Complex Variables Math 313, Spring 2020

Jens Lorenz

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Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 Details 1: First Introduction of the Exponential Function

Recall the Taylor series:

$$\begin{aligned} e^x = \exp(x) &= \sum_{j=0}^{\infty} \frac{x^j}{j!} \\ \cos x &= \sum_{j=0}^{\infty} (-1)^j \, \frac{x^{2j}}{(2j)!} \\ \sin x &= \sum_{j=0}^{\infty} (-1)^{j+1} \, \frac{x^{2j+1}}{(2j+1)!} \end{aligned}$$

We will learn later that these series can be used to define the corresponding functions in the complex plane. For example,

$$e^z = \exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \ .$$

One can then prove that

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \ .$$

Substituting iy for x in the power series for e^x one obtains that

$$\begin{aligned} e^{iy} &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + \dots \\ &= \cos y + i \sin y \end{aligned}$$

This is Euler's important formula:

$$e^{iy} = \cos y + i \sin y \ .$$

Also, for $z = x + iy$:

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) \ .$$

2 Details 2: The Polar Form of a Complex Number

Let $z = x + iy \in \mathbb{C}, z \neq 0$. The absolute value of z is

$$r = |z| = \sqrt{x^2 + y^2}$$

and the number $z/|z|$ lies on the unit circle. Therefore, there is a unique ϕ with

$$\frac{z}{|z|} = \cos\phi + i\sin\phi \quad \text{and} \quad -\pi < \phi \leq \pi \ .$$

One obtains the polar representation of z:

$$z = r(\cos\phi + i\sin\phi) \quad \text{with} \quad r = |z| \ . \tag{2.1}$$

The real number ϕ with (2.1) and $-\pi < \phi \leq \pi$ is called the principle value of the argument of z,

$$\phi = \operatorname{Arg} z \ .$$

For all numbers

$$\phi = \arg z = \operatorname{Arg} z + 2\pi k, \quad k \in \mathbb{Z} \ ,$$

the polar representation (2.1) is valid. One considers the function $\arg z$ as a multivalued function.

3 Details 3: Powers and Roots

Let $z = re^{i\phi} \neq 0$ and let $n \in \mathbb{N}$. Then

$$\begin{aligned} z^n &= r^n e^{in\phi} \\ z^{-n} &= r^{-n} e^{-in\phi} \end{aligned}$$

Here

$$r = |z| \quad \text{and} \quad \phi = \arg z = \operatorname{Arg} z + 2\pi k, \quad k \in \mathbb{Z} \ .$$

Since

$$e^{2\pi kn} = 1 \quad \text{and} \quad e^{-2\pi kn} = 1$$

the value of z^n and z^{-n} does not depend an the choice of k when $\phi = \arg z$ is determined.

Now let $z = re^{i\phi} \neq 0$ with

$$\phi = \phi_0 + 2\pi k, \quad \phi_0 = \operatorname{Arg} z$$

and consider

$$\sqrt{z} = z^{1/2} = \sqrt{r}\, e^{i\pi/2} = \sqrt{r}\, e^{i\pi_0/2} e^{i\pi k} \ .$$

Here $e^{i\pi k} = 1$ if k is an even integer and $e^{i\pi k} = -1$ if k is an odd integer. The root function is double valued:

$$\sqrt{z} = \pm\sqrt{r}\, e^{i\phi_0/2} \ .$$

Roots of Unity: For $z = 1 = e^{2\pi i k}, k \in \mathbb{Z}$, we have

$$\operatorname{Arg} 1 = 0, \quad \arg 1 = 2\pi k$$

and

$$1^{1/n} = e^{2\pi i k/n}, \quad k \in \mathbb{Z} \ .$$

The number

$$\omega_n = e^{2\pi i/n}$$

lies on the unit circle and the n distinct numbers

$$\omega_n,\, \omega_n^2, \ldots, \omega_n^{n-1},\, \omega_n^n = 1$$

are the n distinct n-th roots of 1.

If $z = re^{i\phi_0} e^{2\pi i k}$ with $k \in \mathbb{Z}$ then the n numbers

$$\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\phi_0/n}\, e^{2\pi i k/n}, \quad k = 1, 2, \ldots, n$$

are the n distinct n-th roots of z.

4 Details 4: The Exponential Function and Logarithms; General Powers

The Real Case:

From calculus recall the real exponential function

$$\exp(x) = e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}, \quad x \in \mathbb{R} \ ,$$

which maps $\mathbb{R}$ bijectively onto $(0, \infty)$. We also recall that

$$\exp(0) = 1, \quad \exp'(x) = \exp(x), \quad \exp(a+b) = \exp(a)\exp(b) \ .$$

The function

$$\ln r = \int_1^r \frac{ds}{s}, \quad r > 0 \ ,$$

satisfies

$$\ln 1 = 0 \quad \text{and} \quad \frac{d}{dr} \ln r = \frac{1}{r} \quad \text{for} \quad r > 0 \ .$$

Therefore, if we set $f(x) = \ln(\exp x)$ for $x \in \mathbb{R}$ then

$$f(0) = 0, \quad f'(x) = \frac{\exp x}{\exp x} = x \ ,$$

which yields that

$$\ln(\exp x) = x \quad \text{for all} \quad x \in \mathbb{R} \ .$$

The function

$$\ln : (0, \infty) \to \mathbb{R}$$

is the inverse function of

$$\exp : \mathbb{R} \to (0, \infty) \ .$$

The Complex Case:

For $z = x + iy \in \mathbb{C}$ we have

$$\exp(z) = e^x e^{iy} = \sum_{j=0}^{\infty} \frac{z^j}{j!} \ .$$

The function

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

is onto, but not 1-1 since

$$\exp(z + 2\pi i k) \equiv \exp z \quad \text{for all} \quad k \in \mathbb{Z} \ .$$

Let

$$S_{(-\pi,\pi]} = \{z = x + iy \ : \ x \in \mathbb{R}, -\pi < y \le \pi\} \ .$$

Then the map $z \to \exp z$ from $S_{(-\pi,\pi]}$ to $\mathbb{C} \setminus \{0\}$ is 1-1 and onto. Its inverse

$$\mathrm{Log} \ : \mathbb{C} \setminus \{0\} \to S_{(-\pi,\pi]}$$

is the principle branch of the logarithm. For $w \in \mathbb{C} \setminus \{0\}$ we have

$$w = |w|e^{i\phi} = e^{\ln|w| + i\phi} \quad \text{with} \quad -\pi < \phi = \mathrm{Arg}\, w \le \pi$$

and

$$\mathrm{Log}\, w = \ln|w| + i\mathrm{Arg}\, w \ .$$

One obtains that

$$\exp(\mathrm{Log}\, w) = w \quad \text{for all} \quad w \in \mathbb{C} \setminus \{0\} \ .$$

The function

$$\log w = \ln|w| + i\mathrm{Arg}\, w + 2\pi i k, \quad k \in \mathbb{Z} \ ,$$

is multivalued. We have

$$\log : \mathbb{C} \setminus \{0\} \to \mathbb{C} \ ,$$

and

$$\exp(\log w) = w \quad \text{for all} \quad w \in \mathbb{C} \setminus \{0\} \quad \text{and for all} \quad k \in \mathbb{Z} \ .$$

General Powers:

Let $w \in \mathbb{C} \setminus \{0\}, a \in \mathbb{C}$. We have

$$w = e^{\mathrm{Log}\, w} = e^{\ln|w| + i\phi} \quad \text{with} \quad -\pi < \phi = \mathrm{Arg}\, w \le \pi \ .$$

One defines

$$w^a = e^{a \mathrm{Log}\, w} \ .$$

This is the principle branch of the power function $w \to w^a$ from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$. In general,

$$w = e^{\log w} = e^{\ln|w| + i\phi + 2\pi i k} \quad \text{with} \quad -\pi < \phi = \mathrm{Arg}\, w \le \pi \quad \text{and} \quad k \in \mathbb{Z} \ .$$

Then

$$w \to w^a = e^{a \log w} = e^{a(\ln|w| + i\phi + 2\pi i k)}$$

is the multivalued power function $w \to w^a$.

5 Details 5: Real and Complex Differentiability; the Cauchy–Riemann Equations

Definition: Let $\Omega \subset \mathbb{C}$ be an open set and let $f(z)$ be a complex function defined on Ω. Let $z_0 \in \Omega$. The function $f(z)$ is complex differential at z_0 if the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \ .$$

Relation to Real Differentiability: Let

$$z = x + iy, \quad f(z) = u(x,y) + iv(x,y) \ .$$

The function

$$(x,y) \to \Big(u(x,y), v(x,y)\Big) =: F(x,y)$$

is the function corresponding to $f(z)$ mapping a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$.

If $F(x,y)$ is real differentiable at a point (x,y) then its Jacobian is

$$F'(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix} \ .$$

One can show: If $f(z)$ is complex differentiable at a point $z_0 \in \Omega$ then $F(x,y)$ is real differentiable at the corresponding point (x_0, y_0) and at this point the Cauchy–Riemann equations hold:

$$u_x(x_0,y_0) = v_y(x_0,y_0), \quad u_y(x_0,y_0) = -v_x(x_0,y_0) \ .$$

Conversely, if $F(x,y)$ is continuously real differentiable in the set corresponding to Ω and if the Cauchy–Riemann equations hold in this set,

$$u_x(x,y) = v_y(x,y), \quad u_y(x,y) = -v_x(x,y) \ ,$$

then $f(z)$ is complex differentiable in Ω and

$$\begin{aligned} f'(z) &= u_x(x,y) + iv_x(x,y) \\ &= v_y(x,y) - iu_y(x,y) \end{aligned}$$

The Cauchy-Riemann Equations in Polar Coordinates: Assume that $u(x,y), v(x,y)$ satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x \ .$$

Let

$$\begin{aligned} \tilde{u}(r,\phi) &= u(r\cos\phi, r\sin\phi) \\ \tilde{v}(r,\phi) &= v(r\cos\phi, r\sin\phi) \end{aligned}$$

In the following, we abbreviate $c = \cos\phi, s = \sin\phi$. We have

$$\begin{aligned}\tilde{u}_r &= u_x c + u_y s \\ \tilde{u}_\phi &= r(-u_x s + u_y c)\end{aligned}$$

and, in the same way,

$$\begin{aligned}\tilde{v}_r &= v_x c + v_y s \\ \tilde{v}_\phi &= r(-v_x s + v_y c)\end{aligned}$$

We write these equations in matrix form:

$$\begin{pmatrix} \tilde{v}_r \\ \frac{1}{r}\tilde{v}_\phi \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} .$$

Therefore,

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \tilde{v}_r \\ \frac{1}{r}\tilde{v}_\phi \end{pmatrix} .$$

Thus

$$v_x = c\tilde{v}_r - \frac{s}{r}\tilde{v}_\phi, \quad v_y = s\tilde{v}_r + \frac{c}{r}\tilde{v}_\phi \ . \tag{5.1}$$

In the same way one obtains

$$u_x = c\tilde{u}_r - \frac{s}{r}\tilde{u}_\phi, \quad u_y = s\tilde{u}_r + \frac{c}{r}\tilde{u}_\phi \ . \tag{5.2}$$

Remark: One can obtain the formula (5.1) also from

$$v(x,y) = \tilde{v}\Big(\sqrt{x^2+y^2}, \arctan(y/x)\Big) \ .$$

Using the Cauchy–Riemann equations $u_x = v_y, u_y = -v_x$ and (5.1), one obtains that

$$\begin{aligned}\tilde{u}_r &= u_x c + u_y s \\ &= v_y c - v_x s \\ &= sc\tilde{v}_r + \frac{c^2}{r}\tilde{v}_\phi - sc\tilde{v}_r + \frac{s^2}{r}\tilde{v}_\phi \\ &= \frac{1}{r}\tilde{v}_\phi\end{aligned}$$

and

$$\begin{aligned}\tilde{u}_\phi &= r(-u_x s + u_y c) \\ &= -rsv_y - rcv_x \\ &= -rs^2\tilde{v}_r - sc\tilde{v}_\phi - rc^2\tilde{v}_r + sc\tilde{v}_\phi \\ &= -r\tilde{v}_r\end{aligned}$$

The equations

$$\tilde{u}_r = \frac{1}{r}\,\tilde{v}_\phi, \quad \tilde{v}_r = -\frac{1}{r}\,\tilde{u}_\phi$$

are the Cauchy–Riemann equations in polar coordiates. It is standard the drop the $\tilde{}$–notation,

$$u_r = \frac{1}{r}\,v_\phi, \quad v_r = -\frac{1}{r}\,u_\phi \ .$$

A Formula for $f'(z)$ in Polar Coordinates: Using (5.1) and (5.2) we have:

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= c\tilde{u}_r - \frac{s}{r}\tilde{u}_\phi + i(c\tilde{v}_r - \frac{s}{r}\tilde{v}_\phi) \\ &= c\tilde{u}_r + s\tilde{v}_r + i(c\tilde{v}_r - s\tilde{u}_r) \\ &= (c - is)(\tilde{u}_r + i\tilde{v}_r) \end{aligned}$$

Thus, if $z = r(\cos\phi + i\sin\phi)$ and if $f(z) = \tilde{u}(r,\phi) + i\tilde{v}(r,\phi)$ expresses the complex differentiable function $f(z)$ in polar coordinates, then

$$f'(z) = (\cos\phi - i\sin\phi)\Big(\tilde{u}_r(r,\phi) + i\tilde{v}_r(r,\phi)\Big) \ . \tag{5.3}$$

6 Details 6: The Complex Derivative of $\operatorname{Log} z$

Let

$$\Omega = \{z = x + iy \ : \ x > 0, y \in \mathbb{R}\}$$

denote the open right half–plane. For $z = x + iy \in \Omega$ we have

$$z = |z| e^{i \operatorname{Arg} z} = |z| e^{i\phi}$$

with

$$\tan \phi = \frac{y}{x}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \ .$$

This holds since

$$z = x + iy = |z|(\cos\phi + i \sin\phi) \ .$$

One obtains that

$$z = e^{\ln|z| + i \arctan(y/x)} \ ,$$

thus

$$f(z) := \operatorname{Log} z = \frac{1}{2}\,(x^2 + y^2) + i \arctan(y/x) \ .$$

Writing

$$f(x + iy) = u(x,y) + iv(x,y)$$

obtain that

$$\begin{aligned} u &= \frac{1}{2} \ln(x^2 + y^2) \\ v &= \arctan(y/x) \end{aligned}$$

and

$$\begin{aligned} u_x &= \frac{x}{x^2 + y^2} \\ u_y &= \frac{y}{x^2 + y^2} \\ v_x &= -\frac{y}{x^2} \frac{1}{1 + (y/x)^2} \\ &= -\frac{y}{x^2 + y^2} \\ v_y &= \frac{1}{x} \frac{1}{1 + (y/x)^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

The Cauchy–Riemann equations hold and

$$\begin{aligned} f'(z) &= \text{Log}'(z) \\ &= u_x(x,y) + iv_x(x,y) \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{1}{x+iy} \\ &= \frac{1}{z} \end{aligned}$$

Derivation in Polar Coordinates: We have

$$z = re^{i\phi} = e^{\ln r + i\phi} \ ,$$

thus

$$\text{Log}\,(z) = \ln r + i\phi = \tilde{u}(r,\phi) + i\tilde{v}(r,\phi)$$

with

$$\tilde{u}(r,\phi) = \frac{1}{r}, \quad \tilde{v}(r,\phi) = \phi \ .$$

Clearly, dropping the ˜–notation,

$$u_r = \frac{1}{r}, \quad u_\phi = 0, \quad v_r = 0, \quad v_\phi = 1 \ ,$$

and the Cauchy–Riemann equations

$$u_r = \frac{1}{r} v_\phi, \quad v_r = -\frac{1}{r} u_\phi$$

hold.

Also, using (5.3) and $c = \cos\phi, s = \sin\phi$,

$$\begin{aligned} \text{Log}'(z) &= (c - is)\frac{1}{r} \\ &= \frac{cr - isr}{r^2} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{1}{x+iy} \\ &= \frac{1}{z} \end{aligned}$$

7 Details 7: Parameterized Curves in the Complex Plane

Let $z : [a,b] \to \mathbb{C}$ denote a C^1–map. This means that

$$z(t) = x(t) + iy(t), \quad a \le t \le b ,$$

where $x, y : [a,b] \to \mathbb{R}$ are two functions which have continuous derivatives $x'(t), y'(t)$ for $a \le t \le b$.

Intuitively, we think of the set

$$\{z(t) \ : \ a \le t \le b\}$$

as a curve in $\mathbb{C}$ parameterized by the function $z(t)$.

A curve has different parameterizations. For example, the functions

$$\begin{aligned} z(t) &= e^{it}, \quad 0 \le t \le 2\pi \\ \tilde{z}(t) &= e^{2is}, \quad 0 \le s \le \pi \end{aligned}$$

both parameterize the unit circle.

It is not trivial to define the notion of a curve precisely. One can proceed as follows: Let

$$z : [a,b] \to \mathbb{C} \quad \text{and} \quad \tilde{z} : [c,d] \to \mathbb{C}$$

denote two C^1–maps. Call these two maps equivalent if there exists a C^1–map

$$\phi : [a,b] \to [c,d]$$

(a parameter transformation) which is one–to–one and onto and satisfies

$$\tilde{z}(\phi(t)) = z(t) \quad \text{and} \quad \phi'(t) > 0 \quad \text{for} \quad a \le t \le b .$$

Then an equivalence class of C^1 maps $z = z(t)$ defined on bounded closed intervals is a C^1 curve.

More intuitively, we often identify the image set

$$\Gamma = \{z(t) \ : \ a \le t \le b\}$$

with a C^1 curve if $z = z(t)$ is a C^1 function.

It is also convenient to work with continuous curves which are only piecewise C^1. For example, the boundary of a triangle, parameterized in the counterclockwise direction, can be thought of as a continuous curve which is piecewise C^1.

Length of a Curve: Let

$$z(t) = x(t) + iy(t), \quad a \le t \le b ,$$

parameterize the C^1 curve Γ. Let

$$h = \frac{b-a}{n}, \quad t_j = a + jh \quad \text{for} \quad j = 0, 1, \ldots, n$$

where n is a positive integer. The points $z_j = z(t_j)$ lie a Γ and, for large n,

$$length(\Gamma) \sim \sum_{j=1}^{n} |z_j - z_{j-1}| \; .$$

Here

$$\begin{aligned} |z_j - z_{j-1}|^2 &= \Big(x(t_j) - x(t_{j-1})\Big)^2 + \Big(y(t_j) - y(t_{j-1})\Big)^2 \\ &\sim h^2(x'(t_j))^2 + h^2(y'(t_j))^2 \\ &= h^2|z'(t_j)|^2 \end{aligned}$$

This yields that

$$|z_j - z_{j-1}| \sim h|z'(t_j)|$$

and

$$length(\Gamma) \sim h \sum_{j=1}^{n} |z'(t_j)| \; .$$

For $n \to \infty$ one obtains that

$$length(\Gamma) = \int_a^b |z'(t)| \, dt$$

if $z(t), a \leq t \leq b$, is a C^1 parameterization of Γ.

8 Details 8: Integrals Along Parameterized Curves

Let $g : [a, b] \to \mathbb{C}$ denote a continuous function,

$$g(t) = g_1(t) + ig_2(t), \quad a \le t \le b \ ,$$

where $g_1(t)$ and $g_2(t)$ are real valued, continuous functions defined for $a \le t \le b$.

One defines

$$\int_a^b g(t)\,dt = \int_a^b g_1(t)\,dt + i\int_a^b g_2(t)\,dt \ .$$

Let $z(t), a \le t \le b$, parameterize the curve Γ in the complex plane and let $f(z)$ denote a continuous complex valued function defined on the set of points

$$\{z(t) \ : \ a \le t \le b\} \ .$$

Thus, $f(z)$ is continuous on the curve Γ. One defines the integral of f along Γ by

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt \ .$$

Formally, we have made the substitution $z = z(t), dz = z'(t)\,dt$.

Homework: Assume that $z(t), a \le t \le b$, and $\tilde{z}(s), c \le s \le d$, parameterize the same curve Γ. Prove that the two curve integrals

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$

and

$$\int_\Gamma f(\tilde{z})\,d\tilde{z} = \int_c^d f(\tilde{z}(s))\tilde{z}'(s)\,ds$$

are equal.

The following is a plausible generalization of the fundamental theorem of calculus:

Theorem 8.1 *Assume that the functions $F(z)$ and $f(z)$ are holomorphic in the region $\Omega \subset \mathbb{C}$ and that $F'(z) = f(z)$ in Ω. If Γ is a curve in Ω from point P to point Q, then*

$$\int_\Gamma f(z)\,dz = \int_\Gamma F'(z)\,dz = F(Q) - F(P) \ .$$

In particular, if Γ is a closed curve and if the holomorphic function $f(z)$ has an anti–derivative $F(z)$ in Ω, then

$$\int_\Gamma f(z)\,dz = 0 \ .$$

9 Details 9: Laurent Series

An expression of the form

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

is called a Laurent series centered at z_0.

One calls the series convergent at z if the limits

$$L_1 = \lim_{n\to\infty} \sum_{j=0}^{n} a_j(z-z_0)^j$$

and

$$L_2 = \lim_{n\to\infty} \sum_{j=-n}^{-1} a_j(z-z_0)^j$$

exist. If these limits exist then

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j = L_1 + L_2$$

is the value of the Laurent series at z.

Laurent series are often studied in annuli. If $0 \le r_1 < r_2 \le \infty$ then the set

$$A(z_0, r_1, r_2) = \{z \in \mathbb{C} \ : \ r_1 < |z-z_0| < r_2\}$$

is the annulus centered at z_0 with inner radius r_1 and outer radius r_2.

For simplicity, we will assume that $z_0 = 0$. The following should help you with Problems 5 and 6 of Homework 4.

It will be important to use the geometric sum formula

$$\frac{1}{1-\varepsilon} = \sum_{j=0}^{\infty} \varepsilon^j \quad \text{for} \quad |\varepsilon| < 1 \ .$$

Consider the function

$$f(z) = \frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z}$$

which has poles at $z_1 = 1$ and $z_2 = 2$.

Case 1: Write $f(z)$ as a power series for $|z| < 1$. By the geometric sum formula we have for $|z| < 1$:

$$\begin{aligned}\frac{1}{1-z} &= \sum_{j=0}^{\infty} z^j \\ \frac{1}{2-z} &= \frac{1}{2}\cdot\frac{1}{1-z/2} \\ &= \frac{1}{2}\cdot\sum_{j=0}^{\infty}\frac{z^j}{2^j} \\ &= \sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}}\end{aligned}$$

In this case the Laurent series for $f(z)$ agrees with the power series for $f(z)$:

$$f(x) = \sum_{j=0}^{\infty}\left(1-\frac{1}{2^{j+1}}\right)z^j \quad \text{for} \quad |z| < 1 \ .$$

Case 2: Write $f(z)$ as a Laurent series for $1 < |z| < 2$, i.e., in the annulus $A(0,1,2)$. We have for $|z| > 1$:

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{z}\cdot\frac{1}{\frac{1}{z}-1} \\ &= -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} \\ &= -\frac{1}{z}\cdot\sum_{j=0}^{\infty} z^{-j} \\ &= -\sum_{j=0}^{\infty} z^{-(j+1)}\end{aligned}$$

The series for $\frac{1}{2-z}$ is the same as in Case 1. Therefore, the Laurent series for $f(z)$ in the annulus $A(0,1,2)$ is

$$f(z) = -\sum_{j=0}^{\infty} z^{-(j+1)} - \sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}} \quad \text{for} \quad 1 < |z| < 2 \ .$$

Case 3: Write $f(z)$ as a Laurent series for $|z| > 2$. The series for $\frac{1}{1-z}$ is the same as in Case 1, but we must change the series for $\frac{1}{2-z}$ if $|z| > 2$. We have for $|z| > 2$:

$$\begin{aligned}
\frac{1}{2-z} &= \frac{1}{z} \cdot \frac{1}{\frac{2}{z} - 1} \\
&= -\frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} \\
&= -\frac{1}{z} \cdot \sum_{j=0}^{\infty} \frac{2^j}{z^j} \\
&= -\sum_{j=0}^{\infty} \frac{2^j}{z^{j+1}}
\end{aligned}$$

One obtains the Laurent series

$$f(z) = \sum_{j=0}^{\infty} (2^j - 1) z^{-(j+1)} \quad \text{for} \quad |z| > 2 \ .$$

10 Details 10: Evaluation of Integrals

We first consider two examples. In both examples, Γ denotes the positively oriented unit circle, i.e., the curve with parametrization $z(t) = e^{it}, 0 \le t \le 2\pi$.

Example 1: Compute

$$Int = \int_\Gamma \frac{\sin z}{z^4}\, dz \ .$$

The function

$$f(z) = \frac{\sin z}{z^4}$$

has a pole of order 3 at $z = 0$. We must compute the residue of $f(z)$ at $z = 0$. We know that

$$\sin z = z - \frac{z^3}{6} + z^5 h(z)$$

where $h(z)$ is an entire function. Therefore,

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{6z} + z h(z) \ .$$

One obtains that

$$Res\Big(f(z), z = 0\Big) = -\frac{1}{6} \ ,$$

thus

$$Int = 2\pi i \cdot (-1/6) = -\pi i/3 \ .$$

Example 2: Compute

$$Int = \int_\Gamma \frac{\sin z}{z^4(2 - z^2)}\, dz \ .$$

The function

$$g(z) = \frac{\sin z}{z^4(2 - z^2)}$$

has a pole of order 3 at $z = 0$. The other two poles at $\pm\sqrt{2}$ lie outside Γ.

We must compute the residue of $g(z)$ at $z = 0$. The function $g(z)$ has the form

$$g(z) = \frac{\phi(z)}{z^4} \quad \text{with} \quad \phi(z) = \frac{\sin z}{2 - z^2}$$

and we have

$$\phi(z) = \phi(0) + \phi'(0) z + \frac{1}{2}\, \phi''(0) z^2 + \frac{1}{6}\, \phi'''(0) z^3 + \ldots$$

If follows that

$$Res\Big(g(z), z=0\Big) = \frac{1}{6}\,\phi'''(0)\ .$$

One can compute $\phi'''(0)$ the hard way. But we can also use that

$$\begin{aligned}
\sin z &= z - \frac{z^3}{6} + \mathcal{O}(z^5) \\
\frac{1}{2-z^2} &= \frac{1}{2}\cdot\frac{1}{1-z^2/2} \\
&= \frac{1}{2}\left(1+\frac{z^2}{2}+\mathcal{O}(z^4)\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(z) &= \left(z-\frac{z^3}{6}\right)\frac{1}{2}\left(1+\frac{z^2}{2}\right) + \mathcal{O}(z^5) \\
&= \frac{z}{2} - \frac{z^3}{12} + \frac{z^3}{4} + \mathcal{O}(z^5) \\
&= \frac{z}{2} + \frac{z^3}{6} + \mathcal{O}(z^5)
\end{aligned}$$

One obtains that

$$Res\Big(g(z), z=0\Big) = \frac{1}{6}\ ,$$

thus

$$Int = 2\pi i\cdot(1/6) = \pi i/3\ .$$

Homework: I ask you to compute

$$\int_\Gamma \frac{\cos z}{z^3}\,dz \quad \text{and} \quad \int_\Gamma \frac{\cos z}{z^3(3+z^2)}\,dz\ .$$

Possibly these integrals will come up in your next homework.

11 Details 11: Power Series

Recall that $\operatorname{Log} z$ denotes the principle branch of the complex logarithm. It is a holomorphic function in the open set

$$U = \mathbb{C} \setminus (-\infty, 0] \ .$$

If $z \in U$ then

$$z = |z|e^{i\phi} = e^{\ln|z| + i\phi}$$

for a unique ϕ with $-\pi < \phi < \pi$. We then have

$$\operatorname{Log} z = \ln|z| + i\phi \ .$$

For the following, it is good to remember that

$$\operatorname{Log}{}'(z) = \frac{1}{z} \quad \text{for all} \quad z \in U \ .$$

Problem 1 a) Write the function $\operatorname{Log} z$ as a power series centered at $z_0 = 1$. In other words, determine the coefficients a_j so that

$$\operatorname{Log} z = \sum_{j=0}^{\infty} a_j (z-1)^j \quad \text{for} \quad |z-1| < R \ .$$

Determine the radius of convergence R of the power series.

b) Use your result of a) to write the function

$$f(w) = \operatorname{Log}(1+w)$$

as a power series centered at $w = 0$.

c) Prove the follwoing estimate:

$$\Big|\operatorname{Log}(1+w) - w\Big| \le \frac{1}{2}\frac{|w|^2}{1-|w|} \quad \text{for} \quad |w| < 1 \ .$$

Problem 2 Recall the geometric sum formula

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j \quad \text{for} \quad |z| < 1 \ .$$

By differentiating the formula k times, obtain the power series expansion of

$$\frac{1}{(1-z)^{k+1}}$$

centered at $z = 0$. Use the binomial coefficients

$$\begin{pmatrix} j \\ k \end{pmatrix} = \frac{j!}{k!(j-k)!}$$

in your power series.

12 Details 12: Practice Problems: Integrals and Laurent Series

1) Consider

$$f(z) = \frac{z}{(z-1)(z-3)} \ .$$

Let Γ_0 denote the positively oriented circle centered at $z = 0$ of radius 2.

We want to determine

$$Int = \int_{\Gamma_0} f(z)\, dz \ .$$

Note that the function $f(z)$ has two poles of order one. One pole is at $z_1 = 1$, the other is at $z_2 = 3$.

The pole $z_1 = 1$ lies inside Γ_0; the pole $z_2 = 3$ lies outside Γ_0.

Therefore,

$$Int = \int_{\Gamma_0} f(z)\, dz = 2\pi i \, Res\Big(f(z), z_1 = 1\Big) \ .$$

To determine the residue of $f(z)$ at $z_1 = 1$ we write

$$f(z) = \frac{1}{z-1} \cdot \frac{z}{z-3}$$

and obtain that

$$Res\Big(f(z), z_1 = 1\Big) = \frac{1}{1-3} = -\frac{1}{2} \ .$$

Therefore,

$$Int = \int_{\Gamma_0} f(z)\, dz = 2\pi i \Big(-\frac{1}{2} \Big) = -\pi i \ .$$

2) Homework Problem:

Let Γ_1 denote the positively oriented circle centered at $z = 0$ of radius 4.

Determine

$$Int = \int_{\Gamma_0} f(z)\, dz \ .$$

3) Laurent Series: Let

$$f(z) = \frac{z}{(z-1)(z-3)} \ .$$

We want to write $f(z)$ as a Laurent series centered at $z_1 = 1$. Since $f(z)$ has a pole of order one at $z_1 = 1$ the Laurent series will have the form

$$\begin{aligned} f(z) &= \sum_{j=-1}^{\infty} a_j (z-1)^j \\ &= \frac{a_{-1}}{z-1} + a_0 + a_1(z-1) + a_2(z-1)^2 + \dots \end{aligned}$$

and we expect the series to converge for

$$0 < |z-1| < 2$$

since the distance of the two poles equals 2.

We have

$$f(z) = \frac{1}{z-1} \cdot \frac{z}{z-3}$$

and will expand

$$\frac{z}{z-3} = \frac{z-1}{z-3} + \frac{1}{z-3}$$

as a power series centered at $z = 1$.

Clearly, we must expand

$$\frac{1}{z-3}$$

as a power series centered at $z = 1$. We will use the geometric sum formula and write

$$\frac{1}{z-3} = \frac{1}{(z-1)-2} \; .$$

There are two terms in the denominator: The term $z-1$ and the term 2. Since we will need that $|z-1| < 2$ the term 2 is the larger one in absolute value. We write

$$\begin{aligned} \frac{1}{z-3} &= \frac{1}{(z-1)-2} \\ &= -\frac{1}{2} \cdot \frac{1}{1 - \frac{z-1}{2}} \\ &= -\frac{1}{2} \cdot \sum_{j=0}^{\infty} \Big(\frac{z-1}{2} \Big)^j \end{aligned}$$

Therefore,

$$\frac{1}{z-3} = -\sum_{j=0}^{\infty} \frac{(z-1)^j}{2^{j+1}} \quad \text{for} \quad |z-1| < 2 \; .$$

From

$$\frac{z}{z-3} = \frac{z-1}{z-3} + \frac{1}{z-3}$$

we obtain that

$$\begin{aligned} \frac{z}{z-3} &= -\sum_{j=0}^{\infty} \frac{(z-1)^{j+1}}{2^{j+1}} - \sum_{j=0}^{\infty} \frac{(z-1)^j}{2^{j+1}} \\ &= -\sum_{j=1}^{\infty} \frac{(z-1)^j}{2^j} - \sum_{j=0}^{\infty} \frac{(z-1)^j}{2^{j+1}} \end{aligned}$$

Since

$$\frac{1}{2^j} + \frac{1}{2^{j+1}} = \frac{3}{2^{j+1}}$$

we obtain:

$$\frac{z}{z-3} = -\frac{1}{2} - 3\sum_{j=1}^{\infty} \frac{(z-1)^j}{2^{j+1}}$$

and, finally,

$$\begin{aligned} f(z) &= \frac{1}{z-1} \cdot \frac{z}{z-3} \\ &= -\frac{1}{2} \cdot \frac{1}{z-1} - 3\sum_{j=1}^{\infty} \frac{(z-1)^{j-1}}{2^{j+1}} \\ &= -\frac{1}{2} \cdot \frac{1}{z-1} - 3\sum_{j=0}^{\infty} \frac{(z-1)^j}{2^{j+2}} \end{aligned}$$

This is the Laurent expansion of $f(z)$ valid for $0 < |z-1| < 2$.

4) Homework Problem: Write the function

$$f(z) = \frac{z}{(z-1)(z-3)}$$

as a Laurent series centered at $z_2 = 3$.