Nonlinear Dynamics and Chaos
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Contents

1 Notes on the History of Dynamics

Johannes Kepler (1571-1630) formulated Kepler's laws of planetary motion.

Isaac Newton (1642-1727) used the inverse square law of gravitational attraction to derive Kepler's laws. Invention of calculus. Newton solved the two–body problem. This was an enormous success which lead to a deterministic and mechanical view of the world.

There were many attempts to solve the three body problem in a similar way, by an explicit formula, which gives the positions and velocities of three bodies as functions of time. It turned out that this is not possible.

Jules Henri Poincaré (1854-1912) started the qualitative theory of differential equations. He formulated the first ideas about chaotic motion described by deterministic systems.

KAM theory is named after Andrei Nikolaevich Kolmogorov (1903-1987), Vladimir Igorevich Arnold (1937-2010), amd Jürgen Moser (1928-1999). The theory gives results about invariant tori of perturbed Hamiltonian systems. The origins of KAM theory lie in the question of stability of the solar system. Laplace, Lagrange, Gauss, Poincaré and many others had worked on this.

Edward N. Lorenz (1917-2008), meteorologist, derived a simple deterministic model system with sensitive dependence on initial conditions (1963). Is the butterfly effect real for the weather? The sensitivity of a system can be measured by the largest Lyapunov exponent, α. If the exponent α is positive, then an initial error of size δ grows over time (approximately) like $\delta e^{\alpha t}$, until the size of the system limits further growth. Nevertheless, even for a system with positive Lyapunov exponent, some average quantities may be accurately predictable. Can we compute the climate 30 years in advance though we cannot predict the weather two weeks in advance?

In bifurcation theory one considers parameter dependent systems like

$$u' = f(u, \lambda) \; .$$

As λ changes, the dynamics may change qualitatively, not just quantitatively. If a qualitative change occurs at $\lambda = \lambda_0$ then λ_0 is a bifurcation value.

An interesting bifurcation is the transition from laminar to turbulent flow. This transition was addressed in a paper by Ruelle and Takens, *On the Nature of Turbulence*, 1970, which is still controversial. The term of a strange attractor was introduced.

M. Feigenbaum (1980) studied period doubling bifurcations for maps. He discovered an interesting universality of transition from simple to chaotic motion through repeated period doubling. Feigenbaum's model equations can be used to show that the average behaviour of a chaotic dynamical system may still be well determined and computable even if individual trajectories can be computed accurately only for a short time. We cannot predict the weather in Albuquerque 30 years from today, but the climate (the average weather) may still be predictable.

2 Flows on the Line

2.1 Two Examples with Explicit Solutions

Let $f : \mathbb{R} \to \mathbb{R}$ be a given smooth function and consider the scalar ODE

$$\frac{dx}{dt} = x' = f(x) \ .$$

We can visualize the dynamics qualitatively on the line.

As a first example, consider the initial value problem (IVP)

$$x' = x^2 \quad \text{for} \quad t \geq 0, \quad x(0) = x_0 \ .$$

Obtain

$$\begin{aligned} \frac{dx}{x^2} &= dt \\ \int_{x_0}^{x(t)} \frac{dx}{x^2} &= \int_0^t dt \\ -\frac{1}{x} \Big|_{x_0}^{x(t)} &= t \\ x(t) &= \frac{x_0}{1 - x_0 t} \end{aligned}$$

Thus the solution of the IVP is

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{for} \quad 0 \leq t < \frac{1}{x_0} \quad \text{if} \quad x_0 > 0$$

and

$$x(t) = \frac{x_0}{1 - x_0 t} \quad \text{for} \quad t \geq 0 \quad \text{if} \quad x_0 \leq 0 \ .$$

We can graph the solutions in the (t, x) plane. We can also visualize the dynamics on the x–line.

The next example shows that the explicit solution formula may not be very useful. But it is again easy to visualize the dynamics qualitatively.

Consider

$$x' = \sin x, \quad x(0) = x_0 \ .$$

We have

$$\sin x = 2 \sin(x/2) \cos(x/2) = 2 \tan(x/2) \cos^2(x/2) \ .$$

Also,

$$\frac{d}{d\alpha} \tan \alpha = \frac{1}{\cos^2 \alpha} \ .$$

Therefore, if $\tan(x/2) > 0$:

$$
\begin{aligned}
\frac{d}{dx}\ln\Big(\tan(x/2)\Big) &= \frac{1}{2}\frac{1}{\tan(x/2)}\frac{1}{\cos^2(x/2)} \\
&= \frac{1}{\sin x}
\end{aligned}
$$

We can now solve the initial value problem in an interval where $\tan(x/2) > 0$. The equation $\frac{dx}{dt} = \sin x$ yields that

$$
\begin{aligned}
\frac{dx}{\sin x} &= dt \\
\int_{x_0}^{x(t)} \frac{dx}{\sin x} &= \int_0^t dt \\
t &= \ln\Big(\tan(x/2)\Big)\Big|_{x_0}^{x(t)} \\
&= \ln\Big(\frac{\tan(x(t)/2)}{\tan(x_0/2)}\Big)
\end{aligned}
$$

Solving for $x(t)$ one obtains that

$$
x(t) = 2\arctan\Big(e^t \tan(x_0/2)\Big)\ . \tag{2.1}
$$

Similarly, in an interval where $\tan(x/2) < 0$ we have that

$$
\begin{aligned}
\frac{d}{dx}\ln\Big(-\tan(x/2)\Big) &= \frac{1}{2}\frac{1}{-\tan(x/2)}\frac{-1}{\cos^2(x/2)} \\
&= \frac{1}{\sin x}
\end{aligned}
$$

One again obtains the solution formula (2.1).

It is difficult to make good use of this formula. To get qualitative information about the solution, it is much easier to discuss the equation

$$
x' = \sin x
$$

directly. The graph of the sine function gives us information about the fixed points and their stability.

2.2 Fixed Points and Their Stability

Consider an equation

$$
x' = f(x)
$$

where f is a smooth scalar function. Points x^* with $f(x^*) = 0$ are fixed points (or equilibria) of the evolution. If $f'(x^*) < 0$ then x^* is stable, if $f'(x^*) > 0$ then x^* is unstable.

Definition: Consider an ODE system $x' = f(x)$ where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let $f(x^*) = 0$, i.e., x^* is a fixed point.

a) The fixed point x^* is called stable if for all $\varepsilon > 0$ there exists $\delta > 0$ so that $|x(0) - x^*| < \delta$ implies

$$|x(t) - x^*| < \varepsilon \quad \text{for all} \quad t \ge 0 \ .$$

b) The fixed point x^* is called asymptotically stable if x^* is stable and, in addition, there exists $\delta > 0$ so that $|x(0) - x^*| < \delta$ implies

$$x(t) \to x^* \quad \text{as} \quad t \to \infty \ .$$

One can prove:

Theorem 2.1 *Consider an ODE system $x' = f(x)$ where $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let $f(x^*) = 0$, i.e., x^* is a fixed point. Let*

$$A = f'(x^*) \in \mathbb{R}^{n \times n}$$

denote the Jacobian of f at the fixed point x^*. If all eigenvalues of A have a negative real part then x^* is asymptotically stable.

Example of a linear equation: A simple electric circuit: The circuit has a battery with voltage V_0 producing a direct current, a resistor with resistance R, and a capacitor with capacitance C. The charge on the capacitor is denote by $Q(t)$ and $Q'(t) = I(t)$ is the current. The equation

$$-V_0 + RI + \frac{Q}{C} = 0$$

holds. (By Ohm's law, the voltage drop at the resistor is RI.) One obtains the following linear equation for $Q(t)$:

$$Q' = \frac{V_0}{R} - \frac{Q}{RC} \ .$$

We see that

$$Q^* = CV_0$$

is the only equilibrium, and it is stable. In this case we can obtain an explicit (and useful) solution: The equation

$$u' = a - bu$$

with constants a, b and $b \neq 0$ has the general solution

$$u(t) = \frac{a}{b} + ce^{-bt} \ .$$

One obtains that

$$u(t) = \frac{a}{b} + (u(0) - \frac{a}{b})e^{-bt} \ .$$

For the circuit problem, with $Q(0) = 0$,

$$Q(t) = V_0 C(1 - e^{-t/RC}) \ .$$

Remarks: The unit for voltage is

$$1Volt = \frac{Newton\, meter}{Coulomb} \ .$$

The unit for resistance is

$$1Ohm = \frac{Volt}{Ampere} \ .$$

The unit for capacitance is

$$1farad = \frac{Coulomb}{Volt} \ .$$

Example: A simple nonlinear equation: Consider the equation

$$u' = u - \cos u \ .$$

It is easy to see that there is a unique fixed point u^*, and u^* is unstable. Note that we can classify the stability of u^* though we do not have an explicit formula for u^*.

2.3 Population Growth Models

The simplest model equation is

$$N' = rN$$

modeling exponential growth for $r > 0$, $r = const$. The point $N^* = 0$ is an unstable equilibrium.

The next simplest model is the so–called logistic equation

$$N' = r(1 - \frac{N}{K})N \ .$$

Here $K > 0$ is the carrying capacity. There are two fixed points: $N_1 = 0$ is unstable and $N_2 = K$ is stable.

2.4 Linearization about a Fixed Point

Consider an equation

$$u' = f(u)$$

and let

$$f(u^*) = 0 \ .$$

By Taylor's formula,

$$f(u^* + \varepsilon v) = f'(u^*)\varepsilon v + \mathcal{O}(\varepsilon^2) \ .$$

(Here it is assumed that $v = \mathcal{O}(1)$.) Consider a solution of the differential equation of the form

$$u(t) = u^* + \varepsilon v(t) \ .$$

One obtains that

$$\begin{aligned} u' &= \varepsilon v' \\ &= f(u^* + \varepsilon v) \\ &= f'(u^*)\varepsilon v + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Therefore,

$$v' = f'(u^*)v + \mathcal{O}(\varepsilon) \ .$$

If we formally neglect the $\mathcal{O}(\varepsilon)$ term, we obtain the linear equation

$$v' = f'(u^*)v \ .$$

This equation is called the linearization about u^*.

3 Examples of Bifurcations

Saddle–Node Bifurcation Consider the equation

$$\dot{x} = r + x^2$$

where r is a real parameter. For $r < 0$ there are two fixed points,

$$x_1(r) = -\sqrt{-r}, \quad x_2(r) = \sqrt{-r} \ .$$

The fixed point $x_1(r)$ is stable; $x_2(r)$ is unstable. At $r = 0$ the two fixed points collide. There is no fixed point for $r > 0$.

For systems of equations, a stable fixed point often is a node, an unstable fixed point often is a saddle. Suppose a parameter r changes from $r < r_0$ to $r > r_0$ and at $r = r_0$ a saddle collides with a node. Here it is assumed that the saddle and the node both exist for $r < r_0$, for example. Then one says that a saddle–node bifurcation occurs at $r = r_0$.

Transcritical Bifurcation Consider

$$\dot{x} = x(r - x)$$

For every parameter value $r \in \mathbb{R}$ the equation has two fixed points, the trivial fixed point $x_1(r) = 0$ and the fixed point $x_2(r) = r$. The trivial fixed point is stable for $r < 0$ and unstable for $r > 0$. The fixed point $x_2(r) = r$ is unstable for $r < 0$ and stable for $r > 0$. Thus, at $r = 0$, where the two branches of fixed points cross each other, an exchange of stability from one branch to the other occurs.

Supercritical Pitchfork Bifurcation Consider

$$\dot{x} = x(r - x^2)$$

For every parameter $r \in \mathbb{R}$ the equation has the trivial fixed point $x_1(r) = 0$. The trivial fixed point is stable for $r < 0$, but loses its stability for $r > 0$. For $r > 0$ two new fixed points occur:

$$x_2(r) = \sqrt{r} \quad \text{and} \quad x_3(r) = -\sqrt{r} \ .$$

Both of them are stable. When r crosses from $r < 0$ to $r > 0$, then the trivial stable fixed point $x_1(r) = 0$ is replaced by the two stable fixed points $x_{2,3}(r) = \pm\sqrt{r}$.

Remarks on symmetry: Note that the nonlinear function

$$f(x) = rx - x^3$$

obeys the rule

$$f(-x) = -f(x) \ .$$

The equation has Z_2–symmetry. If S is the operator defined by

$$Sx(t) = -x(t)$$

then

$$S\frac{d}{dt} = \frac{d}{dt}S \quad \text{and} \quad Sf = fS \ .$$

Therefore, if $x(t)$ is a solution of the equation $\dot{x} = f(x)$ then $Sx(t) = -x(t)$ is also a solution.

One says that the group $Z_2 = \{id, S\}$ acts on functions $x(t)$.

Subcritical Pitchfork Bifurcation Consider

$$\dot{x} = x(r + x^2)$$

For every r the equation has the trivial fixed point $x_1(r) = 0$, which is stable for $r < 0$ and unstable for $r > 0$. If $r < 0$ there are the additional unstable fixed point

$$x_2(r) = \sqrt{-r} \quad \text{and} \quad x_3(r) = -\sqrt{-r} \ .$$

When r crosses from $r < 0$ to $r > 0$, the trivial branch $x_1(r) = 0$ loses its stability, but no new stable fixed points occur. The pitchfork bifurcation is subcritical, i.e., the pitchfork occurs for parameter values r below the critical value $r = 0$.

Hysteresis Phenomenon If a subcritical pitchfork bifurcation occurs, additional nonlinear terms may stabilize the dynamics.

For example, consider

$$\dot{x} = rx + x^3 - x^5 \ .$$

This example can be used to illustrate the hysteresis phenomenon.

A simpler equation with hysteresis is

$$x' = r + x - x^3 = f(x, r) \ .$$

Sketch the function

$$g(x) = -x + x^3$$

with zeros at $-1, 0, 1$. The function $g(x)$ attains a local maximum at $x_0 = 1/\sqrt{3}$ and

$$M := g(x_0) = 2/(3\sqrt{3}) \ .$$

For

$$-M < r < M$$

the fixed point equation

$$r = x - x^3$$

has three solutions. The middle fixed point is unstable, the outer two fixed points are stable. Sketch the function $r = -x + x^3$ and switch coordinates to obtain the fixed points x^* over the parameter r. Saddle–node bifurcations occur at $r = -M$ and $r = M$. The stable branches can be used to illustrate the hysteresis phenomenon when r changes slowly between $-M - \varepsilon$ and $M + \varepsilon$.

4 Overdamped Bead on a Rotating Hoop

See Section 3.5 of Strogatz.

Preliminaries on the Centrifugal Force

Consider motion on a circle of radius ρ with position vector

$$R(t) = \rho(\cos\omega t, \sin\omega t) \ .$$

The period of the motion is $T = 2\pi/\omega$. The number $\omega = 2\pi/T$ is called the frequency of the motion.

For the centripetal acceleration obtain

$$R''(t) = -\omega^2 R(t) \ .$$

The force towards the origin is $mR'' = -m\omega^2 R$. The centrifugal force is $m\omega^2 R$ with magnitude $m\omega^2\rho$.

4.1 Description of the Hoop

A hoop of radius r rotates with frequency ω. A bead of mass m, which can slide along the hoop, is positioned at an angle $\phi = \phi(t)$. Thus, if ϕ is fixed, the bead rotates on a circle of radius $\rho = r\sin\phi$.

The gravitational force on the bead is mg in vertical direction, and its tangential component is $-mg\sin\phi$. The horizontal centrifugal force is

$$m\omega^2 r\sin\phi$$

and its tangential component is

$$m\omega^2 r\sin\phi\cos\phi \ .$$

The friction force is assumed to by

$$-b\phi' \quad \text{where} \quad b > 0 \ .$$

Note that $b\phi'$ is a force, thus the dimension of the coefficient b is

$$[b] = \frac{mass * length}{time} \ .$$

One obtains the equation

$$mr\phi'' = -mg\sin\phi + m\omega^2 r\sin\phi\cos\phi - b\phi' \ .$$

First neglect the ϕ''–term to obtain the first order equation

$$b\phi' = mg\sin\phi(\gamma\cos\phi - 1), \quad \gamma = \frac{\omega^2 r}{g} \ .$$

Note that γ is dimensionless.

We can write the equation in the form

$$\phi' = f(\phi,\gamma) \quad \text{with} \quad f(\phi,\gamma) = \frac{mg}{b}\sin\phi(\gamma\cos\phi - 1) \ .$$

For $0 \leq \gamma < 1$ there are exactly two fixed points:

$$\phi_1^* = 0 \quad \text{and} \quad \phi_2^* = \pi \ .$$

Here $\phi_1^* = 0$ is stable and $\phi_2^* = \pi$ is unstable.

If $\gamma > 1$ then $\phi_1^* = 0$ and $\phi_2^* = \pi$ are both unstable. Two new fixed points arise for $\gamma > 1$:

$$\phi_3^* = \arccos(1/\gamma) \quad \text{and} \quad \phi_4^* = -\phi_3^* \ .$$

These bifurcate from the trivial branch $\phi_3^*(\gamma) \equiv 0$ at $\gamma = 1$. The fixed points ϕ_3^* and ϕ_4^* are both stable. A supercritical pitchfork bifurcation from the trivial branch occurs at $\gamma = 1$.

Details: We have

$$f_\phi(\phi, \gamma) = \frac{mg}{b} \cos\phi(\gamma\cos\phi - 1) - \frac{mg}{b}\gamma\sin^2\phi \ .$$

Therefore,

$$\begin{aligned}
f_\phi(0,\gamma) &= \frac{mg}{b}(\gamma - 1) \\
f_\phi(\pi,\gamma) &= -\frac{mg}{b}(-\gamma - 1) \\
&= \frac{mg}{b}(\gamma + 1) \\
f_\phi(\phi_3^*,\gamma) &= -\frac{mg}{b}\gamma\sin^2(\phi_3^*) \\
f_\phi(\phi_4^*,\gamma) &= -\frac{mg}{b}\gamma\sin^2(\phi_4^*)
\end{aligned}$$

It follows that

$$\begin{aligned}
f_\phi(0,\gamma) &< 0 \quad \text{for} \quad 0 \leq \gamma < 1 \\
f_\phi(0,\gamma) &> 0 \quad \text{for} \quad \gamma > 1 \\
f_\phi(\pi,\gamma) &> 0 \quad \text{for} \quad \gamma \geq 0 \\
f_\phi(\phi_3^*,\gamma) &< 0 \quad \text{for} \quad \gamma > 1 \\
f_\phi(\phi_4^*,\gamma) &< 0 \quad \text{for} \quad \gamma > 1
\end{aligned}$$

Note that

$$-\frac{\pi}{2} < \phi_4^* < 0 < \phi_3^* < \frac{\pi}{2} \quad \text{for} \quad \gamma > 1 \ .$$

4.2 Nondimensionalization

Choose a time scale $T > 0$. (The choice of T is important and will be discussed later.)

Let

$$t = T\tau, \quad \tilde{\phi}(\tau) = \phi(T\tau) \ .$$

(Note that Strogatz uses the notation $\phi(\tau)$ instead of $\tilde{\phi}(\tau)$.)

Clearly,

$$\tilde{\phi}_\tau = T\phi_t = T\phi' \ .$$

(A rough idea for the choice of T is that one wants $\tilde{\phi}_\tau = \mathcal{O}(1)$ for the dynamics under consideration.)

Obtain

$$\frac{mr}{T^2}\,\tilde{\phi}_{\tau\tau} = -\frac{b}{T}\,\tilde{\phi}_\tau - mg\sin\tilde{\phi} + m\omega^2 r\sin\tilde{\phi}\cos\tilde{\phi} \ .$$

We now drop the ˜ notation and divide by the force mg to obtain

$$\frac{r}{gT^2}\,\phi_{\tau\tau} = -\frac{b}{mgT}\,\phi_\tau - \sin\phi + \gamma\sin\phi\cos\phi \ . \tag{4.1}$$

Note that the coefficients are all dimensionless. Thus, it makes sense to compare their sizes. Also note that the size of the coefficients depends on the chosen time scale T. For example, assume that we choose $T > 0$ to be very small and we consider (4.1) for $0 \le \tau \le 1$. This means that we consider the motion over the small time interval $0 \le t \le T$. If $T > 0$ is very small then the main terms in (4.1) are the $\phi_{\tau\tau}$ and the ϕ_τ terms. In this case, the nonlinear term

$$f(\phi) = \sin\phi(\gamma\cos\phi - 1)$$

may be negligible.

However, let's choose a different time scale $T > 0$. Since $\sin\phi = \mathcal{O}(1)$ and we want $\phi_\tau = \mathcal{O}(1)$ it is reasonable to choose T so that

$$\frac{b}{mgT} = \mathcal{O}(1) \ .$$

To be precise, let us choose T so that

$$\frac{b}{mgT} = 1 \ ,$$

i.e.,

$$T = \frac{b}{mg} \ .$$

Then the coefficient of $\phi_{\tau\tau}$ becomes

$$\frac{r}{gT^2} = \frac{rgm^2}{b^2} =: \varepsilon \ .$$

If we make the assumption

$$b^2 >> m^2 rg$$

then $0 < \varepsilon << 1$. Under this assumption, the $\phi_{\tau\tau}$ term is negligible on the time scale determined by T.

4.3 A Singular Perturbation Problem

Set

$$f(\phi) = \sin\phi(\gamma\cos\phi - 1) \ .$$

With

$$\frac{r}{gT^2} = \varepsilon \quad \text{and} \quad \frac{b}{mgT} = 1$$

the equation (4.1) becomes

$$\varepsilon\phi_{\tau\tau} = -\phi_\tau + f(\phi) \ . \tag{4.2}$$

If we assume that $0 < \varepsilon << 1$ then $\varepsilon\phi_{\tau\tau}$ is a perturbation term. One calls it a singular perturbation term because, if one neglects the term, the order of the equation changes.

Note the following: If $\varepsilon > 0$ then we can prescribe $\phi(0)$ and $\phi'(0)$. If $\varepsilon = 0$ we can only prescribe $\phi(0)$. For $\varepsilon > 0$ we expect an initial layer of the solution.

We write the equation as a first order system. Introduce

$$\Omega(\tau) = \phi_\tau(\tau) \ .$$

For $\varepsilon > 0$ obtain the system

$$\begin{pmatrix} \phi \\ \Omega \end{pmatrix}_\tau = \begin{pmatrix} \Omega \\ \frac{1}{\varepsilon}(f(\phi) - \Omega) \end{pmatrix}_\tau$$

Sketch the phase–plane arrows.

Important is the curve Γ given by the points (ϕ, Ω) with $\Omega = f(\phi)$. This is the graph of $f(\phi)$. Along Γ, the phase–plane arrows are horizontal, to the right for $\Omega > 0$, to the left for $\Omega < 0$. Away from Γ, the vertical component of the phase–plane arrows are large since $0 < \varepsilon << 1$. At points (ϕ, Ω) above Γ we have $f(\phi) - \Omega < 0$, thus the arrows go down. At points (ϕ, Ω) below Γ we have $f(\phi) - \Omega > 0$, thus the arrows go up. Thus, at every point $P = (\phi, \Omega)$, which is not close to Γ, the phase–plane arrow at P has a large Ω–component and is directed towards Γ.

If at time $t = 0$ we have initial data

$$(\phi(0), \Omega(0))$$

which are not ε–close to Γ, then $\Omega(t)$ will rapidly move towards $f(\phi(0))$. Thus, the solution has an initial layer. After a short time, the solution follows the curve Γ in the phase plane and the dynamics is essentially determined by the first–order equation $\phi_\tau = f(\phi)$, which is obtained by neglecting the $\varepsilon\phi_{\tau\tau}$ term in (4.2).

In singular perturbation theory, one studies systematically how solutions of singularly perturbed problems behave. The occurrence of initial and boundary layers is quite common.

4.4 A Linear Singular Perturbation Problem

Consider the IVP

$$\varepsilon x'' + x' - x = 0, \quad x(0) = 1, \quad x'(0) = 5 \ .$$

We will assume that $0 < \varepsilon << 1$.

We first describe how to obtain an **approximate solution**. Neglecting the term $\varepsilon x''$ and ignoring the condition $x'(0) = 5$ one obtains the problem

$$x' - x = 0, \quad x(0) = 1 \ ,$$

with solution

$$x(t) = e^t \ .$$

Next, we want to add a term which takes the condition $x'(0) = 5$ into account. Since we expect a rapidly changing part of the solution, we neglect the x term in the equation and consider

$$\varepsilon x'' + x' = 0 \ .$$

We are not interested in constant solutions. Thus we must solve

$$\varepsilon x' + x = 0 \ .$$

The general solution is

$$x(t) = a e^{-t/\varepsilon} \ .$$

Consider a function of the form

$$x(t) = a e^{-t/\varepsilon} + e^t$$

where the constant a needs to be determined. We have

$$x'(t) = -\frac{a}{\varepsilon}\, e^{-t/\varepsilon} + e^t \ ,$$

thus

$$x'(0) = -\frac{a}{\varepsilon} + 1 \ .$$

The initial condition $x'(0) = 5$ yields that

$$a = -4\varepsilon \ .$$

We have obtained the following approximate solution of the IVP:

$$x_{app}(t) = -4\varepsilon e^{-t/\varepsilon} + e^t \ .$$

It is easy to check that

$$x_{app}(0) = 1 - 4\varepsilon, \quad x'_{app}(0) = 5 \ ,$$

and

$$\varepsilon x''_{app}(t) + x'_{app}(t) - x_{app}(t) = \varepsilon e^t + 4\varepsilon e^{-t/\varepsilon} \ .$$

Thus, the approximate solution x_{app} satisfies the initial conditions and the differential equation up to order ε.

Computation of the Exact Solution: The ansatz

$$x(t) = e^{\lambda t}$$

for solutions of the differential equation $\varepsilon x'' + x' - x = 0$ leads to

$$\varepsilon\lambda^2 + \lambda - 1 = 0, \quad \lambda^2 + \frac{1}{\varepsilon}\,\lambda - \frac{1}{\varepsilon} = 0 \ .$$

One obtains the roots

$$\lambda_{1,2} = -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1 + 4\varepsilon} \ .$$

The general solution of the differential equation $\varepsilon x'' + x' - x = 0$ is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

with derivative

$$x'(t) = c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \ .$$

The constants $c_{1,2}$ are determined by the initial conditions:

$$c_1 + c_2 = 1, \quad c_1\lambda_1 + c_2\lambda_2 = 5 \ .$$

One obtains the linear system

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

with solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \ ,$$

thus

$$c_1 = \frac{\lambda_2 - 5}{\lambda_2 - \lambda_1}, \quad c_2 = \frac{5 - \lambda_1}{\lambda_2 - \lambda_1} \ .$$

How much does the approximate solution

$$x_{app}(t) = -4\varepsilon e^{-t/\varepsilon} + e^t$$

differ from the exact solution

$$x_{exact}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \ ?$$

Using that

$$\sqrt{1+\eta} = 1 + \frac{\eta}{2} + \mathcal{O}(\eta^2)$$

we can compute the exact solution to leading order in ε.

We have

$$\begin{aligned} \lambda_{1,2} &= -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1+4\varepsilon} \\ &= -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\Big(1 + 2\varepsilon + \mathcal{O}(\varepsilon^2)\Big) \end{aligned}$$

thus

$$\lambda_1 = -\frac{1}{\varepsilon} + \mathcal{O}(1), \quad \lambda_2 = 1 + \mathcal{O}(\varepsilon)$$

and

$$c_1 = -4\varepsilon + \mathcal{O}(\varepsilon^2), \quad c_2 = 1 + \mathcal{O}(\varepsilon) \ .$$

Therefore, if one replaces $\lambda_{1,2}$ and $c_{1,2}$ by their leading order terms, then the exact solutiom

$$x_{exact}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

becomes the approximate solution

$$x_{app}(t) = -4\varepsilon e^{-t/\varepsilon} + e^t \ .$$

5 Imperfect Bifurcations and Catastrophes

See Section 3.6 of Strogatz.

What happens to a bifurcation if the equation is perturbed? As an example, consider the equation

$$\dot{x} = h + rx - x^3 \tag{5.1}$$

with parameters h and r. If $h = 0$ then a supercritical pitchfork bifurcation occurs at $r = 0$. The parameter h unfolds this singularity. It breaks the symmetry $f(-x) = -f(x)$ of the nonlinear term.

5.1 Perturbation of the Pitchfork

It is interesting to make sketches of the fixed points $x^* = x^*(h,r)$ were h is fixed and $-\infty < r < \infty$. For $h = 0$ a pitchfork occurs. Consider the following four curves. They have no tangent at $(x,r) = (0,0)$:

$$\begin{array}{llllllll}
\Gamma_1 & : & x(r) = 0 & \text{for} & r \le 0 & \text{and} & x(r) = \sqrt{r} & \text{for} \quad r \ge 0 \\
\Gamma_2 & : & x(r) = 0 & \text{for} & r \ge 0 & \text{and} & x(r) = -\sqrt{r} & \text{for} \quad r \ge 0 \\
\Gamma_3 & : & x(r) = 0 & \text{for} & r \le 0 & \text{and} & x(r) = -\sqrt{r} & \text{for} \quad r \ge 0 \\
\Gamma_4 & : & x(r) = 0 & \text{for} & r \ge 0 & \text{and} & x(r) = \sqrt{r} & \text{for} \quad r \ge 0
\end{array}$$

The pair Γ_1, Γ_2 as well as the pair Γ_3, Γ_4 make up the pitchfork which occurs for $h = 0$.

We claim: If $h > 0$ then the pair Γ_1, Γ_2 gets perturbed. The perturbed curves Γ_{1h}, Γ_{2h} are smooth. The curve Γ_{2h} has a saddle–node.

If $h < 0$ then the pair Γ_3, Γ_4 gets perturbed. The perturbed curves Γ_{3h}, Γ_{4h} are smooth. The curve Γ_{4h} has a saddle–node.

To see this, we solve the fixed point equation

$$h + rx - x^3 = 0$$

for r and obtain

$$r = r(x) = x^2 - \frac{h}{x} \ .$$

Sketch the graphs of

$$r = x^2, \quad r = -\frac{h}{x}, \quad r = x^2 - \frac{h}{x}$$

for $x > 0$ and for $x < 0$. The cases $h > 0$ and $h < 0$ are different. Then switch the two graphs (for $h > 0$ and for $h < 0$) of $r = x^2 - \frac{h}{x}$ and obtain x as a function of r.

5.2 Saddle–Nodes

Fix $h > 0$. We want to determine the saddle–node (r, x) on the curve Γ_{2h}. We have $r > 0 > x$ and

$$r = x^2 - \frac{h}{x} \quad \text{and} \quad 0 = \frac{dr}{dx} = 2x + \frac{h}{x^2} \ .$$

Obtain

$$\begin{aligned} x &= -(h/2)^{1/3} \\ r &= (h/2)^{2/3} + (2/h)^{1/3} h \\ &= (h/2)^{2/3} + 2(h/2)^{2/3} \\ &= 3(h/2)^{2/3} \end{aligned}$$

Thus, for $h > 0$ fixed, a saddle–node bifurcation occurs at

$$r_{sn} = 3(h/2)^{2/3} \ .$$

The saddle–node is

$$(r_{sn}, x_{sn}) = \Big(3(h/2)^{2/3}, -(h/2)^{1/3}\Big) \ .$$

The equation

$$r = 3(h/2)^{2/3}$$

can also be expressed as

$$h = 2(r/3)^{3/2} \ .$$

One obtains that

$$x_{sn} = -(h/2)^{1/3} = -(r_{sn}/3)^{1/2} \ . \tag{5.2}$$

Similarly, fix $h < 0$. A saddle–node (r, x) lies on Γ_{4h}. We have $r > 0$ and $x > 0$. The saddle–node is

$$(r_{sn}, x_{sn}) = \Big(3(-h/2)^{2/3}, (-h/2)^{1/3}\Big) \ .$$

The equation

$$r = 3(-h/2)^{2/3}$$

can also be expressed as

$$h = -2(r/3)^{3/2} \ .$$

The points in the (r, h)–plane, where a saddle–node bifurcation (w.r.t. r) occurs, lie on the cusp

$$h = \pm 2(r/2)^{3/2} \ . \tag{5.3}$$

5.3 Stability of Fixed Points

Fix r and h. Which fixed points x of the equation (5.1) are stable, which are unstable? Let

$$f(x,r,h) = h + rx - x^3, \quad f_x(x,r,h) = r - 3x^2 \ .$$

A fixed point x is a solution of the equation

$$h + rx - x^3 = 0 \ .$$

It is stable if

$$f_x(x,r,h) = r - 3x^2 < 0$$

and is unstable if

$$f_x(x,r,h) = r - 3x^2 > 0 \ .$$

In the following we fix $h < 0$. If $r \le 0$ and if x is a fixed point, then

$$f_x(x,r,h) = r - 3x^2 < 0 \ ,$$

thus x is stable. Next, let $r > 0$ and let x be a fixed point and let $x > 0$. We have

$$r = x^2 - \frac{h}{x} \ ,$$

thus

$$f_x(x,r,h) = r - 3x^2 = -2x^2 - \frac{h}{x} < 0 \ .$$

Thus, x is stable.

It remains to discuss $r > 0$ and fixed points $x < 0$. We have

$$f_x(x,r,h) = r - 3x^2 < 0 \quad \text{if} \quad x < -\sqrt{r/3}$$

and

$$f_x(x,r,h) = r - 3x^2 > 0 \quad \text{if} \quad -\sqrt{r/3} < x < 0 \ .$$

Recall from (5.2) that

$$-\sqrt{r/3} = x_{sn} \ .$$

It follows that the fixed points x with

$$x < x_{sn} < 0$$

are stable and the fixed points x with

$$x_{sn} < x < 0$$

are unstable. This is expected since a saddle–node bifurcation occurs at

$$(r_{sn}, x_{sn}) = \left(3(h/2)^{2/3}, -(h/2)^{1/3}\right) .$$

5.4 Bifurcation Diagrams for Fixed r

Consider the equation

$$f(x, r, h) = h + rx - x^3 = 0$$

for fixed r. If $r \leq 0$ then the function

$$x \to rx - x^3$$

is strictly monotonically decreasing. The equation

$$f(x, r, h) = 0$$

has a unique solution

$$x^* = x^*(h)$$

for each $h \in \mathbb{R}$. One can sketch the function $h = h(x) = -rx + x^3$ and then switch the axes to obtain the fixed points $x^* = x^*(h)$ as a function of h for $r \leq 0$ fixed. Note that

$$f_x(x, r, h) = r - 3x^2 < 0$$

for $r \leq 0$ and $x \neq 0$. Also, if $h = r = 0$, then the fixed point $x^* = 0$ is stable for the equation $x' = -x^3$. Therefore, if $r \leq 0$ then all fixed points $x^* = x^*(h)$ are stable.

Let $r > 0$. The function

$$g(x) = rx - x^3$$

attains a local maximum at

$$x_1 = \sqrt{r/3}$$

and a local minimum at $x_2 = -x_1$. We have

$$M := g(x_1) = \frac{2}{3\sqrt{3}}\, r^{3/2}$$

and $g(x_2) = -M$. For $-M < h < M$ the equation

$$h + rx - x^3 = 0$$

has three solutions. Two are stable fixed points, the middle one is unstable. For $h = M$ and $h = -M$ two fixed points collide; a saddle–node bifurcation occurs.

One can sketch the function $h = h(x) = -rx + x^3$ and then switch the axes to obtain the branch of fixed points $x^* = x^*(h)$ for $r > 0$ fixed. One obtains a hysteresis loop.

5.5 The 3D Surface of Fixed points

Figure 3.6.5 in Strogatz shows the surface of fixed points

$$x^* = x^*(r, h) \ .$$

One can recognize the hysteresis loops. The saddle–nodes occur for parameters (r, h) on the cusp (5.3). The stable state of the system can change discontinuously if the parameters (r, h) cross the cusp. This may correspond to a catastrophe.

It is more difficult to recognize the pitchfork bifurcation with respect to r at $h = 0$ and the perturbed pitchfork bifurcations, which occur for fixed $h \neq 0$ as a function of r.

6 Flows on the Circle

See Chapter 4 of Strogatz.

So far, we have considered differential equations $x' = f(x)$ where $x(t)$ is a real valued function. For each solution $x(t)$ there are only three possibilities: The function $x(t)$ increases strictly or decreases strictly or is constant, $x(t) \equiv x^*$. The last case occurs if and only if x^* is a fixed point, $f(x^*) = 0$.

We will now consider functions of time $\theta(t)$ which take values in the circle $S^1 = \mathbb{R} \bmod 2\pi$. The values θ and $\theta + 2\pi$ are identified.

A function $f : \mathbb{R} \to M$, which satisfies $f(\theta + 2\pi) \equiv f(\theta)$, may be considered as a function $f : S^1 \to M$. And, conversely, a function $f : S^1 \to M$ may be considered as a function $f : \mathbb{R} \to M$ which is 2π periodic.

6.1 Setup and Simple Examples

Let $f : S^1 \to \mathbb{R}$ denote a C^1 function and consider the equation

$$\dot{\theta} = f(\theta) \ .$$

Any solution $\theta(t)$ will be considered as a function from $\mathbb{R}$ to S^1.

Example 1: Let $\omega \in \mathbb{R}$ be fixed. The equation

$$\dot{\theta} = \omega, \quad \theta(0) = \theta_0 \ ,$$

is solved by

$$\theta(t) = (\omega t + \theta_0) \bmod 2\pi \ .$$

If $\omega > 0$ the period is $T = 2\pi/\omega$.

Example 2: (two runners on a circular track) Let $\omega_1 > \omega_2 > 0$ and consider the two equations

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2 \ .$$

The two runners have periods

$$T_1 = \frac{2\pi}{\omega_1}, \quad T_2 = \frac{2\pi}{\omega_2} \ .$$

How long will it take for runner one to lap runner two?

Let $\phi = \theta_1 - \theta_2$. We have

$$\dot{\phi} = \omega_1 - \omega_2, \quad \phi(0) = 0 \ ,$$

thus

$$\phi(t) = (\omega_1 - \omega_2)t \ .$$

The lap time satisfies

$$(\omega_1 - \omega_2)T_{lap} = 2\pi \ .$$

Therefore,

$$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{\frac{1}{T_1} - \frac{1}{T_2}} \, .$$

6.2 A Nonuniform Oscillator

Consider the equation

$$\dot{\theta} = \omega - a \sin\theta \tag{6.1}$$

where $\omega > 0$ is fixed and $a \geq 0$ is a parameter. For $0 \leq a < \omega$ the fixed point equation

$$\omega - a \sin\theta = 0$$

has no solution θ. The solution $\theta(t)$ of the differential equation (6.1) describes motion around the circle, which is counterclockwise and periodic. We will compute its period to be

$$T(a) = \frac{2\pi}{\sqrt{\omega^2 - a^2}} \, .$$

Note that the period tends to infinity as a approaches ω from below.

If ω is fixed and a increases from $a < \omega$ to $a > \omega$ then a saddle–node bifurcation of fixed points occurs at $a = \omega$. For $a = \omega$ the fixed point is $\theta^*(\omega) = \pi/2$. For $a > \omega$ there are two fixed points $\theta^*_{1,2}(a)$.

Computation of the period $T(a)$: Let $0 < a < \omega$ and let $\theta(t)$ solve

$$\dot{\theta} = \omega - a \sin\theta, \quad \theta(0) = 0 \, .$$

We have

$$\int_0^{\theta(t)} \frac{d\theta}{\omega - a \sin\theta} = \int_0^t dt = t \, .$$

Therefore, the period T is given by

$$T = \int_0^{2\pi} \frac{d\theta}{\omega - a \sin\theta} \, .$$

We evaluate the integral using complex variables.

(For an evaluation using calculus, see Strogatz, Exercise 4.3.2.)

Recall that

$$\sin\theta = \frac{1}{2i} \left(e^{i\theta} - e^{-i\theta} \right) \, .$$

Substituting

$$z = e^{i\theta}, \quad dz = iz d\theta \, ,$$

one obtains (Γ denotes the unit circle):

$$
\begin{aligned}
T &= \int_\Gamma \frac{dz}{iz\Big(\omega - a(z - z^{-1})/2i\Big)} \\
&= \frac{-2}{a}\int_\Gamma \frac{dz}{z^2 - 2i\omega z/a - 1}
\end{aligned}
$$

The quadratic equation

$$z^2 - \frac{2i\omega}{a}\, z - 1 = 0$$

has the solutions

$$z_1 = \frac{i\omega}{a} + i\sqrt{\frac{\omega^2}{a^2} - 1}$$

and

$$z_2 = \frac{i\omega}{a} - i\sqrt{\frac{\omega^2}{a^2} - 1}\ .$$

Clearly, $|z_1| > 1$ and $z_1 z_2 = 1$, thus $|z_2| < 1$. By the residue theorem,

$$T = \frac{-2}{a}\int_\Gamma \frac{dz}{(z - z_1)(z - z_2)} = 2\pi i \frac{-2}{a}\,(z_2 - z_1)^{-1}\ .$$

Since

$$z_2 - z_1 = -2i\sqrt{\frac{\omega^2}{a^2} - 1}$$

one finds that

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}\ .$$

Approximation of the period for $a = \omega - \varepsilon$: If $a = \omega - \varepsilon$ then

$$\omega^2 - a^2 = 2\omega\varepsilon - \varepsilon^2 = (2\omega - \varepsilon)\varepsilon$$

and

$$
\begin{aligned}
T(\omega - \varepsilon) &= \frac{2\pi}{\sqrt{2\omega - \varepsilon}} \cdot \frac{1}{\sqrt{\varepsilon}} \\
&= \frac{2\pi}{\sqrt{2\omega}} \cdot \frac{1}{\sqrt{\varepsilon}} + \mathcal{O}(\sqrt{\varepsilon})
\end{aligned}
$$

Bottleneck and ghost saddle–node: For $a = \omega - \varepsilon$ the periodic solution $\theta(t)$ passes through a bottleneck near $\theta = \pi/2$. The passage through the bottleneck takes up most of the time of the full periodic motion. For $a = \omega + \varepsilon$ the equation has two fixed points $\theta^*_{1,2}(\omega + \varepsilon)$ near $\theta = \pi/2$. They come up through a saddle–node bifurcation occurring at $a = \omega$. It is interesting that the blow–up of the period, $T(a) \to \infty$ as $a \to \omega-$, is related to the occurrence of a ghost saddle–node.

Scaling law for the passage time through a bottleneck: Consider an equation

$$\dot{x} = \varepsilon + \lambda x^2$$

where $\lambda > 0$ is fixed and $0 < \varepsilon << 1$. The equation has no fixed point. Consider the solution with $x(0) = 0$. There are finite times t_0 and t_1 with

$$t_0 < 0 < t_1$$

and

$$\lim_{t \to t_1-} x(t) = \infty \qquad \lim_{t \to t_0+} x(t) = -\infty \; .$$

Then $T = t_1 - t_0$ is the life–time of the solution. We have

$$\begin{aligned}
T &= \int_{-\infty}^{\infty} \frac{dx}{\varepsilon + \lambda x^2} \\
&= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \frac{dx}{1 + \lambda x^2/\varepsilon} \qquad (y = x\sqrt{\lambda/\varepsilon}) \\
&= \frac{1}{\sqrt{\varepsilon\lambda}} \int_{-\infty}^{\infty} \frac{dy}{1 + y^2} \\
&= \frac{1}{\sqrt{\varepsilon\lambda}} \arctan y \Big|_{y=-\infty}^{y=\infty} \\
&= \frac{\pi}{\sqrt{\varepsilon\lambda}} \\
&= \frac{2\pi}{\sqrt{4\lambda}} \cdot \frac{1}{\sqrt{\varepsilon}}
\end{aligned}$$

Relation to the oscillator period: In the oscillator equation

$$\dot{\theta} = \omega - a \sin\theta$$

let $a = \omega - \varepsilon$ and $\theta = \frac{\pi}{2} + x$. The equation becomes

$$\dot{x} = \omega - (\omega - \varepsilon)\sin(\frac{\pi}{2} + x) \; .$$

Consider the equation for $|x| << 1$ and use the approximation

$$\sin(\frac{\pi}{2} + x) \sim 1 - \frac{x^2}{2} \; .$$

Obtain

$$\dot{x} = \omega - (\omega - \varepsilon) + (\omega - \varepsilon)\frac{x^2}{2} \; .$$

Neglecting the εx^2–term, one obtains

$$\dot{x} = \varepsilon + \frac{\omega}{2} x^2 \ .$$

This is the previous equation with

$$\lambda = \frac{\omega}{2} \ .$$

One obtains for the life–time of the solution

$$T = \frac{2\pi}{\sqrt{2\omega}} \cdot \frac{1}{\sqrt{\varepsilon}} \ .$$

This agrees with the period of the oscillator up to a term of order $\mathcal{O}(\sqrt{\varepsilon})$.

6.3 The Overdamped Pendulum

The equation

$$\dot{\theta} = \omega - a \sin\theta$$

can be interpreted by a physical example. Recall the torque vector $\mathbf{r} \times \mathbf{F} = \mathbf{\Gamma}$ of a force $\mathbf{F}$ applied at a point with position vector $\mathbf{r}$. Consider a pendulum with friction and an applied torque:

$$mL\ddot{\theta} = -mg\sin\theta + \frac{\Gamma}{L} - \frac{b}{L}\dot{\theta} \ .$$

Neglect the second–order term and obtain:

$$b\dot{\theta} = \Gamma - mgL\sin\theta \ .$$

Let

$$\tilde{\theta}(\tau) = \theta(T\tau) \ .$$

Choose

$$T = \frac{b}{mgL}, \quad \gamma = \frac{T\Gamma}{b}$$

to obtain

$$\tilde{\theta}' = \gamma - \sin\tilde{\theta} \ .$$

6.4 Pendulum Equations

1. The equation for the unforced, undamped pendulum is

$$mL\ddot{\theta} = -mg\sin\theta \ .$$

One can rescale time so that the equation becomes

$$\ddot{\theta} + \sin\theta = 0 \ .$$

A formula relating the period T to the amplitude α can be obtained. See Strogatz, exercise 6.7.4 on page 192.

2. The equation for the unforced, damped pendulum is

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0, \quad b > 0 \ .$$

3. The equation for the forced, undamped pendulum is

$$\ddot{\theta} + \sin\theta = \gamma \ .$$

Here the constant γ results from a constant torque.

7 Linear Systems

See Chapter 7 of Strogatz.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} .$$

The dynamics of the system $x' = Ax$ depends crucially on the eigenvalues $\lambda_{1,2}$ of A.

The eigenvalues λ of A are the solutions of the equation

$$\begin{aligned} 0 &= det(A - \lambda I) \\ &= (a-\lambda)(d-\lambda) - cb \\ &= \lambda^2 - (a+d)\lambda + det(A) \end{aligned}$$

Therefore,

$$\lambda_{1,2} = \frac{1}{2}\, tr(A) \pm \frac{1}{2}\sqrt{(tr(A))^2 - 4det(A)}\ .$$

Definition: Let $u^* \in \mathbb{R}^n$ denote a fixed point of the system $u' = f(u)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let $A = f'(u^*) \in \mathbb{R}^{n\times n}$ denote the Jacobian of f at u^*. The fixed point u^* is called hyperbolic if all eigenvalues of A have a non–zero real part.

The Hartman–Grobman theorem (1959), named after Philip Hartman and D.M. Grobman (Russian):

Theorem 7.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and let u^* denote a hyperbolic fixed point of the system $u' = f(u)$. Let $A = f'(u^*)$. There exist open subsets U and V in $\mathbb{R}^n$ with*

$$u^* \in U, \quad 0 \in V$$

and there exists a continuous map $H : U \to V$ which is 1-1 and onto so that H^{-1} is also continuous and, for all $u_0 \in U$,

$$H(u(t, u_0)) = e^{At} H(u_0), \quad -\varepsilon \le t \le \varepsilon\ ,$$

where $\varepsilon = \varepsilon(u_0) > 0$.

Roughly speaking: If u^* is a hyperbolic fixed point of $u' = f(u)$, then the local phase portrait for $u' = f(u)$ near u^* is topological equivalent to the local phase portrait of $v' = Av$ (with $A = f'(u^*)$) near $v = 0$. Here the local homeomorphism preserves the parameterization by time of all local orbits.

An example of a system where the linearization about the fixed point zero gives the wrong qualitative picture:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a(x^2+y^2) \begin{pmatrix} x \\ y \end{pmatrix} . \tag{7.1}$$

Details: Let us write the system on polar coordinates. We have

$$x = r\cos\phi, \quad y = r\sin\phi$$

thus

$$r^2 = x^2 + y^2, \quad \phi = \arctan(y/x) \ .$$

One obtains:

$$\begin{aligned} rr' &= xx' + yy' \\ &= x(-y + ar^2x) + y(x + ar^2y) \\ &= ar^4 \end{aligned}$$

thus

$$r' = ar^3 \ .$$

Also,

$$\phi' = \frac{1}{1 + (y/x)^2} \cdot \frac{y'x - yx'}{x^2} = \frac{y'x - yx'}{x^2 + y^2} \ ,$$

where

$$y'x - yx' = (x + ar^2y)x - y(-y + ar^2x) = r^2 \ .$$

Therefore,

$$\phi' = 1 \ .$$

The nonlinear system (7.1) reads in polar coordinates:

$$r' = ar^3, \quad \phi' = 1 \ .$$

The system linearized about zero becomes

$$r' = 0, \quad \phi' = 1 \ .$$

If $a > 0$ all solutions with $r(0) > 0$ blow up it finite time. If $a < 0$ all solutions spiral to zero.

Solution of the linear system

$$u' = Au, \quad A \in \mathbb{R}^{2\times 2}$$

Let $\lambda_{1,2}$ denote the eigenvalues of A.

Case 1: $\lambda_{1,2}$ real and $\lambda_1 \neq \lambda_2$.
Let $Av^{(j)} = \lambda_j v^{(j)}$. The general solution is

$$u(t) = c_1 e^{\lambda_1 t} v^{(1)} + c_2 e^{\lambda_2 t} v^{(2)} \ .$$

Case 2: $\lambda_{1,2} = \alpha \pm i\omega; \alpha, \omega \in \mathbb{R}, \omega > 0$

Let

$$A(a + ib) = (\alpha + i\omega)(a + ib), a + ib \neq 0 \ .$$

Then the matrix $S = (a|b)$ is nonsingular.

Case 3: $\lambda_1 = \lambda_2 = \lambda$

Case 3a: $A = \lambda I$

Case 3b: A is not diagonalizable.

7.1 The Case of Complex Eigenvalues

Set

$$\tau = tr(A) = a + d, \quad \Delta = det(A) = ad - bc \ .$$

The eigenvalues λ of A are the solutions of the quadratic equation

$$\lambda^2 - \tau\lambda + \Delta = 0 \ ,$$

i.e.,

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta} \ .$$

Assuming that

$$\tau^2 - 4\Delta < 0$$

the eigenvalues are

$$\lambda_1 = \alpha + i\omega, \quad \lambda_2 = \alpha - i\omega$$

with

$$\begin{aligned} \alpha &= \frac{1}{2}\tau \\ \omega &= \frac{1}{2}\sqrt{4\Delta - \tau^2} \end{aligned}$$

We want to obtain the general solution of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{7.2}$$

in real form.

First, let's determine the general solution in complex form. Let

$$A(p + iq) = \lambda_1(p + iq) \quad \text{where} \quad p + iq \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p, q \in \mathbb{R}^2 \ ,$$

i.e., $p + iq \in \mathbb{C}^2$ is an eigenvector of A to the eigenvalue $\lambda_1 = \alpha + i\omega$. It follows that $p - iq$ is an eigenvector of A to the eigenvalue $\lambda_2 = \alpha - i\omega$. The general solution of the system (7.2) in complex form is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a_1 e^{\lambda_1 t}(p + iq) + a_2 e^{\lambda_2 t}(p - iq)$$

where $a_1, a_2 \in \mathbb{C}$ are complex parameters.

To determine the general solution in real form we first note that $q \neq 0$. Otherwise,

$$Ap = (\alpha + i\omega)p$$

where

$$p \in \mathbb{R}^2, \quad p \neq 0, \quad \alpha, \omega \in \mathbb{R}, \quad \omega \neq 0 \ ,$$

a contradiction. Assuming that $p = \gamma q, \gamma \in \mathbb{R}$, one obtains that

$$A(\gamma + i)q = \lambda_1(\gamma + i)q \ ,$$

thus

$$Aq = \lambda_1 q \ .$$

As above, this leads to a contradiction. One obtains that the vectors

$$p, q \in \mathbb{R}^2$$

are linearly independent. Consider the complex solution

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{\lambda_1 t}(p + iq) \\ &= e^{\alpha t}\Big(\cos(\omega t) + i\sin(\omega t)\Big)(p + iq) \\ &= e^{\alpha t}\Big(\cos(\omega t)p - \sin(\omega t)q\Big) + ie^{\alpha t}\Big(\cos(\omega t)q + \sin(\omega t)p\Big) \end{aligned}$$

It follows that the real part and the imaginary part of this solution are two real solutions,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}(t) = e^{\alpha t}\Big(\cos(\omega t)p - \sin(\omega t)q\Big), \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}(t) = e^{\alpha t}\Big(\cos(\omega t)q + \sin(\omega t)p\Big) \ .$$

The general solution in real form is

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = a_1 e^{\alpha t}\Big(\cos(\omega t)p - \sin(\omega t)q\Big) + a_2 e^{\alpha t}\Big(\cos(\omega t)q + \sin(\omega t)p\Big)$$

where $a_1, a_2 \in \mathbb{R}$.

If $\alpha < 0$ then the origin is a stable spiral point. If $\alpha > 0$ then the origin is an unstable spiral point.

In the following, let $\alpha = 0$. We claim that all trajectories (except the origin), are ellipses. Set

$$\begin{aligned} c(t) &= \cos(\omega t) \\ s(t) &= \sin(\omega t) \\ \Phi &= (p|q) \in \mathbb{R}^{2\times 2} \end{aligned}$$

We have

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}(t) = c(t)p - s(t)q = \Phi \begin{pmatrix} c \\ -s \end{pmatrix}$$

and

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}(t) = c(t)q + s(t)p = \Phi \begin{pmatrix} s \\ c \end{pmatrix} .$$

The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \Phi \begin{pmatrix} c & s \\ -s & c \end{pmatrix}(t) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Let

$$\Phi = U \Sigma V^T$$

denote the singular value decomposition of the matrix Φ. We have

$$\Sigma^{-1} U^T \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = V^T \begin{pmatrix} c & s \\ -s & c \end{pmatrix}(t) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .$$

It follows that the Euclidean norm of the vector

$$\Sigma^{-1} U^T \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is constant in time:

$$\Big| \Sigma^{-1} U^T \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \Big|^2 = a_1^2 + a_2^2 .$$

Set

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = U^T \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} .$$

Then we have

$$\Big(\frac{\xi(t)}{\sigma_1}\Big)^2 + \Big(\frac{\eta(t)}{\sigma_2}\Big)^2 = a_1^2 + a_2^2 = const .$$

Let

$$U = \left(u^1 | u^2 \right) .$$

The two vectors $u^1, u^2 \in \mathbb{R}^2$ are orthonormal. We have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \xi(t) u^1 + \eta(t) u^2 .$$

The trajectory

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

forms an ellipse with semi–axes in u^1–u^2 directions.

8 Special Classes of Systems

8.1 Conservative Systems

Consider the second order ODE

$$m\ddot{x} = F(x)$$

where $m > 0$ is constant. Define

$$V(x) = -\int_0^x F(\xi)\, d\xi$$

and consider the function

$$E(t) = \frac{m}{2}\, (\dot{x}(t))^2 + V(x(t))$$

for a solution $x(t)$. Obtain that

$$\dot{E}(t) = m\dot{x}(t)\ddot{x}(t) - F(x(t))\dot{x}(t) \equiv 0\ .$$

Therefore, $E(t) \equiv E(0)$. In applications, the term $\frac{m}{2}\dot{x}^2$ often is the kinetic energy and $V(x)$ is the potential energy. Assuming the evolution equation $m\ddot{x} = F(x)$, energy is conserved.

8.2 Reversible Systems

Let $f : \mathbb{R}^n \to \mathbb{R}^n, f \in C^1$. Let $R \in \mathbb{R}^{n\times n}$ with $R^2 = I$. Assume

$$-Rf(x) = f(Rx)$$

for all $x \in \mathbb{R}^n$. Then the following holds: If $x(t)$ is a solution of $\dot{x} = f(x)$ then

$$\tilde{x}(t) = Rx(-t)$$

is also a solution of $\dot{x} = f(x)$.

8.3 Gradient Systems

Consider the first order system $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. If the function $f(x)$ has the form

$$f(x) = -\nabla V(x)$$

for some function $V : \mathbb{R}^n \to \mathbb{R}$, then the system

$$\dot{x} = f(x) = -\nabla V(x)$$

is called a gradient system.

Claim: A gradient system does not have any periodic solutions. To show this, note that

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot \dot{x}(t) = -|\dot{x}(t)|^2 \ .$$

Assume $t_1 < t_2$ and $x(t_1) = x(t_2)$, thus $V(x(t_1)) = V(x(t_2))$. Then obtain that

$$\begin{aligned} 0 &= V(x(t_2)) - V(x(t_1)) \\ &= \int_{t_1}^{t_2} \frac{d}{dt} V(x(t))\, dt \\ &= -\int_{t_1}^{t_2} |\dot{x}(t)|^2\, dt \end{aligned}$$

Therefore, $\dot{x}(t) \equiv 0$ for $t_1 \le t \le t_2$. The point $x(t_1) = x(t_2)$ is a fixed point.

8.4 Systems with a Liapunov Function

Consider the example

$$\ddot{x} + (\dot{x})^3 + x = 0 \ . \tag{8.1}$$

Let

$$E(x, \dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2) \ .$$

If

$$h(t) = E(x(t), \dot{x}(t))$$

then

$$\begin{aligned} \dot{h} &= x\dot{x} + \dot{x}\ddot{x} \\ &= x\dot{x} + \dot{x}(-\dot{x}^3 - x) \\ &= -\dot{x}^4 \\ &\le 0 \end{aligned}$$

This implies that the energy $h(t)$ never grows in time. But more follows: If $T > 0$ then the energy decays strictly for $0 \le t \le T$ (i.e., $h(T) < h(0)$) unless $\dot{x}(t) \equiv 0$ for $0 \le t \le t$. Thus, the energy decays strictly for $0 \le t \le T$ unless

$$(x(t), \dot{x}(t)) = (x(0), 0) \quad \text{for} \quad 0 \le t \le T \ .$$

The energy decays strictly along any orbit which is not a fixed point. It follows that the 2nd order equation (8.1) does not have a periodic solution.

9 Periodic Solutions

Consider the following system in polar coordinates:

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1 \; .$$

Clearly, the solution

$$r(t) \equiv 1, \quad \theta(t) = t \bmod 2\pi \tag{9.1}$$

describes motion on a circle of radius $r^* = 1$. If $r(0) = r_0 > 0$ then $r(t) \to 1$ as $t \to \infty$. The periodic orbit (9.1) is an example of a stable limit cycle.

The Poincaré–Bendixson Theorem gives conditions for the existence of a periodic solution of a two–dimensional system:

Theorem 9.1 *Let $\Omega \subset \mathbb{R}^2$ denote an open set and let $f : \Omega \to \mathbb{R}^2, f \in C^1$. Assume there is a compact set $R \subset \Omega$ with the following properties:*

a) If $u \in R$ then $f(u) \neq 0$, i.e., the system $u' = f(u)$ has no fixed point in R.

b) There exists $u_0 \in R$ so that $u(t, u_0) \in R$ for all $t \geq 0$.

Then: Either the solution $u(t, u_0)$ is periodic or the orbit $u(t, u_0)$ approaches a limit cycle.

We sketch a proof of the Poincaré–Bendixson Theorem below. The proof uses Jordan's Curve Theorem, named after Camille Jordan (1838–1922). (Camille Jordan is also known for the Jordan normal form of a matrix.) Jordan's Curve Theorem is rather plausible, but difficult to prove rigorously.

Definition: Let

$$\Gamma_0 = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$$

denote the unit circle. A set $\Gamma \subset \mathbb{R}^2$ is called a Jordan curve if there exists a map $c : \Gamma_0 \to \Gamma$ which is continuous, 1-1, onto, and which has a continuous inverse. In other words, a set $\Gamma \subset \mathbb{R}^2$ is called a Jordan curve if Γ is homeomorphic to the unit circle Γ_0.

Theorem 9.2 *Let $\Gamma \subset \mathbb{R}^2$ denote a Jordan curve. Then one can write*

$$\mathbb{R}^2 \setminus \Gamma = A \cup B$$

where A and B are nonempty, disjoint, open connected subsets of $\mathbb{R}^2$. One set, A, is bounded, the other, B, is unbounded. Then A is called the interior and B the exterior of Γ.

A proof of Jordan's theorem (and generalizations) is given in algebraic topology.

Sketch of Proof of Poincaré–Bendixson: Write $u(t) = u(t, u_0)$ for the solution of the initial value problem

$$u' = f(u), \quad u(0) = u_0 \ .$$

The sequence $u(n), n = 1, 2, \ldots$ has an accumulation point $P \in R$ and $f(P) \neq 0$. Draw the line L through P perpendicular to $f(P)$. Consider the solution $\tilde{u}(t)$ of $u' = f(u)$ with $\tilde{u}(0) = P$. Given $\varepsilon > 0$ there is a time $T > 0$ so that

$$\|\tilde{u}(T) - P\| < \varepsilon \quad \text{and} \quad \tilde{u}(T) \in L \ .$$

One can now make simple sketches showing that, if $\varepsilon > 0$ is small, then $\tilde{u}(T) \neq P$ leads to contradictions. Here one uses Jordan's Curve Theorem. Therefore, $\tilde{u}(T) = P$. The solution $\tilde{u}(t)$ is periodic.

We will use the following result:

Lemma 9.1 *Let $A \in \mathbb{R}^{2\times 2}$ and let*

$$\Delta = det(A), \quad \tau = tr(A) \ .$$

The eigenvalues of A are

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{1}{2} \ , \sqrt{\tau^2 - 4\Delta} \ .$$

If $\Delta > 0$ and $\tau > 0$ then

$$\lambda_1 \geq \lambda_2 > 0 \quad \textit{or} \quad \mathrm{Re}\,\lambda_{1,2} > 0 \ .$$

If $\Delta > 0$ and $\tau < 0$ then

$$\lambda_1 \leq \lambda_2 < 0 \quad \textit{or} \quad \mathrm{Re}\,\lambda_{1,2} < 0 \ .$$

An application to a system for a biochemical process:

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y = f(x,y) \\ \dot{y} &= b - ay - x^2 y = g(x,y) \end{aligned}$$

(This is a strongly simplified system for glycolysis.)

Here $x(t), y(t)$ are concentrations and a, b are positive parameters. The system has the fixed point

$$P^* = (x^*, y^*)$$

with

$$x^* = b, \quad y^* = \frac{b}{a + b^2} \ .$$

If A denotes the Jacobian at P^* then

$$det(A) = a + b^2 > 0 \ .$$

Also,

$$tr(A) = -\frac{1}{a+b^2}\left(b^4 + (2a-1)b^2 + a + a^2\right) .$$

If

$$tr(A) < 0$$

then P^* is a stable fixed point. If $tr(A) > 0$ then P^* is a repelling fixed point.

One can obtain a region R_1 which a trajectory starting in R_1 cannot leave. Then, if P^* is repelling, the Poincaré–Bendixson theorem applies to the region

$$R = R_1 \setminus B_\varepsilon(P^*) .$$

Draw in the first quadrant of the (a, b) plane the curve determined by the condition

$$tr(A) = 0 .$$

This curve is given by

$$b^4 + (2a-1)b^2 + a + a^2 = 0 .$$

Or:

$$b^2 = \frac{1}{2}(1-2a) \pm \frac{1}{2}\sqrt{1-8a}, \quad 0 < a \leq \frac{1}{8} .$$

10 Perturbation Theory

10.1 Regular Perturbations

Example: Let $A, B \in \mathbb{R}^{n\times n}, det(A) \neq 0$. Let $b \in \mathbb{R}^n$. Consider the linear system

$$(A+\varepsilon B)x = b$$

for $|\varepsilon| << 1$. The term εBx perturbs the unperturbed system $Ax = b$. The unperturbed system has the solution

$$x^{(0)} = A^{-1}b \; .$$

What is the leading order perturbation of $x^{(0)}$ if the matrix εB is added to A?

Write

$$x = x^{(0)} + \varepsilon x^{(1)} + \mathcal{O}(\varepsilon^2)$$

and obtain the equation

$$Ax^{(1)} = -Bx^{(0)} \; .$$

We expect that the solution x of the perturbed system $(A+\varepsilon B)x = b$ is

$$x = x^{(0)} - \varepsilon A^{-1}Bx^{(0)} + \mathcal{O}(\varepsilon^2) \quad \text{where} \quad x^{(0)} = A^{-1}b \; .$$

This is correct. The process can be extended and justified by a geometric–sum argument.

Details: We first consider **complex numbers**. If $q \in \mathbb{C}$ and $|q| < 1$ then

$$q^n \to 0 \quad \text{as} \quad n \to \infty \; .$$

We have

$$(1+q+\ldots+q^n)(1-q) = 1-q^{n+1} \; ,$$

thus

$$\sum_{j=0}^{n} q^j = \frac{1-q^{n+1}}{1-q} \quad \text{for} \quad q \neq 1 \; .$$

As $n \to \infty$ obtain the geometric sum formula

$$\sum_{j=0}^{\infty} q^j = \frac{1}{1-q} \quad \text{for} \quad |q| < 1 \; .$$

For $|q| < 1$ we have

$$\begin{aligned}\sum_{j=0}^{n} q^j - \frac{1}{1-q} &= \sum_{j=n+1}^{\infty} q^j \\ \Big|\sum_{j=0}^{n} q^j - \frac{1}{1-q}\Big| &\le \frac{|q^{n+1}|}{|1-q|}\end{aligned}$$

Therefore,

$$\Big|\sum_{j=0}^{n} q^j - \frac{1}{1-q}\Big| \le 2|q|^{n+1} \quad \text{for} \quad |q| \le \frac{1}{2} \ .$$

One can generalize this from numbers q to **square matrices** $Q \in \mathbb{C}^{m\times m}$ and even to bounded linear operators Q. If $Q \in \mathbb{C}^{m\times m}$ then

$$\sum_{j=0}^{\infty} Q^j = (I-Q)^{-1} \quad \text{if} \quad \|Q\| < 1$$

and

$$\|\sum_{j=0}^{n} Q^j - (I-Q)^{-1}\| \le 2\|Q\|^{n+1} \quad \text{if} \quad \|Q\| \le \frac{1}{2} \ . \tag{10.1}$$

In particular, we have for $n=1$:

$$\|I + Q - (I-Q)^{-1}\| \le 2\|Q\|^2 = 2\varepsilon^2\|A^{-1}B\|^2 \quad \text{if} \quad \varepsilon\|A^{-1}B\| \le \frac{1}{2} \ . \tag{10.2}$$

Now consider the perturbed matrix equation

$$(A + \varepsilon B)x = b \tag{10.3}$$

where $|\varepsilon|$ is small and A is nonsingular. For $\varepsilon = 0$ the solution of the unperturbed equation is

$$x^{(0)} = A^{-1}b \ .$$

Write the perturbed equation (10.3) as

$$A(I + \varepsilon A^{-1}B)x = b$$

and set $Q = -\varepsilon A^{-1}B$. Assuming $\|Q\| < 1$ obtain that the solution of the perturbed equation is

$$x_{exact} = (I-Q)^{-1}A^{-1}b = (I-Q)^{-1}x^{(0)} \ .$$

The first order approximate solution of the perturbed equation is

$$x^{(0)} + \varepsilon x^{(1)} = x^{(0)} - \varepsilon A^{-1}Bx^{(0)} = (I+Q)x^{(0)} \ .$$

Using (10.2) obtain the estimate

$$\|x_{exact} - (x^{(0)} + \varepsilon x^{(1)})\| \le C\varepsilon^2$$

with $C = 2\|A^{-1}B\|^2$ if $\varepsilon\|A^{-1}B\| \le \frac{1}{2}$.

10.2 An Initial–Value Problem: The Regular Perturbation Approach

The main point of this section is to show that the regular perturbation approach may not be useful if differential equations are considered on long time intervals.

Let $0 \le \varepsilon << 1$. Consider the IVP

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1 \ . \tag{10.4}$$

A function $x(t) = e^{\lambda t}$ solves the differential equation if

$$\lambda^2 + 2\varepsilon\lambda + 1 = 0 \ .$$

The roots are

$$\lambda_{1,2} = -\varepsilon \pm i\sqrt{1-\varepsilon^2} \ .$$

The general solution of the differential equation is

$$x(t) = ae^{-\varepsilon t}\cos(\sqrt{1-\varepsilon^2}\,t) + be^{-\varepsilon t}\sin(\sqrt{1-\varepsilon^2}\,t) \ .$$

The exact solution of the IVP is

$$x_{exact}(t,\varepsilon) = \frac{1}{\sqrt{1-\varepsilon^2}}\, e^{-\varepsilon t}\sin\left(\sqrt{1-\varepsilon^2}\,t\right) \ .$$

The solution oscillates almost like $\sin t$, but decays slowly if $0 < \varepsilon << 1$. For $\varepsilon = 0$ the solution is

$$x^{(0)}(t) = \sin t \ .$$

We want to apply the formal process of regular perturbation theory to obtain the next order correction to the solution $\sin t$ of the unperturbed problem. Let

$$x(t) = \sin t + \varepsilon x^{(1)}(t) + \mathcal{O}(\varepsilon^2)$$

Proceeding as in the previous section we obtain the IVP

$$\ddot{x}^{(1)} + x^{(1)} = -2\cos t, \quad x^{(1)}(0) = \dot{x}^{(1)}(0) = 0 \tag{10.5}$$

for the correction term $x^{(1)}$. The solution of (10.5) is

$$x^{(1)}(t) = -t\sin t \ .$$

This yields, formally,

$$x_{app}(t,\varepsilon) = \sin t - \varepsilon t\sin t + \mathcal{O}(\varepsilon^2) \ .$$

One can prove: For every fixed $T > 0$ it holds that

$$\max_{|t|\leq T} |x_{exact}(t,\varepsilon) - (1-\varepsilon t)\sin t) \leq C_T \varepsilon^2 \ . \qquad (10.6)$$

However, the approximate solution $(1-\varepsilon t)\sin t$ is useless for $\varepsilon t = \mathcal{O}(1)$ and is very bad for $\varepsilon t >> 1$.

Remark: The geometric–sum argument of regular perturbation theory fails if one considers the initial value problem (10.4) on the interval $0 \leq t < \infty$. The reason is that the solution operator of the problem

$$\ddot{x} + x = b(t), \quad x(0) = \dot{x}(0) = 0 \qquad (10.7)$$

is unbounded w.r.t $|\cdot|_\infty$. If, for example, $b(t) = 2\cos t$, then the solution is $x(t) = t\sin t$. This shows that one cannot bound the maximum norm of the solution $x(t)$ of the initial value problem (10.7) in terms of the maximum norm of the right–hand side, $b(t)$.

Proof of Estimate (10.6): We have

$$\begin{aligned}
\frac{1}{\sqrt{1-\varepsilon^2}} &= 1 + \mathcal{O}(\varepsilon^2) \\
\sin(\sqrt{1-\varepsilon^2}\, t) &= \sin(t + \mathcal{O}(\varepsilon^2 t)) \\
&= \sin t + \mathcal{O}(\varepsilon^2 t) \\
e^{-\varepsilon t} &= 1 - \varepsilon t + \mathcal{O}(\varepsilon^2 t^2)
\end{aligned}$$

Therefore, for $|t| \leq T$:

$$\begin{aligned}
x_{exact}(t,\varepsilon) &= \Big(1 + \mathcal{O}(\varepsilon^2)\Big)\, e^{-\varepsilon t} \sin(\sqrt{1-\varepsilon^2} t) \\
&= e^{-\varepsilon t} \sin(\sqrt{1-\varepsilon^2}\, t) + \mathcal{O}(\varepsilon^2) \\
&= e^{-\varepsilon t} \sin t + \mathcal{O}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^2) \\
&= (1-\varepsilon t)\sin t + \mathcal{O}(\varepsilon^2 t^2) + \mathcal{O}(\varepsilon^2 t) + \mathcal{O}(\varepsilon^2) \\
&= x_{app}(t,\varepsilon) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

To obtain the last estimate, it is crucial that t is bounded.

10.3 An Initial–Value Problem: Two–Timing

Consider the IVP

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1 \ .$$

Let $\tau = t$ denote the fast time variable and $T = \varepsilon t$ the slow time variable.

We make the ansatz

$$x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \mathcal{O}(\varepsilon^2) \ .$$

We have

$$
\begin{aligned}
\dot{x} &= x_{0\tau} + \varepsilon x_{0T} + \varepsilon x_{1\tau} + \mathcal{O}(\varepsilon^2) \\
\ddot{x} &= x_{0\tau\tau} + \varepsilon x_{0\tau T} + \varepsilon x_{0T\tau} + \varepsilon x_{1\tau\tau} + \mathcal{O}(\varepsilon^2) \\
&= x_{0\tau\tau} + 2\varepsilon x_{0\tau T} + \varepsilon x_{1\tau\tau} + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

Substituting into the differential equation, we obtain

$$x_{0\tau\tau} + 2\varepsilon x_{0\tau T} + \varepsilon x_{1\tau\tau} + 2\varepsilon x_{0\tau} + x_0 + \varepsilon x_1 + \mathcal{O}(\varepsilon^2) = 0 \ .$$

This yields the equations

$$
\begin{aligned}
x_{0\tau\tau} + x_0 &= 0 \\
x_{1\tau\tau} + x_1 &= -2x_{0\tau T} - 2x_{0\tau}
\end{aligned}
$$

Therefore,

$$x_0(\tau, T) = A(T)\sin\tau + B(T)\cos\tau \ ,$$

thus

$$
\begin{aligned}
x_{0\tau} &= A(T)\cos\tau - B(T)\sin\tau \\
x_{0\tau T} &= A'(T)\cos\tau - B'(T)\sin\tau
\end{aligned}
$$

The equation for x_1 becomes

$$x_{1\tau\tau} + x_1 = -2(A'(T) + A(T))\cos\tau + 2(B'(T) + B(T))\sin\tau \ .$$

To avoid resonance terms, we require

$$A'(T) + A(T) = B'(T) + B(T) = 0 \ ,$$

thus

$$A(T) = A(0)e^{-T}, \quad B(T) = B(0)e^{-T} \ .$$

Therefore,

$$x_0(\tau, T) = A(0)e^{-T}\sin\tau + B(0)e^{-T}\cos\tau \ .$$

The initial condition $x(0) = 0$ yields

$$0 = x_0(0, 0) = B(0) \ .$$

Therefore,

$$x_0(\tau, T) = A(0)e^{-T}\sin\tau \ .$$

Furthermore,

$$1 = \dot{x}(0) = x_{0\tau}(0,0) + \varepsilon x_{0T}(0,0)\ .$$

One obtains that

$$A(0) = 1\ .$$

This shows that

$$x_0(\tau, T) = e^{-T} \sin\tau\ .$$

Finally,

$$x(t,\varepsilon) = x_0(t, \varepsilon t) + \mathcal{O}(\varepsilon) = e^{-\varepsilon t}\sin t + \mathcal{O}(\varepsilon)\ .$$

Lemma 10.1 *For the error between the exact solution and the approximate solution* $e^{-\varepsilon t}\sin t$ *we have*

$$\sup_{t\geq 0} |x_{exact}(t,\varepsilon) - e^{-\varepsilon t}\sin t| \leq C\varepsilon\ .$$

10.4 Van der Pol Oscillator: Two–Timing

Consider the equation

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

for $0 \leq \varepsilon << 1$. If $\varepsilon = 0$ then the general solution is

$$x(t) = a\cos t + b\sin t\ ,$$

a regular oscillation. If $0 < \varepsilon << 1$ then the term $\varepsilon(x^2-1)x'$ is a damping term if $x^2(t) - 1 > 0$, but is exciting if $x^2(t) - 1 < 0$. Roughly, if $|x(t)|$ is small, oscillations get excited, but if $|x(t)|$ is large, they get damped.

To better understand the equation, we use two–timing. As above, let $\tau = t, T = \varepsilon t$. We make the ansatz

$$x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \mathcal{O}(\varepsilon^2)\ .$$

We have

$$\begin{aligned}
\dot{x} &= x_{0\tau} + \varepsilon x_{0T} + \varepsilon x_{1\tau} + \mathcal{O}(\varepsilon^2)\\
\ddot{x} &= x_{0\tau\tau} + \varepsilon x_{0\tau T} + \varepsilon x_{0T\tau} + \varepsilon x_{1\tau\tau} + \mathcal{O}(\varepsilon^2)\\
&= x_{0\tau\tau} + 2\varepsilon x_{0\tau T} + \varepsilon x_{1\tau\tau} + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

Substituting into the differential equation, we obtain

$$x_{0\tau\tau} + x_0 + \varepsilon(2x_{0\tau T} + x_{1\tau\tau} + x_1) + \varepsilon(x_0^2 - 1)x_{0\tau} + \mathcal{O}(\varepsilon^2) = 0\ .$$

This yields

$$x_{0\tau\tau} + x_0 = 0$$

and

$$x_{1\tau\tau} + x_1 = -2x_{0\tau T} - (x_0^2 - 1)x_{0\tau} \ .$$

We write

$$x_0(\tau, T) = r(T)\cos(\tau + \phi(T)) \ .$$

Abbreviate

$$s = \sin(\tau + \phi(T)), \quad c = \cos(\tau + \phi(T)) \ .$$

Then we have

$$\begin{aligned} x_{0\tau} &= -rs \\ x_{0\tau T} &= -r's + r\phi' c \end{aligned}$$

The equation for x_1 reads

$$x_{1\tau\tau} + x_1 = -2(-r's + r\phi' c) + (r^2c^2 - 1)rs = (2r' + r^3c^2 - r)s - 2r\phi' c \ .$$

Using the identity

$$\sin\alpha\cos^2\alpha = \frac{1}{4}(\sin\alpha + \sin 3\alpha) \tag{10.8}$$

yields

$$x_{1\tau\tau} + x_1 = (2r' + \frac{1}{4}r^3 - r)s - 2r\phi' c + \frac{1}{4}r^3\sin(3(\tau + \phi)) \ .$$

To avoid resonance, we require the coefficients of s and c to be zero. This yields the following equations

$$\begin{aligned} r' &= \frac{r}{8}(4 - r^2) \\ \phi' &= 0 \end{aligned}$$

Note that the equation for $r(T)$ implies that

$$r(T) \to 2 \quad \text{as} \quad T \to \infty \ .$$

We will solve the r–equation below.

Proof of the Identity (10.8): Recall that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \ .$$

With $\beta = 2\alpha$ obtain that

$$\sin(3\alpha) = \sin\alpha\cos(2\alpha) + \cos\alpha\sin(2\alpha) \ .$$

Abbreviate

$$s = \sin\alpha, \quad c = \cos\alpha \ .$$

Obtain:

$$\begin{aligned}
\sin(3\alpha) &= s(c^2 - s^2) + c2sc \\
&= 3sc^2 - s^3 \\
&= 3sc^2 - s(1 - c^2) \\
&= 4sc^2 - s
\end{aligned}$$

The equation

$$\sin\alpha\cos^2\alpha = sc^2 = \frac{1}{4}\left(\sin\alpha + \sin(3\alpha)\right)$$

follows.

10.5 A Simple Example for Averaging

Consider the equation

$$\dot{x} = \varepsilon x \sin^2 t, \quad x(0) = 1 \ ,$$

where $0 < \varepsilon << 1$. For example, let $\varepsilon = 10^{-4}$ and assume we want to know the solution, or an approximation of the solution, at time $t_0 = \pi * 10^4$. In the time interval,

$$0 \le t \le \pi * 10^4$$

the term $\sin^2 t$ goes through 10^4 oscillations. In the method of averaging, one replaces the oscillatory function $\sin^2 t$ by its time–average. We have

$$\langle \sin^2 t \rangle = \frac{1}{\pi}\int_0^\pi \sin^2 t \, dt = \frac{1}{2} \ .$$

Thus, after averaging, one obtains the much simpler equation

$$\dot{x} = \frac{\varepsilon}{2} x, \quad x(0) = 1$$

with solution

$$x_a(t) = e^{\varepsilon t/2} \ .$$

In the example

$$\varepsilon = 10^{-4}, \quad t_0 = \pi * 10^4$$

we have

$$x_a(t_0) = e^{\pi/2} = 4.810477$$

Let us compute the **exact solution**: We have

$$\int_1^{x(t)} \frac{dx}{x} = \varepsilon \int_0^t \sin^2 \tau \, d\tau \ .$$

Here

$$\begin{aligned} \int \sin^2 \tau d\tau &= \frac{1}{2} \int (1 - \cos^2 \tau + \sin^2 \tau) \, d\tau \\ &= \frac{1}{2} \int (1 - \cos(2\tau)) \, d\tau \\ &= \frac{\tau}{2} - \frac{1}{4} \sin(2\tau) \end{aligned}$$

This yields

$$\ln x(t) = \varepsilon \Big(\frac{t}{2} - \frac{1}{4} \sin(2t) \Big) \ .$$

Therefore, the exact solution is

$$x(t) = e^{\varepsilon t/2} e^{-\frac{\varepsilon}{4} \sin(2t)} \ .$$

For the example, we have $t_0 = \pi * 10^4$, thus $\sin(2t_0) = 0$, and the error between the exact solution and the approximate solution at t_0 is zero.

Error Estimate: The error between the exact solution and the approximate solution is

$$error(t, \varepsilon) = e^{\varepsilon t/2} |e^{-\frac{\varepsilon}{4} \sin(2t)} - 1| \ .$$

Using the simple bound [1]

$$|e^\alpha - 1| \le 2|\alpha| \quad \text{for} \quad |\alpha| \le 1$$

we obtain

$$error(t, \varepsilon) = e^{\varepsilon t/2} \frac{\varepsilon}{2} \quad \text{for} \quad |\varepsilon| \le 4 \ .$$

If $C > 0$ is any constant, then there exists C_1, depending on C, so that

$$error(t, \varepsilon) \le C_1 \varepsilon \quad \text{for} \quad 0 \le t \le \frac{C}{\varepsilon} \ .$$

The error is $\mathcal{O}(\varepsilon)$ in time intervals of length $\mathcal{O}(1/\varepsilon)$.

[1] We have $e^\alpha - 1 = \alpha + \frac{1}{2!}\alpha^2 + \ldots$, thus $|e^\alpha - 1| \le |\alpha|(1 + \frac{1}{2!} + \ldots) \le |\alpha|(e-1) \le 2|\alpha|$.

10.6 Van der Pol Oscillator: Averaging

Consider the equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0 \ .$$

Before we can apply averaging, we must transform the equation to variables which change slowly for small ε. Thus, we will write the equation in amplitude–phase variables. In applications, the transformation to suitable slowly varying variables is often the most difficult part of averaging.

Recall polar coordinates:

$$x = r\cos\alpha, \quad y = r\sin\alpha \ .$$

Since the solution $(x(t), \dot{x}(t))$ moves clockwise and the polar angle increases counterclockwise, we work with $\beta = -\alpha$, obtaining

$$x = r\cos\beta, \quad y = -r\sin\beta \ .$$

If $(x(t), \dot{x}(t))$ solves the van der Pol equation, we write

$$\begin{aligned} x(t) &= r(t)\cos(t + \phi(t)) \\ \dot{x}(t) &= -r(t)\sin(t + \phi(t)) \end{aligned}$$

These equations define $r(t)$ and $\phi(t)$ and one expects that $r(t)$ and $\phi(t)$ vary slowly. We now want to obtain equations for $\dot{r}(t)$ and $\dot{\phi}(t)$.

Abbreviate

$$s = \sin(t + \phi(t)), \quad c = \cos(t + \phi(t)) \ .$$

Then, since $x = rc$, we have

$$\dot{x} = \dot{r}c - rs(1 + \dot{\phi}) \ .$$

This must equal $-rs$. Therefore,

$$c\dot{r} - rs\dot{\phi} = 0 \ .$$

Another equation is obtained from van der Pol's equation. We have

$$\ddot{x} = -s\dot{r} - rc(1 + \dot{\phi}) \ .$$

Van der Pol's equation yields

$$-s\dot{r} - rc(1 + \dot{\phi}) + \varepsilon(r^2c^2 - 1)(-rs) + rc = 0 \ .$$

Therefore,

$$s\dot{r} + rc\dot{\phi} = \varepsilon rs(1 - r^2c^2) \ .$$

We have derived the system

$$\begin{pmatrix} c & -rs \\ -s & rc \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon rs(1-r^2c^2) \end{pmatrix} .$$

Note that the determinant is

$$det = r .$$

One obtains that

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \varepsilon rs^2(1-r^2c^2) \\ \varepsilon sc(1-r^2c^2) \end{pmatrix} .$$

We can also write this as

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \varepsilon(1-r^2c^2) \begin{pmatrix} rs^2 \\ sc \end{pmatrix} .$$

Writing out the abbreviated terms yields the system

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \varepsilon(1-r^2\cos^2(t+\phi)) \begin{pmatrix} r\sin^2(t+\phi) \\ \sin(t+\phi)\cos(t+\phi) \end{pmatrix} .$$

This system is equivalent to the van der Pol equation. The equation is rewritten in the variables r and ϕ, which are (except for a sign) polar coordinates in the $(x, \dot{x})$–plane.

The new system takes the form

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \end{pmatrix} = \varepsilon F(r, \phi, t)$$

where $F(r, \phi, t)$ is 2π–periodic in t. The main point is the ε–factor. The functions $r(t)$ and $\phi(t)$ very slowly. Note that

$$x(t) = r\cos(t+\phi) \quad \text{and} \quad x'(t) = -r\sin(t+\phi)$$

do not vary slowly, even if r and ϕ would be constant.

The method of averaging replaces $F(r, \phi, t)$ by

$$\bar{F}(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} F(r, \phi, t)\, dt$$

We have

$$\begin{aligned} \int_0^{2\pi} \sin^2 t\, dt &= \pi \\ \int_0^{2\pi} \sin^2 t \cos^2 t\, dt &= \pi/4 \\ \int_0^{2\pi} \sin t \cos t\, dt &= 0 \\ \int_0^{2\pi} \sin t \cos^3 t\, dt &= 0 \end{aligned}$$

Therefore,

$$\bar{F}(r,\phi) = \begin{pmatrix} \frac{r}{8}(4-r^2) \\ 0 \end{pmatrix} .$$

In other words, the system obtained by averaging is

$$\dot{r} = \varepsilon \frac{r}{8}(4-r^2), \quad \dot{\phi} = 0 .$$

If one introduces

$$R(T) = r(T/\varepsilon)$$

then $R'(T) = \frac{1}{\varepsilon} r'(T/\varepsilon)$. Thus the equation for $R(T)$ is

$$R'(T) = \frac{R}{8}(4-R^2) .$$

This is exactly the same equation that was obtained by two–timing.

10.7 Solution of the r–Equation of the Averaged System

Obtain

$$\int_{r_0}^{r(T)} \frac{8dr}{r(2+r)(2-r)} = T .$$

Partial fraction decomposition yields

$$\frac{8}{r(2+r)(2-r)} = \frac{2}{r} + \frac{1}{2-r} - \frac{1}{2+r} .$$

Therefore,

$$\int \frac{8dr}{r(2+r)(2-r)} = 2\ln r - \ln(2-r) - \ln(2+r)$$

or

$$\int \frac{8dr}{r(2+r)(2-r)} = \ln \frac{r^2}{4-r^2} .$$

Thus

$$T = \ln \frac{r^2}{4-r^2}\Big|_{r_0}^{r(T)} = \ln\Big(\frac{r^2(T)}{4-r^2(T)} \cdot \frac{1}{Q}\Big)$$

with

$$Q = \frac{r_0^2}{4-r_0^2} .$$

One obtains that

$$Qe^T = \frac{r^2(T)}{4-r^2(T)} .$$

Therefore,

$$\begin{aligned} r^2(T) &= \frac{4Qe^T}{1+Qe^T} \\ &= \frac{4}{1+e^{-T}/Q} \\ &= \frac{4r_0^2}{r_0^2+e^{-T}(4-r_0^2)} \end{aligned}$$

Finally,

$$r(T) = \frac{2r_0}{\sqrt{r_0^2+e^{-T}(4-r_0^2)}} \ .$$

This yields the approximation

$$x(t) \sim \frac{2r_0}{\sqrt{r_0^2+e^{-\varepsilon t}(4-r_0^2)}} \cdot \cos(t+\phi_0)$$

for the solution of van der Pol's equation.

11 More on Bifurcations

11.1 Bifurcations of Stationary Solutions

Example 1: The equations

$$\dot{x} = \mu - x^2, \quad \dot{y} = -y$$

lead to a saddle–node bifurcation when μ changes from $\mu > 0$ to $\mu < 0$.

Example 2: The equations

$$\dot{x} = \mu x + y + \sin x, \quad \dot{y} = x - y$$

lead to a supercritical pitchfork bifurcation at $\mu = -2$.

The trivial solution $(x, y) = (0, 0)$ exists for all μ. We write the two equations as a first order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x, y, \mu)$$

with

$$f(x, y, \mu) = \begin{pmatrix} \mu x + y + \sin x \\ x - y \end{pmatrix} .$$

We have

$$D_{(x,y)} f(x, y, \mu) = \begin{pmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{pmatrix}, \quad f_\mu(x, y, \mu) = \begin{pmatrix} x \\ 0 \end{pmatrix} .$$

At the trivial branch obtain:

$$D_{(x,y)} f(0, 0, \mu) = \begin{pmatrix} \mu + 1 & 1 \\ 1 & -1 \end{pmatrix} =: A(\mu), \quad f_\mu(0, 0, \mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

We see that

$$det A(\mu) = -\mu - 2, \quad tr A(\mu) = \mu .$$

A bifurcation of fixed points from the trivial branch can only occur at values of μ where $A(\mu)$ is singular, thus only at $\mu = -2 =: \mu_c$.

Recall that the eigenvalues of a real 2×2 matrix A are

$$\lambda_{1,2} = \frac{tr}{2} \pm \frac{1}{2} \sqrt{tr^2 - 4\, det}$$

where

$$tr = trace A \quad \text{and} \quad det = det A .$$

For $\mu < -2$ we have

$$det = detA(\mu) = -\mu - 2 > 0, \quad tr = trA(\mu) = \mu < 0 \ .$$

Since $trA(\mu) < 0$ for $\mu < -2$ it follows that the fixed point $(x, y) = (0, 0)$ is stable for $\mu < -2$. Consider

$$tr^2 - 4det = \mu^2 + 4\mu + 8 = (\mu + 2)^2 + 4 > 0 \ .$$

Therefore,

$$4detA(\mu) < (trA(\mu))^2 \ .$$

It follows that the eigenvalues of $A(\mu)$ are real and negative for $\mu < -2$. This yields that the fixed point $(x, y) = (0, 0)$ is a stable node for $\mu < -2$.

If $\mu > -2$ then $detA(\mu) < 0$. The eigenvalues $\lambda_{1,2}$ of $A(\mu)$ are both real and have opposite signs. The fixed point $(x, y) = (0, 0)$ is a saddle for $\mu > -2$.

We expect that either a transcritical bifurcation or a pitchfork bifurcation of fixed points occurs at $\mu = -2$.

Let us determine the fixed points for $\mu = -2 + \varepsilon$ where $0 < \varepsilon << 1$. We have $x = y$ and

$$\begin{aligned}
(\mu + 1)x &= -\sin x \\
\sin x &= -(\mu + 1)x \\
\sin x &= (1 - \varepsilon)x
\end{aligned}$$

We see that a supercritical pitchfork bifurcation occurs at $\mu = -2$.

11.2 Hopf Bifurcations

Supercritical Hopf Bifurcation

$$\begin{aligned}
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= \omega + br^2
\end{aligned}$$

The parameters ω and b are fixed. As μ changes from $\mu < 0$ to $\mu > 0$ a stable fixed point $r = \sqrt{\mu}$ occurs. Since (r, θ) are polar coordinates this corresponds to a supercritical Hopf bifurcation.

Subcritical Hopf Bifurcation

$$\begin{aligned}
\dot{r} &= \mu r + r^3 \\
\dot{\theta} &= \omega + br^2
\end{aligned}$$

The parameters ω and b are fixed. As μ changes from $\mu < 0$ to $\mu > 0$ a a subcritical Hopf bifurcation occurs.

Subcritical Hopf Bifurcation with Hysteresis

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2 \end{aligned}$$

We can write the systems in terms of

$$x = r\cos\theta, \quad y = r\sin\theta \; .$$

Linearize about $(x, y) = 0$. The Jacobian is

$$A(\mu) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2}(\mu) = \mu \pm i\omega \; .$$

Rules of thumb for Hopf bifurcation at $\mu = \mu_c$:

...

Degenerate Hopf Bifurcation

$$\ddot{x} + \mu\dot{x} + \sin x = 0$$

The eigenvalue conditions for a Hopf bifurcation are fulfilled at $\mu = 0$. All periodic orbits occur at $\mu = 0$.

11.3 An Infinite Period Bifurcation

Consider the system

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \mu - \sin\theta \end{aligned}$$

We have $r(t) \to 1$ as $t \to \infty$ if $r(0) > 0$. For $0 < \mu < 1$ the θ–equation has two fixed points. For $\mu > 1$ the θ–equation has a periodic solution. One obtains that the unit circle is a stable limit cycle for $\mu > 1$. As $\mu \to 1+$, the period of the cycle approaches ∞.

In the limit $\mu = 1$, we have a homoclinic orbit. The fixed point, which is approached by the homoclinic orbit as $t \to \infty$ and as $t \to -\infty$, is

$$r = 1, \quad \theta = \frac{\pi}{2} \; .$$

To understand this better, consider the equation

$$\dot{\theta} = 1 - \sin\theta$$

with fixed point $\theta = \frac{\pi}{2}$. If one gives any initial condition

$$\theta(0) = \theta_0 \quad \text{where} \quad \frac{\pi}{2} < \theta_0 < 2\pi + \frac{\pi}{2}$$

then

$$\theta(t) \to 2\pi + \frac{\pi}{2} \quad \text{as} \quad t \to \infty$$

and

$$\theta(t) \to \frac{\pi}{2} \quad \text{as} \quad t \to -\infty \, .$$

Since the angle $2\pi + \frac{\pi}{2}$ is identified with $\frac{\pi}{2}$ one obtains that the solution

$$\Big(r(t), \theta(t)\Big) = \Big(1, \theta(t)\Big)$$

of the above system with $\mu = 1$ describes a homoclinic orbit.

11.4 A Homoclinic Bifurcation

See Section 8.4 of Strogatz.

Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu y + x - x^2 + xy \end{aligned}$$

For all μ the points

$$P_1 = (0,0), \quad P_2 = (1,0)$$

are fixed points.

Let

$$f(x, y, \mu) = \begin{pmatrix} y \\ \mu y + x - x^2 + xy \end{pmatrix} .$$

The Jacobian of $f(x, y, \mu)$ w.r.t. (x, y) is

$$Df(x, y, \mu) = \begin{pmatrix} 0 & 1 \\ 1 - 2x + y & \mu + x \end{pmatrix} .$$

The Jacobian at P_1 is

$$A := \begin{pmatrix} 0 & 1 \\ 1 & \mu \end{pmatrix}$$

and the Jacobian about P_2 is

$$B := \begin{pmatrix} 0 & 1 \\ -1 & \mu + 1 \end{pmatrix} .$$

The eigenvalues of A are

$$\lambda_{1,2} = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} + 1} \; .$$

It follows that for all μ the Jacobian A has two real eigenvalues of opposite sign. Thus, the fixed point $P_1 = (0,0)$ is a saddle point for all μ. If λ is an eigenvalue of A and

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

is a corresponding eigenvector, then the equation

$$\begin{pmatrix} -\lambda & 1 \\ 1 & \mu - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds. Therefore, $y = \lambda x$. If $\lambda > 0$ is the unstable eigenvalue of A, then the unstable direction at the fixed point $P_1 = (0,0)$ points into the first quadrant and the third quadrant. The stable direction, corresponding to $\lambda < 0$, points into the second and fourth quadrant.

The eigenvalues of $B = Df(0,1,\mu)$ are

$$\lambda_{1,2} = \frac{\mu + 1}{2} \pm i\sqrt{1 - (\mu+1)^2/4} \; .$$

If

$$|\mu + 1| < 2$$

then P_2 is a spiral point. Since

$$B \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

the orbits spiral clockwise around P_2. If

$$-1 < \mu < 1$$

then $\mathrm{Re}\,\lambda_{1,2} > 0$ and, therefore, P_2 is an unstable spiral point.

One can now study the unstable manifold of the saddle point $P_1 = (0,0)$ numerically. We are interested in the part of the manifold which starts at P_1 and enters the first quadrant. For $\mu = -0.92$ the manifold spirals first towards P_2, but P_2 is an unstable spiral point. Therefore, there is a periodic orbit around P_2. For $\mu = \mu_2 \sim -0.8645$ the periodic orbit turns into a homoclinic orbit, an orbit which approaches P_1 as $t \to \infty$ and as $t \to -\infty$.

For $\mu > \mu_c$ the unstable manifold of P_1, which starts at P_1 and enters the first quadrant, enters the third quadrant. See Figure 8.4.3 of Strogatz.

12 The Driven Pendulum

Consider the equation

$$mL\ddot{\theta} + \frac{b}{L}\dot{\theta} + mg\sin\theta = \frac{\Gamma}{L} \ . \tag{12.1}$$

Here every term has the dimension of a force. Thus,

$$[b] = \frac{mass * length^2}{time}, \quad [\Gamma] = \frac{mass * length^2}{time^2} \ .$$

Dividing equation (12.1) by mL we obtain

$$\ddot{\theta} + \frac{b}{mL^2}\dot{\theta} + \frac{g}{L}\sin\theta = \frac{\Gamma}{mL^2} \ . \tag{12.2}$$

We now non–dimensionalize the equation. Let $T > 0$ denote a unit of time and let

$$\phi(t) = \theta(Tt) \ .$$

Then t is a dimensionless variable. We have

$$\phi'(t) = T\dot{\theta}(Tt), \quad \phi''(t) = T^2\ddot{\theta}(Tt) \ .$$

Equation (12.2) becomes

$$T^{-2}\phi'' + \frac{b}{TmL^2}\phi' + \frac{g}{L}\sin\phi = \frac{\Gamma}{mL^2} \ . \tag{12.3}$$

This motivates to choose $T^2 = \frac{L}{g}$. Then one obtains

$$\phi'' + \alpha\phi' + \sin\phi = I$$

with

$$\alpha = \frac{bT}{mL^2}, \quad I = \frac{\Gamma T^2}{mL^2} \ .$$

Here α and I are dimensionless. We assume that $\alpha > 0$ is fixed and consider $I > 0$ as bifurcation parameter.

Let $y = \phi'$ and write the equation as a first–order system:

$$\begin{pmatrix} \phi' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ I - \sin\phi - \alpha y \end{pmatrix} =: \begin{pmatrix} y \\ g(\phi, y) \end{pmatrix} \ .$$

There is no fixed point for $I > 1$. We want to show that there is a unique periodic orbit if $I > 1$.

The existence of a periodic orbit will be shown using a Poincaré map.

Note that

$$g(\phi, y) = 0 \quad \text{iff} \quad y = \frac{1}{\alpha}(I - \sin\phi) \ .$$

Choose numbers y_1 and y_2 with

$$0 < y_1 < \frac{I-1}{\alpha} < \frac{I+1}{\alpha} < y_2 \ .$$

If $y \le y_1$ then $g(\phi, y) \ge c > 0$ and if $y \ge y_2$ then $g(\phi, y) \le -c < 0$. Therefore, any trajectory will enter the strip

$$y_1 \le y \le y_2 \ .$$

Furthermore, once a trajectory has entered the strip, it cannot leave the strip in the future.

Definition of the Poincaré Map: Consider the above system with initial condition

$$\phi(0) = 0, \quad y(0) = \beta \ ,$$

where $y_1 \le \beta \le y_2$. For $t \ge 0$ we have $y(t) \ge y_1 > 0$. Therefore, $\dot{\phi}(t) \ge y_1 > 0$. Consequently, there exists a unique time $T = T(\beta)$ with

$$\phi(T) = 2\pi \ .$$

Define the map P by

$$P : \begin{cases} [y_1, y_2] & \to \quad [y_1, y_2] \\ \beta & \to \quad y(T) \end{cases}$$

It is then clear that P is continuous and

$$P(y_1) > y_1, \quad P(y_2) < y_2 \ .$$

By the intermediate–value theorem, there is a value $y^* \in (y_1, y_2)$ with

$$P(y^*) = y^* \ .$$

The solution with initial condition

$$\phi(0) = 0, \quad y(0) = y^*$$

is periodic with period $T(y^*)$.

Uniqueness: Let $(\phi(t), y(t))$ denote any solution of the above system which is periodic in the sense that for some $T > 0$ we have

$$\phi(0) = 0, \quad \phi(T) = 2\pi, \quad y(0) = y(T) \ .$$

From the above, it follows that

$$y_1 < y(0) < y_2 \ .$$

Since $\phi(t)$ increases strictly from 0 to 2π as t changes from 0 to T, we can eliminate time and obtain a function $y(\phi)$ with

$$\frac{dy}{d\phi} = \frac{1}{y}(I - \sin\phi - \alpha y), \quad y(0) = y(2\pi) \ .$$

Consider the function

$$E(\phi) = \frac{1}{2}\,y^2(\phi) - \cos\phi, \quad 0 \le \phi \le 2\pi \ .$$

We obtain that $E(0) = E(2\pi)$ and

$$\frac{dE}{d\phi} = y\frac{dy}{d\phi} + \sin\phi \ .$$

Therefore,

$$\begin{aligned} 0 &= \int_0^{2\pi} E'(\phi) d\phi \\ &= \int_0^{2\pi} (I - \alpha y(\phi))\, d\phi \end{aligned}$$

This implies that for any periodic solution we have

$$\int_0^{2\pi} y(\phi)\, d\phi = \frac{2\pi I}{\alpha} \ .$$

Now suppose that w is a second periodic solution. We may assume that $w(0) > y(0)$. Since

$$\frac{dw}{d\phi} = \frac{1}{w}(I - \sin\phi - \alpha w) \ ,$$

we obtain that $w(\phi) > y(\phi)$ for $0 \le \phi \le 2\pi$. However, this contradicts

$$\int_0^{2\pi} w(\phi)\, d\phi = \frac{2\pi I}{\alpha} = \int_0^{2\pi} y(\phi)\, d\phi \ .$$

Theorem 12.1 *If $\alpha > 0$ and $I > 1$ then equation*

$$\phi'' + \alpha\phi' + \sin\phi = I$$

has a unique solution which is periodic in the sense that for some $T > 0$ we have

$$\phi(0) = 0, \quad \phi(T) = 2\pi, \quad \phi'(0) = \phi'(T) \ .$$

Note that the periodic solution corresponds to a periodic rotation of the pendulum. In fact, rotations occur for rather large values of y. They may be described approximately by neglecting the $\sin(\phi)$–term, i.e., by the linear system

$$x' = y, \quad y' = I - \alpha y \ .$$

For this simplified system, we have $y(t) \to I/\alpha$.

13 Coupled Oscillators: Quasiperiodicity, Ergodicity, Phase Locking

Let S^1 denote the unit circle. We often identify S^1 with $\mathbb{R}$ mod 2π or with $\mathbb{R}$ mod 1. Let $\mathcal{T}^2 = S^1 \times S^1$ denote the 2–torus.

13.1 Uncoupled Oscillators

Example 1: Consider the system of two uncoupled equations

$$\begin{aligned} \dot{\theta}_1 &= 1 \\ \dot{\theta}_2 &= 1 \end{aligned}$$

on $\mathcal{T}^2$. Every solution has the form

$$\theta_j(t) = (\theta_{j0} + t) \text{ mod } 1 \quad \text{for} \quad j = 1, 2 \; .$$

We identify S^1 with $\mathbb{R}$ mod 1. Then every solution has the period $T = 1$.

Example 2: Consider the system

$$\begin{aligned} \dot{\theta}_1 &= 2 \\ \dot{\theta}_2 &= 1 \end{aligned}$$

on $\mathcal{T}^2$. Every solution has the form

$$\theta_1(t) = (\theta_{10} + 2t) \text{ mod } 1, \quad \theta_2(t) = (\theta_{20} + t) \text{ mod } 1 \; .$$

Again, every solution has the period $T = 1$.

Example 3: Consider the system

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2 \end{aligned}$$

where $\omega_j > 0$. Assume that there is a periodic solution

$$\theta_j(t) = (\theta_{j0} + \omega_j t) \text{ mod } 1, \quad j = 1, 2 \; ,$$

of period $T > 0$. Then

$$(\omega_j T) \text{ mod } 1 = 0 \quad \text{for} \quad j = 1, 2 \; ,$$

thus the numbers

$$\omega_1 T = n_1, \quad \omega_2 T = n_2$$

are positive integers. Therefore,

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}$$

is a rational number.

Conversely, assume that

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}$$

is a rational number where p, q are positive integers.

Consider any solution

$$\theta_1(t) = (\theta_{10} + \omega_1 t) \bmod 1, \quad \theta_2(t) = (\theta_{20} + \omega_2 t) \bmod 1 \ .$$

Set $T := p/\omega_1$. Then we have

$$\omega_1 T = p \quad \text{and} \quad \omega_2 T = \frac{\omega_1 q}{p} \cdot \frac{p}{\omega_1} = q \ .$$

This shows that every solution has the period

$$T = \frac{p}{\omega_1} = \frac{q}{\omega_2}$$

if

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}$$

is rational.

We have shown the following:

Lemma 13.1 *Consider the system*

$$\dot{\theta}_1 = \omega_1 \qquad (13.1)$$

$$\dot{\theta}_2 = \omega_2 \qquad (13.2)$$

with $\omega_j > 0$. The following conditions are equivalent:

1. *There is a periodic solution.*
2. *Every solution is periodic.*
3. *$\frac{\omega_1}{\omega_2}$ is rational.*

Now assume that $\frac{\omega_1}{\omega_2}$ is irrational. We will prove:

Lemma 13.2 *If $\frac{\omega_1}{\omega_2}$ is irrational then every trajectory of the system (13.1), (13.2) is dense on the torus $\mathcal{T}^2$.*

Proof: First consider the trajectory starting at

$$\theta_1(0) = 0, \quad \theta_2(0) = \beta \ .$$

Let $T = 1/\omega_1$. Then

$$\theta_1(T) = (0 + \omega_1 T) \bmod 1 = 0$$

and

$$\theta_2(T) = (\beta + \omega_2 T) \bmod 1 = (\beta + c) \bmod 1 \quad \text{where} \quad c = \frac{\omega_2}{\omega 1} \ .$$

This motivates to define the Poincaré map $P : S^1 \to S^1$ by

$$P\beta = (\beta + c) \bmod 1 \ .$$

Lemma 13.3 *For every $\beta \in S^1$ the sequence $P^n\beta, n = 0, 1, 2, \ldots$ is dense in S^1.*

Proof: (See [Hale, Kocak, p. 152].) 1. First note that

$$P^n\beta = (\beta + nc) \bmod 1 \ .$$

Assume that $P^n\beta = P^m\beta$. It follows that

$$((n - m)c) \bmod 1 = 0 \ ,$$

thus $(n - m)c$ is an integer. By assumption, c is irrational. Therefore, $n = m$. We have show that $P^n\beta = P^m\beta$ implies that $n = m$. In other words, the elements of the sequence

$$P^n\beta, n = 0, 1, 2 \ldots$$

are all distinct. The sequence $P^n\beta$ has an accumulation point in S^1,

2. The distance on S^1 is

$$dist(r, s) = \min_{k=-1,0,1} |r - s + k| \ .$$

It is easy to see that P preserves distance,

$$dist(Pr, Ps) = dist(r, s)$$

and that P maps the arc from r to s onto the arc from Pr to Ps.

Since the elements of the sequence $P^n\beta$ have an accumulation point $Q \in S^1$ it follows that for all $\varepsilon > 0$ there are integers $m > 0, k > 0$ with

$$dist(P^k, Q) < \frac{\varepsilon}{2} \quad \text{and} \quad dist(P^{m+k}, Q) < \frac{\varepsilon}{2} \ ,$$

thus

$$dist(P^{m+k}\beta, P^k\beta) < \varepsilon \ .$$

Therefore,

$$
\begin{aligned}
dist(P^m\beta, \beta) &< \varepsilon \\
dist(P^{2m}\beta, P^m\beta) &< \varepsilon \\
dist(P^{3m}\beta, P^2m\beta) &< \varepsilon
\end{aligned}
$$

etc. Here P maps the arc from $P^{jm}\beta$ to $P^{(j+1)m}\beta$ onto the arc from $P^{(j+1)m}\beta$ to $P^{(j+2)m}\beta$. All these arcs have length $< \varepsilon$. They divide up the circle S^1 into pieces of length $< \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the points $P^n\beta$ lie dense in S^1. This completes the proof of Lemma 13.3. $\diamond$

2. We continue the proof of Lemma (13.2). Recall the notation

$$
T = 1/\omega_1, \quad c = \omega_2 T = \frac{\omega_2}{\omega_1} \ .
$$

Consider an arbitrary point $(\alpha_1, \alpha_2) \in \mathcal{T}^2$ and consider the solution $(\theta_1(t), \theta_2(t))$ of (13.1), (13.2) with initial condition

$$
\theta_1(0) = \alpha_1, \quad \theta_2(0) = \alpha_2 \ .
$$

We have

$$
\theta_j(t) = (\alpha_j + \omega_j t) \bmod 1 \quad \text{for} \quad j = 1, 2 \ .
$$

We claim that the points

$$
\Big(\theta_1(t), \theta_2(t)\Big), \quad t \ge 0 \ ,
$$

form a dense subset of $S^1 \times S^1$. More precisely, we claim the following: If $(q_1, q_2) \in S^1 \times S^1$ is an arbitrary point and if $\varepsilon > 0$ is given, then there is a time $t \ge 0$ with

$$
\theta_1(t) = q_1, \quad dist(\theta_2(t), q_2) < \varepsilon \ .
$$

Clearly, there is a time $t_1 \ge 0$ with

$$
\theta_1(t_1) = q_1 \ .
$$

We then have

$$
\begin{aligned}
q_1 &= \theta_1(t_1) \\
&= (\alpha_1 + \omega_1 t_1) \bmod 1 \\
&= (\alpha_1 + \omega_1 t_1 + n) \bmod 1 \\
&= (\alpha_1 + \omega_1(t_1 + n/\omega_1) \bmod 1 \\
&= (\alpha_1 + \omega_1(t_1 + nT) \bmod 1 \\
&= \theta_1(t_1 + nT)
\end{aligned}
$$

Now set

$$\beta = \theta_2(t_1) = \alpha_2 + \omega_2 t_1$$

and define the sequence

$$\begin{aligned} P^n\beta &= \theta_2(t_1 + nT) \\ &= (\alpha_2 + \omega_2(t_1 + nT)) \bmod 1 \\ &= (\alpha_2 + \omega_2 t_1 + nc) \bmod 1 \\ &= (\beta + nc) \bmod 1 \end{aligned}$$

By Lemma (13.3) the sequence $P^n\beta$ is dense in S^1. This completes the proof of Lemma 13.2. ⋄

Remarks on Ergodicity. Let $Q = [a_1, b_1] \times [a_2, b_2]$ denote any square in T^2. Let $\theta(t)$ denote any solution of the system (13.1), (13.2) and let $T > 0$ be arbitrary. Consider the set of times

$$M(Q, T) = \{t \ : \ 0 \le t \le T, \theta(t) \in Q\}$$

and consider

$$\lim_{T\to\infty} \frac{1}{T}\, measure(M(Q,T)) \ .$$

This limit (if it exists) gives the long term time average which the trajectory spends in Q.

One can show the following:

Theorem 13.1 *If $\frac{\omega_1}{\omega_2}$ is irrational, then the above limit exists and*

$$\lim_{T\to\infty} \frac{1}{T}\, measure(M(Q,T)) = area(Q) \ .$$

In fact, the above formula holds for any measurable subset Q of T^2.

Remarks on History: Ludwig Boltzmann (1844-1906) started work on deriving the laws of thermodynamics form the laws of mechanics. He considered a gas as a system of bouncing balls. Since the number N of balls is very large ($N \sim 10^{23}$) one needs statistical arguments. Boltzmann assumed that time averages can be computed as volumes in phase–space. For Boltzmann's system this has never been rigorously justified, but the Boltzmann hypothesis seems very reasonable. It is called the ergodic hypothesis for the Boltzmann system.

In ergodic theory one applies measure theory to dynamics.

There are results about ergodicity for dimension $N = 1$, dynamics of a billiard ball.

Boltzmann's formula:

$$S = k \log W \ .$$

Here S is the entropy of a macrostate, $k = 1.3806505 * 10^{-23} J/K$ (Joule per degree Kelvin) is Boltzmann's constant, and W is the probability of the macrostate. Roughly, one counts the number of microstates leading to a particular macrostate and compares this number with the number of all possible microstates.

13.2 A Time Average Equals A Space Average for a Circle Map

Recall that $S^1 = \mathbb{R} \bmod 1$ denotes the circle. Fix a number $c \in \mathbb{R} \setminus \mathbb{Q}$ and define the circle map $\phi : S^1 \to S^1$ by

$$\phi(\beta) = (\beta + c) \bmod 1 \ .$$

Fix any $\beta_0 \in S^1$ and let I denote any subinterval of $[0, 1)$. Consider the set

$$M(I, N) = \{n \in \mathbb{N} \ : \ 0 \le n \le N, \phi^n(\beta_0) \in I\} \ .$$

Then, for large N,

$$\frac{1}{N+1} \#M(I, N)$$

is the average number of times in which the orbit

$$\beta_n = \phi^n(\beta_0)$$

falls into the interval I.

Theorem 13.2 *Under the above assumptions we have*

$$\lim_{N \to \infty} \frac{1}{N+1} \#M(I, N) = length(I) \ .$$

Proof: First let $f : S^1 \to \mathbb{C}$ denote any function. Then

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\phi^n(\beta_0))$$

denotes the long time average value of the function f along the orbit $\phi^n(\beta_0)$ if the limit exists. If $f = \chi_I$ denotes the characteristic function for the interval I, then we must prove that

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \chi_I(\phi^n(\beta_0)) = length(I) \ .$$

Fix any integer $k \neq 0$ and let

$$f(x) = e^{2\pi i k x}, \quad x \in \mathbb{R} \ .$$

Note that $f(x)$ is one–periodic, thus the function $f(x)$ may be considered as a function defined on S^1. We have

$$\beta_n = \phi^n(\beta_0) = (\beta_0 + nc) \bmod 1 \ ,$$

thus

$$f(\beta_n) = e^{2\pi ik(\beta_0+nc)} = e^{2\pi ik\beta_0}q^n$$

with

$$q = e^{2\pi ikc} \neq 1 \ .$$

Note that $q \neq 1$ follows from the assumption that c is irrational. Therefore,

$$\begin{aligned}
\sum_{n=0}^{N} f(\beta_n) &= e^{2\pi ik\beta_0} \sum_{n=0}^{N} q^n \\
&= e^{2\pi ik\beta_0} \frac{1-q^{N+1}}{1-q}
\end{aligned}$$

Since $|q| = 1$ it follows that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\phi^n(\beta_0)) = 0$$

for any function $f(x)$ of the form

$$f(x) = e^{2\pi ikx} \quad \text{with} \quad k \in \mathbb{Z} \setminus \{0\} \ .$$

On the other hand, if $k = 0$ then $f(x) = e^{2\pi ikx} \equiv 1$ and

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\phi^n(\beta_0)) = 1 \ .$$

We summarize this as

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(\phi^n(\beta_0)) = \delta_{0k} \quad \text{for} \quad f(x) = e^{2\pi ikx} \ .$$

By Fourier expansion, we can write

$$\chi_I(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{2\pi ikx}$$

with

$$\alpha_k = \int_0^1 \chi(x) e^{-2\pi ikx} \, dx \ .$$

In particular, for $k = 0$,

$$\alpha_0 = length(I) \ .$$

In the 3rd equation below we exchange limits and obtain

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N+1} \#M(I,N) &= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \chi_I(\beta_n) \\
&= \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-\infty}^{\infty} \alpha_k e^{2\pi i k \beta_n} \\
&= \sum_{k=-\infty}^{\infty} \alpha_k \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} e^{2\pi i k \beta_n} \\
&= \sum_{k=-\infty}^{\infty} \alpha_k \delta_{0k} \\
&= \alpha_0 \\
&= length(I)
\end{aligned}$$

Rigorous Arguments: As above, let $\beta_n = (\beta_0 + nc) \bmod 1$. If $f : S^1 \to \mathbb{C}$ is any function, we denote

$$A_N f = \frac{1}{N+1} \sum_{n=0}^{N} f(\beta_n) \ .$$

We also let

$$Af = \lim_{N\to\infty} A_N f$$

if the limit exists. We note the trivial estimate

$$|A_N f| \le |f|_\infty$$

if f is bounded.

Lemma 13.4 *Let $f \in C(S^1, \mathbb{C})$. Then we have*

$$Af = \int_0^1 f(x)\,dx \ .$$

Proof: By the considerations above, the formula holds for any trigonometric polynomial

$$p(x) = \sum_{k=-K}^{K} \alpha_k e^{2\pi i k x} \ .$$

Now let $f \in C$. We first show that $A_N f$ is a Cauchy sequence of complex numbers.

Let $\varepsilon > 0$ be given. There exists a trigonometric polynomial p with $|f - p|_\infty \le \varepsilon$.

We have

$$
\begin{aligned}
|A_N f - A_M f| &\leq |A_N f - A_N p| + |A_N p - A_M p| + |A_M p - A_M f| \\
&\leq 2\varepsilon + |A_N p - A_M p| \\
&\leq 3\varepsilon
\end{aligned}
$$

for $N, M \geq N(\varepsilon)$. This shows that Af exists for any continuous f. Now choose a sequence of trigonometric polynomials p_K with $|f - p_K|_\infty \to 0$. We have

$$A_N f = A_N(f - p_K) + A_N p_K \ .$$

In this equation let $N \to \infty$. We obtain

$$Af = A(f - p_K) + \int_0^1 p_K \, dx \ .$$

Here $|A(f - p_K)| = \varepsilon_K \to 0$. In the above equation, let $K \to \infty$ to obtain

$$Af = \int_0^1 f(x) \, dx \ .$$

This proves the lemma. $\diamond$.

Now let I denote any subinterval of $[0, 1)$ and let f denote the characteristic function of I. Thus

$$f(x) = \chi(x) = \begin{cases} 1 & \text{for} \quad x \in I \\ 0 & \text{for} \quad x \in [0,1) \setminus I \end{cases}$$

Set

$$length(I) = l \ .$$

We claim that Af exists and $Af = l$.

For $\varepsilon > 0$ we can choose functions $f_\varepsilon, g_\varepsilon \in C$ with

$$g_\varepsilon(x) \leq f(x) \leq f_\varepsilon(x) \quad \text{for all} \quad x$$

and

$$\int_0^1 g_\varepsilon(x) \, dx = l - \varepsilon, \quad \int_0^1 f_\varepsilon(x) \, dx = l + \varepsilon \ .$$

It is clear that we have for all N:

$$A_N g_\varepsilon \leq A_N f \leq A_N f_\varepsilon \ .$$

This implies that

$$|A_n f - A_N f_\varepsilon| \leq |A_N(f_\varepsilon - g_\varepsilon)| \ .$$

We claim that $A_N f$ is a Cauchy sequence. Note that

$$
\begin{aligned}
|A_N f - A_M f| &\leq |A_N f - A_N f_\varepsilon| + |A_N f_\varepsilon - A_M f_\varepsilon| + |A_M f_\varepsilon - A_M f| \\
&\leq |A_N (f_\varepsilon - g_\varepsilon)| + |A_N f_\varepsilon - A_M f_\varepsilon| + |A_M (f_\varepsilon - g_\varepsilon)|
\end{aligned}
$$

Here

$$A_N(f_\varepsilon - g_\varepsilon) \to 2\varepsilon \quad \text{as} \quad N \to \infty \ .$$

Therefore,

$$|A_N f - A_M f| \leq 5\varepsilon \quad \text{if} \quad N, M \geq N(\varepsilon) \ ,$$

showing that $A_N f$ is Cauchy.

Now recall the bounds

$$A_N g_\varepsilon \leq A_N f \leq A_N f_\varepsilon \ .$$

We let $N \to \infty$ to obtain

$$l - \varepsilon \leq Af \leq l + \varepsilon \ .$$

Since $\varepsilon > 0$ was arbitrary, we have shown that

$$Af = l = length(I) \ .$$

This completes a rigorous proof of Theorem 13.2. $\diamond$

13.3 Coupled Oscillators: Phase Locking

Consider the system

$$
\begin{aligned}
\dot{\theta}_1 &= \omega_1 + K_1 \sin(\theta_2 - \theta_1) \\
\dot{\theta}_2 &= \omega_2 + K_2 \sin(\theta_1 - \theta_2)
\end{aligned}
$$

Assume that

$$\omega_1 > \omega_2 > 0, \quad K = K_1 + K_2 > 0 \ .$$

Let

$$\phi = \theta_1 - \theta_2$$

denote the phase difference. Obtain

$$\dot{\phi} = \omega_1 - \omega_2 - K \sin \phi \ .$$

There are essentially two cases:

Case 1: $K > \omega_1 - \omega_2$ (strong coupling)

The equation

$$\omega_1 - \omega_2 = K \sin\phi$$

has two solutions, $\phi^*_{1,2}$,

$$0 < \phi^*_1 < \frac{\pi}{2} < \phi^*_2 \ .$$

We have

$$\phi(t) \to \phi^*_1 \quad \text{as} \quad t \to \infty \ .$$

Therefore, except for an initial transient, the oscillators approximately solve

$$\begin{aligned}
\dot\theta_1 &= \omega_1 - K_1 \sin(\phi^*_1) \\
\dot\theta_2 &= \omega_2 + K_2 \sin(\phi^*_1)
\end{aligned}$$

Here

$$\omega_1 - K_1 \sin(\phi^*_1) = \frac{K_1\omega_2 + K_2\omega_1}{K} =: \omega^* \ .$$

One also obtains $\dot\theta_2 = \omega^*$.

After an initial transient, the oscillators move (approximately) with the same frequency ω^* and the phase difference

$$\theta_1(t) - \theta_2(t) \sim \phi^*_1 \ .$$

One calls this a phase locked motion.

Case 2: $K < \omega_1 - \omega_2$ (weak coupling) One gets essentially the same behavior as in the case $K = 0$. However, the trajectories are not straight lines, but are curvy.

14 The Lorenz System

Edward Norton Lorenz, 1917–2008.

14.1 Remarks on the Derivation of the Equations

Lorenz (1963) started with a system of PDEs for a velocity field and temperature. Series expansion in space and extreme truncation leads to a system of ODEs for amplitudes depending on time. In the Lorenz system, the variables $x(t), y(t), z(t)$ correspond to amplitudes.

The Lorenz system reads

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

Here σ, b, r are positive parameters. The systems was derived in 1963 by Edward Lorenz from a system of pdes describing a velocity field and temperature.

In the pde system, σ is the Prandtl number,

$$\sigma = \frac{\mu}{k/c_p}$$

suring the ratio of the viscous diffusion rate and the thermal diffusion rate.

Ludwig Prandtl, 1875–1953

The parameter r is the Rayleigh number,

$$r = Ra = Gr Pr$$

where Gr is the Grashof number,

$$Gr = \frac{buoyancy}{viscosity} \; .$$

John Wiliam Strutt=3rd Baron Rayleigh, 1842–1919

14.2 Evolution of Phase Volume

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ denote a C^1 function. Denote the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

by $x(t, x_0)$. Let $V_0 \subset \mathbb{R}^n$ denote a set of finite volume. Let

$$V(t) = \{x(t, x_0) \; : \; x_0 \in V_0\}$$

Then the following formula holds:

$$\frac{d}{dt} vol(V(t)) = \int_{V(t)} \nabla \cdot f(x)\, dx \; .$$

We can make the formula plausible by considering the simple equation $\dot{x} = \lambda x$ and the system

$$\dot{x} = \lambda x, \quad \dot{y} = \mu y \ .$$

Application: The Lorenz system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix} \ .$$

Denote the matrix by A. Then we have

$$f' = A + \begin{pmatrix} 0 & 0 & 0 \\ -z & 0 & -x \\ y & x & 0 \end{pmatrix} \ .$$

It follows that

$$\nabla \cdot f = -(1 + \sigma + b) < -1 \ .$$

Therefore, by th flow of the Lorenz system, phase volumes contract at an exponential rate.

14.3 Symmetry

If

$$(x, y, z)(t)$$

denotes a solution of the Lorenz system, then

$$(-x, -y, z)(t)$$

is also a solution. Therefore, either a solution obeys the symmetry of it has a symmetric partner. A solution that obeys the symmetry satisfies

$$x = y = 0, \quad \dot{z} = -bz \ .$$

14.4 Fixed Points

The origin,

$$P = (0, 0, 0)$$

is a fixed point for all parameter values. Linearization about P leads to

$$f'(0) = A \ .$$

The matrix A has the eigenvalue $-b$ and the submatrix

$$A_0 = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Note that

$$tr = -\sigma - 1 < 0, \quad det = \sigma(1-r) \ .$$

If $0 < r < 1$ then

$$tr < 0, \quad det > 0 \ .$$

Therefore, the origin is asymptotically stable if $0 < r < 1$. Since

$$\begin{aligned} tr^2 - 4det &= (\sigma+1)^2 - 4\sigma + 4\sigma r \\ &= (\sigma-1)^2 + 4\sigma r \\ &> 0 \end{aligned}$$

the origin is a stable node for $0 < r < 1$.

If $r > 1$ then $det = \sigma(1-r) < 0$ and the origin is unstable. We will see that a supercritical pitchfork bifurcation occurs at $r = 1$.

14.5 Global Stability of the Origin for $0 < r < 1$

Set

$$V(x,y,z) = \frac{1}{\sigma}\, x^2 + y^2 + z^2 \ .$$

Let $(x(t), y(t), z(t))$ denote a solution of the Lorenz system and set

$$v(t) = V(x(t), y(t), z(t)) \ .$$

We have

$$\begin{aligned} \frac{1}{2}\dot{v} &= \frac{1}{\sigma} x\dot{x} + y\dot{y} + z\dot{z} \\ &= x(y-x) + y(rx - y - xz) + z(-bz + xy) \\ &= (r+1)xy - x^2 - y^2 - bz^2 \end{aligned}$$

Assume that $0 < r < 1$. The sign of the term $(r+1)xy$ is not known and the term must be bounded in terms of the absolute values of the negative terms.

Set $\alpha = \frac{r+1}{2}$, thus $0 < \alpha < 1$. We have

$$\alpha(x-y)^2 = \alpha x^2 - 2\alpha xy + \alpha y^2 \ .$$

Therefore,

$$\begin{aligned} x^2 + y^2 - 2\alpha xy &= (1-\alpha)(x^2+y^2) + \alpha x^2 + \alpha y^2 - 2\alpha xy \\ &= (1-\alpha)(x^2+y^2) + \alpha(x-y)^2 \end{aligned}$$

This yields

$$
\begin{aligned}
\frac{1}{2}\dot{v} &= -(1-\alpha)(x^2+y^2) - bz^2 - \alpha(x-y)^2 \\
&\le -(1-\alpha)(x^2+y^2) - bz^2 \\
&\le -cv
\end{aligned}
$$

where $c > 0$ only depends on r, σ, b. Therefore,

$$0 \le v(t) \le v(0)e^{-2ct}, \quad t \ge 0 \ .$$

This implies that $v(t) \to 0$ as $t \to \infty$ and, consequently, $(x(t), y(t), z(t)) \to 0$ as $t \to \infty$.

14.6 The Fixed Points C^+ and C^-

For $r > 1$ let

$$x^* = y^* = \sqrt{b(r-1)} \ .$$

The points

$$C^+ = (x^*, y^*, r-1), \quad C^- = (-x^*, -y^*, r-1)$$

are fixed point bifurcating from the origin in a supercritical pitchfork bifurcation as r changes from $r < 1$ to $r > 1$. Linearization about C^+ leads to the matrix

$$f'(C^+) = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1-x^* & \\ x^* & x^* & -b \end{pmatrix} =: A \ .$$

Then we have

$$det(\lambda I - A) = \lambda^3 + (\sigma + b + 1)\lambda^2 + (\sigma + r)b\lambda + 2\sigma b(r-1) \ .$$

Let $\lambda = i\omega$ where ω is real and non-zero. If we require $\lambda = i\omega$ to be an eigenvalue of A we obtain the conditions

$$
\begin{aligned}
2\sigma b(r-1) &= (\sigma + b + 1)\omega^2 \\
(\sigma + r)b &= \omega^2
\end{aligned}
$$

This implies

$$r = r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

assuming that $\sigma > b + 1$. At $r = r_H$ a subcritical Hopf bifurcation occurs. The points C^+ and C^- become unstable for $r > r_H$.

14.7 A Trapping Region

Exercise 9.2.2 with solution on page 462.

Let

$$V = rx^2 + \sigma y^2 + \sigma(z-2r)^2 \ .$$

Let

$$v(t) = V(x(t), y(t), z(t)) \ .$$

After some algebra, one obtains

$$\frac{1}{2}\,\dot{v} = -\sigma(rx^2 + y^2 + b(z-r)^2) + \sigma b r^2 \ .$$

Let $\varepsilon > 0$. Define

$$R_\varepsilon = \{(x,y,z) \ : \ rx^2 + y^2 + b(z-r)^2 \geq br^2 + \varepsilon \ .$$

If $(x,y,z) \in R_\varepsilon$ then

$$\dot{v} \leq -2\sigma\varepsilon \ .$$

For $C > 0$ define the ellipsoid

$$E_C = \{(x,y,z) \ : \ V(x,y,z) \leq C\} \ .$$

Choose $C > 0$ so large that

$$\mathbb{R}^3 \setminus R_\varepsilon \subset E_C \ .$$

Then, if $\mathbf{x}(t) \notin E_C$, we have

$$\dot{v} \leq -2\sigma\varepsilon = -c < 0 \ .$$

Along any trajectory, the V–values decay strictly as long as the trajectory is outside E_C. Therefore, after a finite time the trajectory will enter E_C. Once in E_C, the trajectory cannot leave E_C.

Definition: Consider a system $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$. A set $E \subset \mathbb{R}^n$ is called a trapping region if the following two conditions hold: (a) For every x_0 there is a time $t_0 \geq 0$ so that $x(t_0, x_0) \in E$. (b) If $x_0 \in E$ then $x(t, x_0) \in E$ for all $t \geq 0$.

Our result shows that the Lorenz system has a trapping region of the form $E = E_C$ defined above.

Let S^t denote the solution operator for $\dot{x} = f(x)$ and assume that E is a trapping region. Then we have $S^t(E) \subset E$ for all $t \geq 0$. Therefore,

$$S^t(E) \subset S^s(E) \quad \text{if} \quad t \geq s \geq 0 \ .$$

The sets

$$S^t(E)$$

are nested and the set

$$\mathcal{A} = \cap_{t \geq 0} S^t(E)$$

is of interest to understand the long–time behaviour of the system.

For the Lorenz system, the set $\mathcal{A}$ has measure zero.

14.8 Sensitive Dependence and Prediction Time

Consider a system $\dot{x} = f(x)$. In many cases one observes the following: If one considers two solutions

$$x(t, x_0) \quad \text{and} \quad x(t, x_0 + \delta_0)$$

where δ_0 is a small vector, then the difference

$$\delta(t) = x(t, x_0 + \delta_0) - x(t, x_0)$$

grows approximately exponentially up to some time T:

$$\|\delta(t)\| \sim \|\delta_0\| \, e^{\lambda t} \quad \text{for} \quad 0 \leq t \leq T \ .$$

Here $\lambda > 0$ is often independent of x_0 in some region. After time T, the difference $\|\delta(t)\|$ cannot grow further if the solutions are confined to a trapping region.

One then has

$$\ln \|\delta(t)\| \sim \lambda t + \ln \|\delta_0\| \quad \text{for} \quad 0 \leq t \leq T \ .$$

Prediction Time Suppose we want to know $x(t, x_0)$ with accuracy $a = 10^{-3}$ and only consider the error in x_0. Suppose $\|\delta_0\| = 10^{-7}$ and $\lambda > 0$. How long is $x(t, x_0)$ determined with accuracy $a = 10^{-3}$?

From

$$\|\delta(t)\| \sim \|\delta_0\| \, e^{\lambda t} \leq 10^{-3}$$

and $\|\delta_0\| = 10^{-7}$ we obtain the condition

$$e^{\lambda t} \leq 10^4$$

or

$$t \leq 4 \, \frac{\ln 10}{\lambda} \ .$$

Now assume we increase the accuracy of the initial data by one digit to

$$\|\delta_0\| = 10^{-8} \ .$$

Arguing as above, we see that we can determine the solution with accuracy $a = 10^{-3}$ up to

$$t \leq 5\,\frac{\ln 10}{\lambda}\ .$$

If we denote the maximal prediction time by t_{pred} we obtain that

$$t_{pred}(10^{-5}) = \frac{5}{4}\,t_{pred}(10^{-4})\ .$$

If, for example,

$$\frac{\ln 10}{\lambda} = 1 day$$

then the prediction time increases from four days to five days if we increase the accuracy of the inital data by one digit. The effort to increase the accuracy by one digit may inrease the cost of determining the inital data by a factor 10.

Under the above assumptions, how much accuracy in δ_0 do we need to predict for 20 days? If $\|\delta_0\| = 10^{-N}$ then the condition

$$\|\delta_0\| e^{\lambda t} \leq 10^{-3}$$

yields

$$e^{\lambda t} \leq 10^{N-3}$$

or

$$\lambda t \leq (N-3)\ln 10$$

or

$$20 days = t \leq (N-3)\frac{\ln 10}{\lambda}\ .$$

If

$$\frac{\ln 10}{\lambda} = 1 day$$

then $N = 23$.

15 One Dimensional Maps

15.1 Fixed Points and 2–Cycles

Let $f : \mathbb{R} \to \mathbb{R}$ denote a C^1 map. For any $x_0 \in \mathbb{R}$ a discrete–time trajectory is determined by the iteration

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots \tag{15.1}$$

We denote

$$f^n = f \circ f \circ \ldots \circ f \quad (n \text{ times}) .$$

A point x^* is a fixed point of the evolution if

$$f(x^*) = x^* .$$

A fixed point is asymptotically stable if

$$|f'(x^*)| < 1$$

and is unstable if

$$|f'(x^*)| > 1 .$$

The iteration (15.1) can be visualized by a cobweb.

Two points x_0, x_1 form a 2–cycle if

$$f(x_0) = x_1, \quad f(x_1) = x_0, \quad x_0 \neq x_1 .$$

If x_0, x_1 form a 2–cycle, then each point x_j is a fixed point of f^2.

Suppose that x_0, x_1 form a 2–cycle for f. We have

$$f^2(x) = f(f(x)), \quad (f^2)'(x) = f'(f(x))f'(x) .$$

Therefore,

$$(f^2)'(x_0) = f'(x_1)f'(x_0) = (f^2)'(x_1) .$$

Therefore, the two–cycle x_0, x_1 is asymptotically stable if

$$|f'(x_0)||f'(x_1)| < 1 .$$

It is unstable if

$$|f'(x_0)||f'(x_1)| > 1 .$$

Note the following for later reference: If x_0, x_1 is a 2–cycle for f then

$$(f^2)'(x_0) = (f^2)'(x_1) . \tag{15.2}$$

15.2 The Logistic Map

Let

$$f(x) = rx(1-x), \quad f'(x) = r - 2rx \ ,$$

where $0 \le r \le 4$. Then f maps the interval $[0,1]$ into itself. We consider the discrete–time dynamical system determined by f with the interval $[0,1]$ as state space.

15.2.1 Fixed Points

Note that $x_1^* = 0$ is a fixed point for every parameter value $0 \le r \le 4$. The point $x_1^* = 0$ is globally attracting for $0 \le r \le 1$ and unstable for $1 < r \le 4$ since $f'(0) = r$. For $1 < r \le 4$ there is a second fixed point,

$$x_2^* = 1 - \frac{1}{r} \ .$$

We have

$$f'(1 - 1/r) = 2 - r \ .$$

Therefore, the fixed point x_2^* is unstable for $3 < r \le 4$. For $1 < r \le 3$ the fixed point x_2^* is attracting for every initial value $0 < x_0 < 1$.

The map $f(x)$ has no other fixed points besides $x_1^* = 0$ and $x_2^* = 1 - \frac{1}{r}$. At $r = 1$ a transcritical bifurcation of the trivial branch occurs.

15.2.2 The 2–Cycle Bifurcating From x_2^*

At $r = 3$ the fixed point

$$x_2^* = 1 - \frac{1}{r}$$

loses stability. We claim that a 2-cycle is born at $r = 3$.

We have

$$\begin{aligned} f^2(x) &= rf(x)(1 - f(x)) \\ &= r^2 x(1-x)(1 - rx(1-x)) \end{aligned}$$

The fixed points of $f^2(x)$ are the solutions of the equation

$$f^2(x) = x \ .$$

We can divide by the trivial solution $x = 0$. We also know that the remaining cubic polynomial has the divisor $x - x_2^*$. After division, one obtains a quadratic with roots

$$x_{0,1} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r-3)(r+1)}\right), \quad r > 3 \ .$$

(This should be checked.)

Note that $x_0 \neq x_1$. Also, neiher x_0 nor x_1 is a fixed point of f since f has precisely the fixed points $x_1^* = 0$ and $x_2^* = 1 - \frac{1}{r}$.

Furthermore,

$$f^2(x_0) = x_0$$

implies that

$$f^2(f(x_0)) = f(x_0) \; .$$

Thus, $f(x_0)$ is a fixed point of f^2. It follows that $f(x_0) = x_1$ and $f(x_1) = x_0$.

Let us determine the stability of the 2–cycle: We have

$$\begin{aligned}
(f^2)'(x_0) &= f'(x_0) f'(x_1) \\
&= r^2 (1 - 2x_0)(1 - 2x_1) \\
&= r^2 \Big(1 - \frac{1}{r}(r + 1 + \sqrt{\ldots}) \Big) \Big(1 - \frac{1}{r}(r + 1 - \sqrt{\ldots}) \Big) \\
&= (1 + \sqrt{\ldots})(1 - \sqrt{\ldots}) \\
&= 1 - (r-3)(r+1) \\
&= 4 + 2r - r^2
\end{aligned}$$

Consider the function

$$h(r) = 4 + 2r - r^2, \quad r \geq 3 \; .$$

We have $h(3) = 1$ and $h'(r) = 2 - 2r < 0$ for $r \geq 3$. The equation

$$h(r) = -1$$

has the solution $r_2 = 1 + \sqrt{6} = 3.4495\ldots$ We conclude that the 2-cycle x_0, x_1 is stable for

$$3 = r_1 < r < 1 + \sqrt{6} = r_2 = 3.4495\ldots$$

At $r = r_2$ we have

$$(f^2)'(x_0) = (f^2)'(x_1) = -1 \; .$$

It is then plausible that at $r = r_1$ and at each of the fixed points $x_{0,1}$ of f^2 a 2–cycle of f^2 is born. This leads to a 4–cycle of f for $r_2 < r \leq 4$.

15.2.3 Repeated Period Doubling

We have:

$$r_0 = 1 < r < 3 = r_1 : \quad \text{stable fixed point} x_2^* = 1 - \frac{1}{r}$$

$$r_1 = 3 < r < 1 + \sqrt{6} = r_2 : \quad \text{stable 2–cycle}$$

$$r_2 < r < r_3: \quad \text{stable 4–cycle}$$

$$r_n < r < r_{n+1}: \quad \text{stable } 2^n\text{–cycle}$$

The following is remarkable. (This discovery is essentially due to Feigenbaum.) The bifurcation values r_n converge,

$$r_n \to r_\infty = 3.569946\ldots$$

The convergence is essentially geometric:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

The number $\delta = 4.669\ldots$ is called Fiegenbaum's constant. It turns out to be independent of the family of maps $rx(1-x)$, but occurs universally in period doubling phenomena.

16 The Bernoulli Shift and the Logistic Map for $r = 4$

16.1 The Bernoulli Shift

Define the map $B : [0,1) \to [0,1)$ by

$$B(y) = \begin{cases} 2y & \text{for} \quad 0 \le y < \frac{1}{2} \,, \\ 2y - 1 & \text{for} \quad \frac{1}{2} \le y < 1 \,. \end{cases}$$

We can also write

$$B(y) = (2y) \bmod 1 \,.$$

Let us try to understand the dynamics determined by the iteration $y_{n+1} = B(y_n)$.

Any $y \in [0,1)$ can be written as

$$y = \sum_{j=1}^{\infty} b_j 2^{-j} \tag{16.1}$$

where $b_j \in \{0,1\}$. This representation of y is unique if do not allow that $b_j = 1$ for all sufficiently large j.

Conversely, any series of the above form determines a number $y \in [0,1)$. One also writes

$$y = [0.b_1 b_2 b_3 \ldots]_{base2}$$

and calls this the binary representation of y. Application of B yields

$$B(y) = [0.b_2 b_3 b_4 \ldots]_{base2} \,.$$

Thus, the binary point is moved one place to the right and the digit b_1 is replaced by 0.

Example: Consider the number

$$y_0 = [0.001\,001\,001\,\ldots]_{base2} \,.$$

We have

$$y_1 = [0.010\,010\,010\,\ldots]_{base2}$$

and

$$y_2 = [0.100\,100\,100\,\ldots]_{base2}$$

and $y_3 = y_0$. Thus, we have a 3–cycle.

In standard notation,

$$y_0 = \frac{1}{8} \sum_{j=0}^{\infty} \frac{1}{8^j} = \frac{1}{8} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{7} \,.$$

Application of $B(y) = (2y) \bmod 1$ yields

$$y_0 = \frac{1}{7}, \quad y_1 = \frac{2}{7}, \quad y_2 = \frac{4}{7}, \quad y_3 = \frac{1}{7} \; .$$

Consider the following sets:

$$\begin{aligned} S_1 &= \{y \in [0,1) \; : \; y = \sum_{j=1}^{J} b_j 2^{-j}, J \text{ finite}\} \\ S_2 &= \{y \in [0,1) \; : \; y = \sum_{j=1}^{\infty} b_j 2^{-j}, b_j \text{ becomes periodic}\} \\ S_3 &= [0,1) \setminus S_2 \end{aligned}$$

The sets S_1 and S_2 are infinite and denumerable. All three sets S_j are dense in $[0,1)$. If $y_0 \in S_1$ then $y_n \to 0$. If $y_0 \in S_2$ then y_n becomes periodic. If $y_0 \in S_3$ then y_n does not converge to a periodic cycle. We also note that S_3 has full measure.

16.2 Relation to the Logistic Map for $r = 4$

Let $f(x) = 4x(1-x)$ and let $B(y) = (2y) \bmod 1$. The dynamics determined by $f(x)$ and by $B(y)$ can be related by the transformation $x = \sin^2(\pi y) =: s(y)$.

Lemma 16.1 *Let $y_0 \in [0,1)$ and set $y_1 = B(y_0)$. Also, let $x_0 \in [0,1]$ and set $x_1 = f(x_0)$. With these settings we claim that $x_0 = s(y_0)$ implies $x_1 = s(y_1)$.*

Proof: We have

$$\begin{aligned} x_1 &= 4x_0(1-x_0) \\ &= 4s(y_0)(1-s(y_0)) \\ &= 4\sin^2(\pi y_0)\cos^2(\pi y_0) \\ &= \sin^2(2\pi y_0) \\ &= \sin^2(\pi(2y_0 \bmod 1)) \\ &= \sin^2(\pi y_1) \\ &= s(y_1) \end{aligned}$$

$\diamond$

An application of this lemma and the results on the Bernoulli shift yields the following for the logistic map at $r = 4$:

a) There is a dense denumerable set $T_1 \subset [0,1]$ so that $x_0 \in T_1$ implies $x_n \to 0$.

b) There are periodic cycles of any length.

c) There is a dense set T_3 of full measure so that x_n does not become periodic if $x_0 \in T_3$.

16.3 Invariant Measure for the Logistic Map at $r = 4$

If $x_0 \in [0,1]$ is chosen randomly, can we say something about the probability distribution of the x_n?

More precisely, we try to determine a function $H : (0,1) \to [0,\infty)$ with the property that

$$prob(x_j \in [a,b]) = \int_a^b H(x)\,dx \ .$$

Assume that $H(x)$ has this property. It is reasonable to expect the symmetry $H(x) = H(1-x)$ which we will use below.

We will derive a functional equation for $H(x)$. Let $0 < x < \frac{1}{2}$. Then we have

$$prob(x_j \in [0,x] \cup [1-x,1]) = 2\int_0^x H(p)\,dp \ .$$

The points x_j in $[0,x] \cup [1-x,1]$ are precisely those which get mapped to

$$x_{j+1} \in [0, f(x)] \ .$$

This leads to the requirement

$$2\int_0^x H(p)\,dp = \int_0^{f(x)} H(p)\,dp \ .$$

Differentiation yields the functional equation

$$H(x) = (2-4x)H(4x(1-x)) \ .$$

We also have the normalization condition

$$\int_0^1 H(x)\,dx = 1 \ .$$

Lemma 16.2 *The function*

$$H(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$$

solves the functional equation and the normalization condition.

Proof: We first ignore the normalization condition and set

$$h(y) = \frac{1}{\sqrt{y(1-y)}} \ .$$

Then we have

$$\begin{aligned}
\frac{2-4x}{\sqrt{4x(1-x)(1-4x(1-x))}} &= \frac{1-2x}{\sqrt{x}}\Big((1-x)(1-4x+4x^2)\Big)^{-1/2} \\
&= \frac{1-2x}{\sqrt{x}}\Big((1-x)(1-2x)^2)\Big)^{-1/2} \\
&= \frac{1}{\sqrt{x(1-x)}} \\
&= h(x)
\end{aligned}$$

This shows that $h(x)$ solves the functional equation.

Lemma 16.3

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi$$

Proof: Set

$$g(x) = \arctan\sqrt{\frac{1}{x} - 1} \quad \text{for} \quad 0 < x < 1 \ .$$

We compute

$$\begin{aligned}
g'(x) &= \frac{1}{1+\frac{1}{x}-1}\,\frac{1}{2}\Big(\frac{1}{x}-1\Big)^{-1/2}(-x^{-2}) \\
&= -\frac{1}{2x}\frac{1}{\sqrt{\frac{1}{x}-1}} \\
&= -\frac{1}{2}\frac{1}{\sqrt{x(1-x)}}
\end{aligned}$$

Therefore, the function $-2g(x)$ has the derivative $h(x)$ for $0 < x < 1$.

For $\varepsilon > 0$:

$$\begin{aligned}
\int_\varepsilon^1 \frac{dx}{\sqrt{x(1-x)}} &= -2(g(1) - g(\varepsilon)) \\
&= 2g(\varepsilon) \\
&= 2\arctan\sqrt{\frac{1}{\varepsilon} - 1}
\end{aligned}$$

As $\varepsilon \to 0$, the integral converges to π. $\diamond$

16.4 Programs and Figures

Programs for the Bernoulli Shift

```
% The name of this file is p3.m
% This program plots the function B(x) for
% x between zero and one.
clear
N=100;
 x=linspace(0,0.99,N);
for j=1:N
y(j)=fun3(x(j));
end
plot(x,y)
xlabel('x'), ylabel('B(x)=(2x) mod 1')
title('The Bernoulli Shift Map')

% The name of this file is fun3.m
% This function file evaluates the Bernoulli shift
% map B(x)=(2x)mod 1 for x between zero and one.
function f=fun3(x)
f=2*x-fix(2*x);

% The name of this file is p4.m
% This program plots a numerical orbit
% determined by the Bernoulli shift
clear
N=100;
x(1)=100/777;
for j=1:N
x(j+1)=fun3(x(j));
end
plot(x)
xlabel('n'), ylabel('x_n')
title('Numerical Evolution of the Bernoulli Shift')
```

Programs for the Logistic Map

```
% The name of this file is p2.m
% This program plots the histogram of an
% orbit for the logistic map at parameter r.
N=10000;
r=3.83;
x(1)=0.7;
for j=1:N
x(j+1)=fun1(x(j),r);
end
hist(x,200)
title('Histogram of f(x)=rx(1-x), r=3.83')
```

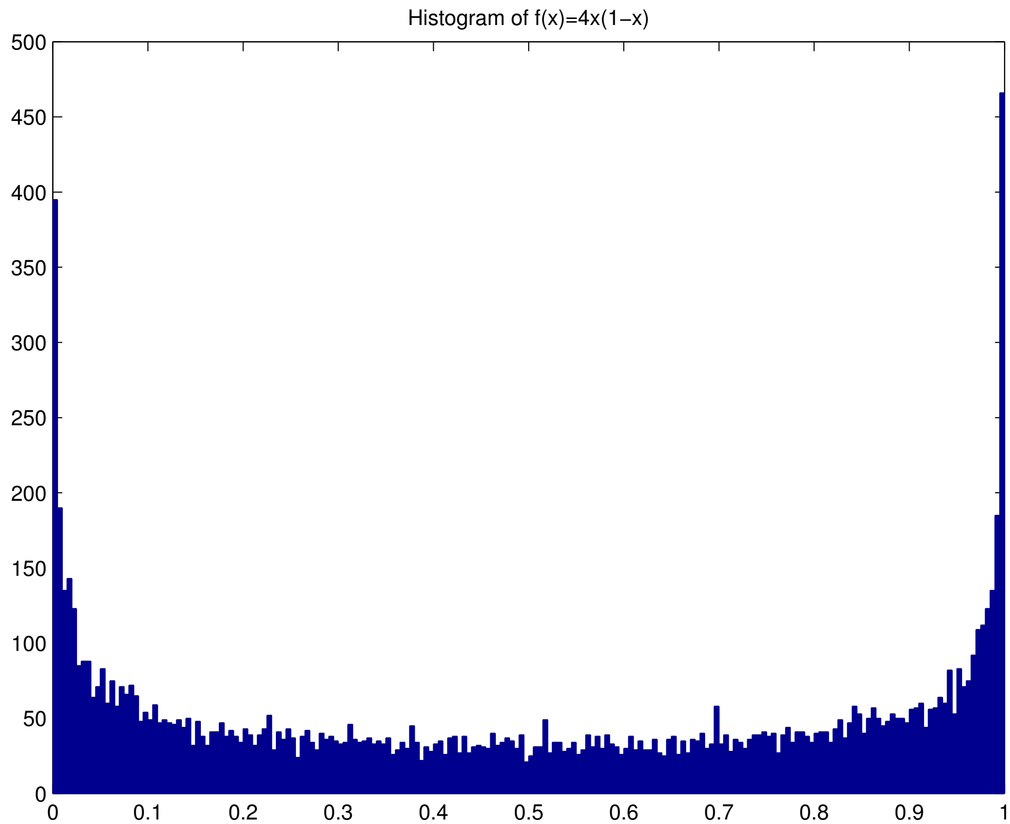

Figure 16.1: Histogram of the Logistic Map for $r = 4$

The output of p2.m is shown in Figure 16.2.

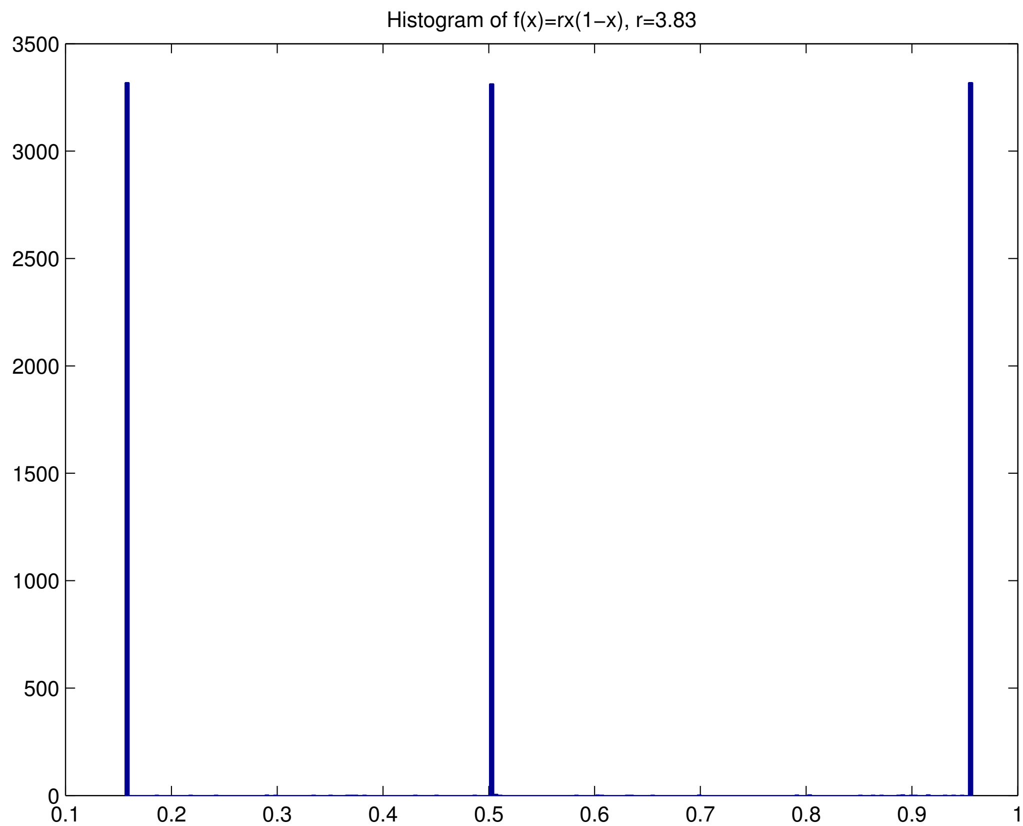

Figure 16.2: Output of p2.m

17 Dimensions of Sets

17.1 The Hausdorff Dimension

Felix Hausdorff (1868–1942)

The Hausdorff dimension is also called the Hausdorff–Besicovitch dimension.

Let $F \subset \mathbb{R}^n$. For $\delta > 0$ and $s > 0$ define

$$H^s_\delta = H^s_\delta(F) = \inf\{\sum_{j=1}^{\infty} |U_j|^s \ : \ (U_j)_{j=1,2,\ldots}\text{is a } \delta\text{–cover of } F \ .$$

Note that

$$0 \le H^s_\delta \le \infty \ .$$

Monotonicity properties:

a) If $0 < \delta_1 < \delta_2$ then $H^s_{\delta_1} \le H^s_{\delta_2}$. Therefore,

$$\lim_{\delta \to 0+} H^s_\delta =: H^s(F) \in [0, \infty]$$

exists. The number $H^s(F)$ is called the s–dimensional Hausdorff measure of F.

b) Fix $\delta > 0$. If $|U_j| \le \delta$ and $0 < s_1 < s_2$ then

$$|U_j|^{s_2} = |U_j|^{s_2 - s_1} \, |U_j|^{s_1} \le \delta^{s_2 - s_1} |U_j|^{s_1} \ .$$

Therefore,

$$H_\delta^{s_2} \le \delta^{s_2 - s_1} \, H_\delta^{s_1} \ .$$

In this estimate, let $\delta \to 0+$. Obtain:

Lemma 17.1 *Let $0 < s_1 < s_2$. If $H^{s_1} < \infty$, then $H^{s_2} = 0$.*

It is not difficult to show:

Lemma 17.2 *Let $F \subset \mathbb{R}^n$ denote any bounded set. Then $H^n(F) < \infty$.*

Consequently, $H^s(F) = 0$ for $s > n$. Define

$$\begin{aligned} dim_{Huasdorff}(F) &= \inf\{s > 0 \ : \ H^s(F) = 0\} \\ &= \inf\{s > 0 \ : \ H^s(F) < 0\} \\ &= \sup\{s > 0 \ : \ H^s(F) = \infty\} \ . \end{aligned}$$