MATH 313, Complex Variables , Spring 2020
Prof. J. Lorenz, Instructor
Homework 2, assigned Feb. 7, due Feb. 21
Solutions

1. Let

$$u(x,y) = 2x^3 - 6xy^2 + x^2 - y^2 - y \ .$$

a) Check if $u(x,y)$ is a harmonic function or not.

b) Find all functions $v(x,y)$ so that

$$f(x+iy) = u(x,y) + iv(x,y)$$

is complex differentiable in $\mathbb{C}$.

Solution: a) We have

$$u_{xx} = 12x + 2 \quad \text{and} \quad u_{yy} = -12x - 2 \ ,$$

thus $\Delta u = u_{xx} + u_{yy} = 0$. The function $u(x,y)$ is harmonic.

b) From $u_x = 6x^2 - 6y^2 + 2x = v_y$ obtain that

$$v = 6x^2y - 2y^3 + 2xy + h(x) \ .$$

Then $v_x = -u_y$ yields that

$$v_x = 12xy + 2y + h'(x) = 12xy + 2y + 1 \ .$$

Obtain that

$$v = 6x^2y - 2y^3 + 2xy + x + const$$

2. Let $f(z) = f(x+iy) = 3x + y + i(3y - x)$.
a) Show that $f(z)$ is an entire function.
b) Determine $f'(z)$.
c) Show that

$$f(z) = cz \quad \text{for all} \quad z \in \mathbb{C} \ ,$$

for a constant c. Determine c.

Solution: a) Let $u = 3x + y, v = 3y - x$. Then we have

$$u_x = 3, \quad u_y = 1, \quad v_x = -1, \quad v_y = 3 \ ,$$

thus $u_x = v_y$ and $u_y = -v_x$. The Cauchy–Riemann equations hold. $f(z)$ is entire.

b) $f'(z) = u_x + iv_x = 3 - i$
c) $(3-i)(x+iy) = 3x + y + i(3y - x) = f(z)$

3. Recall the hyperbolic functions

$$\begin{aligned}\sinh y &= \frac{1}{2}\left(e^{y}-e^{-y}\right)\\ \cosh y &= \frac{1}{2}\left(e^{y}+e^{-y}\right)\end{aligned}$$

and let

$$f(z)=f(x+iy)=\sin x\cosh y+i\cos x\sinh y\ .$$

a) Show that $f(z)$ is an entire function.
b) Determine $f'(z)$.
c) Show that

$$f'(z)=\frac{1}{2}(e^{\alpha z}+e^{\beta z})$$

with constants α,β. Determine the constants α,β.

Solution: a) We have

$$\begin{aligned}u_x &= \cos x\cosh y\\ u_y &= \sin x\sinh y\\ v_x &= -\sin x\sinh y\\ v_y &= \cos x\cosh y\end{aligned}$$

Thus $u_x=v_y, u_y=-v_x$. The Cauchy–Riemann equations hold. $f(z)$ is entire.
b) We have

$$\begin{aligned}f'(z) &= u_x+iv_x\\ &= \cos x\cosh y-i\sin x\sinh y\\ &= \frac{1}{2}\cos x(e^{y}+e^{-y})-\frac{i}{2}\sin x(e^{y}-e^{-y})\\ &= \frac{1}{2}e^{y}(\cos x-i\sin x)+\frac{1}{2}e^{-y}(\cos x+i\sin x)\\ &= \frac{1}{2}e^{y}e^{-ix}+\frac{1}{2}e^{-y}e^{ix}\\ &= \frac{1}{2}e^{-i(x+iy)}+\frac{1}{2}e^{i(x+iy)}\end{aligned}$$

The equation

$$f'(z)=\frac{1}{2}(e^{\alpha z}+e^{\beta z})$$

holds with $\alpha=-i$ and $\beta=i$.

4. a) Prove that the function

$$u(x,y) = \frac{y}{x^2+y^2}, \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}\ ,$$

is harmonic.

b) Determine a holomorphic function $f(z)$ for $z \in \mathbb{C} \setminus \{0\}$ with

$$\operatorname{Re} f(x+iy) = u(x,y)\ .$$

Hint: You may use that the function $f(z)$ has the form $f(z) = \frac{c}{z}$ for a constant $c \in \mathbb{C}$. Determine c.

Solution: a) Let $q(x,y) = (x^2+y^2)^{-1}$, thus $u(x,y) = yq(x,y)$. We have

$$\begin{aligned}
q_x &= -2xq^2 \\
q_{xx} &= -2q^2 - 4xqq_x \\
&= -2q^2 + 8x^2q^3 \\
q_y &= -2yq^2 \\
q_{yy} &= -2q^2 + 8y^2q^3
\end{aligned}$$

Therefore,

$$u_{xx} = yq_{xx} = -2yq^2 + 8x^2yq^3$$

and

$$\begin{aligned}
u_y &= q + yq_y \\
u_{yy} &= 2q_y + yq_{yy} \\
&= -4yq^2 - 2yq^2 + 8y^3q^3
\end{aligned}$$

This yields that

$$\begin{aligned}
u_{xx} + u_{yy} &= yq^2\Big(-2 + 8x^2q - 4 - 2 + 8y^2q\Big) \\
&= yq^2\Big(-8 + 8(x^2+y^2)q\Big) \\
&= 0
\end{aligned}$$

b) Let $a, b \in \mathbb{R}$ and let

$$\begin{aligned}
f(z) &= \frac{a+ib}{x+iy} \\
&= \frac{(a+ib)(x-iy)}{x^2+y^2} \\
&= \frac{ax+by+i(bx-ay)}{x^2+y^2}
\end{aligned}$$

To obtain that $\operatorname{Re} f(z) = u(x,y)$ we must choose

$$a = 0 \quad \text{and} \quad b = 1 \ .$$

This yields that

$$\operatorname{Im} f(z) = v(x,y) = \frac{x}{x^2+y^2} \ .$$

One can then check that $u_x = v_y$ and $u_y = -v_x$.

One obtains that

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ &= \frac{y+ix}{x^2+y^2} \\ &= i\,\frac{x-iy}{x^2+y^2} \\ &= i\,\frac{x-iy}{(x+iy)(x-iy)} \\ &= \frac{i}{z} \end{aligned}$$

This yields that $c = i$.

Remark: Noting that

$$\begin{aligned} \operatorname{Re} \frac{i}{x+iy} &= \operatorname{Re} \frac{i(x-iy)}{x^2+y^2} \\ &= \frac{y}{x^2+y^2} \\ &= u(x,y) \end{aligned}$$

it is clear that $u(x,y)$ is harmonic. This arguments is easier than the computations given under a).

5. Evaluate $\int_0^1 (1+it^2)\,dt$.

Solution:

$$\begin{aligned} \int_0^1 (1+it^2)\,dt &= 1 + i\int_0^1 t^2\,dt \\ &= 1 + \frac{i}{3} \end{aligned}$$

6. Evaluate the integral $\int_0^{\pi/3} e^{it}\,dt$ in two ways: By using the exponential function and by using $e^{it} = \cos t + i \sin t$.

Solution: First way:

$$\begin{aligned}\int_0^{\pi/3} e^{it}\,dt &= \frac{1}{i}\,e^{it}\Big|_0^{\pi/3}\\ &= -i(e^{\pi i/3}-1)\\ &= -i(\cos(\pi/3)+i\sin(\pi/3)-1)\\ &= \sin(\pi/3)-i(\cos(\pi/3)-1)\\ &= \frac{\sqrt{3}}{2}+\frac{i}{2}\end{aligned}$$

Second way:

$$\begin{aligned}\int_0^{\pi/3} (\cos t + i\sin t)\,dt &= (\sin t - i\cos t)\Big|_0^{\pi/3}\\ &= \sin(\pi/3)-i(\cos(\pi/3)-1)\\ &= \frac{\sqrt{3}}{2}+\frac{i}{2}\end{aligned}$$

7. Evaluate $\int_0^\infty e^{-zt}\,dt$ for $\operatorname{Re} z > 0$.

Solution: For $\operatorname{Re} z > 0$:

$$\begin{aligned}\int_0^\infty e^{-zt}\,dt &= \frac{1}{-z}e^{-zt}\Big|_{t=0}^{t=\infty}\\ &= \frac{1}{z}\end{aligned}$$

8. Evaluate $\int_0^{2\pi} e^{ikx}\,dx$ for all integers k and use your result to compute

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx$$

for all positive integers m and n.

Solution: One obtains:

$$\int_0^{2\pi} e^{ikx}\,dx = \begin{cases} 2\pi & \text{for} \quad k=0 \\ 0 & \text{for} \quad k\neq 0 \end{cases}$$

From

$$\sin(kx) = \frac{1}{2i}\,(e^{ikx}-e^{-ikx})$$

obtain that

$$\begin{aligned}\sin(mx)\sin(nx) &= -\frac{1}{4}\,(e^{imx}-e^{-imx})(e^{inx}-e^{-inx})\\ &= -\frac{1}{4}(e^{i(m+n)x}+e^{-i(m+n)x}-e^{i(m-n)x}-e^{i(n-m)x})\end{aligned}$$

The integral is

$$Int = -\frac{1}{4}\begin{cases} -4\pi & \text{for} \quad m = n \\ 0 & \text{for} \quad m \neq n \end{cases}$$

Thus,

$$\int_0^{2\pi} \sin^2(mx)\, dx = \pi$$

and

$$\int_0^{2\pi} \sin(mx)\sin(nx)\, dx = 0 \quad \text{for} \quad m \neq n \ .$$

9. Recall that

$$\begin{aligned} e^{iz} &= \cos z + i \sin z \\ e^{-iz} &= \cos z - i \sin z \end{aligned}$$

Use this to write $\sin z$ in the form

$$\sin z = u(x,y) + iv(x,y) \quad \text{for} \quad z = x + iy \ .$$

Use $\sinh y$ and $\cosh y$ in your answer.

Solution: We have

$$\begin{aligned} \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\ &= \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y}) \\ &= \frac{1}{2i}\Big(e^{-y}(cosx + i\sin x) - e^{y}(\cos x - i \sin x)\Big) \\ &= \frac{1}{2}\sin x(e^{y} + e^{-y}) + \frac{i}{2}\cos x(e^{y} - e^{-y}) \\ &= \sin x \cosh y + \cos x \sinh y \end{aligned}$$

10. Evaluate

$$\int_{\Gamma_j} \frac{z+2}{z}\, dz \quad \text{for} \quad j = 1, 2, 3$$

where
a) Γ_1 is the semicircle with parameterization $z = 2e^{i\phi}, 0 \leq \phi \leq \pi$.
b) Γ_2 is the semicircle with parameterization $z = 2e^{i\phi}, \pi \leq \phi \leq 2\pi$.
c) Γ_3 is the circle with parameterization $z = 2e^{i\phi}, 0 \leq \phi \leq 2\pi$.

Solution:
a) We have $dz = 2ie^{i\phi}$ and

$$\begin{aligned}\int_{\Gamma_1}(1+\frac{2}{z})\,dz &= \int_0^{\pi}(1+e^{-i\phi})2ie^{i\phi}\,d\phi \\ &= 2i\int_0^{\pi}(e^{i\phi}+1)\,d\phi \\ &= 2i\frac{1}{i}(e^{\pi i}-1)+2\pi i \\ &= 2(-1-1)+2\pi i \\ &= -4+2\pi i\end{aligned}$$

b)

$$\begin{aligned}\int_{\Gamma_2}(1+\frac{2}{z})\,dz &= \int_{\pi}^{2\pi}(1+e^{-i\phi})2ie^{i\phi}\,d\phi \\ &= 2i\int_{\pi}^{2\pi}(e^{i\phi}+1)\,d\phi \\ &= 2i\frac{1}{i}(e^{2\pi i}-e^{\pi i})+2\pi i \\ &= 2(1-(-1))+2\pi i \\ &= 4+2\pi i\end{aligned}$$

c) The integral along Γ_3 equals the sum of the integrals along Γ_1 and Γ_2. Therefore,

$$\int_{\Gamma}\frac{z+2}{z}\,dz = 4\pi i\ .$$

This also follows from

$$Res(f(z), z=0) = 2\ .$$

Remark 1: Since $\frac{d}{dz}z = 1$ and since Γ_1 goes from $z_0 = 2$ to $z_1 = -2$ we have

$$\int_{\Gamma_1} 1\,dz = z_1 - z_0 = -4\ .$$

Similarly,

$$\int_{\Gamma_2} 1\,dz = 4, \quad \int_{\Gamma_3} 1\,dz = 0\ .$$

Remark 2: We have

$$\frac{d}{dz}\ \mathrm{Log}\ z = \frac{1}{z}$$

in the upper–half plane. Also,

$$-2 = 2(-1) = 2e^{\pi i} = e^{\ln 2 + \pi i}, \quad \text{Log}\,(-2) = \ln 2 + \pi i \ .$$

Therefore,

$$\begin{aligned}
\int_{\Gamma_1} \frac{2}{z}\, dz &= 2\Big(\text{Log}\,(-2) - \text{Log}\,(2) \Big) \\
&= 2(\ln 2 + \pi i - \ln 2) \\
&= 2\pi i
\end{aligned}$$

11. Let Γ denote the semicircle with parameterization $z = e^{i\phi}, 0 \le \phi \le \pi$. Evaluate $\int_\Gamma \sqrt{z}\, dz =: Int$ and write your result in the form $Int = x + iy$.

Solution: For $z = e^{i\phi}$ we have $\sqrt{z} = e^{i\phi/2}$ and

$$\begin{aligned}
\int_\Gamma \sqrt{z}\, dz &= \int_0^\phi e^{i\phi/2} i e^{i\phi}\, d\phi \\
&= i \int_0^\pi e^{3i\phi/2}\, d\phi \\
&= i \frac{2}{3i} e^{3i\phi/2} \Big|_0^\pi \\
&= \frac{2}{3}\,(e^{3\pi i/2} - 1) \\
&= \frac{2}{3}\,(-i-1) \\
&= -\frac{2}{3}\,(1+i)
\end{aligned}$$

Remark: Let

$$\Omega = \{ z = re^{i\phi} \ : \ r > 0, -\frac{\pi}{2} < \phi < \frac{3\pi}{2} \} \ .$$

For $z = re^{i\phi} \in \Omega$ let

$$f(z) = \frac{2}{3}\, z^{3/2} = \frac{2}{3}\, r^{3/2} e^{3\pi i/2} \ .$$

Then $f(z)$ is holomorphic in Ω and

$$f'(z) = \sqrt{z} \ .$$

(This requires some arguments.)

One obtains that

$$\int_\Gamma \sqrt{z}\, dz = f(-1) - f(1) = \frac{2}{3}(-1)i - \frac{2}{3} = -\frac{2}{3}\,(1+i) \ .$$