MATH 412, Nonlinear Dynamics and Chaos, Spring 2019
Prof. J. Lorenz, Instructor
Homework 2, assigned Feb. 8, 2019, due Feb. 22, 2019

1) Consider the equation

$$x' = rx + x^3 - x^5$$

where r is a real parameter.

a) Derive algebraic expressions for all fixed points as r varies. How many fixed points exists in which r intervals? What is their stability?

b) Sketch the fixed points as a function of r.

c) What bifurcations occur? At which r values?

d) Sketch the flow fields for $r < -\frac{1}{4}$, for $-\frac{1}{4} < r < 0$, for $r > 0$. You may assume that $t \geq 0$.

The flow fields show trajectories $\Big(t, x(t)\Big)$ in the (t,x)–plane where $x(t)$ is a solution of the differential equation.

2) Consider the equation

$$x' = r - x^2 \equiv f(x,r) \ .$$

Compute the potential $V(x,r)$ satisfying

$$-V_x(x,r) = f(x,r), \quad V(0,r) = 0 \ .$$

Sketch the functions

$$x \to f(x,r) \quad \text{and} \quad x \to V(x,r)$$

for different values of r, including the bifurcation value for r, and sketch the bifurcation diagram. What kind of bifurcation occurs? At which value of r?

3) Consider the boundary value problems

$$a) \qquad -\varepsilon x''(t) + x'(t) = 0, \quad x(0) = -2, \quad x(1) = 3$$

and

$$b) \qquad \varepsilon x''(t) + x'(t) = 0, \quad x(0) = -2, \quad x(1) = 3$$

where $\varepsilon > 0$.

Compute the exact solutions. Sketch a solution in the interval $0 \leq t \leq 1$ for a) and for b) assuming $0 < \varepsilon << 1$.

Hint: Each solution has a boundary layer.

4) Exercise 3.6.2 of Strogatz.

Hint: Set $f(x,r,h) = h + rx - x^2$. For which points $(r,h) \in \mathbb{R}^2$ does the system

$$\begin{aligned} f(x,r,h) &= 0 \\ f_x(x,r,h) &= 0 \end{aligned}$$

have a real solution x?