Nonlinear Dynamics and Chaos, Supplements
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Contents

1 Notes on the History of Dynamics

Johannes Kepler (1571-1630) formulated Kepler's laws of planetary motion.

Isaac Newton (1642-1727) used the inverse square law of gravitational attraction to derive Kepler's laws. Invention of calculus. Newton solved the two–body problem. This was an enormous success which lead to a deterministic and mechanical view of the world.

There were many attempts to solve the three body problem in a similar way, by an explicit formula, which gives the positions and velocities of three bodies as functions of time. It turned out that this is not possible.

Jules Henri Poincaré (1854-1912) started the qualitative theory of differential equations. He formulated the first ideas about chaotic motion described by deterministic systems.

KAM theory is named after Andrei Nikolaevich Kolmogorov (1903-1987), Vladimir Igorevich Arnold (1937-2010), and Jürgen Moser (1928-1999). The theory gives results about invariant tori of perturbed Hamiltonian systems. The origins of KAM theory lie in the question of stability of the solar system. Laplace, Lagrange, Gauss, Poincaré and many others had worked on this.

Edward N. Lorenz (1917-2008), meteorologist, derived a simple deterministic model system with sensitive dependence on initial conditions (1963). Is the butterfly effect real for the weather? The sensitivity of a system can be measured by the largest Lyapunov exponent, α. If the exponent α is positive, then an initial error of size δ grows over time (approximately) like $\delta e^{\alpha t}$, until the size of the system limits further growth. Nevertheless, even for a system with positive Lyapunov exponent, some average quantities may be accurately predictable. Can we compute the climate 30 years in advance though we cannot predict the weather two weeks in advance?

In bifurcation theory one considers parameter dependent systems like

$$u'(t) = f(u(t), \lambda)$$

where λ is a parameter. As λ changes, the dynamics may change qualitatively, not just quantitatively. If a qualitative change occurs at $\lambda = \lambda_0$ then λ_0 is called a bifurcation value.

An interesting bifurcation is the transition from laminar to turbulent flow. This transition was addressed in a paper by Ruelle and Takens, *On the Nature of Turbulence*, 1970, which is still controversial. The term of a strange attractor was introduced.

M. Feigenbaum (1980) studied period doubling bifurcations for maps. He discovered an interesting universality of transition from simple to chaotic motion through repeated period doubling. Feigenbaum's model equations can be used to show that the average behaviour of a chaotic dynamical system may still be well determined and computable even if individual trajectories can be computed accurately only for a short time. We cannot predict the weather in Albuquerque 30 years from today, but the climate (the average weather) may still be predictable.

2 Flows on the Line: Supplements

2.1 Rule of Substitution

Let $t_0, t_1, x_0, x_1 \in \mathbb{R}$ and assume that

$$t_0 < t_1 \quad \text{and} \quad x_0 < x_1 \ .$$

Let $\Phi : [x_0, x_1] \to \mathbb{R}$ be a continuous function and let

$$x : [t_0, t_1] \to [x_0, x_1]$$

be one–to–one, onto and let $x \in C^1[t_0, t_1]$.

Then the following rule of substitution holds:

$$\int_{x_0}^{x_1} \Phi(x)\, dx = \int_{t_0}^{t_1} \Phi(x(t)) x'(t)\, dt \ . \tag{2.1}$$

Formally, one replaces dx by $x'(t)dt$.

Proof of (2.1): Assume that the function $Q(x)$ satisfies

$$\frac{d}{dx} Q(x) = Q'(x) = \Phi(x) \quad \text{for} \quad x_0 \le x \le x_1 \ .$$

Then we have

$$\int_{x_0}^{x_1} \Phi(x)\, dx = Q(x_1) - Q(x_0) \ .$$

Also,

$$\frac{d}{dt} Q(x(t)) = Q'(x(t))\, x'(t) = \Phi(x(t))\, x'(t) \quad \text{for} \quad t_0 \le t \le t_1 \ .$$

Therefore,

$$\int_{t_0}^{t_1} \Phi(x(t))\, x'(t)\, dt = Q(x(t_1)) - Q(x(t_0)) = Q(x_1) - Q(x_0) \ .$$

This proves (2.1).

2.2 Application to an IVP

Consider the IVP

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0 \ .$$

Let $x(t)$ denote the solution. Formally,

$$\frac{dx}{f(x)} = dt \ .$$

Apply the rule of substitution where

$$\Phi(x) = \frac{1}{f(x)}$$

to obtain:

$$\begin{aligned} \int_{x_0}^{x(t_1)} \frac{dx}{f(x)} &= \int_0^{t_1} \frac{x'(t)}{f(x(t))}\, dt \\ &= \int_0^{t_1} 1\, dt \\ &= t_1 \end{aligned}$$

Thus, if $x(t)$ solves the IVP then

$$\int_{x_0}^{x(t)} \frac{dx}{f(x)} = t \ . \tag{2.2}$$

Conversely, assume that $x(t)$ solves the equation (2.2). Differentiating the equation (2.2) with respect to t yields that

$$\frac{x'(t)}{f(x(t))} = 1 \ ,$$

thus $x'(t) = f(x(t))$. The function $x(t)$ solves the differential equation $x' = f(x)$.

2.3 A Calculus Rule

Let $s = \sin x, c = \cos x$. We claim that

$$\sin(2x) = 2sc \ .$$

Proof: We have

$$e^{ix} = c + is \quad \text{and} \quad e^{-ix} = c - is \ ,$$

thus

$$2is = e^{ix} - e^{-ix} \ .$$

Therefore,

$$\begin{aligned} 2i\sin(2x) &= (e^{ix})^2 - (e^{-ix})^2 \\ &= (c+is)^2 - (c-is)^2 \\ &= c^2 + 2isc - s^2 - (c^2 - 2isc - s^2) \\ &= 4isc \end{aligned}$$

This proves the rule

$$\sin(2x) = 2\sin x \cos x \ .$$

Also,

$$\sin x = 2\sin(x/2)\cos(x/2) = 2\tan(x/2)\cos^2(x/2)\ .$$

2.4 Units of Physics

unit of length:

$$meter = m$$

unit of time:

$$second = s$$

unit of mass:

$$kilogram = kg$$

unit of force:

$$Newton = N = kg\,m/s^2$$

unit of energy:

$$Newton\,meter = Nm = kg\,m^2/s^2 = joule = J$$

unit of power:

$$watt = joule/s$$

unit of electric charge:

$$Coulomb$$

unit of electric current :

$$Ampere = Coulomb/s$$

unit of voltage:

$$Volt = joule/Coulomb$$

unit of resistance:

$$Ohm = Volt/Ampere$$

unit of capacitance :

$$farad = Coulomb/Volt$$

3 Examples of Bifurcations: Supplements

4 Overdamped Bead on a Rotating Hoop: Supplements

5 Imperfect Bifurcations and Catastrophes: Supplements

6 Flows on the Circle: Supplements

We will use complex variables to show that

$$\int_\Gamma \frac{dz}{(z-z_1)(z-z_2)} = \frac{2\pi i}{z_2 - z_1} \ . \tag{6.1}$$

Here Γ is the unit circle and z_1, z_2 are complex numbers with $|z_2| < 1 < |z_1|$.

First claim:

$$\int_\Gamma \frac{dz}{z} = 2\pi i \tag{6.2}$$

and

$$\int_\Gamma z^j \, dz = 0 \tag{6.3}$$

if $j \in \mathbb{Z}, j \neq -1$.

The unit circle Γ is parameterized by

$$z = e^{it}, \quad 0 \le t \le 2\pi \ .$$

Therefore, $dz = izdt$ and

$$\int_\Gamma z^j \, dz = i \int_0^{2\pi} e^{i(j+1)t} \ dt \ .$$

Clearly, if $j = -1$ then

$$e^{i(j+1)t} \equiv 1$$

and (6.2) follows. If j is an integer and $j \neq -1$ then

$$\int_0^{2\pi} e^{i(j+1)t} \ dt = \frac{1}{i(j+1)} \left(e^{i(j+1)2\pi} - e^0 \right) = 0 \ .$$

Equation (6.3) follows.

Let $h(z)$ be a meromorphic function given by

$$h(z) = \sum_{j=-J}^{\infty} a_j z^j \quad \text{for} \quad 0 < |z| < 1 + \varepsilon$$

where $\varepsilon > 0$ and $J \geq 1$. If $a_{-J} \neq 0$ then $h(z)$ is a meromorphic function with a pole of order J at the origin, $z = 0$. The coefficient a_{-1} is called the residue of the function $h(z)$ at the pole $z = 0$,

$$a_{-1} = Res(h, z = 0) \ .$$

The equations (6.2) and (6.3) yield that

$$\int_\Gamma h(z)\, dz = \sum_{j=-J}^{\infty} a_j \int_\Gamma z^j\, dz = 2\pi i\, a_{-1} \ ,$$

thus

$$\int_\Gamma h(z) dz = 2\pi i\, Res(h, z = 0) \ .$$

Recall the assumption

$$|z_2| < 1 < |z_1| \ .$$

The function

$$g(z) = \frac{1}{z - z_1} \ , \quad z \in \mathbb{C} \setminus \{z_1\} \ ,$$

is holomorphic inside Γ and has the Taylor series expansion

$$g(z) = \sum_{j=0}^{\infty} b_j (z - z_2)^j \quad \text{for} \quad |z - z_2| < |z_1 - z_2| \ ,$$

where

$$b_0 = g(z_2) = \frac{1}{z_2 - z_1} \ .$$

Let

$$h(z) = \frac{1}{(z - z_1)(z - z_2)} = \frac{g(z)}{z - z_2} \ .$$

The Taylor series expansion of $g(z)$ implies that $h(z)$ is a meromorphic function with a pole at $z = z_2$ and

$$Res(h, z = z_2) = g(z_2) = \frac{1}{z_2 - z_1} \ .$$

Let $\Gamma_\varepsilon(z_2)$ denote the circle of radius ε centered at $z = z_2$. Choose $\varepsilon > 0$ so small that $\Gamma_\varepsilon(z_2)$ lies inside Γ. Then one obtains that

$$\begin{aligned} \int_\Gamma h(z)\, dz &= \int_{\Gamma_\varepsilon(z_0)} h(z)\, dz \\ &= 2\pi i\, Res(h, z = z_2) \\ &= \frac{2\pi i}{z_2 - z_1} \ . \end{aligned}$$

This completes the proof of the equation (6.1).