MATH 412, Nonlinear Dynamics and Chaos, Spring 2019
Prof. J. Lorenz, Instructor
Homework 4, assigned March 8, 2019, due March 29, 2019

1) Consider the system

$$r' = r^3 - 4r, \quad \theta' = 1 \ ,$$

where (r, θ) are polar coordinates.

Sketch the phase portrait in the (x, y)–plane.

2a) Consider the system

$$x' = \mu x - x^2, \quad y' = -y$$

where μ is a real parameter.

At which value of μ does a bifurcation of fixed points occur?

Sketch the bifurcation diagram and sketch the phase portrait in the (x, y)–plane for values of μ less than and greater than the bifurcation value.

2b) Do the same for the system

$$x' = \mu x + x^3, \quad y' = -y$$

3) Consider the forced van der Pol equation

$$x'' + \mu(x^2 - 1)x' + x = a$$

where μ and a are parameters. Write the equation as a first order system for x and $y = x'$ and note that the points $P_a = (a, 0)$ are fixed points. Determine the parameter pairs $(\mu, a) \in \mathbb{R}^2$ for which you can expect a Hopf bifurcation to occur at $P_a = (a, 0)$.

4) **Numerical Project:** Consider van der Pol's equation

$$x'' + \varepsilon(x^2 - 1)x' + x = 0$$

with initial condition

$$x(0) = r_0, \quad x'(0) = 0$$

where $r_0 > 0$ and $0 < \varepsilon << 1$. We have derived the approximation

$$x(t) \sim r(t)\cos t, \quad x'(t) \sim -r(t)\sin t$$

with

$$r(t) = \frac{2r_0}{\sqrt{r_0^2 + e^{-\varepsilon t}(4 - r_0^2)}} \ .$$

Idea: Use a numerical code to solve van der Pol's equation and check if the analytical approximation is any good.

Choose $r_0 = 1$, for example, and choose $\varepsilon > 0$ and the time interval $0 \leq t \leq T(\varepsilon)$. Make at least two choices.

For the numerical solution, plot $x(t)$ and $x'(t)$ as functions of $0 \leq t \leq T(\varepsilon)$. Also, plot the points $(x(t), x'(t))$ in the phase plane.

Do the same for the analytical approximation. It is reasonable to neglect the numerical error. Then, how good is the analytical approximation? Express your opinion.