{2}\Big)^{2j} \\ & = & \Big(\frac{2}{\pi x}\Big)^{1/2} \cdot \sum_{j=0}^{\infty} \frac{(-1)^j\, x^{2j+1}}{(2j+1)!} \\ & = & \Big(\frac{2}{\pi x}\Big)^{1/2} \sin x \end{array}$$

It follows that the zeros of $J_{1/2}(x)$ are

$$x_{1/2,n} = n\pi, \quad n = 0, 1, 2\ldots$$

10.6 Sturm's Comparison Theorem

Theorem 10.1 *Let $g_1, g_2 : (a, b) \to$ denote two continuous functions with*

$$g_1(x) < g_2(x) \quad \textit{for} \quad a < x < b \ .$$

Let $y_1, y_2 : (a, b) \to \mathbb{R}$ denote two C^2–functions with

$$\begin{aligned} y_1''(x) + g_1(x)y_1(x) &= 0 \quad \textit{for} \quad a < x < b \\ y_2''(x) + g_2(x)y_2(x) &= 0 \quad \textit{for} \quad a < x < b \end{aligned}$$

Assume that $a < p < q < b$ and

$$y_1(p) = y_1(q) = 0, \quad y_1(x) > 0 \quad \textit{for} \quad p < x < q \ ,$$

i.e., p and q are two consecutive zeros of y_1. Then there exists

$$p < x^* < q \quad \textit{with} \quad y_2(x^*) = 0 \ .$$

Proof: Suppose that $y_2(x) > 0$ for $p < x < q$. We have

$$\begin{aligned} y_1''y_2 + g_1y_1y_2 &= 0 \\ y_2''y_1 + g_2y_2y_1 &= 0 \end{aligned}$$

Therefore,

$$y_1''y_2 - y_2'' = (g_1 - g_2)y_1y_2 > 0 \quad \text{for} \quad p < x < q \ .$$

It follows that

$$\int_p^q (y_1''y_2 - y_2''y_1)\, dx > 0 \ .$$

Also,

$$\begin{aligned} \int_p^q y_1''y_2\, dx &= y_1'y_2|_p^q - \int_p^q y_1'y_2'\, dx \\ \int_p^q y_2''y_2\, dx &= y_2'y_1|_p^q - \int_p^q y_1'y_2'\, dx \end{aligned}$$

Therefore,

$$\begin{aligned} \int_p^q (y_1''y_2 - y_2''y_1)\, dx &= y_1'y_2|_p^q \\ &= y_1'(q)y_2(q) - y_1'(p)y_2(p) \end{aligned}$$

Here $y_1'(q) \le 0 \le y_1'(p)$ and $y_2(p) \ge 0, y_2(q) \ge 0$. One obtains a contradictions.

10.7 The Zeros of $J_m(x)$

Let $m \geq 0$. We show here how to use Sturm's theorem to obtain information about the zeros of $J_m(x)$.

Recall that $J_m(x)$ satisfies Bessel's equation:

$$x^2 J_m''(x) + xJ_m'(x) + (x^2 - m^2)J_m(x) = 0 \ .$$

Fix m and define the function $w_m(x)$ by

$$J_m(x) = x^{-1/2}\, w_m(x), \quad x > 0 \ .$$

(This is the Liouville transform to obtain an equation for $w_m(x)$ without first derivative term. In fact, $p(x) = 1/x$ and $P(x) = \ln x$ and $e^{-P(x)/2} = x^{-1/2}$.)

Lemma 10.3 *The function $w_m(x)$ satisfies the differential equation*

$$w_m''(x) + \Big(1 - \frac{m^2 - \frac{1}{4}}{x^2}\Big) w_m(x) = 0, \quad x > 0 \ .$$

Proof: Assume

$$x^2 f'' + xf' + (x^2 - m^2)f = 0$$

and $f = x^{-1/2}w$ Then we have

$$\begin{aligned} f' &= x^{-1/2}w' - \frac{1}{2}\, x^{-3/2}w \\ f'' &= x^{-1/2}w'' - x^{-3/2}w' + \frac{3}{4}\, x^{-5/2}w \end{aligned}$$

This yields that

$$\begin{aligned} 0 &= x^{3/2}w'' - x^{1/2}w' + \frac{3}{4}\, x^{-1/2}w + x^{1/2}w' - \frac{1}{2}\, x^{-1/2}w + (x^2 - m^2)x^{-1/2}w \\ &= x^{3/2}w'' + \frac{1}{4}\, x^{-1/2}w + x^{-1/2}(x^2 - m^2)w \end{aligned}$$

It follows that

$$w'' + \Big(1 - \frac{m^2 - 1/4}{x^2} w\Big) = 0 \ .$$

$\diamond$

Remark: For $m = 1/2$ the differential equation for w_m has constant coefficients, $w_{1/2}'' + w_{1/2} = 0$. This is consistent with the above result for $J_{1/2}(x) = cx^{-1/2} \sin x$.

Case 1: Let $0 \leq m < \frac{1}{2}$. We apply Sturm's theorem with

$$g_1(x) \equiv 1, \quad g_2(x) = 1 - \frac{m^2 - \frac{1}{4}}{x^2} \ .$$

It is clear that $g_2(x) > 1$ for all $x > 0$. Let

$$y_1(x) = \sin(x - \alpha)$$

and

$$y_2(x) = w_m(x) \ .$$

Here $\alpha \geq 0$ is arbitrary. The function $y_1(x)$ has zeros

$$p = \alpha < q = \alpha + \pi \ .$$

By Sturm's theorem, the function $y_2 = w_m$ has a zero between α and $\alpha + \pi$. Since α is arbitrary, we obtain that the function $J_m(x)$ has a sequence of positive zeros, denoted by $x_{mn}, n = 1, 2, \ldots$:

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

(If $m > 0$ then $x_{m0} = 0$ is also a zero of $J_m(x)$.)

For fixed m, the zeros of $J_m(x)$ cannot accumulate at some finite value $\bar{x}$. Otherwise, one would obtain that

$$J_m(\bar{x}) = J_m'(\bar{x}) = 0 \ ,$$

and $J_m(x) \equiv 0$ would be implied.

Furthermore, we claim that, for $0 \leq m < \frac{1}{2}$:

$$x_{m,n+1} - x_{m,n} < \pi \ .$$

In other words, any two consecutive zeros of $J_m(x)$ have a distance less than π for $0 \leq m < \frac{1}{2}$. This follows from Sturm's theorem applied with

$$y_1(x) = \sin(x - x_{m,n}) \ .$$

We summarize:

Theorem 10.2 *Let $0 \leq m < \frac{1}{2}$. The function $J_m(x)$ has infinitely many positive zeros. These can be ordered as a sequence:*

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We have

$$x_{mn} \to \infty \quad as \quad n \to \infty$$

and

$$x_{m,n+1} - x_{m,n} < \pi, \quad n = 1, 2, \ldots$$

Case 2: $m > \frac{1}{2}$. The function $w_m(x)$ satisfies

$$w_m''(x) + g_2(x) w_m(x) = 0$$

with

$$g_2(x) = 1 - x^{-2}\Big(m^2 - \frac{1}{4}\Big) < 1 \ .$$

For $x > m$ we have

$$g_2(x) > g_2(m) = \frac{1}{4m^2} =: g_1(x) \ .$$

We know that

$$y_1(x) = \sin\Big(\frac{x}{2m} - \alpha\Big)$$

solves

$$y_1'' + \frac{1}{4m^2} y_1 = 0$$

and y_1 has infinitely many positive zeros. It follows that, again, $J_m(x)$ has a sequence of positive zeros,

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We claim that

$$x_{m,n+1} - x_{m,n} > \pi \ .$$

This follows from

$$g_2(x) < 1$$

since the solutions $y_3(x)$ of

$$y_3'' + y_3 = 0$$

have zeros with distance π. We consider

$$y_3(x) = \sin(x - x_{m,n}) \ .$$

Then, by Sturm's theorem, y_3 has a zero strictly between $x_{m,n}$ and $x_{m,n+1}$, which yields $x_{m,n+1} - x_{m,n} > \pi$.

Theorem 10.3 *Let $m > \frac{1}{2}$. The function $J_m(x)$ has infinitely many positive zeros. These can be ordered as a sequence:*

$$0 < x_{m1} < x_{m2} < x_{m3} < \ldots$$

We have

$$x_{mn} \to \infty \quad \textit{as} \quad n \to \infty$$

and

$$x_{m,n+1} - x_{m,n} > \pi, \quad n = 1, 2, \ldots$$

11 Poisson's Equation in $\mathbb{R}^3$

Notations: Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

denote the standard basis of $\mathbb{R}^3$. For

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum_{j=1}^{3} x_j e_j \in \mathbb{R}^3$$

let

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

denote the Euclidean norm. For $r > 0$ let

$$B_r = \{x \in \mathbb{R}^3 \; : \; |x| \leq r\}$$

denote the ball of radius r centered at zero. Let

$$\partial B_r = \{x \in \mathbb{R}^3 \; : \; |x| = r\}$$

denote the surface of the ball B_r.

11.1 Physical Interpretation of Poisson's Equation in $\mathbb{R}^3$

Let Q be a point charge at the point $y \in \mathbb{R}^3$ and let q be a point charge at $x \in \mathbb{R}^3$. By Coulomb's law, the electro static force F of Q on q is

$$F = kQq \, \frac{1}{|x-y|^2} \, \frac{x-y}{|x-y|} \; .$$

Here k is a constant that depends on the units used for charge, force, and length.[1] Therefore, the electric field generated by the charge Q at the point y is

$$E(x) = kQ \, \frac{x-y}{|x-y|^3} \; .$$

The force field $E(x)$ has the potential

$$u(x) = \frac{kQ}{|x-y|} \; .$$

This means that

$$-\nabla \, u(x) = -\mathrm{grad} \, u(x) = E(x), \quad x \neq y \; .$$

The following results about the function $|x|^{-1}$ are useful.

[1] It is common to write $k = \frac{1}{4\pi\varepsilon_0}$, where $\varepsilon_0 = 8.859 \cdot 10^{-12} Coul^2 \cdot N^{-1} \cdot m^{-2}$. The formula for the force F then holds in vacuum.

Lemma 11.1 *Let $r = r(x) = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Then we have, for all $x \neq 0$,*

$$\begin{aligned}
D_j r &= \frac{x_j}{r} \\
D_j\Big(\frac{1}{r}\Big) &= -\frac{x_j}{r^3} \\
\nabla\Big(\frac{1}{r}\Big) &= -\frac{x}{r^3} \\
D_j^2\Big(\frac{1}{r}\Big) &= -|x|^{-5}(|x|^2 - 3x_j^2) \\
D_i D_j\Big(\frac{1}{r}\Big) &= 3|x|^{-5} x_i x_j \quad \textit{for} \quad i \neq j \\
\Delta\Big(\frac{1}{r}\Big) &= 0
\end{aligned}$$

It is easy to show that $|x|$ may be replaced by $|x-y|$ with y fixed. For example,

$$\nabla_x \frac{1}{|x-y|} = -\frac{x-y}{|x-y|^3} \ .$$

This yields $-\nabla\, u(x) = E(x)$, as claimed above.

By the above lemma, the potential $u(x)$ satisfies Laplace's equation, $\Delta u(x) = 0$, at every point $x \neq y$. At $x = y$, the potential $u(x)$ has a singularity, marking the point y where the charge Q, which generates the field, is located.

It is not difficult to generalize to N point charges: Assume that there are N point charges $Q_1, \ldots, Q_N$ located at $y^{(1)}, \ldots, y^{(N)}$. They generate the electric field

$$E(x) = k \sum_i Q_i \frac{x - y^{(i)}}{|x - y^{(i)}|^3}$$

with potential

$$u(x) = k \sum_i \frac{Q_i}{|x - y^{(i)}|} \ .$$

At each point $x = y^{(i)}$ the potential has a singularity, but in the set

$$\mathbb{R}^3 \setminus \{y^{(1)}, \ldots, y^{(N)}\}$$

the function $u(x)$ satisfies Laplace's equation, $\Delta u = 0$.

It is plausible that Coulomb's law can be extended from point charges to continuously distributed charges. Let $V \subset \mathbb{R}^3$ denote a bounded region and let $f : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function with $f(y) = 0$ for $y \in \mathbb{R}^3 \setminus V$. Assume that f describes a smooth charge distribution in V. What is the electric field $E(x)$ generated by the charge distribution f?

Suppose that $V_i \subset V$ denotes a small volume about the point $y^{(i)}$. Let

$$Q_i = \int_{V_i} f(y)dy \sim f(y^{(i)})\, vol(V_i)$$

denote the charge in V_i. Assume that

$$V = V_1 \cup V_2 \cup \ldots \cup V_N$$

where the V_i are disjoint.

For the electric field $E(x)$ generated by the charge distribution f obtain:

$$\begin{aligned} E(x) &\sim k \sum_i Q_i \frac{x - y^{(i)}}{|x - y^{(i)}|^3} \\ &\sim k \sum_i f(y^{(i)}) \frac{x - y^{(i)}}{|x - y^{(i)}|^3}\, vol(V_i) \\ &\sim k \int_{\mathbb{R}^3} f(y) \frac{x - y}{|x - y|^3}\, dy \end{aligned}$$

An extension of Coulomb's law for point charges says that the charge distribution f generates the elctric field

$$E(x) = k \int_{\mathbb{R}^3} f(y) \frac{x - y}{|x - y|^3}\, dy\ .$$

The function

$$u(x) = k \int_{\mathbb{R}^3} \frac{f(y)}{|x - y|}\, dy \tag{11.1}$$

satisfies

$$-\nabla u(x) = E(x), \quad x \in \mathbb{R}^3\ ,$$

i.e., $u(x)$ is a potential of the electric field $E(x)$.

(The formula for $E(x)$ cannot be deduced rigorously from Coulomb's law for point charges, but may be considered as a reasonable extension of the law.)

If u is defined by (11.1), we expect that $\Delta u = 0$ in regions where $f = 0$. We also expect a simple relation between Δu and f in regions where f is not zero. We will show that, under suitable assumptions on the function $f(y)$, we have

$$-\Delta u = 4\pi k f = \frac{1}{\varepsilon_0} f$$

if u is defined by (11.1). To summarize, an interpretation of Poisson's equation

$$-\Delta u = f$$

is the following: The function f is a charge distribution, generating an electric field, and (modulo a constant factor $4\pi k$) the solution u of Poisson's equation $-\Delta\, u = f$ is a potential of this field.

11.2 Use of the Potential

Let $E(x)$ denote an electric field with potential $u(x)$, i.e.,

$$-\nabla u(x) = E(x) \; .$$

Let q denote a point charge. The electric force on q is

$$F(x) = qE(x) \; .$$

Let

$$\Gamma \; : \; x(s) \in \mathbb{R}^3, \quad 0 \le s \le L \; ,$$

denote a smooth path in $\mathbb{R}^3$, parametrized by arclength s. The path goes from $P_0 = x(0)$ to $P_L = x(L)$. The unit tangent vector to Γ at the point $x(s)$ is $x'(s)$.

If q moves from P_0 to P_L along Γ then the work done by the field F is

$$W = q \int_0^L E(x(s)) \cdot x'(s) \, ds \; .$$

We have

$$\frac{d}{ds} u(x(s)) = \nabla u(x(s)) \cdot x'(s) = -E(x(s)) \cdot x'(s) \; ,$$

thus

$$W = -q \int_0^L \frac{d}{ds} \, u(x(s)) \, ds = qu(P_0) - qu(P_L) \; .$$

This is the work done by the field $F(x) = qE(x)$.

The work done against the field is

$$-W = q\Big(u(P_L) - u(P_0)\Big) \; .$$

Thus

$$qu(P)$$

represents the potential energy of the point charge q at the point $P \in \mathbb{R}^3$.

11.3 Poisson's Equation in $\mathbb{R}^3$

In this section we consider Poisson's equation in $\mathbb{R}^3$,

$$-\Delta u = f(x), \quad x \in \mathbb{R}^3 \; ,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a given function. We will prove that the fundamental solution

$$\Phi(x) = \frac{1}{4\pi|x|}$$

can be used to obtain the only decaying solution u of Poisson's equation.

11.3.1 The Newtonian Potential

Let $f \in C_c(\mathbb{R}^3)$. This means that $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and there exists $R > 0$ with $f(z) = 0$ if $|z| \geq R$.

Define[2]

$$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^3} \Phi(y) f(x-y)\, dy, \quad x \in \mathbb{R}^3 \ . \tag{11.2}$$

Note: The function $\Phi(y)$ is not integrable over $\mathbb{R}^3$ since it decays too slowly. However, the above integral is finite since f has compact support. All integrals in this section effectively extend over a finite region only. We call this region $U = B_M$; here M depends on x.

The function $u = \Phi * f$ is called the Newtonian potential (or Coulomb's potential) of f.

We will show:

Theorem 11.1 *If $f \in C^2_c(\mathbb{R}^3)$ then $u = \Phi * f \in C^2(\mathbb{R}^3)$ and*

$$-\Delta u(x) = f(x) \quad \textit{for all} \quad x \in \mathbb{R}^3 \ .$$

Auxiliary results:

Theorem 11.2 *(Gauss–Green theorem) Let $U \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary ∂U and unit outward normal*

$$n = (n_1, n_2, n_3), \quad n_j = n_j(y), \quad y \in \partial U \ .$$

If $u \in C^1(\bar{U})$ then

$$\int_U D_j u \, dx = \int_{\partial U} u n_j \, dS \ . \tag{11.3}$$

Replacing u with uv one obtains:

Theorem 11.3 *(integration by parts) If $u, v \in C^1(\bar{U})$ then*

$$\int_U (D_j u) v \, dx = -\int_U u D_j v \, dx + \int_{\partial U} u v n_j \, dS \ . \tag{11.4}$$

Proof of Theorem 11.1:

1. By Taylor,

$$f(z + he_j) = f(z) + h D_j f(z) + R_j(z,h)$$

with

$$|R_j(z,h)| \leq Ch^2 \ .$$

Therefore,

$$u(x + he_j) - u(x) = h \int_{\mathbb{R}^3} \Phi(y) D_j f(x-y)\, dy + \mathcal{O}(h^2) \ .$$

[2] The function $(\Phi * f)(x)$ given by equation (11.2) is called the convolution of Φ and f.

Divide by h and let $h \to 0$. Obtain that $D_j u(x)$ exists and

$$D_j u(x) = \int_{\mathbb{R}^3} \Phi(y) D_j f(x-y)\, dy \; .$$

To summarize, we have justified to differentiate equation (11.2) under the integral sign.

2. Let us write $D_{y_j} = \partial/\partial y_j$. Since $D_j f(x-y) = -D_{y_j} f(x-y)$ we have

$$D_j u(x) = -\int \Phi(y) D_{y_j} f(x-y)\, dy \; .$$

We want to integrate by parts and move D_{y_j} to $\Phi(y)$. However $\Phi(y)$ is not smooth at $y = 0$. Therefore, we first remove a small ball B_ε about the origin.

Let $U = B_M$ where M is large, and let $U_\varepsilon = B_M \setminus B_\varepsilon$. We have

$$\begin{aligned} D_j u(x) &= -\int_{B_M} \Phi(y) D_{y_j} f(x-y)\, dy \\ &= -\int_{B_\varepsilon} \ldots - \int_{U_\varepsilon} \ldots \end{aligned}$$

As $\varepsilon \to 0$, the first integral goes to zero. (Note that $|\Phi(y)| \le C|y|^{-1}$. Therefore, the integral over B_ε is $\le C\varepsilon^2$.) In the second integral we can integrate by parts. Note that the unit outward[3] normal on ∂B_ε is

$$n(y) = -\frac{y}{\varepsilon}, \quad |y| = \varepsilon \; .$$

Obtain that

$$-\int_{U_\varepsilon} \Phi(y) D_{y_j} f(x-y)\, dy = \int_{U_\varepsilon} D_j \Phi(y) f(x-y)\, dy - \int_{\partial B_\varepsilon} \Phi(y) f(x-y)(-y_j/\varepsilon)\, dS(y) \; .$$

As $\varepsilon \to 0$, the boundary term goes to zero since $\Phi(y) = \frac{1}{4\pi\varepsilon}$ for $y \in \partial B_\varepsilon$ and $area(\partial B_\varepsilon) = 4\pi\varepsilon^2$. As $\varepsilon \to 0$, we obtain

$$D_j u(x) = \int_{\mathbb{R}^3} D_j \Phi(y) f(x-y)\, dy \; .$$

3. As above, we can differentiate again under the integral sign and put the derivative on f,

$$\begin{aligned} D_j^2 u(x) &= \int_{\mathbb{R}^3} D_j \Phi(y) D_j f(x-y)\, dy \\ &= -\int_{\mathbb{R}^3} D_j \Phi(y) D_{y_j} f(x-y)\, dy \\ &= -\int_{B_\varepsilon} \ldots - \int_{U_\varepsilon} \ldots \\ &=: -I_{1,\varepsilon,j} - I_{2,\varepsilon,j} \end{aligned}$$

[3] Here outward refers to the domain U_ε.

We obtain that $|I_{1,\varepsilon,j}| \le \varepsilon$. In the second integral, we integrate by parts,

$$-I_{2,\varepsilon,j} = \int_{U_\varepsilon} D_j^2\Phi(y)f(x-y)\,dy - \int_{\partial B_\varepsilon} D_j\Phi(y)f(x-y)(-y_j/\varepsilon)\,dS(y)\ .$$

Since $\Delta\Phi(y) = 0$ in U_ε, the first terms sum to zero if we sum over $j = 1,2,3$. It remains to discuss the boundary terms: We have

$$D_j\Phi(y) = -\frac{1}{4\pi}\frac{y_j}{\varepsilon^3} \quad \text{for} \quad y \in \partial B_\varepsilon\ .$$

Therefore, the above boundary term is

$$BT(\varepsilon, j) = -\frac{1}{4\pi}\int_{\partial B_\varepsilon}\frac{y_j^2}{\varepsilon^4}\,f(x-y)\,dS(y)\ .$$

Summation yields

$$\begin{aligned}\sum_{j=1}^{3} BT(\varepsilon,j) &= -\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon} f(x-y)\,dS(y)\\ &= -f(x-y_\varepsilon)\end{aligned}$$

where

$$y_\varepsilon \in \partial B_\varepsilon\ .$$

As $\varepsilon \to 0$ we obtain that $\Delta u(x) = -f(x)$. $\diamond$

11.3.2 Uniqueness of a Decaying Solution

There are many harmonic functions on $\mathbb{R}^n$. For example,

$$v(x_1,x_2) = a + bx_1 + cx_2, \quad v(x_1,x_2) = x_1^2 - x_2^2, \quad v(x_1,x_2) = e^{x_1}\cos(x_2)$$

are solutions of $\Delta v = 0$. Therefore, the solution

$$u(x) = \int_{\mathbb{R}^3}\Phi(y)f(x-y)\,dy = \int_{\mathbb{R}^3}\Phi(x-y)f(y)\,dy \tag{11.5}$$

of the equation $-\Delta u = f$ is not unique. We will show, however, that (11.5) is the *only* decaying solution.

We first prove a decay estimate for the function u defined in (11.5).

Lemma 11.2 *Let $f \in C_c$ be supported in B_R. Then, if $|x| \ge 2R$, $u = \Phi * f$ satisfies the bound*

$$|u(x)| \le \frac{\|f\|_{L^1}}{2\pi}\,\frac{1}{|x|}$$

where

$$\|f\|_{L^1} = \int_{\mathbb{R}^3}|f(y)|\,dy$$

is the L^1–norm of f.

Proof: If $|x| \geq 2R$ and $|y| \leq R$, then

$$|x-y| \geq |x| - |y| \geq \frac{1}{2}\,|x| \; .$$

Therefore,

$$\Phi(x-y) \leq \frac{1}{2\pi}\,\frac{1}{|x|} \; .$$

It follows that

$$\begin{aligned} |u(x)| &\leq \int_{|y|\leq R} \Phi(x-y)|f(y)|\,dy \\ &\leq \frac{1}{2\pi}\,\frac{1}{|x|}\,\|f\|_{L^1} \quad \text{for} \quad |x| \geq 2R \; . \end{aligned}$$

This proves the lemma. $\diamond$

We next show an important property of harmonic functions, the **mean–value property**.

Let ω_n denote the surface area of the unit ball in $\mathbb{R}^n$. For example, $\omega_3 = 4\pi$.

Theorem 11.4 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in C^2(\Omega)$ and let $\Delta u = 0$ in Ω. Let $\bar{B}_R(x) \subset \Omega$. Then we have*

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} u(y)\,dS(y) \; . \tag{11.6}$$

Proof: We may assume $x=0$. For $0 < r \leq R$ define the function

$$\phi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r} u(y)\,dS(y) \; .$$

It is clear that $\lim_{r\to 0}\phi(r) = u(0)$ and $\phi(R)$ is the right–hand side in (11.6). We now show that $\phi'(r) = 0$ for $0 < r \leq R$. This implies that ϕ is constant and shows (11.6).

Changing variables, $y = rz$, in the right–hand side of (11.6), we have

$$\phi(r) = \frac{1}{\omega_n} \int_{\partial B_1} u(rz)\,dS(z) \; .$$

Therefore,

$$\phi'(r) = \frac{1}{\omega_n} \int_{\partial B_1} \sum_j D_j u(rz) z_j\,dS(z) \; .$$

Since $n(z) = z$ for $z \in \partial B_1$ one obtains by Theorem 11.2 (the Gauss–Green theorem):

$$\int_{\partial B_1} v(z) z_j\,dS(z) = \int_{B_1} \frac{\partial}{\partial z_j} v(z)\,dz \; .$$

Since

$$\frac{\partial}{\partial z_j} D_j u(rz) = r(D_j^2 u)(rz)$$

and $\sum_j D_j^2 u(rz) = 0$, it follows that $\phi'(r) = 0$. $\diamond$

We can now extend Theorem 11.1 by a uniqueness statement.

Theorem 11.5 *Let $f \in C_c^2(\mathbb{R}^3)$. Then the Newtonian potential of f, $u = \Phi * f$, satisfies $-\Delta u = f$ and $|u(x)| \to 0$ as $|x| \to \infty$. Furthermore, if $v \in C^2$ is any function with $-\Delta v = f$ and*

$$v(x) \to 0 \quad as \quad |x| \to \infty \ , \tag{11.7}$$

then $v = u$.

Proof: It remains to show the uniqueness statement, $v = u$. By definition, (11.7) means that for any $\varepsilon > 0$ there is $R > 0$ with $|v(x)| < \varepsilon$ if $|x| \geq R$. Set $w = v - u$. Then $\Delta w = 0$ and $w(x) \to 0$ as $|x| \to \infty$. Suppose that there exists $x \in \mathbb{R}^3$ with $|w(x)| > 0$. Choose $0 < \varepsilon < |w(x)|$. By Theorem 11.4 (with $n = 3$) we know that

$$w(x) = \frac{1}{\omega_3 R^2} \int_{\partial B_R(x)} w(y)\, dS(y) \ . \tag{11.8}$$

However, if R is large enough, then $|w(y)| < \varepsilon$ for $y \in \partial B_R(x)$. For such R the right–hand side of (11.8) is $< \varepsilon$ in absolute value, a contradiction. $\diamond$

11.3.3 Remarks on the Relation $-\Delta\Phi = \delta_0$

We want to explain what it means that the function $\Phi(x) = \frac{1}{4\pi}\frac{1}{|x|}, x \in \mathbb{R}^3, x \neq 0$, satisfies the equation

$$-\Delta\Phi = \delta_0$$

in the sense of distributions. Here δ_0 denotes Dirac's δ–distribution with unit mass at $x = 0$.

The distributional equation $-\Delta\Phi = \delta_0$ means that for every test function $v \in C_c^\infty$ we have

$$\langle -\Delta\Phi, v\rangle = \langle \delta_0, v\rangle \ , \tag{11.9}$$

and, by definition of the distributions $-\Delta\Phi$ and δ_0, this means that

$$-\int_{\mathbb{R}^3} \Phi(y)(\Delta v)(y)\, dy = v(0) \quad \text{for all} \quad v \in C_c^\infty \ . \tag{11.10}$$

To show (11.10), let $v \in C_c^\infty$ and set $f = -\Delta v$. Then $f \in C_c^\infty$; let $u = \Phi * f$, as above. We know that $-\Delta u = f$, and $v = u$ by Theorem 11.5. Therefore,

$$\begin{aligned} v(0) &= u(0) \\ &= \int_{\mathbb{R}^3} \Phi(0-y) f(y)\, dy \\ &= \int_{\mathbb{R}^3} \Phi(y) f(y)\, dy \\ &= -\int_{\mathbb{R}^3} \Phi(y)(\Delta v)(y)\, dy \end{aligned}$$

Thus we have proved (11.10), i.e., we have proved that $-\Delta\Phi = \delta_0$ in the sense of distributions.

If one has shown that

$$-\Delta\Phi(y) = \delta_0(y)$$

then one can give the following intuitive argument showing that the function

$$u(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\, dy \tag{11.11}$$

satisfies

$$-\Delta u(x) = f(x) \ .$$

Apply the operator $-\Delta$ to (11.11) and assume that one can apply $-\Delta$ under the integral sign. Obtain:

$$\begin{aligned} -\Delta u(x) &= \int_{\mathbb{R}^3} -\Delta\Phi(x-y) f(y)\, dy \\ &= \int_{\mathbb{R}^3} \delta_0(x-y) f(y)\, dy \\ &= f(x) \end{aligned}$$

11.3.4 Remarks on Coulomb's Law for Point Charges

Let $Q_1, \ldots, Q_N$ denote point charges at the points $y^{(1)}, \ldots, y^{(N)} \in \mathbb{R}^3$. They generate an electric field $E(x)$ which has the potential

$$u(x) = \frac{1}{4\pi\varepsilon_0} \sum_{j=1}^{N} \frac{Q_j}{|x - y^{(j)}|} \ .$$

In the sense of distributions, we have

$$-\Delta\Big(\frac{1}{4\pi} \frac{1}{|x - y^{(j)}|}\Big) = \delta_0(x - y^{(j)}) \ ,$$

thus

$$-\Delta u(x) = \sum_{j=1}^{N} \frac{Q_j}{\varepsilon_0}\, \delta_0(x - y^{(j)}) \ .$$

On the other hand, if $\rho(y)$ is a smooth charge distribution, we have argued that it generates an electric field with potential

$$u(x) = \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy$$

which satisfies

$$-\Delta u(x) = \frac{\rho(x)}{\varepsilon_0} \ .$$

This suggests that we should assign to the point Q_j at the point $y^{(j)}$ the charge distribution function

$$Q_j\,\delta_0(x-y^{(j)})\ .$$

11.4 Application of the Potential $u(x)$

Let $E(x)$ denote the electric field determined by a charge distribution $f(x)$. Let Γ denote a curve in $\mathbb{R}^3$ from point $x^{(1)}$ to point $x^{(2)}$. Assume that the point charge q moves from $x^{(1)}$ to $x^{(2)}$ along Γ. What is the work done by the field $E(x)$ during this motion? The force at q at the point $x\in\Gamma$ is $E(x)$.

If q is displaced by a vector D and a constant force F acts on q during this displacement the work done by F is

$$W=F\cdot D=\sum_{j=1}^3 F_jD_j\ .$$

Assume that the curve Γ is parameterized by

$$\gamma(s)=\Big(\gamma_1(s),\gamma_2(s),\gamma_3(s)\Big),\quad 0\le s\le L\ .$$

The tangent vector to the curve Γ at the point $\gamma(s)$ is $\gamma'(s)$. The work done by the force $qE(x)$ during the motion of q along Γ is

$$W=q\int_0^L E(\gamma(s))\cdot\gamma'(s)\,ds\ .$$

Assume that the function $u(x)$ is a potential of $E(x)$, i.e.,

$$-\nabla\,u(x)=E(x),\quad x\in\mathbb{R}^3\ .$$

Set

$$\phi(s)=u(\gamma(s)),\quad 0\le s\le L\ .$$

Then

$$\begin{aligned}\phi'(s)&=\nabla\,u(\gamma(s))\cdot\gamma'(s)\\&=-E(\gamma(s))\cdot\gamma'(s)\end{aligned}$$

Therefore,

$$\begin{aligned}W&=-q\int_0^L\phi'(s)\,ds\\&=-q\Big(\phi(L)-\phi(0)\Big)\\&=-q\Big(u(\gamma(L))-u(\gamma(0))\Big)\\&=q\Big(u(x^{(1)})-u(x^{(2)})\Big)\end{aligned}$$

This is an application of the potential $u(x)$.

Example: Assume the point charge $Q = 1Coulomb$ is located at the origin and the point charge $q = 1Coulomb$ moves from the distance $1meter$ to the distance $2meter$ away from the origin. The work done by the electric field generated by Q is

$$
\begin{aligned}
W &= q\left(\frac{kQ}{1meter} - \frac{kQ}{2meter}\right) \\
&= \frac{1}{2}\frac{kqQ}{meter} \\
&= \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\frac{Coul^2}{meter} \\
&= \frac{1}{8\pi}\cdot\frac{10^{12}}{8.859}\,Newton\,meter \\
&= \frac{1}{8\pi}\cdot\frac{10^{12}}{8.859}\,Joule
\end{aligned}
$$

12 Auxiliary Results

12.1 Term–by–Term Differentiation of a Series

In Chapter 6 we considered the Poisson kernel

$$P(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \;, \quad 0 \le r < 1, \quad \phi \in \mathbb{R} \;.$$

As shown in Chapter 6, every term

$$u_n(r,\theta) = r^{|n|}\, e^{in\theta}$$

satisfies the Laplace equation

$$\Delta u_n(r,\theta) = 0 \quad \text{for} \quad r \ge 0, \quad \phi \in \mathbb{R} \;.$$

We want to show that $P(r,\theta)$ satisfies the Laplace equation for

$$0 \le r < 1, \quad \phi \in \mathbb{R} \;.$$

To show this, we must justify that we can exchange differentiation and summation.

In general, one cannot exchange two limit process as the following example shows:

Example: Consider the functions

$$f_j(x) = x^j \quad \text{for} \quad 0 \le x < 1, \quad j = 1,2,3,\ldots$$

We have

$$\lim_{j\to\infty} x^j = 0 \quad \text{for} \quad 0 \le x < 1 \;,$$

thus

$$\lim_{x\to 1-} \lim_{j\to\infty} x^j = 0 \;.$$

On the other hand,

$$\lim_{x\to 1-} x^j = 1 \quad \text{for all} \quad j = 1,2,\ldots$$

thus

$$\lim_{j\to\infty} \lim_{x\to 1-} x^j = 1 \;.$$

If the two limit process, $x \to 1-$ and $j \to \infty$, are exchanged in order, one obtains two different results.

Notation: If $f : [a,b] \to \mathbb{C}$ is a continuous function then

$$|f|_\infty = \max_{a\le x\le b} |f(x)|$$

denotes the maximum norm of f.

Theorem 12.1 *For $j = 1, 2, \ldots$ let $f_j \in C^2[a, b]$ and assume*

$$\sum_{j=1}^{\infty} |f_j|_\infty \quad < \quad \infty \tag{12.1}$$

$$\sum_{j=1}^{\infty} |f_j'|_\infty \quad < \quad \infty \tag{12.2}$$

$$C := \sum_{j=1}^{\infty} |f_j''|_\infty \quad < \quad \infty \tag{12.3}$$

Then the function

$$f(x) = \sum_{j=1}^{\infty} f_j(x)$$

is differentiable for $a \leq x \leq b$ and

$$f'(x) = \sum_{j=1}^{\infty} f_j'(x) \quad \textit{for} \quad a \leq x \leq b \ .$$

The theorem says that, under the given assumptions, the limit process of summation and differentiation can be exchanged in order,

$$\frac{d}{dx} \sum_{j=1}^{\infty} f_j(x) = \sum_{j=1}^{\infty} \frac{d}{dx} f_j(x) \ .$$

Proof: The estimates (12.1) and (12.2) imply convergence of the series

$$f(x) := \sum_{j=1}^{\infty} f_j(x) \quad \text{and} \quad g(x) := \sum_{j=1}^{\infty} f_j'(x) \quad \text{for} \quad 0 \leq x \leq 1 \ .$$

In the following, let x and $x + h$ denote two distinct points in the interval $[a, b]$. We have

$$\begin{aligned} f_j(x+h) &= f(x_j) + \int_0^h f_j'(x+s)\, ds \\ f_j'(x+s) &= f_j'(x) + \int_0^s f_j''(x+\tau)\, d\tau \end{aligned}$$

Therefore,

$$f_j(x+h) - f_j(x) = h f_j'(x) + \int_0^h \int_0^s f_j''(x+\tau)\, d\tau ds \ .$$

The double integral can be estimated:

$$\Big|\int_0^h \int_0^s f_j''(x+\tau)\,d\tau ds\Big| \le \frac{1}{2}\,h^2\,|f_j''|_\infty \ .$$

Using (12.3) this yields that

$$\Big|\sum_{j=1}^\infty f_j(x+h) - \sum_{j=1}^\infty f_j(x) - h\sum_{j=1}^\infty f_j'(x)\Big| \le \frac{1}{2}\,h^2 C \ .$$

Dividing by h we obtain that

$$\Big|\frac{1}{h}\,(f(x+h)-f(x)) - g(x)\Big| \le \frac{|h|}{2}\,C \ .$$

As $h \to 0$ obtain that $f'(x)$ exists and

$$f'(x) = g(x) = \sum_{j=1}^\infty f_j(x) \ .$$

$\diamond$

It is not difficult to generalize the result of Theorem 12.1 to sequences of functions f_j of two or more variables.

The following lemma will be used for application to the Poisson kernel.

Lemma 12.1 *Let $0 < r_0 < 1$ and let $m \in \mathbb{N}$. Then the following series converges:*

$$\sum_{j=1}^\infty j^m r_0^j < \infty \ .$$

Proof: Set $y_j = j^m r_0^j$ and fix $\varepsilon > 0$ with

$$r_0 < r_0 + \varepsilon < 1 \ .$$

We have

$$\frac{y_{j+1}}{y_j} = \Big(\frac{j+1}{j}\Big)^m r_0 \le r_0 + \varepsilon < 1$$

for all large j since

$$\Big(\frac{j+1}{j}\Big)^m \to 1 \quad \mbox{as} \quad j \to \infty \ .$$

The estimate

$$y_{j+1} \le (r_0+\varepsilon) y_j \quad \mbox{for} \quad j \ge J$$

follows for some $J \in \mathbb{N}$. Therefore,

$$y_{J+j} \le y_J (r_0 + \varepsilon)^j \quad \mbox{for} \quad j = 0,1,2,\ldots$$

Convergence of the given series follows since the geometric series

$$\sum_{j=0}^\infty (r_0+\varepsilon)^j$$

converges. $\diamond$

The lemma can be use to show that the sequence

$$\sum_{n=1}^\infty r^n e^{in\theta}$$

converges absolutely for

$$0 \le r \le r_0 < 1 \ .$$

Also, if one differentiates term–by–term w.r.t. θ and considers

$$\sum_{n=1}^\infty inr^n e^{in\theta}$$

the maximum norm over

$$0 \le r \le r_0$$

converges if $0 < r_0 < 1$ is fixed. This hold for all derivatives taken term–by–term.

It follows that the Poisson kernel

$$P(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^\infty r^{|n|} e^{in\theta} \ , \quad 0 \le r < 1, \quad \phi \in \mathbb{R} \ ,$$

is a C^∞–function and all derivatives can be obtained using term–by–term differentiation. Since the functions

$$r^n e^{in\theta} \quad \text{and} \quad r^n e^{-in\theta}$$

are harmonic, it follows that

$$\Delta P(r,\theta) = 0 \quad \text{for} \quad 0 \le r < 1, \quad \theta \in \mathbb{R} \ .$$

12.2 Some Integrals

Let

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2}$$

denote the Euclidean norm of the vector $x \in \mathbb{R}^n$. Let $\lambda \in \mathbb{R}$ denote a parameter and let $r_0 > 0$.

Lemma 12.2 *In the following integrals, the function $x \to 1/|x|^\lambda$ is a function on $\mathbb{R}^3 \setminus \{0\}$.*

a) The integral

$$I(\lambda) = \int_{|x|\le r_0} \frac{dx}{|x|^\lambda}$$

is finite if and only if $\lambda < 3$.

b) The integral

$$J(\lambda) = \int_{|x|\ge r_0} \frac{dx}{|x|^\lambda}$$

is finite if and only if $\lambda > 3$.

Proof: The surface area of a ball of radius r is $4\pi r^2$. Therefore,

$$I(\lambda) = 4\pi \int_0^{r_0} r^{2-\lambda}\, dr \ .$$

For $\lambda \neq 3$ obtain that

$$I(\lambda) = \frac{4\pi}{3-\lambda}\, r^{3-\lambda}\Big|_0^{r_0} \ .$$

If $\lambda < 3$ obtain the finite value

$$I(\lambda) = \frac{4\pi}{3-\lambda}\, r_0^{3-\lambda} \ .$$

For $\lambda > 3$ obtain that $I(\lambda) = \infty$ since

$$r^{3-\lambda} \to \infty \quad \text{as} \quad r \to 0+ \ .$$

For $\lambda = 3$ obtain

$$\begin{aligned} I(3) &= 4\pi \int_0^{r_0} \frac{dr}{r} \\ &= 4\pi\, (\ln r)\Big|_0^{r_0} \\ &= \infty \end{aligned}$$

since

$$\ln r \to -\infty \quad \text{as} \quad r \to 0+ \ .$$

b) The proof for $J(\lambda)$ is similar. $\diamond$

Remarks: a) If $\lambda \ge 3$ then $I(\lambda) = \infty$ since the singularity of the function

$$\frac{1}{|x|^\lambda}, \quad x \in \mathbb{R}^3 \setminus \{0\} \ ,$$

at $x = 0$ is too strong. If $0 < \lambda < 3$ then one says that the singularity of the function $1/|x|^\lambda$ at $x = 0$ is integable.

b) If $\lambda \le 3$ then $J(\lambda) = \infty$ since the function $1/|x|^\lambda$ decays too slowly as $|x| \to \infty$.

It is easy to generalize the results of the lemma to integrals of $1/|x|^\lambda$ in $\mathbb{R}^n$. The value $\lambda = 3$ becomes $\lambda = n$.

12.3 The Green–Gauss Theorem and Integration by Parts

First recall the fundamental theorem of calculus and integration by parts.

For $f \in C^1[a,b]$ we have

$$\int_a^b f'(y)\,dy = f|_a^b = f(b) - f(a) \ .$$

If $f, g \in C^1[a,b]$ then

$$\int_a^b (fg)'(y)\,dy = (fg)|_a^b = f(b)g(b) - f(a)g(a) \ ,$$

thus

$$\int_a^b fg'\,dy = (fg)|_a^b - \int_a^b f'g\,dy \ .$$

(Integration by parts)

The Green–Gauss Theorem: Let $D \subset \mathbb{R}^k$ be an open, bounded set with smooth boundary ∂D. Let $n(y) \in \mathbb{R}^k$ denote the unit outward normal at the boundary point $y \in \partial D$. For $f \in C^1(\bar{D})$ we have

$$\int_D D_j f(y)\,dy = \int_{\partial D} f(y) n_j(y)\,dS(y), \quad j = 1, 2, \ldots, k \ .$$

Here

$$D_j = \frac{\partial}{\partial y_j}$$

and $n_j(y)$ is the j–th component of $n(y)$.

The Green–Gauss Theorem is a generalization of the fundamental theorem of calculus.

In the following simple example, the result of the Green–Gauss theorem is deduced from the fundamental theorem of calculus.

Let $D \subset \mathbb{R}^2$ denote the rectangle

$$D = (0, L) \times (0, H) \ .$$

For the first component of the unit outward normal we have

$$n_1(y) = \begin{cases} \ \ 0 & \text{for} \quad y = (y_1, 0) \\ \ \ 0 & \text{for} \quad y = (y_1, H) \\ \ \ 1 & \text{for} \quad y = (L, y_2) \\ -1 & \text{for} \quad y = (0, y_2) \end{cases}$$

If $f \in C^1(\bar{D})$ then

$$\begin{aligned}
\int_D D_1 f(y)\, dy &= \int_0^H \int_0^L D_1 f(y_1, y_2)\, dy_1 dy_2 \\
&= \int_0^H \Big(f(L, y_2) - f(0, y_2) \Big)\, dy_2 \\
&= \int_{\partial D} f(y) n_1(y)\, dS(y)
\end{aligned}$$

12.4 Binomial Coefficients and Leibniz Rule

Let $j, n \in \mathbb{N}_0$ with $0 \le j \le n$. The binomial coefficients are defined by

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{n!}{j!(n-j)!} = \frac{n(n-1)\cdots(n+1-j)}{j!} \ .$$

A set of n elements has

$$\left(\begin{array}{c} n \\ j \end{array} \right), \quad n \text{ choose } j \ ,$$

subsets of j elements.

It is easy to check that

$$\left(\begin{array}{c} n \\ j-1 \end{array} \right) + \left(\begin{array}{c} n \\ j \end{array} \right) = \left(\begin{array}{c} n+1 \\ j \end{array} \right) \ .$$

Therefore, the binomial coefficients appear in Pascal's triangle.

Proof:

$$\begin{aligned}
\left(\begin{array}{c} n \\ j-1 \end{array} \right) + \left(\begin{array}{c} n \\ j \end{array} \right) &= \frac{n!}{(j-1)!(n+1-j)!} + \frac{n!}{j!(n-j)!} \\
&= \frac{n!j + n!(n+1-j)}{j!(n+1-j)!} \\
&= \left(\begin{array}{c} n+1 \\ j \end{array} \right)
\end{aligned}$$

Leibniz Rule: Let $f, g \in C^n[a,b]$. Let $D = d/dx$. Then

$$D^n(fg) = \sum_{j=0}^n \left(\begin{array}{c} n \\ j \end{array} \right) (D^j f)(D^{n-j} g) \ .$$

Proof by induction in n. Using the induction assumption, we have

$$
\begin{aligned}
D^{n+1}(fg) &= \sum_{j=0}^{n} \binom{n}{j} (D^{j+1}f)(D^{n-j}g) + \sum_{j=0}^{n} \binom{n}{j} (D^{j}f)(D^{n+1-j}g) \\
&= \sum_{j=1}^{n+1} \binom{n}{j-1} (D^{j}f)(D^{n+1-j}g) + \sum_{j=0}^{n} \binom{n}{j} (D^{j}f)(D^{n+1-j}g) \\
&= (D^{n+1}f)g + \sum_{j=1}^{n} \binom{n}{j-1} (D^{j}f)(D^{n+1-j}g) \\
&\quad + f(D^{n+1}g) + \sum_{j=1}^{n} \binom{n}{j} (D^{j}f)(D^{n+1-j}g) \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} (D^{j}f)(D^{n+1-j}g)
\end{aligned}
$$

This completes the induction argument.

13 The Heat Equation and the Wave Equation in a Rectangle

Let $D \subset \mathbb{R}^2$ denote the rectangle

$$D = (0, L) \times (0, H)$$

and let ∂D denote its boundary.

13.1 The Heat Equation in a Rectangle

Consider the IBVP

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \quad \text{for} \quad (x,y) \in D, \quad t \geq 0 \ , \\ u(x,y,t) &= 0 \quad \text{for} \quad (x,y) \in \partial D, \quad t \geq 0 \ , \\ u(x,y,0) &= \alpha(x,y) \quad \text{for} \quad (x,y) \in D \ . \end{aligned}$$

The ansatz in separated variables

$$u(x,y,t) = \phi(x,y)h(t)$$

leads to

$$\frac{h'(t)}{h(t)} = \frac{\Delta\phi(x,y)}{\phi(x,y)} = -\lambda \ .$$

One obtains the eigenvalue problem

$$-\Delta\phi(x,y) = \lambda\phi(x,y) \quad \text{in} \quad D, \quad \phi(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{13.1}$$

and the equation

$$h'(t) = -\lambda h(t) \ .$$

To solve the eigenvalue problem, let

$$\phi(x,y) = f(x)g(y)$$

and obtain

$$-f''(x)g(y) - f(x)g''(y) = \lambda f(x)g(y) \ ,$$

thus

$$\frac{-f''(x)}{f(x)} - \frac{g''(y)}{g(y)} = \lambda \ .$$

Also,

$$f(0) = f(L) = 0 \quad \text{and} \quad g(0) = g(H) = 0 \ .$$

The eigenvalue problem

$$-f''(x) = \mu f(x), \quad f(0) = f(L) = 0$$

has the eigenvalues and eigenfunctions

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad f_n(x) = \sin(n\pi x/L) \quad \text{for} \quad n = 1, 2, \ldots$$

Similarly, the eigenvalue problem

$$-g''(y) = \kappa g(y), \quad g(0) = g(H) = 0$$

has the eigenvalues and eigenfunctions

$$\kappa_m = \left(\frac{m\pi}{H}\right)^2, \quad g_m(y) = \sin(m\pi y/H) \quad \text{for} \quad m = 1, 2, \ldots$$

The eigenvalue problem (13.1) has the eigenvalues

$$\lambda_{nm} = \mu_n + \kappa_m = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

and eigenfunctions

$$\phi_{nm}(x, y) = \sin(n\pi x/L)\ \sin(m\pi y/H$$

for $(n, m) \in \mathbb{N} \times \mathbb{N}$.

If one can write the initial function

$$u(x, y, 0) = \alpha(x, y)$$

as a series

$$\alpha(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} \phi_{nm}(x, y)$$

then one obtains the solution

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} \phi_{nm}(x, y)\ e^{-\lambda_{nm} t}\ .$$

Note that

$$\int_D \phi_{n_1 m_1}(x, y) \phi_{n_2 m_2}(x, y)\, dxdy = 0$$

unless $n_1 = n_2$ and $m_1 = m_2$. Also,

$$\int_D \phi_{nm}^2(x, y)\, dxdy = \frac{LH}{4}\ .$$

Therefore,

$$\alpha_{nm} = \frac{4}{LH} \int_D \phi_{nm}(x, y) \alpha(x, y)\, dxdy\ .$$

Example 1: Let $D = (0\pi) \times (0, \pi)$ and consider the IBVP

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \quad \text{for} \quad (x,y) \in D, \quad t \geq 0 \ , \\ u(x,y) &= 0 \quad \text{for} \quad (x,y) \in \partial D \ , \\ u(x,y,0) &= \sin x \sin(2y) \quad \text{for} \quad (x,y) \in D \ . \end{aligned}$$

We have $L = H = \pi$ and

$$\alpha(x,y) = u(x,y,0) = \sin x \sin(2y) \ .$$

Since $L = H = \pi$ obtain that

$$\phi_{nm}(x,y) = \sin(nx)\sin(my) \quad \text{and} \quad \lambda_{nm} = n^2 + m^2 \ .$$

Therefore,

$$\alpha(x,y) = \phi_{12}(x,y), \quad \lambda_{12} = 5 \ .$$

The solution of the IBVP is

$$u(x,y,t) = \sin x \ \sin(2y) \ e^{-5t} \ .$$

13.2 The Wave Equation in a Rectangle

See Haberman, Sect. 7.3.

Consider the IBVP

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy}) \quad \text{for} \quad (x,y) \in D, \quad t \geq 0 \ , \\ u(x,y) &= 0 \quad \text{for} \quad (x,y) \in \partial D \ , \\ u(x,y,0) &= \alpha(x,y) \quad \text{for} \quad (x,y) \in D \ , \\ u_t(x,y,0) &= \beta(x,y) \quad \text{for} \quad (x,y) \in D \ . \end{aligned}$$

The ansatz in separated variables

$$u(x,y,t) = \phi(x,y)h(t)$$

leads to the eigenvalue problem

$$-\Delta\phi(x,y) = \lambda\phi(x,y) \quad \text{in} \quad D, \quad \phi(x,y) = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{13.2}$$

and the equation

$$h''(t) + c^2\lambda h(t) = 0 \ . \tag{13.3}$$

The eigenvalue problem is the same as for the heat equation. One obtains the eigenfunctions

$$\phi_{nm}(x,y) = \sin(n\pi x/L)\ \sin(m\pi y/H)$$

and the corresponding eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 .$$

With $\lambda = \lambda_{nm} = k_{nm}^2$ the equation (13.3) has the general solution

$$h_{nm}(t) = a_{nm}\cos(k_{nm}ct) + b_{nm}\sin(k_{nm}ct) .$$

The functions

$$u_{nm}(x,y,t) = \phi_{nm}(x,y)h_{nm}(t)$$

solve the wave equation and satisfy the boundary condition

$$u_{nm}(x,y,t) = 0 \quad \text{for} \quad (x,y) = \partial D .$$

Let

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(x,y)h_{nm}(t) .$$

Then the initial condition for u requires that

$$\alpha(x,y) = u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(x,y)\ a_{nm}$$

and the initial condition for u_t requires that

$$\beta(x,y) = u_t(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(x,y)\ b_{nm}k_{nm}c .$$

These series expansions of the initial functions $\alpha(x,y)$ and $\beta(x,y)$ determine the coefficients a_{nm} and b_{nm}.

Example 2: Consider the wave equation

$$u_{tt} = c^2\Delta u$$

in the square $D = (0,\pi)\times(0,\pi)$ with boundary condition

$$u(x,y,t) = 0 \quad \text{for} \quad (x,y) \in \partial D$$

and initial conditions

$$u(x,y,0 = 0 \quad \text{for} \quad (x,y) \in D$$

and

$$u_t(x,y,0 = \sin(2x)\sin(3y) \quad \text{for} \quad (x,y) \in D .$$

We have

$$\phi_{23}(x,y) = \sin(2x)\sin(3y) = u_t(x,y,0)$$

and

$$\lambda_{23} = 2^2 + 3^2 = 13, \quad k_{23} = \sqrt{13}\ .$$

One obtains

$$b_{23}\ k_{23}\ c = 1, \quad b_{23} = \frac{1}{\sqrt{13}\ c}$$

and

$$b_{nm} = 0 \quad \text{for} \quad (n,m) \neq (2,3)\ .$$

The solution of the IBVP is

$$u(x,y,t) = \frac{1}{\sqrt{13}\ c}\ \sin(2x)\sin(3y)\sin(\sqrt{13}\ ct)\ .$$

It is easy to check that the boundary and initial conditions are satisfied. Also,

$$\begin{aligned} u_{tt} &= -13c^2 u\ , \\ \Delta u &= -(4+9)u = -13u \end{aligned}$$

thus $u_{tt} = c^2 \Delta u$.

14 Poisson's Equation in $\mathbb{R}^2$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote a C^2–function with compact support, i.e., there exists $R > 0$ so that $f(y) = 0$ for $|y| \geq R$. Here

$$y = (y_1, y_2) \in \mathbb{R}^2, \quad |y|^2 = y_1^2 + y_2^2 \ .$$

Let

$$\Phi(y) = \ln |y| \quad \text{for} \quad y \in \mathbb{R}^2, \quad y \neq 0 \ .$$

Theorem 14.1 *The function*

$$u(x) = \int_{\mathbb{R}^2} \Phi(y) f(x-y)\, dy, \quad x \in \mathbb{R}^2 \ ,$$

satisfies

$$\Delta u(x) = 2\pi f(x) \quad \textit{for} \quad x \in \mathbb{R}^2 \ .$$

Proof: We have for $j = 1, 2$:

$$\frac{\partial}{\partial y_j} \Phi(y) = \frac{y_j}{y_1^2 + y_2^2}$$

and

$$\frac{\partial^2}{\partial y_j^2} \Phi(y) = \frac{1}{y_1^2 + y_2^2} - \frac{2y_j^2}{(y_1^2 + y_2^2)^2}$$

thus

$$\Delta \Phi(y) = 0 \quad \text{for} \quad y \in \mathbb{R}^2, \quad y \neq 0 \ .$$

Fix $x \in \mathbb{R}$ and choose $R > 0$ so large that

$$f(x-y) = 0 \quad \text{for} \quad |y| \geq R \ .$$

Set

$$B_R = \{ y \in \mathbb{R}^2 \ : \ |y| < R \} \quad \text{and} \quad B_\varepsilon = \{ y \in \mathbb{R}^2 \ : \ |y| < \varepsilon \} \ .$$

With

$$\Gamma_\varepsilon = \partial B_\varepsilon = \{ y \in \mathbb{R}^2 \ : \ |y| = \varepsilon \}$$

we denote the boundary curve of B_ε. Clearly, Γ_ε is the circle of radius ε centered at $y = 0$.

The singularities of the functions

$$\Phi(y) = \ln |y| \quad \text{and} \quad \frac{\partial}{\partial y_j} \Phi(y) = \frac{y_j}{y_1^2 + y_2^2}$$

at $y=0$ are integrable. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x_j}u(x) &= \int_{\mathbb{R}^2}\Phi(y)\,\frac{\partial}{\partial x_j}f(x-y)\,dy\\ &= -\int_{\mathbb{R}^2}\Phi(y)\,\frac{\partial}{\partial y_j}f(x-y)\,dy\\ &= \int_{\mathbb{R}^2}\Big(\frac{\partial}{\partial y_j}\Phi(y)\Big)\,f(x-y)\,dy\end{aligned}$$

Differentiating again,

$$\begin{aligned}\frac{\partial^2}{\partial x_j^2}u(x) &= -\int_{\mathbb{R}^2}\Big(\frac{\partial}{\partial y_j}\Phi(y)\Big)\,\frac{\partial}{\partial y_j}f(x-y)\,dy\\ &= -\int_{B_\varepsilon}\ldots-\int_{B_R\setminus B_\varepsilon}\ldots\\ &=: Int_1+Int_2\end{aligned}$$

We have

$$\Big|\frac{\partial}{\partial y_j}\Phi(y)\Big|\le\frac{1}{|y|}$$

and

$$\int_{B_\varepsilon}\frac{dy}{|y|}=\int_0^\varepsilon\int_0^{2\pi}\frac{1}{r}\,rdrd\phi=2\pi\varepsilon\ .$$

Therefore,

$$|Int_1|=\mathcal{O}(\varepsilon)\ .$$

Also,

$$\begin{aligned}Int_2 &= \int_{B_R\setminus B_\varepsilon}\Big(\frac{\partial^2}{\partial y_j^2}\Phi(y)\Big)\,f(x-y)\,dy-\int_{\Gamma_\varepsilon}\Big(\frac{\partial}{\partial y_j}\Phi(y)\Big)\,f(x-y)n_j(y)\,dS(y)\\ &=: Int_{3,j}+Int_{4,j}\end{aligned}$$

Here

$$n(y)=\Big(n_1(y),n_2(y)\Big)=-\frac{y}{|y|}\quad\text{for}\quad y\in\Gamma_\varepsilon$$

is the unit outward normal on Γ_ε. (*Outward* refers to the domain $B_R\setminus B_\varepsilon$.)

Since $\Delta\Phi(y)=0$ for $y\neq 0$ one obtains that

$$\sum_{j=1}^2 Int_{3,j}=0\ .$$

Also, since

$$n_j(y) = -\frac{y_j}{|y|}$$

we have

$$-\Big(\frac{\partial}{\partial y_j}\Phi(y)\Big)\, n_j(y) = \frac{y_j^2}{|y|^3}$$

and

$$-\sum_{j=1}^2 \Big(\frac{\partial}{\partial y_j}\Phi(y)\Big)\, n_j(y) = \frac{1}{|y|} = \frac{1}{\varepsilon} \quad \text{for} \quad y \in \Gamma_\varepsilon \ .$$

It follows that

$$\sum_{j=1}^2 Int_{4,j} = \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon} f(x-y)\, dy = \frac{1}{\varepsilon}\, 2\pi\varepsilon f(x-y_\varepsilon)$$

for some $y_\varepsilon \in \Gamma_\varepsilon$. Therefore,

$$\Delta u(x) = \mathcal{O}(\varepsilon) + 2\pi f(x-y_\varepsilon) \ .$$

For $\varepsilon \to 0$ one obtains that

$$\Delta u(x) = 2\pi f(x) \ .$$

$\diamond$

Theorem 14.1 yields: If $f \in C_0^2(\mathbb{R}^2)$ then the function

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|y|) f(x-y)\, dy$$

satisfies

$$\Delta u(x) = f(x) \quad \text{for} \quad x \in \mathbb{R}^2 \ .$$

15 Green's Functions for BVPs

15.1 An Ordinary BVP

Consider the BVP

$$-u''(x) = f(x), \quad u(0) = u(1) = 0$$

where $f \in C[0,1]$ is a given function. It is easy to prove that there exists a unique solution.

We construct a solution as follows: Let

$$\tilde{u}(x) = -\int_0^x (x-y) f(y)\, dy \ .$$

We have

$$\tilde{u}'(x) = -\int_0^x f(y)\, dy \ ,$$

thus

$$-\tilde{u}''(x) = f(x) \ .$$

To obtain a solution of the BVP let

$$u(x) = \tilde{u}(x) + cx \ .$$

We determine the constant c so that $u(1) = 0$. This yields

$$c = -\tilde{u}(1) = \int_0^1 (1-y) f(y)\, dy \ .$$

Obtain

$$\begin{aligned} u(x) &= \int_0^x (y-x) f(y)\, dy + \int_0^1 x(1-y) f(y)\, dy \\ &= \int_0^x (y - x + x - xy) f(y)\, dy + \int_x^1 x(1-y) f(y)\, dy \\ &= \int_0^x y(1-x) f(y)\, dy + \int_x^1 x(1-y) f(y)\, dy \end{aligned}$$

The Green's function for the BVP is

$$G(x,y) = \begin{cases} y(1-x) & \text{for} \quad 0 \le y \le x \le 1 \\ x(1-y) & \text{for} \quad 0 \le x \le y \le 1 \end{cases}$$

The solution of the BVP is

$$u(x) = \int_0^1 G(x,y) f(y)\, dy \quad \text{for} \quad 0 \le x \le 1 \ .$$

By our construction, it is clear that $u(x)$ solves the BVP.

But we want to check this formally in a different way. Since

$$G(0,y) = G(1,y) = 0 \quad \text{for} \quad 0 \le y \le 1$$

it follows that $u(0) = u(1) = 0$. Also,

$$-G_x(x,y) = \begin{cases} y & \text{for} \quad 0 \le y < x \le 1 \\ y-1 & \text{for} \quad 0 \le x < y \le 1 \end{cases}$$

Therefore,

$$\begin{aligned} -u'(x) &= \int_0^1 -G_x(x,y) f(y)\, dy \\ &= \int_0^x y f(y)\, dy + \int_x^1 (y-1) f(y)\, dy \end{aligned}$$

Differentiation yields that

$$-u''(x) = x f(x) - (x-1) f(x) = f(x) \ .$$

A more formal argument: We have

$$-G_{xx}(x,y) = \begin{cases} 0 & \text{for} \quad x \ne y \\ \infty & \text{for} \quad x = y \end{cases}$$

and

$$\int_0^1 -G_{xx}(x,y)\, dx = -G_x(1,y) + G_x(0,y) + (1-y) = 1 \ .$$

Therefore,

$$-G_{xx}(x,y) = \delta(x-y)$$

and

$$u(x) = \int_0^1 G(x,y) f(y)\, dy$$

yields that

$$\begin{aligned} -u''(x) &= \int_0^1 -G_{xx}(x,y) f(y)\, dy \\ &= \int_0^1 \delta(x-y) f(y)\, dy \\ &= f(x) \end{aligned}$$

15.2 The Dirichlet Problem for Poisson's Equation in the Half–Space $\mathbb{R}^3_+$

Let

$$U = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \ : \ x_3 > 0\} \ .$$

The boundary of the half–space U is the plane

$$\partial U = \{x \in \mathbb{R}^3 \ : \ x_3 = 0\}$$

and the unit outward normal on ∂U is the vector

$$n(y) = (0, 0, -1), \quad y \in \partial U \ .$$

Consider the Dirichlet Problem

$$-\Delta u(x) = f(x) \quad \text{for} \quad x \in U, \quad u(x) = g(x) \quad \text{for} \quad x \in \partial U \ . \tag{15.1}$$

We assume that $f(x)$ and $g(x)$ are given smooth functions with compact support.

Recall that

$$\Phi(x) = \frac{1}{4\pi|x|}, \quad x \in \mathbb{R}^3, \quad x \neq 0 \ ,$$

is the fundamental solution of Poisson's equation $-\Delta u(x) = f(x)$ in $\mathbb{R}^3$.

For $x = (x_1, x_2, x_3) \in \bar{U}$ let $\tilde{x} = (x_1, x_2, -x_3)$. Consider the Green's function

$$G(x, y) = \Phi(x - y) - \Phi(\tilde{x} - y)$$

defined for $x \in \bar{U}, y \in \bar{U}, y \neq x$.

We claim that the solution of the Dirichlet problem (15.1) is given by

$$u(x) = \int_U G(x, y) f(y) \, dy + \int_{\partial U} K(x, y) g(y) \, dS(y)$$

where

$$K(x, y) = -\frac{\partial}{\partial n(y)} G(x, y) = \frac{\partial}{\partial y_3} G(x, y) \ .$$

We have $u(x) = u_1(x) + u_2(x)$ with

$$u_1(x) = \int_U G(x, y) f(y) \, dy \quad \text{for} \quad x \in \bar{U}$$

and

$$u_2(x) = \begin{cases} \int_{\partial U} K(x, y) g(y) \, dS(y) & \text{for} \quad x \in U \\ g(x) & \text{for} \quad x \in \partial U \end{cases}$$

Lemma 15.1 *Define*

$$u_1(x) = \int_U G(x,y) f(y)\, dy \quad \textit{for} \quad x \in \bar{U} \ .$$

Then

$$-\Delta u_1(x) = f(x) \quad \textit{for} \quad x \in U, \quad u_1(x) = 0 \quad \textit{for} \quad x \in \partial U \ .$$

Proof: We have

$$\Delta \Phi(x-y) = 0 \quad \text{for} \quad x \neq y \ .$$

Therefore, if $x \in U$ then

$$\Delta \Phi(\tilde{x} - y) = 0 \quad \text{for all} \quad y \in U \ .$$

It follows that

$$\Delta_x \int_U \Phi(\tilde{x} - y) f(y)\, dy = 0 \quad \text{for} \quad x \in U \ .$$

Therefore,

$$-\Delta u_1(x) = -\Delta \int_U \Phi(x-y) f(y)\, dy \quad \text{for} \quad x \in U \ .$$

The proof that

$$-\Delta u_1(x) = f(x) \quad \text{for} \quad x \in U$$

follows in the same way as in the case of the equation $-\Delta u(x) = f(x)$ in $\mathbb{R}^3$. In a formal sense,

$$-\Delta \Phi(x) = \delta(x), \quad -\Delta \Phi(x-y) = \delta(x-y) \ .$$

This argument yields that

$$\begin{aligned}
-\Delta u_1(x) &= \int_U -\Delta \Phi(x-y) f(y)\, dy \\
&= \int_U \delta(x-y) f(y)\, dy \\
&= f(x)
\end{aligned}$$

Next, let us show that $u_1(x) = 0$ for $x \in \partial U$. If $x \in \partial U$ then $\tilde{x} = x$ and the two functions

$$\Phi(x-y) \quad \text{and} \quad \Phi(\tilde{x} - y)$$

are identical as functions of $y \in \partial U, y \neq x$. Therefore,

$$G(x,y) = 0 \quad \text{for} \quad x, y \in \partial U, \quad x \neq y \ .$$

At $y = x$ the two functions $\Phi(x, y)$ and $\Phi(\tilde{x}, y)$ are not defined. But the singularities are integrable. It follows that

$$u_1(x) = 0 \quad \text{for} \quad x \in \partial u \ .$$

$\diamond$

Lemma 15.2 *We have*

$$K(x,y) = \frac{\partial}{\partial y_3} G(x,y) = \frac{x_3}{2\pi |x-y|^3} \quad \textit{for} \quad x \in U, \quad y \in \partial U \ .$$

Proof: For $x \in U, y \in \partial U$ it holds that

$$\begin{aligned}
\frac{1}{|x-y|} &= \Big(\sum_{j=1}^{3} (x_j - y_j)^2 \Big)^{-1/2} \\
\frac{\partial}{\partial y_3} \frac{1}{|x-y|} &= \frac{x_3 - y_3}{|x-y|^3} \\
\frac{\partial}{\partial y_3} \frac{1}{|\tilde{x}-y|} &= \frac{\tilde{x}_3 - y_3}{|\tilde{x}-y|^3}
\end{aligned}$$

Since $|x - y| = |\tilde{x} - y|$ one obtains that

$$K(x,y) = \frac{1}{4\pi |x-y|^3} \Big(x_3 - y_3 - (\tilde{x}_3 - y_3) \Big) = \frac{x_3}{2\pi |x-y|^3} \ .$$

$\diamond$

Note: For $x \in U, y \in \partial U$ we have

$$K(x,y) = \frac{1}{2\pi} \frac{x_3}{\Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2 \Big)^{3/2}} \ .$$

Therefore, if $(y_1, y_2) \neq (x_1, x_2)$, then

$$K(x,y) \to 0 \quad \text{as} \quad x_3 \to 0 \ .$$

If $x \in U, y \in \partial U$ and

$$(y_1, y_2) = (x_1, x_2)$$

then

$$K(x,y) = \frac{1}{2\pi} \frac{x_3}{x_3^3} = \frac{1}{2\pi x_3^2} \to \infty \quad \text{as} \quad x_3 \to 0 \ .$$

Lemma 15.3 *We have*

$$\int_{\mathbb{R}^2} K(x,y)\, dy = 1 \quad \textit{for all} \quad x \in U \ .$$

Proof: Fix $x \in U$ and consider the integral

$$I = \int_{\mathbb{R}^2} \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2 \Big)^{-3/2} \, dy \ .$$

Introducing polar coordinates in the plane ∂U, centered at (x_1, x_2), we have

$$\begin{aligned}
I &= \int_0^{2\pi} \int_0^\infty (r^2 + x_3^2)^{-3/2} r \, dr d\phi \\
&= 2\pi \int_0^\infty (r^2 + x_3^2)^{-3/2} r \, dr \quad (\text{ substitute } r = x_3 \rho) \\
&= 2\pi x_3^2 \int_0^\infty (x_3^2 \rho^2 + x_3^2)^{-3/2} \rho \, d\rho \\
&= \frac{2\pi}{x_3} \int_0^\infty \frac{\rho}{(\rho^2 + 1)^{3/2}} \, d\rho \quad (\text{ substitute } \rho^2 + 1 = v, \, 2\rho d\rho = dv) \\
&= \frac{\pi}{x_3} \int_1^\infty \frac{dv}{v^{3/2}} \\
&= \frac{\pi}{x_3} \, (-2 v^{-1/2})|_1^\infty \\
&= \frac{2\pi}{x_3}
\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^2} K(x,y) \, dy = \frac{x_3}{2\pi} \, I = 1 \ .$$

◇

Lemma 15.4 *Let $g : \partial U \to \mathbb{R}$ denote a continuous, bounded function. Set*

$$u_2(x) = \int_{\partial U} K(x,y) g(y) \, dS(y)$$

for $x \in U$. Then

$$\Delta u_2(x) = 0 \quad \textit{for all} \quad x \in U \ .$$

Also, the function given by

$$u_2(x) = \begin{cases} \int_{\partial U} K(x,y) g(y) \, dS(y) & \textit{for} \quad x \in U \\ g(x) & \textit{for} \quad x \in \partial U \end{cases}$$

for $x \in \bar{U}$ is continuous

Proof: Let $x \in U$ and let $\varepsilon = x_3/2$. We have

$$(\Delta \Phi)(x - y) = 0 \quad \text{if} \quad |y_3| < \varepsilon$$

and

$$(\Delta \Phi)(\tilde{x} - y) = 0 \quad \text{if} \quad |y_3| < \varepsilon$$

Therefore,

$$\Delta_x G(x,y) = 0 \quad \text{for} \quad x \in U, \quad |y_3| < \frac{x_3}{2} \ .$$

It follows that

$$\frac{\partial}{\partial y_3} \Delta_x G(x,y) = 0 \quad \text{for} \quad x \in U \quad \text{and} \quad y \in \partial U$$

and

$$\Delta_x K(x,y) = 0 \quad \text{for} \quad x \in U \quad \text{and} \quad y \in \partial U \ .$$

Therefore,

$$\Delta u_2(x) = \Delta_x \int_{\partial U} K(x,y) g(y) \, dS(y) = 0 \ .$$

To show continuity of the function $u_2(x)$ at the points $x \in \partial U$ we fix a point

$$x^0 = (x_1^0, x_2^0, 0) \in \partial U \ .$$

We have for $x \in U$:

$$u_2(x) - g(x^0) = \int_{\partial U} K(x,y)(g(y) - g(x^0)) \, dS(y) \ .$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ with

$$|g(y) - g(x^0)| \le \varepsilon \quad \text{if} \quad |y - x^0| \le \delta, \quad y \in \partial U \ .$$

Set

$$M_\delta = \{ y \in \partial U \ : \ |y - x^0| \le \delta \} \ .$$

In the following, let $x \in U$ and let

$$|x - x^0| \le \frac{\delta}{2} \ .$$

We have

$$\begin{aligned} |u_2(x) - g(x^0)| &\le \int_{\partial U} K(x,y) |g(y) - g(x^0)| \, dS(y) \\ &= \int_{M_\delta} \ldots + \int_{\partial U \setminus M_\delta} \ldots \\ &=: Int_1 + Int_2 \end{aligned}$$

Clearly, $Int_1 \le \varepsilon$ since $|g(y) - g(x^0)| \le \varepsilon$ for $y \in M_\delta$.

To estimate Int_2 note that $y \in \partial U \setminus M_\delta$ implies that

$$\begin{aligned} |y-x| &= |y-x^0+x^0-x| \\ &\geq |y-x^0|-|x^0-x| \\ &\geq \frac{1}{2}\,|y-x^0| \end{aligned}$$

In the last estimate we have used that $|y-x|\geq\delta$ and $|x-x^0|\leq\frac{\delta}{2}$.

One obtains:

$$K(x,y)=\frac{x_3}{2\pi}\,\frac{1}{|x-y|^3}\leq\frac{x_3}{2\pi}\,\frac{8}{|x^0-y|^3}\ .$$

Therefore,

$$Int_2\leq 2|g|_\infty\,\frac{8x_3}{2\pi}\,Int_3$$

with

$$Int_3=\int_{\partial U\setminus M_\delta}\frac{1}{|x^0-y|^3}\,dS(y)\ .$$

Let

$$U_\delta=\{y\in\mathbb{R}^2\ :\ |y|\geq\delta\}\ .$$

We have, using polar coordinates,

$$\begin{aligned} Int_3 &= \int_{U_\delta}\frac{dy}{|y|^3} \\ &= \int_0^{2\pi}\int_\delta^\infty\frac{1}{r^3}r\,drd\phi \\ &= 2\pi\int_\delta^\infty\frac{dr}{r^2} \\ &= \frac{2\pi}{\delta} \end{aligned}$$

It follows that

$$Int_2\leq\frac{Cx_3}{\delta}$$

where C is a constant independent of x. There exists $\tilde{\delta}>0$ so that

$$\frac{Cx_3}{\delta}\leq\varepsilon\quad\text{for}\quad 0<x_3\leq\tilde{\delta}\ .$$

Therefore,

$$|u_2(x)-g(x^0)|\leq 2\varepsilon\quad\text{if}\quad x\in U\quad\text{and}\quad |x-x^0|\leq\frac{\delta}{2}\quad\text{and}\quad 0<x_3\leq\tilde{\delta}\ .$$

$\diamond$

16 The Wave Equation in Spherical Coordinates

16.1 The Lapacian in Spherical Coordinates

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\phi &= \arctan(y/x) \\
\theta &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)
\end{aligned}
$$

The angle ϕ is the azimuthal angel; θ is the polar angle.

The Laplacian:

$$\Delta v(r,\theta,\phi) = \frac{1}{r^2}\,(r^2 v_r)_r + \frac{1}{r^2 \sin\theta}\,(\sin\theta\ v_\theta)_\theta + \frac{1}{r^2 \sin^2\theta}\,v_{\phi\phi}$$

16.2 The Wave Equation in a Ball

We consider the wave equation

$$u_{tt} = c^2 \Delta u$$

in a ball of radius a. We require the boundary condition

$$u(a,\theta,\phi,t) = 0\ .$$

Let

$$u(r,\theta,\phi,t) = v(r,\theta,\phi)h(t)\ .$$

One obtains

$$\frac{h''(t)}{h(t)} = c^2 \frac{\Delta v(r,\theta,\phi)}{v(r,\theta,\phi)} = -\omega^2 = -(ck)^2 \quad \text{with} \quad k = \omega/c\ .$$

The equation

$$h''(t) + k^2 c^2 h(t) = 0$$

has the solutions

$$h(t) = A\cos(kct) + B\sin(kct)\ .$$

Consider the eigenvalue problem

$$\Delta v + k^2 v = 0, \quad v(a,\theta,\phi) = 0\ .$$

Let

$$v(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \; .$$

Substitute this ansatz into

$$\Delta v + k^2 v = 0$$

and divide by v to obtain

$$\frac{1}{Rr^2}(r^2 R_r)_r + \frac{1}{\Theta r^2 \sin\theta}(\sin\theta\,\Theta_\theta)_\theta + \frac{1}{\Phi r^2 \sin^2\theta}\Phi_{\phi\phi} + k^2 = 0 \; . \tag{16.1}$$

Multiply by $r^2 \sin^2\theta$.

Obtain that

$$\frac{\Phi''}{\Phi} = const =: -m^2 \; .$$

Here m must be an integer to make $\Phi(\phi)$ periodic with period 2π.

Substituting $\Phi''/\Phi = -m^2$ into (16.1) one obtains

$$\frac{1}{R}(r^2 R_r)_r + \frac{1}{\Theta\sin\theta}(\sin\theta\,\Theta_\theta)_\theta - \frac{m^2}{\sin^2\theta} + r^2 k^2 = 0 \; . \tag{16.2}$$

There are two terms depending only on r and two terms depending only on θ. Call the separation constant Q. Obtain

$$\frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0 \; . \tag{16.3}$$

and

$$(r^2 R')' + (r^2 k^2 - Q)R = 0 \; . \tag{16.4}$$

The R–equation

$$r^2 R'' + 2rR' + (k^2 r^2 - Q)R = 0 \tag{16.5}$$

is called a spherical Bessel equation. (A difference to the R–equation, that one obtains in 2D polar coordinates, is the factor 2 in the equation above. Also, in the 2D equation the constant Q becomes m^2, where $\Phi'' + m^2\Phi = 0$. In 3D, the constant Q comes from the Θ-equation.)

16.2.1 The Spherical Bessel Equation

First consider (16.5) for $k = 0$. One obtains an Euler equation, and the ansatz

$$R(r) = r^\lambda$$

leads to the indicial equation

$$\lambda(\lambda+1) = Q \; .$$

The θ–equation will require to choose

$$Q = Q_n = n(n+1), \quad n = 0, 1, \ldots$$

For $Q = n(n+1)$ the indicial equation has the roots

$$\lambda_1 = n, \quad \lambda_2 = -n-1 \ .$$

This yields the general solution

$$R(r) = \alpha r^n + \frac{\beta}{r^{n+1}}$$

of (16.5) for $k = 0$ and $Q = n(n+1)$. The boundary conditions

$$R(0) \ \text{ finite}, \quad R(a) = 0$$

yield $R \equiv 0$.

Consider (16.5) for $k > 0$. One can transform to Bessel's equation as follows: Define

$$x = kr, \quad y(x) = y(kr) = r^{1/2} R(r) \ .$$

(Note that the factor $r^{1/2}$ was not present in 2D.) Obtain:

$$\begin{aligned} R(r) &= r^{-1/2} y(kr) \\ R'(r) &= -\frac{1}{2} r^{-3/2} y(kr) + k r^{-1/2} y'(kr) \\ R''(r) &= \frac{3}{4} r^{-5/2} y(kr) - k r^{-3/2} y'(kr) + k^2 r^{-1/2} y''(kr) \end{aligned}$$

Therefore, if $R(r)$ satisfies (16.5), then we have

$$\begin{aligned} 0 &= r^{1/2}\Big(r^2 R'' + 2rR' + (k^2 r^2 - Q) R \Big) \\ &= k^2 r^2 y''(kr) - rky'(kr) + \frac{3}{4} y(kr) + 2kry'(kr) - y(kr) + (k^2 r^2 - Q) y(kr) \\ &= x^2 y''(x) + xy'(x) + (x^2 - Q - \frac{1}{4}) y(x) \end{aligned}$$

We have derived the equation

$$x^2 y''(x) + xy'(x) + \Big(x^2 - Q - \frac{1}{4} \Big) y(x) = 0 \ ,$$

which is Bessel's equation of index $\sqrt{Q + \frac{1}{4}}$.

If $Q = n(n+1)$ then

$$Q + \frac{1}{4} = (n + \frac{1}{2})^2 \ ,$$

i.e., we obtain Bessel's equation of index $n + \frac{1}{2}$.

Summary: Using polar coordinates in 2D, Helmholtz equation $\Delta v + k^2 v = 0$ leads to Bessel's equation of index m for the R–equation, where $m = 0, 1, 2 \ldots$ The index m is determined by the Φ–equation.

Using spherical coordinates, the 3D Helmholtz equation $\Delta v + k^2 v = 0$ leads to Bessel's equation of index $n + \frac{1}{2}$ for the R–equation, where $n = 0, 1, 2 \ldots$ The index n is determined by the Θ–equation.

In 3D, the index m, coming from the Φ–equation, is present in the Θ–equation, but is not present in the R–equation.

16.2.2 Legendre's Equation

The Θ equation (16.3) reads

$$\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0 \; . \tag{16.6}$$

Recall that $0 < \theta < \pi$. Thus we may write

$$\begin{aligned} \Theta(\theta) &= P(\cos\theta) \\ \Theta'(\theta) &= -\sin\theta\, P'(\cos\theta) \\ \Theta''(\theta) &= -\cos\theta\, P'(\cos\theta) + \sin^2\theta\, P''(\cos\theta) \end{aligned}$$

If $\Theta(\theta)$ solves (16.6) and if $P(\cos\theta) = \Theta(\theta)$ then obtain

$$\sin^2\theta\, P''(\cos\theta) - 2\cos\theta\, P'(\cos\theta) + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)P(\cos\theta) = 0 \; . \tag{16.7}$$

Set $x = \cos\theta$. Obtain

$$(1 - x^2)\, P''(x) - 2x\, P'(x) + \Big(Q - \frac{m^2}{1 - x^2}\Big)P(x) = 0 \; . \tag{16.8}$$

This equation is called an **associated Legendre equation**. The points $x = \pm 1$ are regular singular points.

We may assume m to be integer. Then one can show that (16.8) has nontrivial solutions that are bounded for $-1 < x < 1$ if only if $Q = n(n+1)$ and $-n \le m \le n$ with integers n.

We now assume

$$Q = n(n+1)$$

with integer n, $n \ge 0$. Then, for $m = 0$, one obtains Legendre's equation

$$(1 - x^2)\, P''(x) - 2x\, P'(x) + n(n+1)P(x) = 0 \; . \tag{16.9}$$

Remark: If $m = 0$ then $\Phi(\phi) = const$, i.e., we consider solutions $v(r, \theta, \phi)$ of Helmholtz's equation that are independent of ϕ.

16.3 Legendre Polynomials

Lemma 16.1 *The n–th degree polynomial*

$$P(x) = D^n\Big((x^2-1)^n\Big), \quad D = \frac{d}{dx} \ ,$$

solves Legendre's equation (16.9).

Proof: Let

$$w(x) = (x^2-1)^n, \quad w'(x) = 2nx(x^2-1)^{n-1} \ ,$$

thus

$$(1-x^2)w' + 2nxw = 0 \ . \tag{16.10}$$

Recall Leibniz' rule,

$$\begin{aligned} D^{n+1}(fg) &= \sum_{j=0}^{n+1} \left(\begin{array}{c} n+1 \\ j \end{array} \right) (D^j f)(D^{n+1-j}g) \\ &= fD^{n+1}g + (n+1)(Df)(D^n g) + \frac{1}{2}n(n+1)(D^2 f)(D^{n-1}g) + \ldots \end{aligned}$$

Apply D^{n+1} to (16.10),

$$(1-x^2)D^{n+2}w - 2x(n+1)D^{n+1}w - n(n+1)D^n w + 2nxD^{n+1}w + 2n(n+1)D^n w = 0$$

thus

$$(1-x^2)D^{n+2}w - 2xD^{n+1}w + n(n+1)D^n w = 0$$

This shows that $D^n w$ solves Legendre's equation and completes the proof. $\diamond$

The polynomial

$$P_n(x) = \frac{1}{n!2^n}\, D^n\Big((x^2-1)^n\Big) \tag{16.11}$$

is called the n–th Legendre polynomial. Formula (16.11) is called Rodrigues' formula for the Legendre polynomial $P_n(x)$ of degree n.

We claim that the normalization factor $1/(n!2^n)$ is chosen so that $P_n(1) = 1$. In other words, we have

Lemma 16.2 *The n–th Legendre polynomial, defined by (16.11), satisfies*

$$P_n(1) = 1 \ .$$

Proof: We have

$$D^n\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) D^j\Big((x+1)^n\Big)D^{n-j}\Big((x-1)^n\Big)$$

Evaluate at $x = 1$. Note that, for $j \geq 1$, the term $D^{n-j}((x-1)^n)$ is zero at $x = 1$. For $j = 0$ obtain:

$$\begin{aligned}
\Big(D^j((x+1)^n)\, D^{n-j}((x-1)^n) \Big)\Big|_{x=1} &= \Big((x+1)^n D^n((x-1)^n) \Big)\Big|_{x=1} \\
&= 2^n\, n!\ .
\end{aligned}$$

This is the value of the above sum at $x = 1$. The lemma is proved. $\diamond$

The following theorem of the French mathematician Michel Rolle (1652–1719) is a special case of the mean value theorem:

Theorem 16.1 *Let $f \in C[a,b] \cap C^1(a,b)$ denote a real–valued function. If $f(a) = f(b)$ then there exists $a < \xi < b$ with $f'(\xi) = 0$.*

Using Rolle's theorem, it is easy to show:

Lemma 16.3 *The n–th Legendre polynomial $P_n(x)$ has n simple zeros in the open interval $-1 < x < 1$.*

Proof: The function

$$f(x) = (x^2-1)^n = (x+1)^n(x-1)^n$$

satisfies

$$D^j f(-1) = D^j f(1) = 0 \quad \text{for} \quad j = 0, 1, \ldots, n-1\ .$$

Since $f(-1) = f(1)$ there exists $-1 < \xi < 1$ with $Df(\xi) = 0$. Since

$$Df(-1) = Df(\xi) = Df(1) = 0$$

there exist ξ_1, ξ_2 with

$$-1 < \xi_1 < \xi < \xi_2 < 1$$

so that

$$D^2 f(\xi_2) = D^2 f(\xi_2) = 0\ .$$

If $n \geq 3$ then

$$D^2 f(-1) = D^2 f(\xi_1) = D^2 f(\xi_2) = D^2 f(1) = 0\ .$$

Therefore, $D^3 f$ has three distinct zeros in the interval $(-1, 1)$, etc. $\diamond$

For series expansions in terms of the polynomials $P_n(x)$ one has to know orthogonality in the L_2–inner product.

Lemma 16.4 *The sequence of Legendre polynomials,*

$$P_n(x) = \frac{1}{n!2^n} D^n((x^2-1)^n), \quad n = 0, 1, \ldots$$

satisfies

$$\int_{-1}^{1} P_m(x) P_n(x) = \frac{2\delta_{mn}}{2n+1}, \quad m, n = 0, 1, \ldots$$

Proof: Orthogonality: For $m < n$ it follows through integration by parts that

$$\int_{-1}^{1} D^m\Big((x^2-1)^m\Big) D^n\Big((x^2-1)^n\Big)\, dx = 0 \ .$$

(Move D^n to the first factor through integration by parts.)

Normalization: We claim that

$$\int_{-1}^{1} \Big(P_n(x)\Big)^2 dx = \frac{2}{2n+1} \ . \tag{16.12}$$

For the left–hand side in (16.12) we have

$$\begin{aligned} lhs &= \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} D^n\Big((x^2-1)^n\Big) D^n\Big((x^2-1)^n\Big)\, dx \\ &= \frac{(2n)!}{2^{2n}(n!)^2} J \end{aligned}$$

with

$$J = \int_{-1}^{1} (1-x^2)^n \, dx \ .$$

To obtain the last equation we have used n fold integration by parts, noting that

$$D^{2n}\Big((x^2-1)^n\Big) = D^{2n} x^{2n} = (2n)! \ .$$

It remains to compute J. We will prove:

$$\int_{-1}^{1} (1-x^2)^n \, dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \ . \tag{16.13}$$

To show this, we will use Euler's *Beta* function and its relation to the Γ function.

By definition,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,$$

for $p > 0, q > 0, z > 0$.

Lemma 16.5 *For all $p > 0, q > 0$,*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ .$$

A proof is given below.

Using the substitution

$$x^2 = y, \quad 2x\,dx = dy, \quad dx = \frac{1}{2}y^{-1/2}\,dy$$

we have

$$\begin{aligned}
J &= 2\int_0^1 (1-x^2)^n\,dx \\
&= \int_0^1 y^{-1/2}(1-y)^n\,dy \\
&= B(\frac{1}{2}, n+1) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}
\end{aligned}$$

Here $\Gamma(n+1) = n!$. Also, using the fundamental functional equation for the Γ function, $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned}
\Gamma(\frac{1}{2}+1) &= \frac{1}{2}\Gamma(\frac{1}{2}) \\
\Gamma(\frac{1}{2}+2) &= \Gamma(\frac{3}{2}+1) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) \\
\Gamma(\frac{1}{2}+n+1) &= \frac{1\cdot 3\cdot \ldots \cdot (2n+1)}{2^{n+1}}\Gamma(\frac{1}{2})
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} &= \frac{2^{n+1}}{1\cdot 3\cdot \ldots \cdot (2n+1)} \\
&= \frac{2^{2n+1}n!}{(2n+1)!}
\end{aligned}$$

Obtain, with *lhs* the left–hand side of (16.12),

$$\begin{aligned}
lhs &= \frac{(2n)!J}{2^{2n}(n!)^2} \\
&= \frac{(2n)!2^{2n+1}}{2^{2n}(2n+1)!} \\
&= \frac{2}{2n+1}
\end{aligned}$$

This completes the proof of (16.12).
Proof of Lemma 16.5: Using the substitution

$$x^2 = t, \quad 2x\,dx = dt\ ,$$

one obtains that

$$\begin{aligned}\int_0^\infty x^{2p-1} e^{-x^2}\,dx &= \frac{1}{2}\int_0^\infty t^{p-1} e^{-t}\,dt \\ &= \frac{1}{2}\Gamma(p)\end{aligned}$$

We will evaluate the integral

$$I = \int_0^\infty \int_0^\infty x^{2p-1} y^{2q-1} e^{-x^2-y^2}\,dxdy$$

in two ways: (a) using Fubini's theorem, (b) using polar coordinates. Obtain (a):

$$\begin{aligned}I &= \Big(\int_0^\infty x^{2p-1} e^{-x^2}\,dx\Big)\Big(\int_0^\infty y^{2q-1} e^{-y^2}\,dy\Big) \\ &= \frac{1}{4}\,\Gamma(p)\Gamma(q)\end{aligned}$$

Also, (b), using $x = r\cos\phi, y = r\sin\phi$,

$$\begin{aligned}I &= \int_{\phi=0}^{\pi/2}\int_{r=0}^\infty r^{2p+2q-2}(\cos^{2p-1}\phi)\,(\sin^{2q-1}\phi)\,e^{-r^2}\,r\,drd\phi \\ &= \Big(\int_0^\infty r^{2p+2q-1} e^{-r^2}\,dr\Big)\Big(\int_0^{\pi/2} \cos^{2p-1}\phi\,\sin^{2q-1}\phi\,d\phi\Big) \\ &=: \frac{1}{2}\,\Gamma(p+q) I_1\end{aligned}$$

To evaluate I_1 we will use the substitution

$$t = \cos^2\phi, \quad dt = -2\sin\phi\,\cos\phi\,d\phi\ .$$

We have

$$\begin{aligned}I_1 &= \frac{1}{2}\int_0^{\pi/2}(\cos^{2p-2}\phi)\,(\sin^{2q-2}\phi\,)2\sin\phi\,\cos\phi\,d\phi \\ &= \frac{1}{2}\int_0^{\pi/2}(\cos^{2p-2}\phi)\,(1-\cos^2\phi)^{q-1}\,2\sin\phi\,\cos\phi\,d\phi \\ &= \frac{1}{2}\int_0^1 t^{p-1}(1-t)^{q-1}\,dt \\ &= \frac{1}{2}\,B(p,q)\end{aligned}$$

We have shown that

$$I = \frac{1}{4}\,\Gamma(p)\Gamma(q) = \frac{1}{4}\,\Gamma(p+q)B(p,q)\ ,$$

which proves the lemma. $\diamond$

16.4 Expansion in Terms of Legendre Polynomials

In the function space $H = L_2(-1,1)$ one defines the inner product and norm by

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,dx, \quad \|f\|^2 = (f,f)\ .$$

Then the sequence of Legendre polynomials,

$$P_n(x), \quad n = 0, 1, \ldots$$

is an orthogonal sequence in H with

$$\|P_n\|^2 = \frac{2}{2n+1}\ .$$

Let

$$\Pi_n = span\{1, x, \ldots, x^n\} = span\{P_0, P_1, \ldots, P_n\}$$

denote the subspace of H consisting of all polynomials of degree $\le n$. If $f \in H$ is given, denote by

$$f_n = \sum_{j=0}^{n} \alpha_j P_j$$

the best approximation to f in Π_n. One can show that f_n is characterized by the orthogonality condition

$$f - f_n \perp \Pi_n\ .$$

In other words, one needs to find $f_n \in \Pi_n$ so that

$$(f - f_n, P_k) = 0, \quad k = 0, \ldots, n\ .$$

This yields

$$\alpha_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx\ .$$

One can prove that

$$\|f - f_n\| \to 0 \quad \text{as} \quad n \to \infty\ .$$

This follows, essentially, from the Weierstrass Approximation Theorem.

16.5 All Eigenvalues of Legendre's Equation

Consider the equation

$$(1-x^2)P''(x) - 2xP'(x) = -QP(x), \quad P(-1) \quad \text{and} \quad P(1) \quad \text{finite} .$$

We claim that the eigenvalues $Q_n = n(n+1)$ are the only eigenvalues.

Suppose that

$$(1-x^2)S''(x) - 2xS'(x) = -\lambda S(x), \quad S(-1) \quad \text{and} \quad S(1) \quad \text{finite}$$

and $\lambda \neq Q_n$ for all n. We claim that $S \equiv 0$.

We have

$$\begin{array}{rcl} ((1-x^2)S')' + \lambda S &=& 0 \\ ((1-x^2)P')' + QP &=& 0 \end{array}$$

for $Q = Q_n$ and $P = P_n$. Obtain

$$\begin{array}{rcl} ((1-x^2)S')'P + \lambda SP &=& 0 \\ ((1-x^2)P')'S + QPS &=& 0 \end{array}$$

Integration over $-1 \leq x \leq 1$ and integration by parts implies that

$$(S,P) = \int_{-1}^{1} S(x)P(x)\,dx = 0$$

for every Legendre polynomial $P = P_n$

Let

$$\Pi_n = span\{1, x, \ldots, x^n\} = span\{P_0, P_1, \ldots, P_n\}$$

denote the vector space of all polynomials of degree $\leq n$. If $f \in \Pi_n$ is arbitrary then

$$(S, f) = 0 \; .$$

Therefore,

$$\begin{array}{rcl} \|S-f\|^2 &=& (S-f, S-f) \\ &=& \|S\|^2 + \|f\|^2 \end{array}$$

thus

$$\|S\| \leq \|S-f\| \quad \text{for all} \quad f \in \Pi_n \; .$$

Let $\varepsilon > 0$ be given. By Weierstrass' approximation theorem there exists a polynomial f with

$$|S - f|_\infty \le \varepsilon \ .$$

It follows that

$$\|S\|^2 \le 2\varepsilon^2 \ .$$

Since $\varepsilon > 0$ is arbitrary, one obtains that $S = 0$.

16.6 Harmonic Functions in Spherical Coordinates

Recall the Laplacian in spherical coordinates

$$\Delta v(r,\theta,\phi) = \frac{1}{r^2}\,(r^2 v_r)_r + \frac{1}{r^2 \sin\theta}\,(\sin\theta\ v_\theta)_\theta + \frac{1}{r^2\sin^2\theta}\,v_{\phi\phi} \ .$$

The ansatz

$$v(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

leads to

$$\frac{1}{Rr^2}(r^2 R_r)_r + \frac{1}{\Theta r^2 \sin\theta}(\sin\theta\,\Theta_\theta)_\theta + \frac{1}{r^2\sin^2\theta}\,\frac{\Phi_{\phi\phi}}{\Phi} = 0 \ . \tag{16.14}$$

Obtain

$$\Phi'' + m^2\Phi = 0 \ .$$

Multiply the equation (16.14) by r^2 and call the separation constant Q. Obtain

$$\begin{aligned} (r^2R')' - QR &= 0 \\ \frac{1}{\sin\theta}\,(\sin\theta\Theta')' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta &= 0 \ . \end{aligned}$$

The Θ–equation will lead to

$$Q = n(n+1) \quad \text{for} \quad n \in \mathbb{N}_0 \ .$$

Then the R–equation becomes

$$r^2R'' + 2rR' - n(n+1)R = 0 \ .$$

The ansatz

$$R(r) = r^\lambda$$

leads to the quadratic

$$\lambda^2 + \lambda - n(n+1) = 0$$

with solutions

$$\lambda_1 = n, \quad \lambda_2 = -n - 1 \ .$$

The general solution of the R–equation is

$$R(r) = Ar^n + Br^{-(n+1)} \ .$$

The Θ–equation reads

$$\Theta'' + \frac{\cos\theta}{\sin\theta}\,\Theta' + \Big(Q - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0 \ . \tag{16.15}$$

Using the transformation

$$\Theta(\theta) = P(\cos\theta)$$

one obtains the associated Legendre equation for $P = P(x)$:

$$(1-x^2)\,P''(x) - 2x\,P'(x) + \Big(Q - \frac{m^2}{1-x^2}\Big)P(x) = 0 \ . \tag{16.16}$$

For $m = 0$ and $Q = n(n+1)$ one obtains Legendre's equation

$$(1-x^2)\,P''(x) - 2x\,P'(x) + n(n+1)P(x) = 0 \ . \tag{16.17}$$

Lemma 16.6 *The n–th degree polynomial*

$$P(x) = D^n\Big((x^2-1)^n\Big), \quad D = \frac{d}{dx} \ ,$$

solves Legendre's equation.

The proof is given in Section 16.3.
The polynomial

$$P_n(x) = \frac{1}{n!2^n}\,D^n\Big((x^2-1)^n\Big) \tag{16.18}$$

is called the n–th Legendre polynomial. The above formula is called Rodrigues' formula for the Legendre polynomial $P_n(x)$ of degree n.

Examples:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \end{aligned}$$

Note that the functions

$$\Big(Ar^n + Br^{-(n+1)}\Big)P_n(\cos\theta)$$

are harmonic functions in $\mathbb{R}^3 \setminus \{0\}$. These functions do not depend on ϕ.

Consider the associated Legendre equation with $Q = n(n+1)$:

$$(1-x^2)y'' - 2xy' + \Big(n(n+1) - \frac{m^2}{1-x^2}\Big)y = 0 \tag{16.19}$$

Here n and m are integers. It is remarkable that a solution $P(x) = P_n(x)$ of the equation for $m = 0$ leads, in a simple way, to a nontrivial solution for any integer m with $1 \le m \le n$.

Lemma 16.7 *Let $1 \le m \le n$ with integers m and n. The function*

$$y(x) = (1-x^2)^{m/2} D^m P(x), \quad -1 < x < 1 \ ,$$

solves (16.19) if $P(x)$ solves Legendre's equation,

$$(1-x^2)\, P''(x) - 2x\, P'(x) + n(n+1)P(x) = 0 \ . \tag{16.20}$$

16.7 Solutions of the Associated Legendre Equation

The associated Legendre equation reads

$$(1-x^2)y'' - 2xy' + \Big(n(n+1) - \frac{m^2}{1-x^2}\Big)y = 0 \tag{16.21}$$

Here n and m are integers. It is remarkable that a solution $P(x) = P_n(x)$ of the equation for $m = 0$ leads, in a simple way, to a nontrivial solution for any integer m with $1 \le m \le n$.

Lemma 16.8 *Let $1 \le m \le n$ with integers m and n. The function*

$$y(x) = (1-x^2)^{m/2} D^m P(x), \quad -1 < x < 1 \ ,$$

solves (16.21) if $P(x)$ solves Legendre's equation,

$$(1-x^2)\, P''(x) - 2x\, P'(x) + n(n+1)P(x) = 0 \ . \tag{16.22}$$

Proof: Let $u = D^m P$. We derive an equation satisfied by u by differentiating (16.22) m times:

We have

$$\begin{aligned} D^m((1-x^2)P'') &= (1-x^2)u'' - 2mxu' - m(m-1)u \\ D^m(-2xP') &= -2xu' - 2mu \end{aligned}$$

Therefore,

$$(1-x^2)u'' - 2x(m+1)u' + (n^2 + n - m^2 - m)u = 0 \ .$$

We have

$$
\begin{aligned}
u &= (1-x^2)^{-m/2} y \\
u' &= (1-x^2)^{-m/2} y' + mx(1-x^2)^{-\frac{m}{2}-1} y \\
u'' &= (1-x^2)^{-m/2} y'' + 2mx(1-x^2)^{-\frac{m}{2}-1} y' \\
&\quad + \Big(m(1-x^2)^{-\frac{m}{2}-1} + m(m+2)x^2(1-x^2)^{-\frac{m}{2}-2} \Big) y
\end{aligned}
$$

Substitute these expressions for u, u', u'' into the equation for u and multiply by $(1-x^2)^{m/2}$. Obtain for $y = (1-x^2)^{m/2} u$:

$$(1-x^2)y'' + Q_1 y' + Q_2 y = 0$$

where

$$Q_1 = (1-x^2) 2mx (1-x^2)^{-1} - 2x(m+1) = -2x$$

and

$$
\begin{aligned}
Q_2 &= m + m(m+2)x^2(1-x^2)^{-1} \quad \text{(from } u''\text{)} \\
&\quad - 2(m+1)mx^2(1-x^2)^{-1} \quad \text{(from } u'\text{)} \\
&\quad + n(n+1) - m^2 - m \quad \text{from} u \\
&= n(n+1) + (1-x^2)^{-1} \Big(m(m+2)x^2 - 2m(m+1)x^2 - m^2(1-x^2) \Big) \\
&= n(n+1) - \frac{m^2}{1-x^2}
\end{aligned}
$$

This proves the lemma. $\diamond$

If $P = P_n$ is the n–th Legendre polynomial, then the function $y(x)$ defined in the previous lemma is nontrivial for $1 \le m \le n$. In fact, if m is even, then y is a polynomial of degree n. If m is odd, then y is a polynomial of degree $n-1$ multiplied by $\sqrt{1-x^2}$.

For our discussion of spherical harmonics below, it will be convenient to introduce the following functions:

$$P_n^m(x) = \frac{(-1)^m}{2^n\, n!} (1-x^2)^{m/2} D^{m+n} X^n, \quad X = x^2 - 1 \ , \tag{16.23}$$

for $-n \le m \le n$. The functions $P_n^m(x)$ are called associated Legendre functions of order m. Since

$$P_n(x) = \frac{1}{2^n\, n!} D^n X^n, \quad X = x^2 - 1$$

is the Legendre polynomial of degree n, we have

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} D^m P_n(x) \quad \text{for} \quad 0 \le m \le n \ .$$

However, the formula (16.23) makes sense also for $-n \le m \le -1$.

Example:

$$
\begin{aligned}
P_4^4(x) &= \frac{1}{2^4\,4!}\,(1-x^2)^2 D^{4+4}\Big((x^2-1)^4\Big) \\
&= \frac{8!}{2^4\,4!}\,(1-x^2)^2 \\
&= 105\,(1-x^2)^2
\end{aligned}
$$

Therefore,

$$P_4^4(\cos\theta) = 105\,\sin^4\theta\ .$$

In general, if $x = \cos\theta$, then $(1-x^2)^{m/2} = \sin^m\theta$. Since $D^{m+n}X^n$ is a polynomial in x, the function $P_n^m(\cos\theta)$ is a polynomial in $\sin\theta$ and $\cos\theta$ if $0 \le m \le n$:

$$P_n^m(\cos\theta) = \frac{(-1)^m}{2^n\,n!}\,\sin^m\theta\left(\frac{d}{dx}\right)^{m+n}(x^2-1)^n\Big|_{x=\cos\theta}\ .$$

It turns out that for $-n \le m \le -1$, the function $P_n^m(\theta)$ is also a polynomial in $\sin\theta$ and $\cos\theta$. This follows from the next lemma.

We claim that, for $1 \le m \le n$, the function $P_n^{-m}(x)$ is a multiple of the function $P_n^m(x)$. Precisely:

Lemma 16.9 *For $1 \le m \le n$ we have*

$$P_n^{-m}(x) = (-1)^m\,\frac{(n-m)!}{(n+m)!}\,P_n^m(x), \quad -1 < x < 1\ .$$

Before proving the lemma, consider the example $m = n = 4$. We have seen above that

$$P_4^4(x) = \frac{8!}{2^4\,4!}\,(1-x^2)^2\ .$$

Also,

$$P_4^{-4} = \frac{1}{2^4\,4!}\,(1-x^2)^2\ .$$

This yields that

$$P_4^{-4} = \frac{1}{8!}\,P_4^4(x)\ ,$$

which agrees with the claim of the lemma.

Proof of Lemma 16.9: One must show that

$$\frac{(n-m)!}{(n+m)!}\,X^m D^{m+n}X^n = D^{-m+n}X^n \quad \text{where} \quad X = x^2-1\ .$$

Essentially, this can be shown by applying Leibniz' rule of differentiation to

$$X^n = (x+1)^n(x-1)^n\ .$$

We have

$$D^{n-m}\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n-m} \left(\begin{array}{c} n-m \\ j \end{array} \right) \Big(D^j(x+1)^n\Big)\Big(D^{n-m-j}(x-1)^n\Big) \ .$$

Here

$$D^j(x+1)^n = \frac{n!}{(n-j)!}\,(x+1)^{n-j}$$

and

$$D^{n-m-j}(x-1)^n = \frac{n!}{(m+j)!}\,(x-1)^{m+j} \ .$$

One obtains:

$$D^{n-m}\Big((x+1)^n(x-1)^n\Big) = \sum_{j=0}^{n-m} c_{mnj}(x+1)^{n-j}(x-1)^{m+j}$$

with

$$c_{mnj} = \frac{(n-m)!n!n!}{j!(n-m-j)!(n-j)!(m+j)!} \ .$$

Similarly,

$$D^{n+m}\Big((x+1)^n(x-1)^n\Big) = \sum_{k=0}^{n+m} \left(\begin{array}{c} n+m \\ k \end{array} \right) \Big(D^k(x+1)^n\Big)\Big(D^{n+m-k}(x-1)^n\Big) \ .$$

Note that the term in the sum is zero unless $m \le k \le n$. Therefore, with $k = m+j$,

$$\begin{aligned} D^{n+m}\Big((x+1)^n(x-1)^n\Big) &= \sum_{k=m}^{n} \left(\begin{array}{c} n+m \\ k \end{array} \right) \Big(D^k(x+1)^n\Big)\Big(D^{n+m-k}(x-1)^n\Big) \\ &= \sum_{j=0}^{n-m} \left(\begin{array}{c} n+m \\ m+j \end{array} \right) \Big(D^{m+j}(x+1)^n\Big)\Big(D^{n-j}(x-1)^n\Big) \\ &= \sum_{j=0}^{n-m} \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!}\,(x+1)^{n-m-j}(x-1)^j \end{aligned}$$

Therefore,

$$X^m D^{n+m}\Big((x{+}1)^n(x{-}1)^n\Big) = \sum_{j=0}^{n-m} \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!}\,(x{+}1)^{n-j}(x{-}1)^{m+j} \ .$$

Finally,

$$\frac{(n-m)!}{(n+m)!} \cdot \frac{(n+m)!n!n!}{(m+j)!(n-j)!(n-m-j)!j!} = \frac{(n-m)!n!n!}{j!(n-m-j)!(n-j)!(m+j)!} = c_{mnj} \; .$$

The lemma is proved. $\diamond$

From the previous two lemmas, it is clear that both functions, $P_n^{-m}(x)$ and $P_n^m(x)$, satisfy the associated Legendre equation (16.21).

The functions $P_n^m(x)$ for $-n \le m \le n$ are introduced in order to define the spherical harmonics,

$$Y_n^m(\theta, \phi) = \gamma_n^m P_n^m(\cos\theta) e^{im\phi}, \quad -\pi < \theta < \pi, \quad 0 < \pi < 2\pi \; ,$$

with

$$\gamma_n^m = \sqrt{\frac{2n+1}{4\pi} \, \frac{(n-m)!}{(n+m)!}}$$

for $-n \le m \le n$ with integers m, n. We can think of $Y_n^m(\theta, \phi)$ as a function defined on the unit sphere in $\mathbb{R}^3$. We will consider spherical harmonic in the next section.

Let us first discuss the functions $P_n^m(x)$ further.

Lemma 16.10 *We claim that*

$$\int_{-1}^{1} P_p^m(x) P_q^m(x) \, dx = 0 \quad \textit{for} \quad p \neq q \; .$$

Here we may assume $|m| \le p < q$.

Proof: The functions

$$f_p(x) = P_p^m(x) \quad \text{and} \quad f_q(x) = P_q^m(x)$$

satisfy

$$((1-x^2) f_p')' + \Big(p(p+1) - \frac{m^2}{1-x^2} \Big) f_p = 0 \tag{16.24}$$

and

$$((1-x^2) f_q')' + \Big(q(q+1) - \frac{m^2}{1-x^2} \Big) f_q = 0 \tag{16.25}$$

Multiply equation (16.24) by f_q and multiply (16.25) by f_p. Take the difference of the resulting equations and obtain that

$$\int_{-1}^{1} f_p(x) f_q(x) \, dx = 0$$

since $p(p+1) \neq q(q+1)$. $\diamond$

Lemma 16.11 *For $|m| \le n$ we have*

$$\int_{-1}^{1} P_n^m(x) P_n^m(x)\, dx = \frac{(n+m)!}{(n-m)!} \, \frac{2}{2n+1} \; .$$

Proof: With $c_n = \frac{1}{2^n n!}$ and $X = x^2 - 1$ we have

$$\begin{aligned}
\int_{-1}^{1} P_n^m(x) P_n^m(x)\, dx &= (-1)^m c_n^2 \int_{-1}^{1} X^m (D^{m+n} X^n)(D^{m+n} X^n)\, dx \\
&= (-1)^n c_n^2 \int_{-1}^{1} D^{m+n} \Big(X^m (D^{m+n} X^n) \Big) X^n \, dx \\
&=: Int
\end{aligned}$$

Apply Leibniz's rule,

$$D^{m+n} \Big(X^m (D^{m+n} X^n) \Big) = \sum_{j=0}^{m+n} \left(\begin{array}{c} m+n \\ j \end{array} \right) (D^{m+n-j} X^m)(D^{m+n+j} X^n) \; .$$

If $n - j > m$ then the first term is zero. If $n - j < m$ then $m + j > n$ and $m + n + j > 2n$. Therefore, the second term is zero. We must only consider the term in the above sum that is obtained for $j = n - m$. The integral in Int becomes

$$\left(\begin{array}{c} n+m \\ n-m \end{array} \right) \int_{-1}^{1} (D^{2m} X^m)(D^{2n} X^n) X^n \, dx \; .$$

Here $D^{2m} X^m = (2m)!$ and $D^{2n} X^n = (2n)!$. Obtain:

$$Int = (-1)^n c_n^2 (2m)!(2n)! \, \frac{(n+m)!}{(n-m)!(2m)!} \, J \qquad (16.26)$$

with

$$\begin{aligned}
J &= \int_{-1}^{1} X^n dx \\
&= (-1)^n \int_{-1}^{1} (1 - x^2)^n dx \\
&= (-1)^n \, \frac{2^{2n+1} (n!)^2}{(2n+1)!}
\end{aligned}$$

In the last equation we have used (16.13).

Together with (16.26):

$$\begin{aligned}
Int &= \frac{(2m)!(2n)!(n+m)! 2^{2n+1} n! n!}{2^{2n} n! n! (n-m)!(2m)!(2n+1)!} \\
&= \frac{(n+m)!}{(n-m)!} \, \frac{2}{2n+1}
\end{aligned}$$

This proves the lemma. $\diamond$

16.8 Spherical Harmonics as Eigenfunctions of the Laplace–Beltrami Operator

Let r, θ, ϕ denote the usual spherical coordinates. Recall the expression for the Laplacian in spherical coordinates,

$$\Delta v = \frac{1}{r^2}\,(r^2 v_r)_r + \frac{1}{r^2}\Big(v_{\theta\theta} + \cot\theta\, v_\theta + \frac{1}{\sin^2\theta}\, v_{\phi\phi}\Big) \; .$$

If $Y(\theta, \phi)$ is a function on the unit sphere, then define the Laplace–Beltrami operator L by

$$LY = Y_{\theta\theta} + \cot\theta\; Y_\theta + \frac{1}{\sin^2\theta}\, Y_{\phi\phi} \; .$$

If $v(r, \theta, \phi) = R(r)Y(\theta, \phi)$ then we can write the Helmholtz' equation

$$\Delta v + k^2 v = 0$$

as

$$\frac{1}{r^2}\,(r^2 R')'Y + \frac{R}{r^2}\, LY + k^2 RY = 0 \; .$$

Equivalently,

$$\frac{1}{R}\,(r^2 R')' + k^2 r^2 + \frac{1}{Y}\, LY = 0 \; .$$

Denoting the separation constant by Q we obtain

$$\begin{aligned} (r^2 R')' + (k^2 r^2 - Q)R &= 0 \\ -LY &= QY \end{aligned}$$

As we have seen, the R–equation is an Euler–equation for $k = 0$ and leads to Bessel's equation for $k > 0$. The equation

$$-LY(\theta, \phi) = QY(\theta, \phi)$$

is an eigenvalue problem for the Laplace–Beltrami operator L. It turns out that we have already determined eigenvalues and eigenfunctions of L.

Theorem 16.2 *Let m, n denote integers with $-n \le m \le n$ and let*

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}\,(1 - x^2)^{m/2}\, D^{m+n}\Big((x^2 - 1)^n\Big)$$

denote the solution of the associated Legendre equation constructed above. Then the function

$$Y(\theta, \phi) = P_n^m(\cos\theta)e^{im\phi}$$

is an eigenfunction of $-L$ to the eigenvalue $Q = Q_n = n(n+1)$.

Proof: If one substitutes $Z(\theta,\phi)=\Theta(\theta)\Phi(\phi)$ into the equation

$$LZ+QZ=0$$

then one obtains

$$\Theta''\Phi+(\cot\theta)\Theta'\Phi+\frac{1}{\sin^2\theta}\,\Theta\Phi''+Q\Theta\Phi=0\ .$$

Equivalently,

$$\frac{1}{\Theta}\Big(\Theta''+(\cot\theta)\Theta'\Big)+\frac{1}{\sin^2\theta}\,\frac{\Phi''}{\Phi}+Q=0\ .$$

If

$$\Phi''+m^2\Phi=0$$

then the equation for Θ becomes

$$\Theta''+(\cot\theta)\Theta'+\Big(Q-\frac{m^2}{\sin^2\theta}\Big)\,\Theta=0\ .$$

The function

$$\Theta(\theta)=P_n^m(\cos\theta)$$

satisfies this equation with $Q=n(n+1)$ and the function $\Phi(\phi)=e^{im\phi}$ satisfies the equation $\Phi''+m^2\Phi=0$. This proves the theorem. $\diamond$

One can show that the $2n+1$ eigenfunctions

$$Z_n^m(\theta,\phi)=P_n^m(\cos\theta)e^{im\phi},\quad -n\le m\le n\ ,$$

to the eigenvalue $Q_n=n(n+1)$ of $-L$ are linearly independent. This follows since these functions are orthogonal with respect to the L_2–inner product on the unit sphere. Therefore, the eigenvalue $Q_n=n(n+1)$ has multiplicity at least $2n+1$.

One can prove that the system of functions

$$Z_n^m(\theta,\phi),\quad -n\le m\le n,\quad n=0,1,\ldots$$

is complete in L_2.

One can prove that the operator $-L$ has the eigenvalues $Q_n=n(n+1)$ where $n=0,1,2,\ldots$. Each eigenvalue $n(n+1)$ has multiplicity $2n+1$. We have constructed an orthonormal basis of eigenfunctions for L,

$$Y_n^m(\theta,\phi)=\gamma_n^m P_n^m(\cos\theta)e^{im\phi},\quad -n\le m\le n\ ,$$

where γ_n^m is defined as above,

$$\gamma_n^m=\sqrt{\frac{2n+1}{4\pi}\,\frac{(n-m)!}{(n+m)!}}\ .$$

Example: The spherical harmonic Y_2^1 is

$$Y_2^1(\theta,\phi) = -\frac{3}{2}\sqrt{\frac{5}{6\pi}}\,\cos\theta\sin\theta\, e^{i\phi}\ .$$

Let us continue the discussion of harmonic functions

$$v(r,\theta,\phi) = R(r)Y(\theta,\phi)\ .$$

With $Q = n(n+1)$, the R–equation is an Euler equation with general solution

$$R(r) = c_1 r^n + \frac{c_2}{r^{n+1}}\ .$$

Assume $Q = n(n+1)$. Substitute

$$Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$$

into

$$Y_{\theta\theta} + \cot\theta\, Y_\theta + \frac{1}{\sin^2\theta}\, Y_{\phi\phi} + n(n+1)Y = 0\ .$$

Divide by Y. Obtain

$$\frac{1}{\Theta}\Big(\Theta'' + \cot\theta\,\Theta'\Big) + \frac{1}{\sin^2\theta}\,\frac{\Phi''}{\Phi} + n(n+1) = 0\ .$$

With $\Phi''/\Phi = -m^2$ obtain

$$\Theta'' + \cot\theta\,\Theta' + \Big(n(n+1) - \frac{m^2}{\sin^2\theta}\Big)\Theta = 0\ .$$

Let $\Theta(\theta) = P(\cos\theta)$,

$$\Theta' = -\sin\theta\, P', \quad \Theta'' = -\cos\theta\, P' + \sin^2\theta P''\ .$$

Obtain

$$\sin^2\theta\, P'' - 2\cos\theta\, P' + \Big(n(n+1) - \frac{m^2}{\sin^2\theta}\Big)P = 0\ .$$

If $x = \cos\theta$, then $\sin^2\theta = 1 - x^2$, thus

$$(1-x^2)P'' - 2xP' + \Big(n(n+1) - \frac{m^2}{1-x^2}\Big)P = 0\ .$$

The functions $P_n^m(x)$ satisfy this equation for $-n \le m \le n$. Then

$$Z_n^m(\theta,\phi) := P_n^m(\cos\theta)e^{im\phi}$$

satisfies

$$-LZ_n^m = n(n+1)Z_n^m\ .$$

We claim that the system of functions

$$Y_n^m(\theta,\phi), \quad -n \le m \le n, \quad n = 0, 1, \ldots$$

is an orthonormal system in $L_2(\mathcal{S})$ where $\mathcal{S}$ is the unit sphere in $\mathbb{R}^3$. Recall that the element of area for the unit sphere $\mathcal{S}$ is

$$dS = \sin\theta \, d\theta d\phi \ .$$

Integration over the sphere $\mathcal{S}$:

$$\int_{\mathcal{S}} Z \, dS = \int_0^{2\pi} \int_0^{\pi} Z(\theta,\phi) \sin\theta \, d\theta d\phi$$

The L_2–inner–product of two functions defined on $\mathcal{S}$ is:

$$(Z_1, Z_2) = \int_{\mathcal{S}} \bar{Z}_1 Z_2 \, dS$$

The L_2–norm is:

$$\|Z\| = (Z, Z)^{1/2}$$

Orthogonality: If $m_1 \neq m_2$ then

$$\int_0^{2\pi} e^{-im_1\phi} e^{im_2\phi} \, d\phi = 0 \ .$$

If $n_1 \neq n_2$ then

$$\begin{aligned} \int_0^{\pi} P_{n_1}^m(\cos\theta) P_{n_2}^m(\cos\theta) \sin\theta \, d\theta &= \int_{-1}^{1} P_{n_1}^m(x) P_{n_2}^m(x) \, dx \\ &= 0 \end{aligned}$$

Therefore, if $m_1 \neq m_2$ or $n_1 \neq n_2$ then

$$\left(Z_{n_1}^{m_1}, Z_{n_2}^{m_2} \right) = 0 \ .$$

Normalization:

$$\begin{aligned} \|Z_n^m\|^2 &= 2\pi \int_{-1}^{1} (P_n^m)^2(x) \, dx \\ &= \frac{4\pi}{2n+1} \frac{(n+m)!}{(n-m)!} \\ &= (1/\gamma_n^m)^2 \end{aligned}$$

Therefore, the functions

$$Y_n^m(\theta,\phi), \quad -n \le m \le n, \quad n = 0, 1, \ldots$$

form an orthonormal system in $L_2(\mathcal{S})$. One can prove that this systen is complete, i.e., if $y = y(\theta,\phi)$ is any function in $L_2(\mathcal{S})$ then the series

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} Y_n^m(\theta,\phi) \quad \text{with} \quad a_{mn} = (Y_n^m, y)$$

converges to y w.r.t. the L_2–norm on $\mathcal{S}$.

16.9 Spherical Harmonics are Restrictions of Harmonic Polynomials to the Unit Sphere

We have

$$Y_n^m(\theta,\phi) = \gamma_n^m \, P_n^m(\cos\theta) e^{im\phi}$$

where

$$P_n^m(x) = \frac{1}{2^n n!} \, (1-x^2)^{m/2} D^{m+n} X^n, \quad X = x^2 - 1 \; .$$

Set

$$v(r,\theta,\phi) = r^n Y_n^m(\theta,\phi)$$

to obtain a function defined in all space.

Claim:

$$\Delta v = 0$$

Proof: We have

$$\Delta v = \frac{1}{r^2} \, (r^2 v_r)_r + \frac{1}{r^2} \, Lv$$

Here

$$L(r^n Y_n^m) = -r^n n(n+1) Y_n^m$$

Also,

$$\begin{aligned} (r^n)_r &= nr^{n-1} \\ r^2 (r^n)_r &= nr^{n+1} \\ (r^2 v_r)_r &= n(n+1) r^n Y_n^m \end{aligned}$$

This shows that $\Delta v = 0$.

Definition: A function

$$p(x,y,z), \quad (x,y,z) \in \mathbb{R}^3 \; ,$$

is called homogeneous of degree n if

$$p(\lambda x, \lambda y, \lambda z) = \lambda^n p(x,y,z)$$

for all $(x,y,z) \in \mathbb{R}^3$ and all $\lambda \in \mathbb{R}$.

Clearly, any function of the form

$$T(x,y,z) = x^{n_1} y^{n_2} z^{n_3}$$

with nonnegative integers n_j is a polynomial which is homogeneous of degree $n = n_1 + n_2 + n_3$. Any sum of such terms, of the same degree n, is a polynomial that is homogeneous of degree n.

Harmonic polynomials that are homogeneous of degree n can be studied with algebraic methods. We want to show:

Theorem 16.3 *If one writes the function*

$$v(r,\theta,\phi) = r^n Y_n^m(\theta,\phi)$$

in Cartesian coordinates x,y,z, then one obtains a polynomial in x,y,z which is homogeneous of degree n.

Proof: It suffices to prove this for $0 \le m \le n$. We have

$$v = cr^n \sin^m\theta \, (D^{m+n} X^n)|_{x=\cos\theta} \, e^{im\phi}$$

Here

$$e^{im\phi} = (\cos\phi + i\sin\phi)^m$$

is a sum of terms

$$\cos^l\phi \, \sin^{m-l}\phi, \quad 0 \le l \le m \; .$$

Also,

$$X^n = (x^2-1)^n = \sum_{j=0}^{n} c_{jn} x^{2n-2j} = x^{2n} + c_{1n} x^{2n-2} + \ldots \; .$$

Therefore, $D^{m+n}X^n$ is a sum of terms

$$x^{n-m-2j}, \quad n-m \ge n-m-2j \ge 0 \; .$$

It follows that v is a sum of terms

$$H = r^n \sin^m\theta \cos^{n-m-2j}\theta \cos^l\phi \sin^{m-l}\phi$$

where

$$0 \le l \le m \quad \text{and} \quad n-m-2j \ge 0 \; .$$

Recall that

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Therefore,

$$
\begin{aligned}
H &= r^n \sin^l \theta \cos^l \phi \sin^{m-l} \theta \sin^{m-l} \phi \cos^{n-m-2j} \theta \\
&= x^l y^{m-l} r^{n-m} \cos^{n-m-2j} \theta \\
&= x^l y^{m-l} r^{2j} z^{n-m-2j} \\
&= x^l y^{m-l} (x^2 + y^2 + z^2)^j z^{n-m-2j}
\end{aligned}
$$

This is a polynomial which is homogeneous of degree n.

2D Analogy. Consider

$$p(x,y) = (x+iy)^n, \quad q(x,y) = (x-iy)^n \ .$$

It is easy to see that $p(x,y)$ and $q(x,y)$ are harmonic polynomials which are homogeneous of degree n. The corresponding restrictions to the unit circle are

$$
\begin{aligned}
p(\cos\phi, \sin\phi) &= e^{in\phi} \\
q(\cos\phi, \sin\phi) &= e^{-in\phi}
\end{aligned}
$$

The functions $e^{\pm in\phi}$ are the 'spherical harmonics' on the unit circle in 2D. They are restrictions to the unit circle of the homogeneous polynomials p and q.

3D Generalization. The functions $Y_n^m(\theta,\phi)$, defined on the unit sphere in $\mathbb{R}^3$, are the generalizations of the functions $e^{\pm in\phi}$ defined on the unit circle in $\mathbb{R}^2$. The expansion in terms of the spherical harmonics $Y_n^m(\theta,\phi)$ generalizes the Fourier expansion in terms of the functions $e^{\pm in\phi}$.

17 Review I

17.1 Scalar ODEs

Solve the IVP

$$\alpha'(t) = \lambda\alpha(t) + h(t), \quad \alpha(0) = \alpha_0 \; .$$

The solution is

$$\alpha(t) = e^{\lambda t}\alpha_0 + \int_0^t e^{\lambda(t-s)}\, h(s)\, ds \; .$$

17.2 A Matrix Equation

Let $A \in \mathbb{C}^{n\times n}$. Let $\phi \in \mathbb{C}^n$ be an eigenvector of A,

$$A\phi = \lambda\phi \; .$$

Consider the IVP

$$u'(t) = Au(t) + \phi, \quad u(0) = 0 \; .$$

Ansatz:

$$u(t) = \alpha(t)\phi$$

where $\alpha(t)$ is a scalar function.

The equation $u'(t) = Au(t) + \phi$ becomes

$$\alpha'(t)\phi = \alpha(t)\lambda\phi + \phi \; ,$$

thus

$$\alpha'(t) = \lambda\alpha(t) + 1, \quad \alpha(0) = 0 \; .$$

Assuming $\lambda \neq 0$, the solution is

$$\alpha(t) = \int_0^t e^{\lambda(t-s)}\, ds = \frac{1}{\lambda}\left(e^{\lambda t} - 1\right) \; .$$

Therefore,

$$u(t) = \frac{1}{\lambda}\left(e^{\lambda t} - 1\right)\phi \; .$$

17.3 An Inhomogeneous Heat Equation

Consider the IBVP

$$\begin{aligned} u_t &= u_{xx} + \sin(3x) \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 0 \end{aligned}$$

Ansatz:

$$u(x,t) = \alpha(t)\sin(3x)$$

Note that $\sin(3x)$ is an eigenfunction of the operator $\partial^2/\partial x^2$.
Obtain

$$\begin{aligned} u_t(x,t) &= \alpha'(t)\sin(3x) \\ u_{xx}(x,t) &= -9\alpha(t)\sin(3x) \end{aligned}$$

The equation $u_t = u_{xx} + \sin(3x)$ leads to

$$\alpha'(t) = -9\alpha(t) + 1 \ .$$

Since $\alpha(0) = 0$ obtain

$$\alpha(t) = \int_0^t e^{-9(t-s)}\, ds = \frac{1}{9}\,(1 - e^{-9t})$$

and

$$u(x,t) = \frac{1}{9}\,(1 - e^{-9t})\sin(3x) \ .$$

17.4 An Inhomogeneous Wave Equation

Consider the IBVP

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + \sin(3x) \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{aligned}$$

Ansatz:

$$u(x,t) = \alpha(t)\sin(3x)$$

Note that $\sin(3x)$ is an eigenfunction of the operator $\partial^2/\partial x^2$.
Obtain the IVP

$$\alpha''(t) + 9c^2\alpha(t) = 1, \quad \alpha(0) = \alpha'(0) = 0 \ .$$

The constant function

$$\alpha_s(t) \equiv \frac{1}{9c^2}$$

is a special solution. The general solution is

$$\alpha(t) = \frac{1}{9c^2} + A\cos(3ct) + B\sin(3ct) \ .$$

The condition $\alpha(0) = 0$ yields that $A = -\frac{1}{9c^2}$ and the condition $\alpha'(0) = 0$ yields that $B = 0$. Therefore,

$$\begin{aligned}
\alpha(t) &= \frac{1}{9c^2}\,(1 - \cos(3ct)) \\
u(x,t) &= \frac{1}{9c^2}\,(1 - \cos(3ct))\sin(3x)
\end{aligned}$$

17.5 An Inhomogeneous Heat Equation with Time–Dependent Source

Consider the IBVP

$$\begin{aligned}
u_t &= u_{xx} + e^{\varepsilon t}\sin(3x) \\
u(0,t) &= u(\pi,t) = 0 \\
u(x,0) &= 0
\end{aligned}$$

Ansatz:

$$u(x,t) = \alpha(t)\sin(3x)$$

Note that $\sin(3x)$ is an eigenfunction of the operator $\partial^2/\partial x^2$.

Proceeding as in Sect. 17.3 leads to the IVP

$$\alpha'(t) = -9\alpha(t) + e^{\varepsilon t}, \quad \alpha(0) = 0\ .$$

The solution is

$$\begin{aligned}
\alpha(t) &= \int_0^t e^{-9(t-s)}e^{\varepsilon s}\,ds \\
&= e^{-9t}\int_0^t e^{(9+\varepsilon)s}\,ds \\
&= \frac{1}{9+\varepsilon}\,(e^{\varepsilon t} - e^{-9t})
\end{aligned}$$

Obtain the solution

$$u(x,t) = \frac{1}{9+\varepsilon}\,(e^{\varepsilon t} - e^{-9t})\sin(3x)\ .$$

If $\varepsilon > 0$, the growing forcing term $e^{\varepsilon t}\sin(3x)$ in the heat equation leads to a growing solution $u(x,t)$.

17.6 The Wave Equation in a Rectangle

Let $D = (0,L) \times (0,H)$.

Consider the IBVP

$$
\begin{aligned}
u_{tt} &= c^2(u_{xx}+u_{yy}) \quad \text{in} \quad D\times[0,\infty) \\
u(x,y,t) &= 0 \quad \text{on} \quad \partial D\times[0,\infty) \\
u(x,y,0) &= 0 \quad \text{in} \quad D \\
u_t(x,y,0) &= \sin(2x)\sin(3y) \quad \text{in} \quad D
\end{aligned}
$$

Note that

$$\phi_{23}(x,y)=\sin(2x)\sin(3y)$$

is an eigenfunction of the Laplace operator Δ:

$$\Delta\phi_{23}=(-4-9)\phi_{23}=-13\phi_{23}\ .$$

Ansatz:

$$u(x,y,t)=h(t)\phi_{23}(x,y)$$

Obtain the IVP

$$h''(t)+13c^2h(t)=0,\quad h(0)=0,\quad h'(0)=1\ .$$

The solution of the IVP is

$$h(t)=\frac{1}{\sqrt{13}c}\,\sin(\sqrt{13}ct)\ .$$

The solution of the wave equation is

$$u(x,t)=\frac{1}{\sqrt{13}c}\,\sin(\sqrt{13}ct)\sin(2x)\sin(3y)\ .$$

17.7 First Order PDEs

Consider the IVP

$$u_t(x,t)+au_x(x,t)=F(x,t),\quad u(x,0)=f(x)\ .$$

Assume that $u(x,t)$ is a solution.

Fix x_0 and set

$$h(t)=u(x_0+at,t)\ .$$

Obtain

$$h'(t)=F(x_0+at,t),\quad h(0)=f(x_0)\ .$$

Therefore,

$$u(x_0+at,t)=h(t)=f(x_0)+\int_0^t F(x_0+as,s)\,ds\ .$$

If $x_0 + at = x$ then $x_0 = x - at$. Therefore, the solution is

$$u(x,t) = h(t) = f(x - at) + \int_0^t F(x - at + as, s)\, ds \ .$$

If $F \equiv 0$ then

$$u(x,t) = f(x - at) \ .$$

If $a > 0$ then the profile $u(x,0) = f(x)$ moves at speed a to the right.

Variable Signal Speed:

Consider the IVP

$$u_t(x,t) + a(x,t)u_x(x,t) = 0, \quad u(x,0) = f(x) \ .$$

Fix x_0 and let

$$\xi'(t) = a(\xi(t),t), \quad \xi(0) = x_0 \ .$$

The characteristic line starting at the point $(x,t) = (x_0, 0)$ consists of the points

$$(\xi(t),t), \quad t \geq 0 \ .$$

If $u(x,t)$ is a solution and

$$h(t) := u(\xi(t),t)$$

then $h(0) = u(x_0, 0) = f(x_0)$ and

$$h'(t) = u_t(\xi(t),t) + \xi'(t)u_x(\xi(t),t) \equiv 0 \ .$$

Therefore, the solution $u(x,t)$ carries the value $f(x_0)$ along the characteristic line starting at $(x_0, 0)$.

Example:

$$u_t(x,t) + tu_x(x,t) = 0, \quad u(x,0) = f(x) \ .$$

The IVP

$$\xi'(t) = t, \quad \xi(0) = x_0$$

has the solution

$$\xi(t) = \frac{t^2}{2} + x_0 \ .$$

Obtain

$$u(\frac{t^2}{2} + x_0, t) = f(x_0) \ ,$$

thus

$$u(x,t) = f(x - \frac{t^2}{2}) \ .$$

17.8 Laplace's Equation in a Rectangle

Let $D = (0, \pi) \times (0, 1)$. Consider the equation $\Delta u = 0$ in D with boundary conditions

$$
\begin{aligned}
u(0, y) &= u(\pi, y) = 0 \quad \text{for} \quad 0 < y < 1 \\
u(x, 0) &= \sin x \quad \text{for} \quad 0 < x < \pi \\
u(x, 1) &= \sin(2x) \quad \text{for} \quad 0 < x < \pi
\end{aligned}
$$

To solve the problem, let $u = u_1 + u_2$ where u_1 and u_2 are harmonic functions satisfying

$$
\begin{aligned}
u_1(0, y) &= u_1(\pi, y) = 0 \quad \text{for} \quad 0 < y < 1 \\
u_1(x, 0) &= \sin x \quad \text{for} \quad 0 < x < \pi \\
u_1(x, 1) &= 0 \quad \text{for} \quad 0 < x < \pi
\end{aligned}
$$

and

$$
\begin{aligned}
u_2(0, y) &= u_1(\pi, y) = 0 \quad \text{for} \quad 0 < y < 1 \\
u_2(x, 0) &= 0 \quad \text{for} \quad 0 < x < \pi \\
u_1(x, 1) &= \sin(2x) \quad \text{for} \quad 0 < x < \pi
\end{aligned}
$$

To solve the problems, let

$$u_1(x, t) = (\sin x) h(y)$$

and

$$u_2(x, t) = (\sin 2x) g(y) \ .$$

Obtain

$$0 = \Delta u_1 = -(\sin x) h(y) + (\sin x) h''(y) \ ,$$

thus

$$h''(y) = h(y) \ .$$

The boundary conditions are

$$h(0) = 1 \quad \text{and} \quad h(1) = 0 \ .$$

Obtain

$$h(y) = a e^y + b e^{-y}$$

where

$$a + b = 1, \quad ae + b/e = 1 \ .$$

The solution of the linear system for a and b is

$$a = -\frac{1}{e^2 - 1}, \quad b = \frac{e^2}{e^2 - 1} \ .$$

Therefore,

$$h(y) = -\frac{e^y}{e^2 - 1} + \frac{e^{2-y}}{e^2 - 1}$$

and

$$u_1(x, y) = (\sin x)\, \frac{e^{2-y} - e^y}{e^2 - 1} \ .$$

Similarly,

$$0 = \Delta u_2 = -4(\sin 2x)g(y) + (\sin 2x)g''(y) \ ,$$

thus

$$g''(y) = 4g(y) \ .$$

The boundary conditions are

$$g(0) = 0 \quad \text{and} \quad g(1) = 1 \ .$$

Obtain

$$g(y) = ae^{2y} + be^{-2y}$$

where

$$a + b = 0, \quad ae^2 + b/e^2 = 0 \ .$$

The solution of the linear system for a and b is

$$a = \frac{1}{e^2 - 1/e^2}, \quad b = \frac{-1}{e^2 - 1/e^2} \ .$$

Therefore,

$$g(y) = \frac{e^{2y}}{e^2 - 1/e^2} - \frac{e^{-2y}}{e^2 - 1/e^2}$$

and

$$u_2(x, y) = (\sin 2x) \left(\frac{e^{2y}}{e^2 - 1/e^2} - \frac{e^{-2y}}{e^2 - 1/e^2} \right) \ .$$

The solution of the given problem is

$$u(x, y) = u_1(x, y) + u_2(x, y) \ .$$

17.9 Laplace's Equation in a Disk

Let

$$B_5 = \{(x,y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 < 25\}$$

denote the disk of radius 5, centered at the origin. We want to determine a function $u(r,\phi)$ with $\Delta u = 0$ in B_5 and

$$u(5,\phi) = \cos(2\phi) + 7\sin(3\phi) \quad \text{for} \quad \phi \in \mathbb{R} \ .$$

We know that the functions

$$r^n \Big(A\cos(n\phi) + B\sin(n\phi) \Big)$$

are harmonic. Determine

$$u_1(r,\phi) = Ar^2\cos(2\phi)$$

with

$$u_1(5,\phi) = \cos(2\phi) \ .$$

This requires $A = \frac{1}{25}$, thus

$$u_1(r,\phi) = (r/5)^2\cos(2\phi) \ .$$

Determine

$$u_2(r,\phi) = Br^3\sin(3\phi)$$

with

$$u_2(5,\phi) = 7\sin(3\phi) \ .$$

This requires $B = \frac{7}{125}$, thus

$$u_2(r,\phi) = 7(r/5)^3\sin(3\phi) \ .$$

Together, the solution of the given problem is

$$u(r,\phi) = (r/5)^2\cos(2\phi) + 7(r/5)^3\sin(3\phi) \ .$$

17.10 Laplace's Equation in an Annulus

Let

$$A = \{(x,y) \in \mathbb{R}^2 \ : \ 1 < x^2 + y^2 < 25\} \ .$$

We want to determine a function $u(r,\phi)$ for which $\Delta u = 0$ in A and

$$\begin{aligned} u(1,\phi) &= 1 + \sin(2\phi) \quad \text{for} \quad \phi \in \mathbb{R} \ , \\ u(5,\phi) &= 0 \quad \text{for} \quad \phi \in \mathbb{R} \ . \end{aligned}$$

Determine harmonic functions u_1 and u_2 with

$$\begin{aligned} u_1(1,\phi) &= 1 \\ u_1(5,\phi) &= 0 \end{aligned}$$

and

$$\begin{aligned} u_2(1,\phi) &= \sin(2\pi) \\ u_2(5,\phi) &= 0 \end{aligned}$$

Then $u = u_1 + u_2$.

Obtain

$$u_1(r,\phi) = A + B\ln r \ .$$

The boundary conditions require that

$$1 = u_1(1,\phi) = A$$

and

$$0 = u_1(5,\phi) = A + B\ln 5 \ .$$

Therefore,

$$u_1(r,\phi) = 1 - \frac{\ln r}{\ln 5} \ .$$

For $u_2(r,\phi)$ obtain that

$$u_2(r,\phi) = Ar^2\sin(2\phi) + B\frac{1}{r^2}\sin(2\phi) \ .$$

The boundary conditions require that

$$A + B = 1 \quad \text{and} \quad 25A + \frac{B}{25} = 0 \ .$$

Obtain that

$$A = \frac{-1}{(25)^2 - 1}, \quad B = \frac{(25)^2}{(25)^2 - 1} \ ,$$

thus

$$u_2(r,\phi) = \frac{1}{(25^2) - 1}\left(\frac{(25)^2}{r^2} - r^2\right)\sin(2\phi) \ .$$

The solution of the given problem is

$$u(r,\phi) = u_1(r,\phi) + u_2(r,\phi) \ .$$

17.11 The Heat Equation in a Rectangle

Let

$$D = (0,\pi) \times (0,\pi) \ .$$

Consider the IBVP

$$\begin{aligned} u_t &= \Delta u + \sin x \sin(2y) \quad \text{for} \quad D \times [0,\infty) \\ u &= 0 \quad \text{for} \quad (x,y,t) \in \partial D \times [0,\infty) \\ u &= 0 \quad \text{for} \quad t = 0 \end{aligned}$$

Note: The function

$$\phi(x,y) = \sin x \sin(2y)$$

satisfies the boundary condition and is an eigenfunction of Δ.

Ansatz:

$$u(x,y,t) = \alpha(t) \sin x \sin(2y) \ .$$

Obtain

$$\alpha'(t) = -5\alpha(t) + 1, \quad \alpha(0) = 0 \ .$$

$$u(x,y,t) = \frac{1}{5}\,(1 - e^{-5t}) \sin x \sin(2y) \ .$$

17.12 The Heat Equation with Inhomogeneous Term and Initial Condition

Consider the equation

$$u_t = u_{xx} + \sin x \quad \text{for} \quad 0 < x < \pi, \quad t \geq 0$$

with boundary condition

$$u(0,t) = u(\pi,t) = 0$$

and initial condition

$$u(x,0) = \sin(2x) \ .$$

Solution:

$$u(x,t) = e^{-4t} \sin(2x) + (1 - e^{-t}) \sin(2x) \ .$$

18 Review II

18.1 2D Laplace Equation in Polar Coordinates

$$\Delta v = v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\phi\phi} = 0$$

Let

$$v = R(r)\Phi(\phi) \ .$$

Obtain

$$\Phi(\phi) = e^{im\phi}, \quad m \in \mathbb{Z} \ .$$

The equation

$$r^2 R'' + rR' - m^2 R = 0$$

has the solution

$$R(r) = r^{|m|} \ .$$

Obtain the harmonic functions:

$$v(r, \phi) = r^{|m|} e^{im\phi}, \quad m \in \mathbb{Z} \ .$$

18.2 2D Wave Equation in Polar Coordinates

Consider the wave equation

$$u_{tt} = c^2 \Delta u \ .$$

Let

$$u(r, \phi, t) = v(r, \phi) T(t) \ .$$

Obtain

$$T''(t) + \omega^2 T(t) = 0, \quad \omega = kc \ ,$$

and

$$\Delta v + k^2 v = 0 \ .$$

Let

$$v(r, \phi) = R(r)\Phi(\phi) \ .$$

Obtain

$$\Phi_m(\phi) = e^{im\phi}, \quad m \in \mathbb{Z} \ ,$$

and

$$r^2R'' + rR' + (r^2k^2 - m^2)R = 0 \ .$$

Introduce the function $f(x)$ by

$$f(rk) = R(r) \ .$$

Obtain Bessel's differential equation for $f(x)$:

$$x^2 f''(x) + xf'(x) + (x^2 - m^2)f(x) = 0 \ .$$

Bessel functions of the first kind are solutions which are regular at $x = 0$. These are the Bessel functions

$$J_m(x) = \Big(\frac{x}{2}\Big)^m \sum_0^\infty \frac{(-1)^j}{j!\Gamma(j+m+1)}\Big(\frac{x}{2}\Big)^{2j} \quad \text{for} \quad m \in Z \ .$$

(Note that $J_{-m}(x) = (-1^m J_m(x)$.) Obtain

$$R(r) = J_m(rk) \quad \text{for} \quad m \in \mathbb{Z}, \quad k > 0 \ .$$

Solutions of the 2D–Helmholtz equation:

$$v(r, \phi) = J_m(rk)e^{im\phi}, \quad m \in \mathbb{Z}, \quad k > 0 \ .$$

Solutions of the 2D–Wave equation:

$$u(r, \phi, t) = J_m(rk)e^{im\phi}\Big(A\cos(\omega t) + B\sin(\omega t)\Big) \quad \text{where} \quad \omega = ck \ .$$

The parameter k becomes discrete,

$$k = x_{mj}/a \quad \text{for} \quad m \in \mathbb{Z}, \quad j \in N \ ,$$

if the boundary condition

$$u(a, \phi, t) = 0$$

is required.

18.3 3D Laplace Equation in Spherical Coordinates

$$\Delta v = \frac{1}{r^2}(r^2 v_r)_r + \frac{1}{r^2 \sin\theta}\,(\sin\theta v_\theta)_\theta + \frac{1}{r^2\sin^2\theta}\, v_{\phi\phi} = 0$$

Let

$$v(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) \ .$$

Obtain

$$\Phi(\phi) = e^{im\phi}, \quad m \in \mathbb{Z} \ .$$

Let

$$\Theta(\theta) = P(\cos\theta), \quad x = \cos\theta \ .$$

Obtain the associate Legendre equation

$$(1-x^2)P''(x) - 2xP'(x) + \Big(Q - \frac{m^2}{1-x^2}\Big)P(x) = 0$$

and

$$(r^2 R')' - QR = 0 \ .$$

The equation for $P(x)$ yields that

$$Q = n(n+1), \quad n = 0, 1, 2 \dots$$

It is remarkable that the eigenvalues $Q = n(n+1)$ do not depend on m. Solutions for which $P(\pm 1)$ is finite are the Legendre functions

$$P_n^m(x) = \frac{(-1)^m}{2^n n!}\,(1-x^2)^{m/2} D^{n+m}((x^2-1)^n) \quad \text{for} \quad -n \le m \le n \ .$$

(Note that $P_n^{-m}(x)$ is a multiple of $P_n^m(x)$.)

Obtain the harmonic functions:

$$v(r,\theta,\phi) = r^n P_n^m(\cos\theta) e^{im\phi} \quad \text{for} \quad -n \le m \le n, \quad n = 0, 1, \dots$$

Remark: In 2D the R–equation depends on the parameter $m \in \mathbb{Z}$, where $\Phi'' + m^2\Phi = 0$. In 3D the R–equation depends on the parameter $n \in \mathbb{N}_0$, which appears the Θ–equation.

18.4 3D Wave Equation in Spherical Coordinates

Let

$$u(r,\theta,\phi,t) = v(r,\theta,\phi)T(t) \ .$$

Obtain

$$T''(t) + \omega^2 T(t) = 0, \quad \omega = kc \ ,$$

and

$$\Delta v + k^2 v = 0 \ .$$

Let

$$v(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \ .$$

The equations for Θ and for Φ are the same as for the Laplace equation: Obtain

$$\Theta(\theta) = P(\cos\theta), \quad x = \cos\theta \ ,$$

and

$$P_n^m(x) = \frac{(-1)^m}{2^n n!} (1-x^2)^{m/2} D^{n+m}((x^2-1)^n) \quad \text{for} \quad -n \le m \le n \ .$$

For $R(r)$ obtain

$$(r^2R')' + \Big(r^2k^2 - n(n+1)\Big)R = 0 \ .$$

Let

$$y(rk) = \sqrt{r}\, R(r), \quad x = rk \ .$$

Obtain the Bessel equation

$$x^2y''(x) + xy'(x) + \Big(x^2 - (n+\frac{1}{2})^2\Big)y(x) = 0 \ .$$

This yields that

$$y(x) = J_{n+1/2}(x), \quad n = 0, 1, 2, \ldots$$

Obtain

$$R(r) = \frac{1}{\sqrt{r}}\, J_{n+1/2}(rk) \ .$$

Solutions of the 3D–Helmholtz equation:

$$v(r,\theta,\phi) = \frac{1}{\sqrt{r}}\, J_{n+1/2}(rk)P_n^m(\cos\theta)e^{im\phi}, \quad -n \le m \le n, \quad k > 0 \ .$$

Solutions of the 3D–Wave equation:

$$u(r,\theta,\phi,t) = \frac{1}{\sqrt{r}}\, J_{n+1/2}(rk)P_n^m(\cos\theta)e^{im\phi}\Big(A\cos(\omega t) + B\sin(\omega t)\Big) \quad \text{where} \quad \omega = ck \ .$$