Introduction to Partial Differential Equations, Math 463/513, Spring 2015

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Contents

1 First Order Scalar PDEs

First order scalar PDEs can be solved by solving families of ODEs. This is not true anymore for higher order PDEs or for *systems* of first order PDEs.

The general quasilinear first order equation for an unknown function u of two independent variables x, y is

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \ .$$

In applications, y is often the time variable, $y = t$, and $b(x,y,u) \neq 0$ for all arguments of interest. Then we can divide by b and achieve the standard form

$$u_t + a(x,t,u)u_x = c(x,t,u) \ .$$

We start with some simpler special forms and discuss the following types of linear equations first:

$$\begin{aligned} u_t + au_x &= 0, \quad a \in \mathbb{R} \\ u_t + au_x &= F(x,t) \\ u_t + au_x &= c(x,t)u + F(x,t) \\ u_t + a(x,t)u_x &= 0 \\ u_t + a(x,t)u_x &= c(x,t)u + F(x,t) \end{aligned}$$

It is not difficult to generalize to more than two independent variables, e.g., to pdes for $u(x,y,t)$.

1.1 Linear Problems

1. The simplest problem. Consider the Cauchy problem [1]

$$u_t + au_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \ , \tag{1.1}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R} \ , \tag{1.2}$$

where $f \in C^1$ is a given function and a is a real constant. It is not difficult to check that the problem is solved by

$$u(x,t) = f(x - at) \tag{1.3}$$

since $u_t(x,t) = -af'(x-at)$ and $u_x(x,t) = f'(x-at)$. The solution describes the propagation of the initial data $f(x)$ at speed a.

Consider a so–called projected characteristic, parameterized by t,

$$\Gamma_{x_0} = \Big\{ (x_0 + at, t) \ : \ t \geq 0 \Big\} \ . \tag{1.4}$$

If we evaluate the solution $u(x,t) = f(x-at)$ on Γ_{x_0} we obtain that

$$u(x_0 + at, t) = f(x_0) \ ,$$

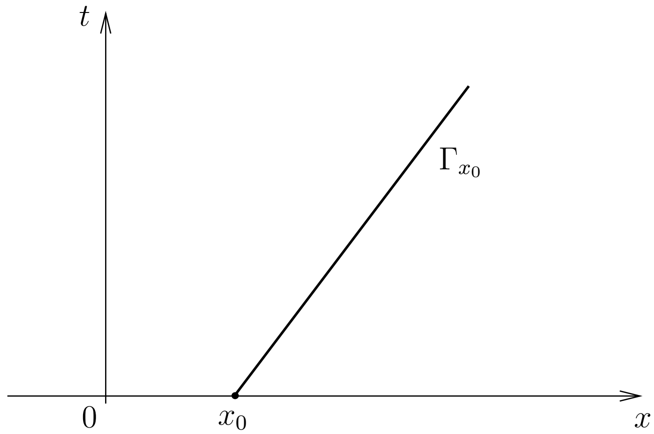

Figure 1: Projected characteristic Γ_{x_0} for $a > 0$

i.e., the solution carries the initial value $f(x_0)$ along the projected characteristic Γ_{x_0}.

We want to show that the above Cauchy problem does not have another solution. To this end, let $v(x,t)$ denote any C^1 solution. Fix any $x_0 \in \mathbb{R}$ and consider the function

$$h(t) = v(x_0 + at, t) \ ,$$

i.e., consider v along the projected characteristic Γ_{x_0}. Obtain

$$h'(t) = (v_t + av_x)(x_0 + at, t) = 0 \ .$$

Since $h(0) = v(x_0, 0) = f(x_0)$ we obtain that v equals $f(x_0)$ along the projected characteristic Γ_{x_0}, and therefore v agrees with the solution (1.3).

2. Addition of a forcing. Consider the Cauchy problem

$$u_t + au_x = F(x,t), \quad x \in \mathbb{R}, \quad t \geq 0 \ , \tag{1.5}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R} \ . \tag{1.6}$$

As above, let $f \in C^1, a \in \mathbb{R}$; we also assume $F, F_x \in C$. Assume that $u \in C^1$ is a solution. Define

$$h(t) = u(x_0 + at, t) \ ,$$

i.e., consider u along the projected characteristic Γ_{x_0}. Obtain

$$h'(t) = (u_t + au_x)(x_0 + at, t) = F(x_0 + at, t) \ .$$

Also, $h(0) = f(x_0)$. Therefore,

$$h(t) = f(x_0) + \int_0^t F(x_0 + as, s) ds \ .$$

[1] A PDE together with an initial condition is called a Cauchy problem.

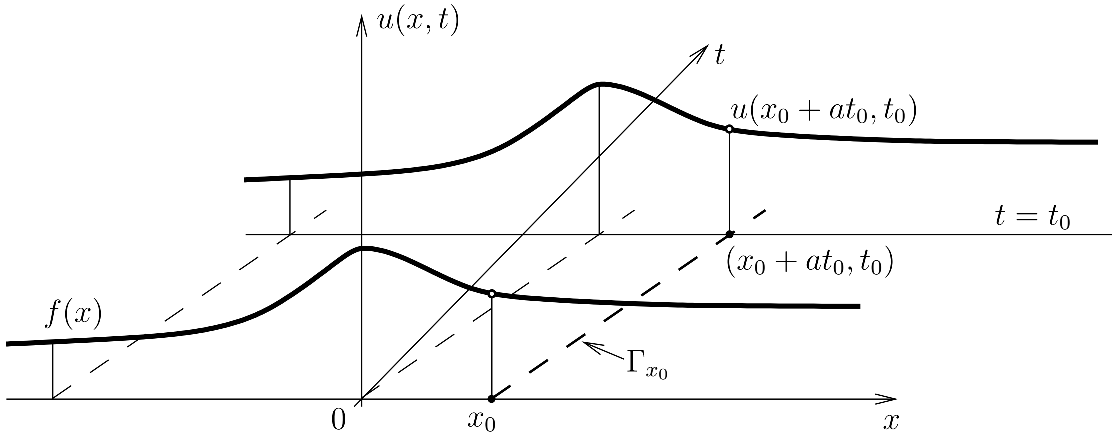

Figure 2: Solution of $u_t + au_x = 0$, $u(x,0) = f(x)$, $a > 0$

Since $u(x_0 + at, t) = h(t)$ we can obtain $u(x,t)$ as follows: Given (x,t), first determine x_0 with $x_0 + at = x$, i.e.,

$$x_0 = x - at \ .$$

(This determines the projected characteristic Γ_{x_0} which passes through x at time t.) Second,

$$\begin{aligned} u(x,t) &= h(t) \\ &= f(x-at) + \int_0^t F(x - at + as, s)ds \ . \end{aligned} \tag{1.7}$$

So far, we have assumed that u solves the problem (1.5), (1.6) and have derived the above formula (1.7).

Exercise: Show that the function given by (1.7) solves the problem (1.5), (1.6).

Example 1.1: Solve

$$u_t + au_x = 1, \quad u(x,0) = \sin x \ .$$

The formula (1.7) yields

$$u(x,t) = \sin(x - at) + t \ .$$

We note that the forcing term 1 on the right–hand side of the pde $u_t + au_x = 1$ leads to the growing term t in the solution.

3. Addition of a zero–order term. Consider the Cauchy problem

$$u_t + au_x = c(x,t)u + F(x,t), \quad x \in \mathbb{R}, \quad t \geq 0 \ , \tag{1.8}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R} \ . \tag{1.9}$$

Assume $f \in C^1, c, c_x, F, F_x \in C, a \in \mathbb{R}$. As before, we first assume that u is a C^1 solution. This will lead us to a formula for u. We can then use the formula to show that the problem actually has a solution.

Consider the function

$$h(t) = u(x_0 + at, t) \ .$$

Obtain $h(0) = f(x_0)$ and

$$\begin{aligned} h'(t) &= (u_t + au_x)(x_0 + at, t) \\ &= c(x_0 + at, t)h(t) + F(x_0 + at, t) \ . \end{aligned}$$

Thus $h(t)$ satisfies a 1st order linear ODE.

Recall that the ODE IVP

$$h'(t) = \gamma(t)h(t) + g(t), \quad h(0) = h_0 \ ,$$

is solved by

$$h(t) = h_0 \exp(\int_0^t \gamma(\tau)d\tau) + \int_0^t \exp(\int_s^t \gamma(\tau)d\tau)g(s)ds \ .$$

Thus we can obtain the solution $u(x, t)$ in explicit form by solving linear ODE initial value problems.

Exercise: Derive the resulting formula for $u(x, t)$ and show that it defines a solution of the Cauchy problem (1.8), (1.9).

Example 1.2:

$$u_t + au_x = -u + 1, \quad u(x, 0) = \sin x \ .$$

Obtain

$$\begin{aligned} h(t) &= f(x_0)e^{-t} + \int_0^t e^{-(t-s)}ds \\ &= \sin(x_0)e^{-t} + e^{-t}(e^t - 1) \ , \end{aligned}$$

thus

$$u(x, t) = \sin(x - at)e^{-t} + 1 - e^{-t} \ .$$

We note that the zero–order term $-u$ on the right–hand side of the pde leads to the exponetially decaying factor e^{-t}.

4. Variable signal speed $a(x, t)$**.** Consider the Cauchy problem

$$u_t + a(x, t)u_x = 0, \quad u(x, 0) = f(x) \ , \tag{1.10}$$

where $a, f \in C^1$. Consider a line parameterized by t of the form

$$\Gamma_{x_0} = \Big\{(\xi(t), t), t \geq 0\Big\}, \quad \xi \in C^1 \ , \tag{1.11}$$

where $\xi : [0, \infty) \to \mathbb{R}$ is a C^1 function with $\xi(0) = x_0$. We first assume that $\xi(t)$ is an arbitrary C^1 function with $\xi(0) = x_0$, but we will determine $\xi(t)$ below.

Assume that $u(x,t)$ solves the Cauchy problem (1.10) and consider the function

$$h(t) = u(\xi(t), t) \ ,$$

which is the solution u along the line Γ_{x_0}. Then we obtain $h(0) = f(x_0)$ and

$$h'(t) = (u_t + \xi'(t) u_x)(\xi(t), t) \ .$$

Therefore, $h'(t) \equiv 0$ if

$$\xi'(t) = a(\xi(t), t), \quad t \geq 0 \ . \tag{1.12}$$

This motivates the following definition.

Definition 1.1 *The parameterized line (1.11) is called a projected characteristic for the equation $u_t + a(x,t)u_x = 0$ if the function $\xi(t)$ satisfies (1.12).*

Our considerations suggest to solve the problem (1.10) as follows: Let (x,t) be a given point. Determine the real number $x_0 = x_0(x,t)$ so that the solution $\xi(t) = \xi(t; x_0)$ of the IVP

$$\xi'(t) = a(\xi(t), t), \quad \xi(0) = x_0 \ , \tag{1.13}$$

satisfies

$$\xi(t; x_0) = x \ .$$

Then set

$$u(x,t) = f(x_0(x,t)) \ .$$

Example 1.3:

$$u_t + x u_x = 0, \quad u(x,0) = f(x) \ .$$

We consider the IVP

$$\xi'(t) = \xi(t), \quad \xi(0) = x_0 \ .$$

The solution is

$$\xi(t; x_0) = x_0 e^t \ .$$

Let (x,t) be a given point. We solve

$$x_0 e^t = x$$

for $x_0 = x_0(x,t)$. This yields

$$x_0(x,t) = x e^{-t} \ .$$

Then we obtain

$$u(x,t) = f(x_0(x,t)) = f(x e^{-t}) \ .$$

Thus, our solution formula is

$$u(x,t) = f(x e^{-t}) \ .$$

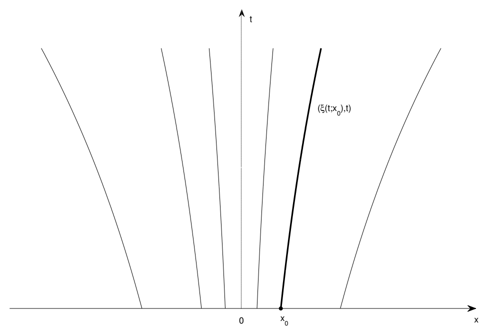

Figure 3: Characteristics $\xi(t;x_0) = e^t x_0$

5. Add a zero–order term plus forcing. Consider the Cauchy problem

$$u_t + a(x,t)u_x = c(x,t)u + F(x,t), \quad u(x,0) = f(x) \ .$$

To solve the problem, first determine the projected characteristics of the problem $u_t + a(x,t)u_x = 0$, i.e., solve the IVPs (1.13). Denote the solution of (1.13) by $\xi(t) = \xi(t;x_0)$. Assuming that $u(x,t)$ solves the given problem, we set

$$h(t) = u(\xi(t;x_0),t) \ .$$

Then $h(t)$ sayisfies $h(0) = f(x_0)$ and

$$h'(t) = c(\xi(t;x_0),t)h(t) + F(\xi(t;x_0),t) \ .$$

This allows us to compute $h(t) = h(t;x_0)$. If (x,t) is a given point, we determine $x_0 = x_0(x,t)$ with $\xi(t;x_0) = x$. Then we obtain

$$u(x,t) = h(t;x_0(x,t)) \ .$$

Though the given Cauchy problem is linear in u, there is no guarantee that it is solvable in some time interval $0 \le t \le T$ with $T > 0$. The reason is that the equation $\xi'(t) = a(\xi(t),t)$ for $\xi(t)$ (which determines the projected characteristics) maybe nonlinear nonlinear in ξ.

Example 1.4: In this example we let $(x,t) \in \mathbb{R}^2$, i.e., we allow t to be negative.

Consider the Cauchy problem

$$u_t + x^2 u_x = 0, \quad u(x,0) = f(x) \ , \tag{1.14}$$

where $f \in C^1$. The projected characteristics

$$\Gamma_{x_0} = \Big\{ (\xi(t;x_0),t) \ : \ t \in I_{x_0} \Big\}$$

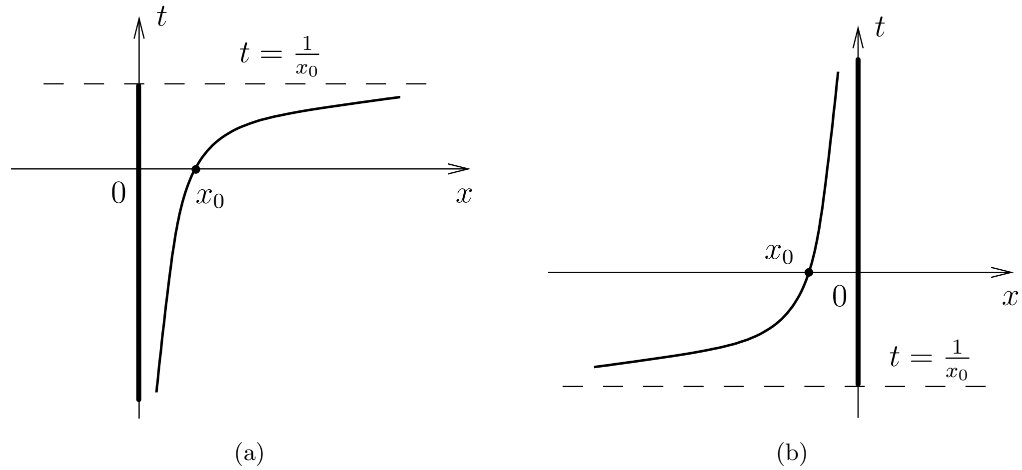

Figure 4: Time interval I_{x_0} for positive (a) and negative (b) values of x_0. Recall that $\xi(t;x_0) = \frac{x_0}{1-x_0 t}$

are determined by

$$\xi' = \xi^2, \quad \xi(0) = x_0 \ , \tag{1.15}$$

thus

$$\xi(t;x_0) = \frac{x_0}{1-x_0 t} \ .$$

Here I_{x_0} is the maximal interval of existence of the solution $\xi(t) = \xi(t;x_0)$ of the IVP (1.15).

Clearly, $\xi(t;x_0)$ becomes singular for $t = 1/x_0$. Therefore, $\xi(t;x_0)$ solves the initial value problem (1.15) in the following time intervals:

If $x_0 > 0$ we require $-\infty < t < \frac{1}{x_0}$; if $x_0 < 0$ we require $\frac{1}{x_0} < t < \infty$; if $x_0 = 0$ then $t \in \mathbb{R}$ is not restricted, i.e., $I_0 = \mathbb{R}$.

The solution $u(x,t)$ is determined in the region R of the (x,t) plane which is covered by the projected characteristics. This is the region consisting of all points

$$\Big(\xi(t;x_0), t\Big) \tag{1.16}$$

where $x_0 \in \mathbb{R}, t \in I_{x_0}$ with

$$\begin{aligned} I_{x_0} &= (-\infty, \frac{1}{x_0}) \quad \text{for} \quad x_0 > 0 \\ I_{x_0} &= (\frac{1}{x_0}, \infty) \quad \text{for} \quad x_0 < 0 \\ I_{x_0} &= \mathbb{R} \quad \text{for} \quad x_0 = 0 \end{aligned}$$

Exercise: Show that the region R covered by the projected characteristics equals the region which lies strictly between the two branches of the hyperbola

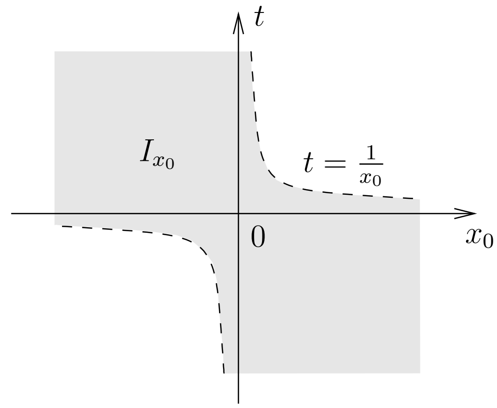

Figure 5: Region of points (x_0, t) with $x_0 \in \mathbb{R}$ and $t \in I_{x_0}$

$$1 + xt = 0, \quad \text{i.e.,} \quad t = -\frac{1}{x} \ .$$

Thus R is the shaded region in Figure 6.

To determine the solution $u(x,t)$, fix any point $(x,t) \in R$. Then determine $x_0 \in \mathbb{R}$ with

$$\frac{x_0}{1 - x_0 t} = x \ .$$

Obtain that

$$x_0 = \frac{x}{1 + xt}$$

and note that $1 + xt \neq 0$ for $(x,t) \in R$. This suggests that the solution of the Cauchy problem (1.14) is

$$u(x,t) = f\Big(\frac{x}{1+xt}\Big) \quad \text{for} \quad (x,t) \in R \ .$$

Exercise: Show that the above function $u(x,t)$ solves the Cauchy problem.

One considers the function $u(x,t) = f(x/(1+xt))$ as a solution of the initial value problem (1.14) only in the region R covered by the projected characteristics (1.16), i.e., in the region between the two branches of the hyperbola

$$1 + xt = 0 \quad \text{or} \quad t = -\frac{1}{x} \ .$$

In the region R^+ above the left branch of the hyperbola and in the region R^- below the right branch of the hyperbola, the formula $u(x,t) = f(x/(1+xt))$ still defines a solution of the PDE $u_t + x^2 u_x = 0$. However, one does not consider u in R^+ or R^- as a solution of the initial value problem (1.14) since the projected characteristics of the equation which start at points $(x_0, 0)$ do not enter the regions R^+, R^-. Thus, in R^+ and R^-, any solution $u(x,t)$ of the PDE $u_t + x^2 u_x = 0$ is completely unrelated to the initial condition $u(x,0) = f(x)$.

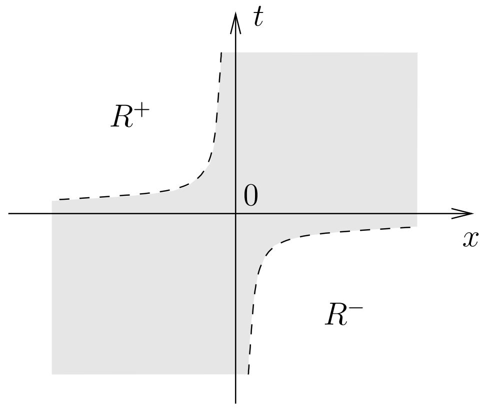

Figure 6: Regions R^+ and R^-

1.2 Semi–Linear Problems

The process described in Section 1.1 can be generalized to solve semi–linear problems

$$u_t + a(x,t)u_x = c(x,t,u), \quad u(x,0) = f(x) \ .$$

The projected characteristics are the same as for $u_t + a(x,t)u_x = 0$. The IVP for $h(t) = u(\xi(t;x_0),t)$ becomes

$$h'(t) = c(\xi(t;x_0),t,h(t)), \quad h(0) = f(x_0) \ .$$

Example 1.5:

$$u_t + au_x = u^2, \quad u(x,0) = f(x) = \sin x \ .$$

We let

$$h(t) = u(x_0 + at, t)$$

and obtain

$$h'(t) = h^2(t), \quad h(0) = f(x_0) \ .$$

Using separation of variables one obtains

$$h(t) = \frac{f(x_0)}{1 - f(x_0)t} \ .$$

Therefore,

$$u(x,t) = \frac{\sin(x - at)}{1 - t\sin(x - at)} \ .$$

The solution is C^∞ for $0 \le t < 1$, but blows up at certain x values when $t \to 1$.

1.3 Burgers' Equation: A First Study of Shocks

The equation $u_t + uu_x = 0$, called the inviscid Burgers' equation, is a simple model for shock formation. Consider the Cauchy problem

$$u_t + uu_x = 0, \quad u(x,0) = \sin x \ .$$

Assume that $u(x,t)$ is a solution. The projected characteristic Γ_{x_0} is determined by

$$\xi'(t) = u(\xi(t),t), \quad \xi(0) = x_0 \ ,$$

and u carries the value $\sin x_0$ along Γ_{x_0}. Therefore,

$$\xi'(t) = \sin x_0, \quad \xi(0) = x_0 \ .$$

One obtains that the projected characteristics Γ_{x_0} are straight lines $(\xi(t),t)$ given by

$$\xi(t) = x_0 + t\sin x_0 \ .$$

The slope of the line Γ_{x_0} depends on the initial value at x_0.

Given a point (x,t) we must solve

$$x_0 + t\sin x_0 = x \tag{1.17}$$

for x_0.

Lemma 1.1 *If $0 \le t < 1$ then for each $x \in \mathbb{R}$ the equation (1.17) has a unique solution $x_0 = x_0(x,t)$. The function $x_0(x,t)$ is C^∞ for $x \in \mathbb{R}, 0 \le t < 1$.*

Proof: Let $H(y) = y + t\sin y$. Then $H(y) \to -\infty$ as $y \to -\infty$ and $H(y) \to \infty$ as $y \to \infty$. Existence of a solution x_0 of (1.17) follows from the intermediate value theorem. Furthermore, $H'(y) \ge 1 - t > 0$. Therefore, the solution x_0 is unique. Smoothness of the function $x_0(x,t)$ follows from the implicit function theorem. $\diamond$

Let us determine the first derivatives x_{0x} and x_{0t} in terms of the function $x_0(x,t)$ through implicit differentiation: Differentiating (1.17) w.r.t x we obtain

$$x_{0x} + t\cos(x_0)x_{0x} = 1 \ ,$$

thus

$$x_{0x}(x,t) = \frac{1}{1 + t\cos x_0(x,t)} \ . \tag{1.18}$$

Similarly,

$$x_{0t}(x,t) = \frac{-\sin x_0(x,t)}{1 + t\cos x_0(x,t)} \ . \tag{1.19}$$

Our considerations suggest that the solution of Example 1.6 is

$$u(x,t) = \sin x_0(x,t) \ ,$$

where $x_0(x,t)$ is the solution of (1.17). We can check this: Clearly, $x_0(x,0) = x$ for all x, thus $u(x,0) = \sin x_0(x,0) = \sin x$. Furthermore, $u_t = (\cos x_0) x_{0t}$ and $u_x = (\cos x_0) x_{0x}$. The formulas (1.18) and (1.19) then imply that $u_t + u u_x = 0$.

Remarks on shock formation and weak solutions: Consider the Cauchy problem

$$u_t + u u_x = 0, \quad u(x,0) = \sin x \ .$$

We have shown that the problem has a C^∞ solution $u(x,t)$ defined for $x \in \mathbb{R}, 0 \le t < 1$. However, if $t > 1$, then the projected characteristics

$$\Gamma_{x_0} = \Big\{ (x_0 + t \sin x_0, t) \ : \ t \ge 0 \Big\}$$

intersect. For example, consider

$$\Gamma_\pi = \Big\{ (\pi, t) \ : \ t \ge 0 \Big\}$$

and

$$\Gamma_{\pi+\varepsilon} = \Big\{ (\pi + \varepsilon + t \sin(\pi+\varepsilon), t) \ : \ t \ge 0 \Big\}$$

for $0 < \varepsilon << 1$. By Taylor expansion,

$$\sin(\pi + \varepsilon) = -\varepsilon + \frac{\varepsilon^3}{6} + \mathcal{O}(\varepsilon^5) \ .$$

The two projected characteristics Γ_π and $\Gamma_{\pi+\varepsilon}$ intersect at time $t > 1$ if

$$\pi = \pi + \varepsilon + t \sin(\pi + \varepsilon) \ ,$$

i.e., if

$$t = \frac{\varepsilon}{-\sin(\pi+\varepsilon)} = \frac{1}{1 - \frac{\varepsilon^2}{6} + \mathcal{O}(\varepsilon^4)} \ .$$

We obtain that intersection occurs for times $t > 1$, where t is arbitrarily close to the time 1. The constructed solution $u(x,t)$ carries the value $u(\pi,t) = \sin \pi = 0$ along Γ_π and carries the different value $\sin(\pi+\varepsilon)$ along $\Gamma_{\pi+\varepsilon}$. This shows that a smooth solution $u(x,t)$ of the above Cauchy problem does not exist in $\mathbb{R} \times [0,T)$ if $T > 1$. A shock forms at $t = 1$.

It is interesting, however, to broaden the solution concept and consider so–called weak solutions. The idea is the following: Write Burgers' equation as

$$u_t + \frac{1}{2} (u^2)_x = 0 \ .$$

Let $\phi \in C_0^\infty(\mathbb{R}^2)$ denote a test function. First assume that $u(x,t)$ is a smooth solution of Burgers' equation with $u(x,0) = f(x)$. Multiply Burgers' equation by $\phi(x,t)$ and integrate to obtain

$$0 = \int_0^\infty \int_{-\infty}^\infty \Big(u_t \phi + \frac{1}{2}(u^2)_x \phi \Big) \, dxdt \ .$$

Now move the derivative operators from u to ϕ through integration by parts and obtain

$$\int_0^\infty \int_{-\infty}^\infty \Big(u \phi_t + \frac{1}{2}(u^2) \phi_x \Big) \, dxdt + \int_{-\infty}^\infty f(x)\phi(x,0)\, dx = 0 \ . \qquad (1.20)$$

One calls u a weak solution of Burgers' equation with initial condition $u(x,0) = f(x)$ if the above equality (1.20) holds for all test functions $\phi \in C_0^\infty(\mathbb{R}^2)$. The integrals in (1.20) all make sense if, for example, $u \in L^\infty(\mathbb{R} \times [0,\infty))$. Thus, weak solutions do not necessarily have classical derivatives.

Of special interest are piecewise smooth weak solutions. These satisfy the differential equation classically in regions where they are smooth and satisfy the so–called Rankine–Hugoniot jump condition along shocks, which are lines of discontinuity.

Auxiliary Results: Green–Gauss Theorem

Let U denote a bounded open subset of $\mathbb{R}^k$ with smooth boundary ∂U. Denote the unit outword normal on ∂U by $n(x)$ for $x \in \partial U$. Let $g : U \to \mathbb{R}$ denote a smooth function, which can be smoothly extended to the boundary ∂U. We denote the extended function again by g. Then the following formula holds:

$$\int_U \frac{\partial}{\partial x_j} g(x)\, dx = \int_{\partial U} g(x) n_j(x)\, dS(x) \quad \text{for} \quad j = 1, \ldots, k \ .$$

Here $dS(x)$ denotes the surface measure on the boundary ∂U.

Let us specialize the result to a region $U \in \mathbb{R}^2$ and let us assume that the boundary curve Γ of U has the parametrization

$$(x_1(s), x_2(s)), \quad 0 \le s \le L \ ,$$

by arclength s. We assume that this parametrization is counterclockwise. Then the unit outward normal is

$$n(x_1(s), x_2(s)) = (x_2'(s), -x_1'(s)) \ .$$

We obtain the formulas

$$\begin{aligned} \int_U \frac{\partial}{\partial x_1} g(x)\, dx &= \int_\Gamma g(x) n_1(x)\, dS(x) \\ &= \int_0^L g(x_1(s), x_2(s)) \frac{dx_2(s)}{ds}\, ds \end{aligned}$$

$$= \int_\Gamma g(x_1, x_2)\, dx_2$$

and

$$\begin{aligned}
\int_U \frac{\partial}{\partial x_2} g(x)\, dx &= \int_\Gamma g(x) n_2(x)\, dS(x) \\
&= -\int_0^L g(x_1(s), x_2(s)) \frac{dx_1(s)}{ds}\, ds \\
&= -\int_\Gamma g(x_1, x_2)\, dx_1
\end{aligned}$$

If we use the notation

$$(x_1, x_2) = (x, t)$$

then the resulting formulas become

$$\int_U g_x(x,t)\, dxdt = \int_\Gamma g(x,t)\, dt \tag{1.21}$$

$$\int_U g_t(x,t)\, dxdt = -\int_\Gamma g(x,t)\, dx \tag{1.22}$$

Derivation of the Rankine–Hugoniot Jump Condition

Let $F : \mathbb{R} \to \mathbb{R}$ denote a smooth function, a so–called flux function. In the case of Burgers' equation, we have $F(u) = \frac{1}{2} u^2$. Let $u(x,t)$ denote a weak solution of

$$u_t + F(u)_x = 0, \quad u(x,0) = f(x) \ .$$

This means that

$$\int_0^\infty \int_{-\infty}^\infty \Big(u\phi_t + F(u)\phi_x \Big)\, dxdt + \int_{-\infty}^\infty f(x)\phi(x,t)\, dx = 0 \tag{1.23}$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$.

Let $U \subset \mathbb{R} \times (0, \infty)$ denote an open bounded set and let Γ denote a curve crossing U, which has a parametrization

$$(x(t), t), \quad t_0 \le t \le t_1 \ .$$

Write

$$U = U_l \cup U_r$$

where U_l lies to the left and U_r lies to the right of Γ.

Let $\phi \in C_0^\infty(U)$. We have

$$\begin{aligned} 0 &= \int_U (u\phi_t + F(u)\phi_x)\,dxdt \\ &= \int_{U_l} \ldots + \int_{U_r} \ldots \end{aligned}$$

Assume that the weak solution $u(x,t)$ is a smooth classical solution of

$$u_t + F(u)_x = 0$$

in U_l and in U_r which has a jump discontinuity along Γ. Using the Green–Gauss formulas (1.21) and (1.22) we have

$$\begin{aligned} \int_{U_l} (F(u)\phi)_x(x,t)\,dxdt &= \int_\Gamma (F(u_l)\phi)(x,t)\,dt \\ \int_{U_l} (F(u)\phi)_t(x,t)\,dxdt &= -\int_\Gamma (F(u_l)\phi)(x,t)\,dx \end{aligned}$$

Since $u_t + F(u)_x = 0$ in U_l we obtain

$$\int_{U_l} (u\phi_t + F(u)\phi_x)(x,t)\,dxdt = \int_\Gamma (F(u_l)\,dt - u_l\,dx)\phi\ .$$

If we integrate over U_r instead of U_l we obtain in the same way:

$$\int_{U_r} (u\phi_t + F(u)\phi_x)(x,t)\,dxdt = -\int_\Gamma (F(u_r)\,dt - u_r\,dx)\phi\ .$$

The reason for the minus sign is that the direction of Γ agrees with the orientation of ∂U_l, but is opposite to the orientation ∂U_r.

Adding the last two equations and recalling that u is a weak solution, we obtain that

$$\int_\Gamma (F(u_l)\,dt - u_l\,dx)\phi = \int_\Gamma (F(u_r)\,dt - u_r\,dx)\phi\ .$$

Since ϕ is an arbitray function in $C_0^\infty(U)$ the above equation implies that

$$F(u_l)\,dt - u_l\,dx = F(u_r)\,dt - u_r\,dx$$

along Γ. Therefore,

$$(u_r - u_l)\frac{dx}{dt} = F(u_r) - F(u_l)$$

along Γ. This is the Ranking–Hugoniot jump condition. It couples the shock speed $\frac{dx}{dt}$ to the jump of u and the jump of $F(u)$ along a shock curve Γ.

We have shown:

Theorem 1.1 *Let u denote a piecewise smooth weak solution of*

$$u_t + F(u)_x = 0$$

which has a jump discontinuity along a curve Γ with parametrization

$$(x(t), t), \quad t_0 \le t \le t_1 \ .$$

Then we have

$$(u_r - u_l)(x(t), t)\frac{dx(t)}{dt} = (F(u_r) - F(u_l))(x(t), t), \quad t_0 \le t \le t_1 \ .$$

1.4 Quasi–Linear Problems

We now consider equations where the signal speed a also depends on u. The general form of the equation is

$$u_t + a(x, t, u)u_x = c(x, t, u) \ . \tag{1.24}$$

Example 1.7:

$$u_t + uu_x = -u, \quad u(x, 0) = \sin x \ .$$

Assume that $u(x, t)$ is a solution. The projected characteristic Γ_{x_0} is determined by

$$\xi'(t) = u(\xi(t), t), \quad \xi(0) = x_0 \ .$$

Set

$$h(t) = u(\xi(t), t) \ .$$

Then we have $h(0) = u(x_0, 0) = \sin x_0$ and

$$\begin{aligned} h'(t) &= u_t + \xi'(t)u_x \\ &= (u_t + uu_x)(\xi(t), t) \\ &= -h(t) \ . \end{aligned}$$

Thus we must solve the system

$$\xi'(t) = h(t), \quad h'(t) = -h(t)$$

with initial condition

$$\xi(0) = x_0, \quad h(0) = \sin x_0 \ .$$

Obtain

$$h(t; x_0) = e^{-t} \sin x_0$$

and

$$\xi(t; x_0) = x_0 + (1 - e^{-t}) \sin x_0 \ .$$

Given a point (x,t) we must solve

$$x_0 + (1 - e^{-t}) \sin x_0 = x \tag{1.25}$$

for x_0. Denote the solution by $x_0(x,t)$. Then

$$\begin{aligned} u(x,t) &= h(t;x_0) \\ &= e^{-t} \sin x_0(x,t) \end{aligned}$$

solves the problem.

Note: If $H(y) = y + (1 - e^{-t}) \sin y$ then

$$H'(y) \geq 1 - (1 - e^{-t}) = e^{-t} > 0$$

for all $t \geq 0$. Therefore, (1.25) is uniquely solvable for x_0 for any given $t \geq 0$ and all $x \in \mathbb{R}$. The solution $x_0(x,t)$ is C^∞ by the implicit function theorem, for all $t \geq 0$. Therefore, $u(x,t)$ is C^∞ for all $t \geq 0$.

Exercise: Use implicit differentiation to compute the derivatives x_{0t} and x_{0x} in terms of $x_0(x,t)$. Then show that the function $u(x,t) = e^{-t} \sin x_0(x,t)$ solves the problem.

In principle, the general equation (1.24) with initial condition $u(x,0) = f(x)$ can be solved in the same way: We consider the system of ODEs

$$\xi'(t) = a(\xi(t), t, h(t)), \quad h'(t) = c(\xi(t), t, h(t)) \tag{1.26}$$

with initial condition

$$\xi(0) = x_0, \quad h(0) = f(x_0) \ .$$

Denote the solution by

$$\xi(t;x_0), \quad h(t;x_0) \ .$$

Given a point (x,t) we solve

$$\xi(t;x_0) = x$$

for x_0. Assume that there is a unique solution x_0 whenever $0 \leq t \leq T$ and $x \in \mathbb{R}$. Denote the solution by

$$x_0(x,t) \ .$$

Then we expect that

$$u(x,t) = h(t; x_0(x,t)) \tag{1.27}$$

solves the problem.

We have $\xi(0;x_0) = x_0$ for all x_0. Therefore, if we solve

$$\xi(0;x_0) = x$$

for x_0, then the solution is $x_0 = x$. This ensures that

$$x_0(x,0) = x$$

for all x. Obtain

$$\begin{aligned} u(x,0) &= h(0;x_0(x,0)) \\ &= h(0;x) \\ &= f(x) \ . \end{aligned}$$

Thus the function (1.27) satisfies the initial condition.

We want to discuss now under what assumptions $u(x,t)$ (given by (1.27)) satisfies the PDE $u_t + a(x,t,u)u_x = c(x,t,u)$. To this end, let us assume that the implicitly defined function $x_0(x,t)$ is C^1. We make the following formal computations:

$$u_x = h_{x_0} x_{0x}, \quad u_t = h_t + h_{x_0} x_{0t} \ ,$$

thus

$$\begin{aligned} u_t + a u_x &= h_t + h_{x_0} x_{0t} + a h_{x_0} x_{0x} \\ &= c + h_{x_0}(x_{0t} + a x_{0x}) \ . \end{aligned}$$

The function x_0 satisfies

$$\xi(t; x_0(x,t)) = x \ ,$$

thus

$$\xi_{x_0} x_{0x} = 1$$

and

$$\xi_t + \xi_{x_0} x_{0t} = 0 \ .$$

This yields

$$x_{0x} = \frac{1}{\xi_{x_0}}$$

and

$$x_{0t} = \frac{-\xi_t}{\xi_{x_0}} \ .$$

Since $\xi_t = a$ it follows that

$$x_{0t} + a x_{0x} = 0 \ ,$$

which implies $u_t + a u_x = c$. These computations can be justified as long as

$$\xi_{x_0}(t; x_0) \neq 0$$

for $0 \le t \le T, x_0 \in \mathbb{R}$.

Remark: Reconsider Example 1.4, $u_t + x^2 u_x = 0$, where

$$\xi(t; x_0) = \frac{x_0}{1 - x_0 t}$$

and

$$\xi_{x_0}(t;x_0) = \frac{1}{(1-x_0 t)^2} \ .$$

Though the condition $\xi_{x_0} \neq 0$ is always satisfied, there is no time interval of existence for $u(x,t)$ since there is no time interval $0 \le t \le T, T > 0$, where $\xi(t;x_0)$ is defined for all $x_0 \in \mathbb{R}$. The trouble is due to the strong growth of the coefficient $a = x^2$.

1.5 Characteristics for General Quasilinear Equations

Consider the equation

$$a(x,t,u)u_x + b(x,t,u)u_t = c(x,t,u) \tag{1.28}$$

where $a, b, c \in C^1$.

A parametrized C^1 curve (in $\mathbb{R}^3$)

$$\Big(\bar{x}(s), \bar{t}(s), \bar{u}(s)\Big), \quad a \le s \le b \ , \tag{1.29}$$

is called a characteristic of equation (1.28) if

$$\frac{d\bar{x}}{ds} = a(\bar{x}, \bar{t}, \bar{u}) \tag{1.30}$$

$$\frac{d\bar{t}}{ds} = b(\bar{x}, \bar{t}, \bar{u}) \tag{1.31}$$

$$\frac{d\bar{u}}{ds} = c(\bar{x}, \bar{t}, \bar{u}) \tag{1.32}$$

The corresponding curve

$$\Big(\bar{x}(s), \bar{t}(s)\Big), \quad a \le s \le b \ , \tag{1.33}$$

in the (x,t)–plane is called a projected characteristic.

Characteristics are important because of the following result.

Theorem 1.2 *Suppose that $u(x,t)$ is a solution of (1.28) defined in a neighborhood of the projected characteristic (1.33). Also, assume that*

$$u(\bar{x}(a), \bar{t}(a)) = \bar{u}(a) \ ,$$

i.e., the starting point of the characteristic (1.29) lies in the solution surface $\mathcal{S} \subset \mathbb{R}^3$ determined by u. Under these assumptions the whole characteristic (1.29) lies in $\mathcal{S}$.

Proof: In the following the argument s of the functions $\bar{x}$ etc. is often suppressed in the notation.

Set

$$U(s) = \bar{u}(s) - u(\bar{x}(s), \bar{t}(s)) \ .$$

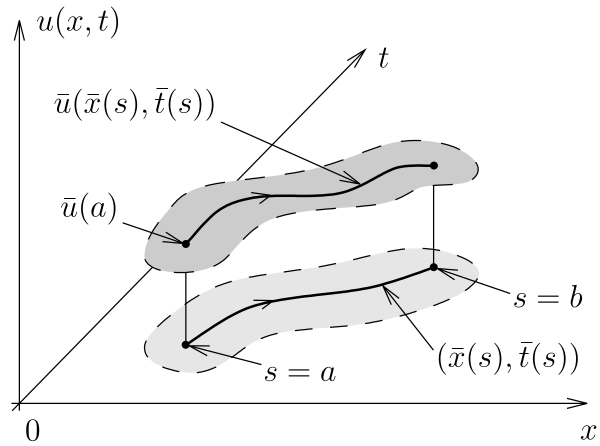

Figure 7: Theorem 1.1

By assumption, $U(a) = 0$. Also,

$$\begin{aligned}
\frac{dU}{ds} &= \bar{u}' - u_x(\bar{x},\bar{t})\bar{x}' - u_y(\bar{x},\bar{t})\bar{t}' \\
&= c(\bar{x},\bar{t},\bar{u}) - u_x(\bar{x},\bar{t})a(\bar{x},\bar{t},\bar{u}) - u_y(\bar{x},\bar{t})b(\bar{x},\bar{t},\bar{u}) \\
&= c(\bar{x},\bar{t},U+u(\bar{x},\bar{t})) - u_x(\bar{x},\bar{t})a(\bar{x},\bar{t},U+u(\bar{x},\bar{t})) - u_y(\bar{x},\bar{t})b(\bar{x},\bar{t},U+u(\bar{x},\bar{t}))
\end{aligned}$$

Given the solution $u(x,t)$ and the curve $(\bar{x}(s),\bar{t}(s))$, we may consider the above equation as an ODE for the unknown function $U = U(s)$. Note that $U \equiv 0$ solves this ODE since u solves (1.28). Since $U(a) = 0$, *uniqueness* of solutions of ordinary initial value problems implies that $U \equiv 0$, thus $u(\bar{x}(s),\bar{t}(s)) \equiv \bar{u}(s)$. $\diamond$

The theorem says that all solution surfaces

$$\mathcal{S} = \Big\{(x,t,u(x,t))\Big\}$$

of the PDE (1.28) are made up of characteristic curves. Therefore, the solutions of the PDE (1.28) can be constructed by solving the ODE system (1.30) to (1.32).

Previously we have considered the special case $b \equiv 1$. If $b \equiv 1$, then equation (1.31) reads

$$\frac{d\bar{t}}{ds} = 1$$

and we can choose the variable t as parameter of any characteristic; we do not need the abstract parameter s. With $b \equiv 1$ and $t = s$ the characteristic system (1.30) to (1.32) becomes

$$\frac{d\bar{x}}{dt} = a(\bar{x},\bar{t},\bar{u}), \quad \frac{d\bar{u}}{dt} = c(\bar{x},\bar{t},\bar{u}) \ .$$

This is exactly the same system as

$$\xi' = a(\xi, t, h), \quad h' = c(\xi, t, h)$$

in our previous notation; see (1.26).

1.6 General Nonlinear Scalar PDEs

Let $F(x, y, u, p, q)$ be a smooth function of five variables. We denote the partial derivatives of F by F_x etc.

Consider the PDE

$$F(x, y, u, u_x, u_y) = 0 \ . \tag{1.34}$$

Suppose that $u(x, y)$ is a solution and set

$$p(x, y) = u_x(x, y), \quad q(x, y) = u_y(x, y) \ .$$

For later reference we derive the equations (1.36) and (1.37) below.

If one differentiates the identity

$$F(x, y, u(x, y), p(x, y), q(x, y)) = 0 \tag{1.35}$$

w.r.t. x, then one obtains

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0 \ ,$$

thus

$$-p_x F_p - q_x F_q = F_x + pF_u \ . \tag{1.36}$$

Similarly, by differentiating (1.35) w.r.t. y,

$$-p_y F_p - q_y F_q = F_y + qF_u \ . \tag{1.37}$$

We want to show that, in principle, the PDE (1.34) can again be solved by solving a system of ODEs.

The characteristic system for (1.34) consists of the following five coupled ODEs:

$$\frac{d\bar{x}}{ds} = F_p \tag{1.38}$$

$$\frac{d\bar{y}}{ds} = F_q \tag{1.39}$$

$$\frac{d\bar{u}}{ds} = \bar{p}F_p + \bar{q}F_q \tag{1.40}$$

$$\frac{d\bar{p}}{ds} = -F_x - \bar{p}F_u \tag{1.41}$$

$$\frac{d\bar{q}}{ds} = -F_y - \bar{q}F_u \tag{1.42}$$

where

$$F_p = F_p(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}) \ ,$$

etc. An instructive example is given by the so–called **Eikonal equation** of geometrical optics:
Example 1.8:

$$u_x^2 + u_y^2 = 1$$

Here

$$F(p, q) = \frac{1}{2}(p^2 + q^2 - 1) \ ,$$

thus

$$F_x = F_y = F_u = 0, \quad F_p = p, \quad F_q = q \ .$$

The characteristic system becomes

$$\begin{aligned} \frac{d\bar{x}}{ds} &= \bar{p} \\ \frac{d\bar{y}}{ds} &= \bar{q} \\ \frac{d\bar{u}}{ds} &= \bar{p}^2 + \bar{q}^2 \\ \frac{d\bar{p}}{ds} &= 0 \\ \frac{d\bar{q}}{ds} &= 0 \end{aligned}$$

We discuss this system below.

Let us first continue the discussion of the general case. Suppose that

$$\Big(\bar{x}(s), \bar{y}(s), \bar{u}(s), \bar{p}(s), \bar{q}(s)\Big), \quad a \le s \le b \ , \tag{1.43}$$

is a solution of the characteristic system (1.38) to (1.42). Such a solution is called a characteristic for (1.34). Let $u(x, y)$ be a solution of (1.34) and recall the settings $p = u_x, q = u_y$. Suppose that $u(x, y)$ is defined in a neighborhood of the projected characteristic

$$\Big(\bar{x}(s), \bar{y}(s)\Big), \quad a \le s \le b \ .$$

Theorem 1.3 *Under the above assumptions, suppose that*

$$\begin{aligned} u(\bar{x}(a), \bar{y}(a)) &= \bar{u}(a) \\ p(\bar{x}(a), \bar{y}(a)) &= \bar{p}(a) \\ q(\bar{x}(a), \bar{y}(a)) &= \bar{q}(a) \end{aligned}$$

Then the whole characteristic (1.43) lies in the solution surface corresponding to u, and we have for $a \le s \le b$,

$$\begin{aligned} u(\bar{x}(s), \bar{y}(s)) &= \bar{u}(s) \\ p(\bar{x}(s), \bar{y}(s)) &= \bar{p}(s) \\ q(\bar{x}(s), \bar{y}(s)) &= \bar{q}(s) \end{aligned}$$

Proof: The proof is similar to the proof of Theorem 1.2. Set

$$\begin{aligned} U(s) &= \bar{u}(s) - u(\bar{x}(s), \bar{y}(s)) \\ P(s) &= \bar{p}(s) - p(\bar{x}(s), \bar{y}(s)) \\ Q(s) &= \bar{q}(s) - q(\bar{x}(s), \bar{y}(s)) \end{aligned}$$

By assumption,

$$U(a) = P(a) = Q(a) = 0 \ .$$

We calculate (note that $p_y = u_{xy} = u_{yx} = q_x$)

$$\begin{aligned} U' &= \bar{u}' - u_x \bar{x}' - u_y \bar{y}' \\ &= \bar{p} F_p + \bar{q} F_q - p F_p - q F_q \\ P' &= \bar{p}' - p_x \bar{x}' - p_y \bar{y}' \\ &= -F_x - \bar{p} F_u - p_x F_p - p_y F_q \\ &= -F_x - \bar{p} F_u - p_x F_p - q_x F_q \\ Q' &= \bar{q}' - q_x \bar{x}' - q_y \bar{y}' \\ &= -F_y - \bar{q} F_u - q_x F_p - q_y F_q \\ &= -F_y - \bar{q} F_u - p_y F_p - q_y F_q \end{aligned}$$

Here all functions F_x etc. are evaluated at

$$\Big(\bar{x}(s), \bar{y}(s), \bar{u}(s), \bar{p}(s), \bar{q}(s)\Big) \ .$$

Recall that, by (1.36) and (1.37),

$$-p_x F_p - q_x F_q = F_x + p F_u \quad \text{and} \quad -p_y F_p - q_y F_q = F_y + q F_u \ .$$

The left–hand side $-p_x F_p - q_x F_q$ appears on the right–hand side of the equation for P' if one ignores the difference between variables with and without bars. Similarly, $-p_y F_p - q_y F_q$ appears on the right–hand side for Q'. If one could ignore the difference between variables with and without bars, then the above equations would read

$$U' = P' = Q' = 0 \ .$$

Using the same argument as in the proof of Theorem 1.2 (based on the *unique* solvability of ordinary initial value problems), it follows that

$$U \equiv P \equiv Q = 0 \ .$$

◇

Example 1.8 (continued)

The characteristic equations are solved by

$$\begin{aligned}
\bar{x}(s) &= \bar{x}(0)+\bar{p}(0)s \\
\bar{y}(s) &= \bar{y}(0)+\bar{q}(0)s \\
\bar{u}(s) &= \bar{u}(0)+(\bar{p}^2(0)+\bar{q}^2(0))s \\
\bar{p}(s) &= \bar{p}(0) \\
\bar{q}(s) &= \bar{q}(0)
\end{aligned}$$

Suppose that

$$\Gamma \ : \ (f(\sigma), g(\sigma)), \quad 0 \le \sigma \le L \ ,$$

is a smooth curve in the (x, y) plane parametrized (for simplicity) by arclength σ, thus

$$\Big(f'(\sigma)\Big)^2 + \Big(g'(\sigma)\Big)^2 = 1 \ .$$

We want to determine a solution $u(x, y)$ of the eikonal equation satisfying $u = 0$ on Γ, i.e.,

$$u(f(\sigma), g(\sigma)) = 0, \quad 0 \le \sigma \le L \ . \tag{1.44}$$

Suppose first that $u(x, y)$ is a solution of the PDE $u_x^2 + u_y^2 = 1$ satisfying the side condition (1.44); set $p = u_x, q = u_y$. Furthermore, set

$$\phi(\sigma) = p(f(\sigma), g(\sigma)), \quad \psi(\sigma) = q(f(\sigma), g(\sigma)) \ .$$

Differentiating (1.44) w.r.t. σ we obtain

$$f'(\sigma)\phi(\sigma) + g'(\sigma)\psi(\sigma) = 0 \ . \tag{1.45}$$

Also, the PDE $u_x^2 + u_y^2 = 1$ requires

$$\phi^2(\sigma) + \psi^2(\sigma) = 1 \ . \tag{1.46}$$

We consider $\phi(\sigma)$ and $\psi(\sigma)$ as unknowns in (1.45), (1.46), and note that there are two solutions of (1.45), (1.46), namely

$$(\phi, \psi) = (-g', f')$$

and

$$(\phi, \psi) = (g', -f') \ .$$

For definiteness, let us assume that the solution $u(x, y)$ under consideration corresponds to the first solution,

$$\phi(\sigma) = -g'(\sigma), \quad \psi(\sigma) = f'(\sigma) \ .$$

The vector $(\phi(\sigma), \psi(\sigma))$ is a unit normal to the curve Γ in the point

$$P_\sigma = (f(\sigma), g(\sigma)) \ .$$

Fix the point P_σ on the curve Γ. We want to start a projected characteristic at this point and want the whole characteristic to lie in the solution surface. Then the corresponding initial condition for the characteristic system is

$$\begin{aligned}\bar{x}(0) &= f(\sigma)\\ \bar{y}(0) &= g(\sigma)\\ \bar{u}(0) &= 0\\ \bar{p}(0) &= \phi(\sigma)\\ \bar{q}(0) &= \psi(\sigma)\end{aligned}$$

The solution of the characteristic system with this initial condition is

$$\begin{aligned}\bar{x}(s) &= f(\sigma)+\phi(\sigma)s\\ \bar{y}(s) &= g(\sigma)+\psi(\sigma)s\\ \bar{u}(s) &= s\\ \bar{p}(s) &= \phi(\sigma)\\ \bar{q}(s) &= \psi(\sigma)\end{aligned}$$

For the solution $u(x,y)$ under consideration, this implies that

$$u\Big(f(\sigma)+\phi(\sigma)s, g(\sigma)+\psi(\sigma)s\Big) = s\ . \tag{1.47}$$

This says that, along any straight line,

$$\Big(f(\sigma)+\phi(\sigma)s, g(\sigma)+\psi(\sigma)s\Big), \quad s\in\mathbb{R}\ , \tag{1.48}$$

the solution u equals s, as long as the solution u exists. The straight line (1.48) is a projected characteristic. As in the quasilinear case, projected characteristics can intersect, which generally leads to breakdown of the solution $u(x,y)$. In regions where the projected characteristics do not intersect, they can be used to determine a solution $u(x,y)$.

If, in the above example, we fix s and vary σ, we obtain the curve

$$\Gamma_s\ :\ \Big(f(\sigma)+\phi(\sigma)s, g(\sigma)+\psi(\sigma)s\Big), \quad 0\le\sigma\le L\ , \tag{1.49}$$

which is roughly parallel to the given curve Γ. On Γ_s the solution u carries the value s, i.e., Γ_s is a level curve of u. If we think of s as time, then the mapping

$$\Gamma\to\Gamma_s$$

describes a motion of the given initial curve Γ in the (x,y)–plane.
A Particular Case: Let

$$\Gamma\ :\ (f(\sigma),g(\sigma)) = (\cos\sigma,\sin\sigma), \quad 0\le\sigma\le 2\pi\ ,$$

be the parametrized unit circle. Then Γ_s is the circle

$$(1-s)(\cos\sigma, \sin\sigma), \quad 0 \le \sigma \le 2\pi \ ,$$

of radius $1-s$. On this circle, u equals s. Setting $r = \sqrt{x^2+y^2}$ and noting that $1-s=r$ implies $s = 1-r$, one obtains

$$u(x,y) = 1 - \sqrt{x^2+y^2} \ .$$

If instead of $(\phi,\psi) = (-g, f)$ the solution $u(x,y)$ corresponds to $(\phi,\psi) = (g,-f)$, then one obtains

$$u(x,y) = \sqrt{x^2+y^2} - 1 \ .$$

This example shows that the solution of a nonlinear problem

$$F(x,y,u,u_x,u_y) = 0$$

with a side condition

$$u(f(\sigma), g(\sigma)) = h(\sigma)$$

is generally not unique, not even locally near the curve $\Gamma : (f(\sigma), g(\sigma))$.

1.7 The Eikonal Equation and the Wave Equation

1.7.1 Derivation of the Eikonal Equation

Consider the 2D wave equation

$$v_{tt} = c^2(v_{xx} + v_{yy})$$

where $c = c(x,y)$. Let

$$v(x,y,t) = V(x,y)e^{-i\omega t} \ ,$$

i.e., we consider a solution of the wave equation which oscillates in time at the frequency ω. One obtains Helmholtz' equation for the amplitude function $V(x,y)$:

$$V_{xx} + V_{yy} + \frac{\omega^2}{c^2} V = 0 \ .$$

Let c_0 denote a constant reference value for the wave speed. For example, c_0 may be the speed of light in vacuum and $c(x,y)$ the speed of light in a material. The number

$$n(x,y) = \frac{c_0}{c(x,y)}$$

is called the index of refraction. We have

$$\frac{\omega}{c(x,y)} = \frac{\omega n(x,y)}{c_0} = kn(x,y) \quad \text{with} \quad k = \frac{\omega}{c_0} \ .$$

Here $k = \frac{\omega}{c_0}$ is a wave number. The equation for $V(x,y)$ becomes

$$V_{xx} + V_{yy} + k^2 n^2(x,y) V = 0 \ . \tag{1.50}$$

The following values are approximate values for visible light:

$$\begin{aligned} c_0 &= 3 * 10^8 \frac{m}{sec} \\ \lambda &= 4 * 10^{-7} m \quad \text{(wavelength)} \\ \omega &= \frac{c_0}{\lambda} = 7.5 * 10^{14} sec^{-1} \\ k &= \frac{\omega}{c_0} = \frac{1}{\lambda} = 2.5 * 10^6 m^{-1} \end{aligned}$$

Thus, if we choose $1m$ as the unit of length, then the wave number k is very large. We try to find solutions of (1.50) of the form

$$V(x,y) = A(x,y) e^{iku(x,y)}$$

where $u(x,y)$ is the phase function and $A(x,y)$ the amplitude. Setting $E = e^{iku(x,y)}$ we have

$$\begin{aligned} V_x &= A_x E + AEiku_x \\ V_{xx} &= A_{xx} E + 2A_x Eiku_x + AE(-k^2)(u_x)^2 + AEiku_{xx} \end{aligned}$$

and similar equations hold for V_y, V_{yy}. After dividing by E equation (1.50) becomes

$$\Delta A + ik\Big(2A_x u_x + 2A_y u_y + A\Delta u\Big) - k^2 A\Big((u_x)^2 + u_y)^2 - n^2(x,y)\Big) = 0 \ .$$

If $k >> 1$ it is reasonable to neglect the term ΔA and to solve the two equations

$$\begin{aligned} (u_x)^2 + (u_y)^2 &= n^2(x,y) \\ 2A_x u_x + 2A_y u_y + A\Delta u &= 0 \end{aligned}$$

For the phase function $u(x,y)$ we have obtained the eikonal equation. If $u(x,y)$ is known, then the second equation is a transport equation for $A(x,y)$. If one does not neglect the term ΔA, then one obtains a singular perturbation problem for A.

1.7.2 The Approximate Solution of the Wave Equation

For simplicity, let

$$c(x,y) = c_0 = c$$

be constant. The eikonal equation is

$$(u_x)^2 + (u_y)^2 = 1 \ .$$

Let $u(x,y)$ denote a solution with $u = 0$ on Γ where Γ has the arclength parametrization

$$\Gamma \ : \ (f(\sigma), g(\sigma)), \quad 0 \le \sigma \le L \ .$$

Let Γ_s denote the curve with parametrizaton

$$\Gamma_s \ : \ (f(\sigma), g(\sigma)) + s(-g'(\sigma), f'(\sigma)), \quad 0 \le \sigma \le L \ .$$

We have obtained that (Case 1):

$$u(f(\sigma), g(\sigma)) + s(-g'(\sigma), f'(\sigma)) = s$$

or (Case 2):

$$u(f(\sigma), g(\sigma)) + s(-g'(\sigma), f'(\sigma)) = -s \ .$$

For definiteness, assume Case 1. The corresponding approximated solution of the wave equation is

$$v(x,y,t) = A(x,y) e^{ik(u(x,y)-ct)} \ .$$

Here $A(x,y)$ does not depend on k and we may expect that $A(x,y)$ varies slowly on the length scale $1/k$.

Consider the real part of the exponential,

$$a(x,y,t) = \cos(ku(x,y) - ct)) \ .$$

Fix a straight line

$$(x(s), y(s)) = (f(\sigma), g(\sigma)) + s(-g'(\sigma), f'(\sigma)), \quad -s_0 \le s \le s_0 \ , \tag{1.51}$$

where σ is fixed. On this line we have

$$u(x(s), y(s)) = s \ ,$$

thus

$$\phi(s,t) := a(x(s), y(s), t) = \cos(k(s - ct)) \ .$$

Since $k >> 1$, the function $\phi(s,t)$ is highly oscillator. It moves at speed c in positive s–direction. Therefore, the function $a(x,y,t)$ moves at speed c along the straight line (1.51). The lines (1.51) are the light rays. The lines Γ_s are the wave fronts.

1.7.3 Terminology for Waves

Plane Waves: Consider the wave equation

$$u_{tt} = c^2 \Delta u$$

in n space dimensions. Let $F : \mathbb{R} \to \mathbb{C}$ denote a smooth function and fix

$$k \in \mathbb{R}^n \quad \text{with} \quad |k| = 1 \ .$$

It is easy to check that the function

$$u(x,t) = F(k \cdot x - ct), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \ , \tag{1.52}$$

solves the wave equation.

We want to explain that the above function $u(x,t)$ describes a plane wave which moves at speed c in direction k. Let

$$H_0 = \{\xi \in \mathbb{R}^n \ : \ k \cdot \xi = 0\}$$

denote the hyperplane through the origin orthogonal to k and for any real positive α let

$$\begin{aligned} H_\alpha &= \{x \in \mathbb{R}^n \ : \ k \cdot \xi = \alpha\} \\ &= \{x = \xi + \alpha k \ : \ \xi \in H_0\} \end{aligned}$$

denote the hyperplane parallel to H_0 with distance α from H_0. At time $t = 0$ we have

$$u(x,0) = F(k \cdot x) = F(\alpha) \quad \text{for} \quad x \in H_\alpha$$

and at time $t > 0$ we have

$$u(x,t) = F(k \cdot x - ct) = F(\beta - ct) \quad \text{for} \quad x \in H_\beta \ .$$

If $\beta = \alpha + ct$ then we have $F(\beta - ct) = F(\alpha)$. We can interprete this as follows: At time $t = 0$ the function (1.52) has the value $F(\alpha)$ on the plane H_α and at time $t > 0$ it has the same value $F(\alpha)$ on the plane $H_{\alpha+ct}$. The value $F(\alpha)$ has moved from H_α to $H_{\alpha+ct}$, i.e., it has moved at speed c in direction k.

Since the function $u(x,t) = F(k \cdot x - ct)$ is constant on each plane H_α (at fixed t), it is called a plane wave.

Plane Waves With Periodic F: Consider the function

$$u(x,t) = \cos(\kappa(k \cdot x - ct)) \tag{1.53}$$

where, as above, $k \in \mathbb{R}^n, |k| = 1$, and let $\kappa > 0$.

Choose $\kappa = 5$, for example. Then we have

$$u(x,0) = \cos(5\alpha) \quad \text{for} \quad x \in H_\alpha \; .$$

For $0 \le \alpha \le 2\pi$ the function

$$\cos(5\alpha)$$

has 5 waves with wave length

$$\lambda = \frac{2\pi}{5} \; .$$

In general, the number κ in (1.53) is called the wave number. The wave length is

$$\lambda = \frac{2\pi}{\kappa} \; .$$

We can also write the function (1.53) in the form

$$u(x,t) = \cos(\kappa k \cdot x - \omega t)) \quad \text{with} \quad \omega = \kappa c \; . \tag{1.54}$$

Here $\omega = \kappa c$ is the circular frequency of the wave. The time period of the oscillation (1.54) is

$$T = \frac{2\pi}{\omega} \; .$$

The frequency is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

and we can write

$$u(x,t) = \cos(\kappa k \cdot x - 2\pi f t)) \; .$$

Data For Visible Light: The following approximate values hold for visible light in vacuum:

$$\text{Speed:} \quad c = 3 * 10^8 \; \frac{meter}{sec}$$

$$\text{Frequency:} \quad f = 6 * 10^{14} \; \frac{1}{sec}$$

In fact, visible light has (approximately) frequencies in the rather narrow band

$$4 * 10^{14} \; \frac{1}{sec} \le f \le 8 * 10^{14} \; \frac{1}{sec} \; .$$

$$\text{Time Period:} \quad T = \frac{1}{f} = \frac{1}{6} * 10^{-14} \; sec$$

$$\text{Wave Number:} \quad \kappa = \frac{\omega}{c} = \frac{2\pi f}{c} = 4\pi * 10^6 \ \frac{1}{meter}$$

The number of waves in meter length is $2 * 10^6$.

$$\text{Wave Length:} \quad \lambda = \frac{2\pi}{\kappa} = 0.5 * 10^{-6} \ meter = 500 \ nanometer$$

1.8 Generalizations

More than two independent variables. There are no difficulties to generalize the approach via characteristics to problems where the unknown function u depends on more than two independent variables.

Example 1.9: Consider

$$u_t + a(x,y,t)u_x + b(x,y,t)u_y = 0$$

with initial condition

$$u(x,y,0) = f(x,y) \ .$$

Projected characteristics are now parametrized lines of the form

$$(\xi_1(t), \xi_2(t), t), \quad t \geq 0 \ ,$$

where

$$\xi_1' = a(\xi_1, \xi_2, t), \quad \xi_2' = b(\xi_1, \xi_2, t) \ .$$

Let $u(x,y,t)$ solve the differential equation and consider u along a projected characteristic:

$$h(t) = u(\xi_1(t), \xi_2(t), t) \ .$$

Clearly,

$$h'(t) = u_t + \xi_1'(t)u_x + \xi_2'(t)u_y = 0 \ .$$

Thus u carries the initial data along the projected characteristics. This can again be used conversely to construct solutions.

Data not given at $t = 0$. If the data are not given at $t = 0$ it is often good (and sometimes necessary) to parameterize the projected characteristics by some parameter s, not by t, even if the PDE can be solved for u_t.

Example 1.10: Consider

$$u_t + u_x = u^2$$

for $u = u(x,t)$ with side condition

$$u(x,-x) = x \quad \text{for all} \quad x \in \mathbb{R} \ .$$

The projected characteristics are the straight lines

$$(x_0 + s, -x_0 + s), \quad s \in \mathbb{R} \ .$$

Assume that u is a solution and consider u along a projected characteristic,

$$h(s) = u(x_0 + s, -x_0 + s) \ .$$

One finds that $h' = h^2$ and

$$h(0) = u(x_0, -x_0) = x_0 \ .$$

Therefore,

$$h(s) = \frac{x_0}{1 - s x_0} \ .$$

If (x, t) is a given point we determine x_0 and s with

$$x_0 + s = x, \quad -x_0 + s = t \ .$$

This yields

$$x_0 = \frac{1}{2}\,(x - t), \quad s = \frac{1}{2}\,(x + t) \ ,$$

and therefore

$$u(x, t) = \frac{2(x - t)}{4 + t^2 - x^2} \ .$$

Note that u blows up at all points of the hyperbola $t^2 = x^2 - 4$.

1.9 Supplement: The Implicit Function Theorem

Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ denote nonempty open sets and let

$$\Phi : U \times V \to \mathbb{R}^m$$

be a C^p function, $p \geq 1$. For every fixed $u \in U$ the system

$$\Phi(u, v) = 0 \tag{1.55}$$

consists of m equations for m variables, the m components of v. It is reasonable to expect that — under certain assumptions — for any fixed u there is a unique solution v, and that $v = v(u)$ changes smoothly as a function of u. Then the equation (1.55) defines v implicitly as a function of u. The implicit function theorem gives a condition ensuring this to be correct *locally* near a solution $(u_0, v_0) \in U \times V$ of the equation $\Phi(u, v) = 0$.

Theorem 1.4 *(Implicit Function Theorem) Under the above assumptions, let $(u_0, v_0) \in U \times V$ and assume the following two conditions:*

$$\Phi(u_0, v_0) = 0, \quad det(\Phi_v(u_0, v_0)) \neq 0 \ . \tag{1.56}$$

(Note that $\Phi_v(u_0, v_0)$ is an $m \times m$ matrix.) Then there are open sets U_0, V_0 with

$$u_0 \in U_0 \subset U, \quad v_0 \in V_0 \subset V$$

so that the following holds: For all $u \in U_0$ the equation (1.55) has a solution $v \in V_0$, which is unique in V_0. In addition, the function $u \to v$, which assigns to each $u \in U_0$ the corresponding solution $v \in V_0$, is C^p.

Application: Consider the equation

$$x_0 + t \sin x_0 = x$$

for $x \in \mathbb{R}$ and $|t| < 1$. We have already shown that there is a unique solution $x_0 = x_0(x,t)$ and want to argue that the assignment $(x,t) \to x_0(x,t)$ is C^∞. This follows from the implicit function theorem with

$$U = \mathbb{R} \times (-1,1), \quad V = \mathbb{R} \ ,$$

and

$$\Phi(x,t,x_0) = x_0 + t \sin x_0 - x \ .$$

Note that

$$\Phi_{x_0} = 1 + t \cos x_0 \neq 0 \ .$$

1.10 Notes

The PDE $F(x,u,\nabla u)$ where $x \in \mathbb{R}^n$ is treated in Evans, Partial Differential Equations, section 3.2. A side condition $u = h$ on Γ is assumed given. Here Γ is a submanifold of $\mathbb{R}^n$ of dimensions $n-1$. Using a transformation, the side condition and PDE are transformed so that the new Γ becomes a subset of the hyperplane $x_n = 0$. Then, under the assumption that the new boundary Γ is noncharacteristic, a local solution is constructed via the implicit function theorem. For details, see Evans.

A geometric interpretation of the PDE $F(x,y,u,u_x,u_y)$ using Monge cones is in John, Partial Differential Equations.

2 Laplace's Equation and Poisson's Equation

2.1 Terminology

Let $D_j = \partial/\partial x_j$. Then

$$\Delta = \sum_{j=1}^{n} D_j^2$$

is the Laplace operator acting on functions $u(x), x \in \Omega \subset \mathbb{R}^n$. Here, typically, Ω is an open subset of $\mathbb{R}^n$.

The equation

$$\Delta u = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

is called Laplace's equation; its solutions are called harmonic functions in Ω. The inhomogeneous equation

$$-\Delta u = f(x) \quad \text{in} \quad \Omega \tag{2.2}$$

is called Poisson's equation. Typically, the equations (2.1) and (2.2) have to be solved together with boundary conditions for u, or with decay conditions if Ω is unbounded. For example, the problem

$$-\Delta u = f(x) \quad \text{in} \quad \Omega, \quad u = g(x) \quad \text{on} \quad \partial\Omega\ , \tag{2.3}$$

is called a Dirichlet problem for Poisson's equation. Here $\partial\Omega$ is the boundary of the set Ω.

2.2 Volumes and Surface Areas of Balls in $\mathbb{R}^n$

The fundamental solution $\Phi(x)$ for the Laplace operator Δ in $\mathbb{R}^n$ will be discussed below; $\Phi(x)$ depends on a constant ω_n, the surface area of the unit sphere in $\mathbb{R}^n$. In this section we compute ω_n and the volume of the unit ball in $\mathbb{R}^n$.

Notations: For $x, y \in \mathbb{R}^n$ let the Euclidean inner product be denoted by

$$\langle x, y\rangle = x \cdot y = \sum_j x_j y_j\ .$$

The Euclidean norm is

$$|x| = (x \cdot x)^{1/2}\ .$$

Let

$$B_r = B_r(\mathbb{R}^n) = \{x \in \mathbb{R}^n\ :\ |x| < r\}$$

denote the open ball of radius r centered at 0. Its surface is

$$\partial B_r = \partial B_r(\mathbb{R}^n) = \{x \in \mathbb{R}^n\ :\ |x| = r\}\ .$$

We will use Euler's Γ–function, which may be defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \quad \mathrm{Re}\, s > 0\ .$$

(We will need the function only for real $s > 0$.) Let

$$\omega_n = area(\partial B_1(\mathbb{R}^n)) \ ,$$

i.e., ω_n is the surface area of the unit ball in $\mathbb{R}^n$.

For example,

$$\omega_2 = 2\pi, \quad vol(B_1(\mathbb{R}^2)) = \pi = \omega_2/2$$

and

$$\omega_3 = 4\pi, \quad vol(B_1(\mathbb{R}^3)) = \frac{4\pi}{3} = \omega_3/3 \ .$$

Theorem 2.1 *The following formulas are valid:*

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{2.4}$$

$$vol(B_1(\mathbb{R}^n)) = \frac{\omega_n}{n} \tag{2.5}$$

$$area(\partial B_r(\mathbb{R}^n)) = \omega_n r^{n-1} \tag{2.6}$$

$$vol(B_r(\mathbb{R}^n)) = \frac{\omega_n}{n} r^n \tag{2.7}$$

Formula (2.6) says that the area of the $(n-1)$–dimensional surface ∂B_r scales like r^{n-1}. Similarly, (2.7) says that the n–dimensional volume of B_r scales like r^n. These are results of integration theory.

We will prove (2.4) and (2.5) below. First we evaluate an important integral and show some simple properties of the Γ–function.

Lemma 2.1

$$I_1 := \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \tag{2.8}$$

$$\Gamma(s+1) = s\Gamma(s), \quad \mathrm{Re}\, s > 0 \tag{2.9}$$

$$\Gamma(m+1) = m!, \quad m = 0, 1, 2, \ldots \tag{2.10}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{2.11}$$

Proof: The proof of (2.8) uses a trick and polar coordinates:

$$\begin{aligned} I_1^2 &= \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy \\ &= \int_0^{2\pi} \int_0^\infty re^{-r^2} drd\phi \\ &= \pi \int_0^\infty 2re^{-r^2} dr \\ &= \pi \int_0^\infty e^{-\rho} d\rho \\ &= \pi \end{aligned}$$

Note that we have used the result that the circumference of the unit circle is 2π, which may be taken as the definition of π.

Equation (2.9), which is the fundamental functional equation of the Γ–function, follows through integrating by parts:

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty t^s e^{-t} dt \\ &= t^s(-e^{-t})|_0^\infty + s\int_0^\infty t^{s-1}e^{-t}dt \\ &= s\Gamma(s) \end{aligned}$$

This proves (2.9). Also,

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1 \ ,$$

and then (2.10) follows from (2.9) by induction. Finally, (2.11) follows from (2.8) by substituting

$$t = x^2, \quad dt = 2xdx \ ,$$

$$\begin{aligned} \Gamma(\frac{1}{2}) &= \int_0^\infty t^{-1/2}e^{-t}dt \\ &= 2\int_0^\infty \frac{1}{x}xe^{-x^2}dx \\ &= \sqrt{\pi} \end{aligned}$$

This completes the proof of Lemma 2.1. ⋄

We now calculate ω_n by considering the integral of $e^{-|x|^2}$ over $\mathbb{R}^n$. By Fubini's theorem and (2.8) we know that the result is $\pi^{n/2}$:

$$\begin{aligned} \pi^{n/2} &= \int_{\mathbb{R}^n} e^{-|x|^2}\,dx \\ &= \int_0^\infty area(\partial B_r)e^{-r^2}\,dr \\ &= \omega_n \int_0^\infty r^{n-1}e^{-r^2}\,dr \\ &= \frac{\omega_n}{2}\int_0^\infty r^{n-2}\,2re^{-r^2}\,dr \\ &= \frac{\omega_n}{2}\int_0^\infty t^{\frac{n}{2}-1}\,e^{-t}\,dt \\ &= \frac{\omega_n}{2}\,\Gamma(n/2) \end{aligned}$$

This proves (2.4). Also,

$$\begin{aligned} vol(B_1(\mathbb{R}^n)) &= \int_0^1 area(\partial B_r)\,dr \\ &= \omega_n \int_0^1 r^{n-1}\,dr \\ &= \omega_n/n \end{aligned}$$

Note that one can use the functional equation $\Gamma(s+1) = s\Gamma(s)$ and the two equations $\Gamma(1/2) = \sqrt{\pi}, \Gamma(1) = 1$ to evaluate $\Gamma(n/2)$ for every integer n. For example,

$$\begin{aligned} \Gamma\Big(\frac{3}{2}\Big) &= \Gamma\Big(\frac{1}{2}+1\Big) \\ &= \frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) \\ &= \frac{1}{2}\sqrt{\pi} \\ \Gamma\Big(\frac{5}{2}\Big) &= \Gamma\Big(\frac{3}{2}+1\Big) \\ &= \frac{3}{2}\Gamma\Big(\frac{3}{2}\Big) \\ &= \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi} \\ \Gamma\Big(\frac{7}{2}\Big) &= \Gamma\Big(\frac{5}{2}+1\Big) \\ &= \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi} \end{aligned}$$

etc.

2.3 Integration of $|x|^\lambda$ Over Circular Regions in $\mathbb{R}^n$

For $\lambda < 0$ the function $|x|^\lambda, x \in \mathbb{R}^n, x \neq 0$, has a singularity at $x = 0$. It is important to understand when the singularity is locally integrable. It is also important to know when the function $|x|^\lambda$ decays fast enough as $|x| \to \infty$ to make $|x|^\lambda$ integrable over an outer region $|x| \geq a > 0$.

Let $0 < a < b < \infty$ and consider the n–dimensional integral

$$\begin{aligned} I(a,b,\lambda,n) &:= \int_{a<|x|<b} |x|^\lambda\,dx \\ &= \omega_n \int_a^b r^{\lambda+n-1}\,dr \\ &= \begin{cases} \omega_n \frac{b^{\lambda+n}-a^{\lambda+n}}{\lambda+n} & \text{if} \quad \lambda+n \neq 0 \\ \omega_n \log\frac{b}{a} & \text{if} \quad \lambda+n = 0 \end{cases} \end{aligned}$$

Consider the limit $a \to 0$ with $b > 0$ fixed. One obtains

$$\int_{|x|<b} |x|^\lambda \, dx = \begin{cases} \omega_n \frac{b^{\lambda+n}}{\lambda+n} & \text{if} \quad \lambda > -n \\ \infty & \text{if} \quad \lambda \leq -n \end{cases}$$

For $0 > \lambda > -n$ the singularity of the function $|x|^\lambda$ at $x = 0$ is integrable; for $\lambda \leq -n$ the singularity is so strong that $|x|^\lambda$ is not integrable over B_b.

Now consider the limit of $I(a, b, \lambda, n)$ as $b \to \infty$ with $a > 0$ fixed. One obtains

$$\int_{|x|>a} |x|^\lambda \, dx = \begin{cases} \omega_n \frac{a^{\lambda+n}}{|\lambda+n|} & \text{if} \quad \lambda < -n \\ \infty & \text{if} \quad \lambda \geq -n \end{cases}$$

If $\lambda < -n$, then, as $|x| \to \infty$, the function $|x|^\lambda$ decays fast enough to be integrable over the domain $\mathbb{R}^n \setminus B_a$; for $\lambda \geq -n$ the decay is too slow to make the function integrable over $\mathbb{R}^n \setminus B_a$.

Note that there is no value of λ for which the function $|x|^\lambda$ is integrable over $\mathbb{R}^n$: Either the singularity at $x = 0$ is too strong (for $\lambda \leq -n$) or the decay as $|x| \to \infty$ is too slow (for $\lambda \geq -n$). If $\lambda = -n$ then the singularity is too strong and the decay is too slow.

2.4 Rotational Invariance of Δ; Radial Harmonic Functions

A remarkable property of the Laplace operator $\Delta = \sum_{j=1}^n D_j^2$ is its invariance under an orthogonal change of coordinates. We make precise what this means and prove it. This invariance motivates to look for radial solutions of Laplace's equation, $\Delta u = 0$. Here $u(x)$ is called a radial function, if there is a function $v(r)$, depending only on a single variable $0 < r < \infty$, with $u(x) = v(|x|)$ for all $x \in \mathbb{R}^n, x \neq 0$. The radial solutions of Laplace's equation will lead to the fundamental solution.

The Laplacian is invariant under orthogonal transformations of the coordinate system. A precise statement is the following.

Theorem 2.2 *Let $u \in C^2(\mathbb{R}^n)$ and let S denote an orthogonal $n \times n$ matrix, $SS^T = I$. Let*

$$v(y) = u(Sy), \quad y \in \mathbb{R}^n \ ,$$

i.e., $v(y)$ is obtained from $u(x)$ by changing from x coordinates to y coordinates. Then we have

$$(\Delta v)(y) = (\Delta u)(Sy) \quad \textit{for all} \quad y \in \mathbb{R}^n \ .$$

The theorem says that the following two operations lead to the same result:

A: First change from x coordinates to y coordinates (where $x = Sy, SS^T = I$) and then take the Laplacian;

B: First take the Laplacian and then change from x coordinates to y coordinates.

Proof: First note that

$$(Sy)_j = \sum_l s_{jl} y_l \ ,$$

thus

$$D_k(Sy)_j = s_{jk} \quad \text{where} \quad D_k = \partial/\partial y_k$$

Since $v(y) = u(Sy)$ we have by the chain rule:

$$\begin{aligned} D_k v(y) &= \sum_j (D_j u)(Sy) D_k (Sy)_j \\ &= \sum_j (D_j u)(Sy) s_{jk} \end{aligned}$$

Therefore,

$$D_k^2 v(y) = \sum_j \sum_i (D_i D_j u)(Sy) s_{ik} s_{jk} \ .$$

Summation over k yields

$$\Delta v(y) = \sum_k \sum_j \sum_i (D_i D_j u)(Sy) s_{ik} s_{jk} \ .$$

Here

$$\sum_k s_{ik} s_{jk} = \delta_{ij}$$

and the claim follows. $\diamond$

The Laplacian applied to radial functions. Let $v(r), r > 0$, be a C^2 function and let

$$u(x) = v(|x|), \quad x \in \mathbb{R}^n \ .$$

Then

$$\Delta u(x) = v''(r) + \frac{n-1}{r}\, v'(r) \quad \text{with} \quad r = |x| \ .$$

Proof: We have

$$\begin{aligned} D_j u(x) &= v'(r)\, \frac{x_j}{r} \\ D_j^2 u(x) &= v''(r)\, \frac{x_j^2}{r^2} + v'(r)\Big(\frac{1}{r} - \frac{x_j^2}{r^3}\Big) \end{aligned}$$

Summation over j from 1 to n yields the result. $\diamond$

We can now determine all solutions of $\Delta u = 0$ in $\mathbb{R}^n \setminus \{0\}$ which are radially symmetric. We must solve

$$v'' + \frac{n-1}{r}\, v' = 0 \ ,$$

thus $v' = w + const$ where

$$w' + \frac{n-1}{r} w = 0 \ .$$

Separation of variables yields

$$\int \frac{dw}{w} = \int \frac{1-n}{r}\, dr \ ,$$

thus

$$\ln w = \ln r^{1-n} + c_1 \ ,$$

thus

$$w(r) = c r^{1-n} \ .$$

Integration yields

$$v(r) = \begin{cases} c_1 \ln r + c_2, & n = 2 \\ c_1 r^{2-n} + c_2, & n = 1 \quad \text{or} \quad n \geq 3 \end{cases} \tag{2.12}$$

The 2D Laplacian in polar coordinates. Let $u(x,y)$ be a C^2 function. Express this function in polar coordinates,

$$U(r,\phi) = u(r\cos\phi, r\sin\phi) \ .$$

Then

$$\Delta u(x,y) = (U_{rr} + \frac{1}{r}\, U_r + \frac{1}{r^2} U_{\phi\phi})(r,\phi)$$

if

$$x = r\cos\phi, \quad y = r\sin\phi \ .$$

Proof: Homework.

2.5 Physical Interpretation of Poisson's Equation in $\mathbb{R}^3$

Let Q be a point charge at (the point with coordinates) y and let q be a point charge at x. By Coulomb's law, the electro static force F of Q on q is

$$F = kQq\, \frac{1}{|x-y|^2}\, \frac{x-y}{|x-y|} \ .$$

Here k is a constant that depends on the units used for charge, force, and length.[2] Therefore, the electric field generated by Q is

$$E(x) = kQ\, \frac{x-y}{|x-y|^3} \ .$$

The force field $E(x)$ has the potential

$$u(x) = \frac{kQ}{|x-y|} \ .$$

This means that

$$-grad\, u(x) = E(x), \quad x \neq y \ .$$

The following results about the function $|x|^{-1}$ are useful.

[2]It is common to write $k = \frac{1}{4\pi\varepsilon_0}$, where $\varepsilon_0 = 8.859 \cdot 10^{-12} Coul^2 \cdot N^{-1} \cdot m^{-2}$. The formula for the force F then holds in vacuum.

Lemma 2.2 *Let $r = r(x) = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Then we have, for all $x \neq 0$,*

$$\begin{aligned} D_j r &= \frac{x_j}{r} \\ D_j\Big(\frac{1}{r}\Big) &= -\frac{x_j}{r^3} \\ grad\,\Big(\frac{1}{r}\Big) &= -\frac{x}{r^3} \\ D_j^2\Big(\frac{1}{r}\Big) &= -|x|^{-5}(|x|^2 - 3x_j^2) \\ D_i D_j\Big(\frac{1}{r}\Big) &= 3|x|^{-5}x_i x_j \quad for \quad i \neq j \\ \Delta\Big(\frac{1}{r}\Big) &= 0 \end{aligned}$$

It is easy to show that $|x|$ may be replaced by $|x-y|$ with y fixed. For example,

$$grad_x \, \frac{1}{|x-y|} = -\frac{x-y}{|x-y|^3} \; .$$

This yields $-grad\, u(x) = E(x)$, as claimed above.

By the above lemma, the potential $u(x)$ satisfies Laplace's equation, $\Delta u(x) = 0$, at every point $x \neq y$. At $x = y$, the potential has a singularity, marking the point y where the charge Q, which generates the field, is located.

It is not difficult to generalize to N point charges: Assume that there are N point charges $Q_1, \ldots, Q_N$ located at $y^{(1)}, \ldots, y^{(N)}$. They generate an electric field with potential

$$u(x) = k \sum_i \frac{Q_i}{|x - y^{(i)}|} \; .$$

At each point $x = y^{(i)}$ the potential has a singularity, but in the set

$$\mathbb{R}^3 \setminus \{y^{(1)}, \ldots, y^{(N)}\}$$

the function $u(x)$ satisfies Laplace's equation.

It is plausible that Coulomb's law can be extended from point charges to continuously distributed charges. Suppose that V_i denotes a small volume about the point $y^{(i)}$ and assume the point charge Q_i is uniformly distributed over V_i. Then the charge density in V_i is $Q_i/vol(V_i)$. (We assume that the V_i do not overlap.) The following function is the charge density in space:

$$f(y) = \begin{cases} Q_i/vol(V_i), & y \in V_i \\ 0, & \text{outside all } V_i \end{cases}$$

It is plausible that, for small V_i, the charge density f generates approximately the same field as the N point charges. Thus, if $u(x)$ is the potential of the field generated by the charges with density f, we expect that

$$\begin{aligned} u(x) &\approx k \sum_i \frac{Q_i/vol(V_i)}{|x-y^{(i)}|}\, vol(V_i) \\ &\approx k \int \frac{f(y)}{|x-y|}\, dy \end{aligned}$$

We expect the approximation to become an equality if the V_i become infinitesimal. Thus, if $f = f(y)$ denotes a charge density in space, we expect that the generated electric field has the potential

$$u(x) = k \int \frac{f(y)}{|x-y|}\, dy \ . \tag{2.13}$$

(This cannot be deduced rigorously from Coulomb's law for point charges, but may be considered as a reasonable extension of the law.) If u is defined by (2.13), we expect that $\Delta u = 0$ in regions where $f = 0$. We also expect a simple relation between Δu and f in regions where f is not zero. We will show that, under suitable assumptions on the function $f(y)$, we have

$$-\Delta u = 4\pi k f = \frac{1}{\varepsilon_0}\, f$$

if u is defined by (2.13). To summarize, an interpretation of Poisson's equation

$$-\Delta u = f$$

is the following: The function f is a charge distribution, generating an electric field, and (modulo a constant factor $4\pi k$) the solution u is the potential of this field.

2.6 Poisson's Equation in $\mathbb{R}^3$

In this section we consider Poisson's equation in $\mathbb{R}^3$,

$$-\Delta u = f(x), \quad x \in \mathbb{R}^3 \ ,$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a given function. The fundamental solution

$$\Phi(x) = \frac{1}{4\pi|x|}$$

can be used to obtain the only decaying solution u.

2.6.1 The Newtonian Potential

Let $f \in C_c(\mathbb{R}^3)$. This means that $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and there exists $R > 0$ with $f(z) = 0$ if $|z| \geq R$.

Define

$$u(x) = \int \Phi(y) f(x-y)\, dy, \quad x \in \mathbb{R}^3 \ . \tag{2.14}$$

Note: The function $\Phi(y)$ is not integrable over $\mathbb{R}^3$ since it decays too slowly. However, the above integral is finite since f has compact support. All integrals in this section effectively extend over a finite region only. We call this region $U = B_M$; here M depends on x.

The function $u = \Phi * f$ is called the Newtonian potential (or Coulomb's potential) of f.

We want to show:

Theorem 2.3 *If $f \in C_c^2(\mathbb{R}^3)$ then $u = \Phi * f \in C^2(\mathbb{R}^3)$ and*

$$-\Delta u(x) = f(x) \quad \text{for all} \quad x \in \mathbb{R}^3 \ .$$

Auxiliary results:

Theorem 2.4 *(Gauss–Green theorem) Let $U \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary ∂U and unit outward normal*

$$n = (n_1, n_2, n_3), \quad n_j = n_j(y), \quad y \in \partial U \ .$$

If $u \in C^1(\bar{U})$ then

$$\int_U D_j u \, dx = \int_{\partial U} u n_j \, dS \ . \tag{2.15}$$

Replacing u with uv one obtains:

Theorem 2.5 *(integration by parts) If $u, v \in C^1(\bar{U})$ then*

$$\int_U (D_j u) v \, dx = -\int_U u D_j v \, dx + \int_{\partial U} u v n_j \, dS \ . \tag{2.16}$$

Proof of Theorem 2.3:

1. By Taylor,

$$f(z + h e_j) = f(z) + h D_j f(z) + R_j(z, h)$$

with

$$|R_j(z, h)| \le C h^2 \ .$$

Therefore,

$$u(x + h e_j) - u(x) = h \int \Phi(y) D_j f(x - y) \, dy + \mathcal{O}(h^2) \ .$$

Divide by h and let $h \to 0$. Obtain that $D_j u(x)$ exists and

$$D_j u(x) = \int \Phi(y) D_j f(x - y) \, dy \ .$$

To summarize, we have justified to differentiate equation (2.14) under the integral sign.

2. Let us write $D_{y_j} = \partial / \partial y_j$. We have

$$D_j u(x) = -\int \Phi(y) D_{y_j} f(x - y) \, dy \ .$$

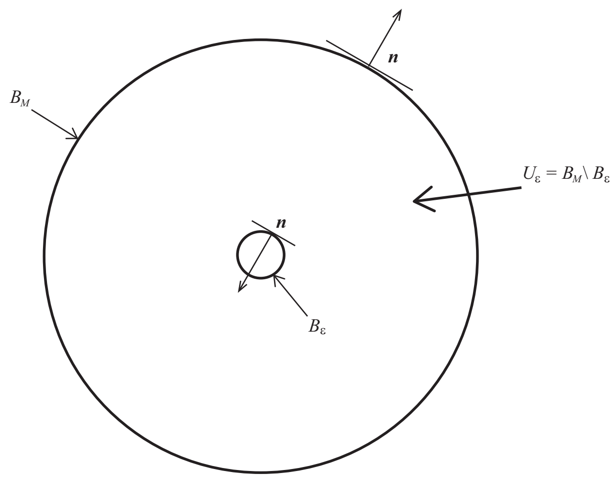

Figure 8: Region U_ϵ and outward normals

We want to integrate by parts and move D_{y_j} to $\Phi(y)$. However $\Phi(y)$ is not smooth at $y = 0$. Therefore, we first remove a small ball B_ε about the origin.

Let $U = B_M$ where M is large, and let $U_\varepsilon = B_M \setminus B_\varepsilon$. We have

$$\begin{aligned} D_j u(x) &= -\int_{B_M} \Phi(y) D_{y_j} f(x-y)\, dy \\ &= -\int_{B_\varepsilon} \ldots - \int_{U_\varepsilon} \ldots \end{aligned}$$

As $\varepsilon \to 0$, the first integral goes to zero. (Note that $|\Phi(y)| \le C|y|^{-1}$. Therefore, the integral over B_ε is $\le C\varepsilon^2$.) In the second integral we can integrate by parts. Note that the unit outward[3] normal on ∂B_ε is

$$n(y) = -\frac{y}{\varepsilon}, \quad |y| = \varepsilon\ .$$

Obtain that

$$-\int_{U_\varepsilon} \Phi(y) D_{y_j} f(x-y)\, dy = \int_{U_\varepsilon} D_j \Phi(y) f(x-y)\, dy - \int_{\partial B_\varepsilon} \Phi(y) f(x-y)(-y_j/\varepsilon)\, dS(y)\ .$$

As $\varepsilon \to 0$, the boundary term goes to zero since $\Phi(y) = \frac{1}{4\pi\varepsilon}$ for $y \in \partial B_\varepsilon$ and $area(\partial B_\varepsilon) = 4\pi\varepsilon^2$. As $\varepsilon \to 0$, we obtain

$$D_j u(x) = \int D_j \Phi(y) f(x-y)\, dy\ .$$

[3]Here outward refers to the domain U_ε.

3. As above, we can differentiate again under the integral sign and put the derivative on f,

$$\begin{aligned} D_j^2 u(x) &= \int D_j\Phi(y) D_j f(x-y)\,dy \\ &= -\int D_j\Phi(y) D_{y_j} f(x-y)\,dy \\ &= -\int_{B_\varepsilon} \ldots - \int_{U_\varepsilon} \ldots \\ &=: -I_{1,\varepsilon,j} - I_{2,\varepsilon,j} \end{aligned}$$

We obtain that $|I_{1,\varepsilon,j}| \le \varepsilon$. In the second integral, we integrate by parts,

$$-I_{2,\varepsilon,j} = \int_{U_\varepsilon} D_j^2\Phi(y) f(x-y)\,dy - \int_{\partial B_\varepsilon} D_j\Phi(y) f(x-y)(-y_j/\varepsilon)\,dS(y) \ .$$

Since $\Delta\Phi(y) = 0$ in U_ε, the first terms sum to zero if we sum over $j = 1, 2, 3$. It remains to discuss the boundary terms: We have

$$D_j\Phi(y) = -\frac{1}{4\pi}\frac{y_j}{\varepsilon^3} \quad \text{for} \quad y \in \partial B_\varepsilon \ .$$

Therefore, the above boundary term is

$$BT(\varepsilon, j) = -\frac{1}{4\pi}\int_{\partial B_\varepsilon} \frac{y_j^2}{\varepsilon^4} f(x-y)\,dS(y) \ .$$

Summation yields

$$\begin{aligned} \sum_{j=1}^3 BT(\varepsilon, j) &= -\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon} f(x-y)\,dS(y) \\ &= -f(x - y_\varepsilon) \end{aligned}$$

where

$$y_\varepsilon \in \partial B_\varepsilon \ .$$

As $\varepsilon \to 0$ we obtain that $\Delta u(x) = -f(x)$. $\diamond$

2.6.2 Uniqueness of Decaying Solutions

There are many harmonic functions on $\mathbb{R}^n$. For example,

$$v(x_1, x_2) = a + bx_1 + cx_2, \quad v(x_1, x_2) = x_1^2 - x_2^2, \quad v(x_1, x_2) = e^{x_1}\cos(x_2)$$

are solutions of $\Delta v = 0$. Therefore, the solution

$$u(x) = \int \Phi(y) f(x-y)\,dy = \int \Phi(x-y) f(y)\,dy \tag{2.17}$$

of the equation $-\Delta u = f$ is not unique. We will show, however, that (2.17) is the *only* decaying solution.

We first prove a decay estimate for the function u defined in (2.17).

Lemma 2.3 *Let $f \in C_c$ be supported in B_R. Then, if $|x| \geq 2R$, $u = \Phi * f$ satisfies the bound*

$$|u(x)| \leq \frac{\|f\|_{L^1}}{2\pi} \frac{1}{|x|}$$

where

$$\|f\|_{L^1} = \int |f(y)| \, dy$$

is the L^1–norm of f.

Proof: If $|x| \geq 2R$ and $|y| \leq R$, then

$$|x - y| \geq |x| - |y| \geq \frac{1}{2} |x| \; .$$

Therefore,

$$\Phi(x-y) \leq \frac{1}{2\pi} \frac{1}{|x|} \; .$$

It follows that

$$\begin{aligned} |u(x)| &\leq \int_{|y| \leq R} \Phi(x-y) |f(y)| \, dy \\ &\leq \frac{1}{2\pi} \frac{1}{|x|} \|f\|_{L^1} \quad \text{for} \quad |x| \geq 2R \; . \end{aligned}$$

This proves the lemma. $\diamond$

We next show an important property of harmonic functions, the **mean–value property**.

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u \in C^2(\Omega)$ and let $\Delta u = 0$ in Ω. Let $\bar{B}_R(x) \subset \Omega$. Then we have*

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} u(y) \, dS(y) \; . \tag{2.18}$$

Proof: We may assume $x = 0$. For $0 < r \leq R$ define the function

$$\phi(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r} u(y) \, dS(y) \; .$$

It is clear that $\lim_{r \to 0} \phi(r) = u(0)$ and $\phi(R)$ is the right–hand side in (2.18). We now show that $\phi'(r) = 0$ for $0 < r \leq R$. This implies that ϕ is constant and shows (2.18).

Changing variables, $y = rz$, in the right–hand side of (2.18), we have

$$\phi(r) = \frac{1}{\omega_n} \int_{\partial B_1} u(rz)\, dS(z) \ .$$

Therefore,

$$\phi'(r) = \frac{1}{\omega_n} \int_{\partial B_1} \sum_j D_j u(rz) z_j \, dS(z) \ .$$

Stokes' theorem yields

$$\int_{\partial B_1} v(z) z_j \, dS(z) = \int_{B_1} \frac{\partial}{\partial z_j} v(z)\, dz \ .$$

Since

$$\frac{\partial}{\partial z_j} D_j u(rz) = r(D_j^2 u)(rz)$$

and $\sum_j D_j^2 u(rz) = 0$, it follows that $\phi'(r) = 0$. $\diamond$

We can now extend Theorem 2.3 by a uniqueness statement.

Theorem 2.7 *Let $f \in C_c^2(\mathbb{R}^3)$. Then the Newtonian potential of f, $u = \Phi * f$, satisfies $-\Delta u = f$ and $|u(x)| \to 0$ as $|x| \to \infty$. Furthermore, if $v \in C^2$ is any function with $-\Delta v = f$ and*

$$v(x) \to 0 \quad as \quad |x| \to \infty \ , \tag{2.19}$$

then $v = u$.

Proof: It remains to show the uniqueness statement, $v = u$. By definition, (2.19) means that for any $\varepsilon > 0$ there is $R > 0$ with $|v(x)| < \varepsilon$ if $|x| \geq R$. Set $w = v - u$. Then $\Delta w = 0$ and $w(x) \to 0$ as $|x| \to \infty$. Suppose that there is $x \in \mathbb{R}^3$ with $|w(x)| > 0$. Choose $0 < \varepsilon < |w(x)|$. By Theorem 2.6 (with $n = 3$) we know that

$$w(x) = \frac{1}{\omega_3 R^2} \int_{\partial B_R(x)} w(y)\, dS(y) \ . \tag{2.20}$$

However, if R is large enough, then $|w(y)| < \varepsilon$ for $y \in \partial B_R(x)$. For such R the right–hand side of (2.20) is $< \varepsilon$ in absolute value, a contradiction. $\diamond$

2.6.3 Remarks on the Relation $-\Delta\Phi = \delta_0$

We want to explain what it means that the function $\Phi(x) = \frac{1}{4\pi} \frac{1}{|x|}, x \in \mathbb{R}^3, x \neq 0$, satisfies the equation

$$-\Delta\Phi = \delta_0$$

in the sense of distributions. Here δ_0 denotes Dirac's δ–distribution with unit mass at $x = 0$.

The distributional equation $-\Delta\Phi = \delta_0$ means that for every test function $v \in C_c^\infty$ we have

$$\langle -\Delta\Phi, v \rangle = \langle \delta_0, v \rangle \ , \tag{2.21}$$

and, by definition of the distributions $-\Delta\Phi$ and δ_0, this means that

$$-\int \Phi(y)(\Delta v)(y)\,dy = v(0) \quad \text{for all} \quad v \in C_c^\infty \ . \tag{2.22}$$

To show (2.22), let $v \in C_c^\infty$ and set $f = -\Delta v$. Then $f \in C_c^\infty$; let $u = \Phi * f$, as above. We know that $-\Delta u = f$, and $v = u$ by Theorem 2.7. Therefore,

$$\begin{aligned} v(0) &= u(0) \\ &= \int \Phi(0-y)f(y)\,dy \\ &= \int \Phi(y)f(y)\,dy \\ &= -\int \Phi(y)(\Delta v)(y)\,dy \end{aligned}$$

Thus we have proved (2.22), i.e., we have proved that $-\Delta\Phi = \delta_0$ in the sense of distributions.

If one has shown that

$$-\Delta\Phi(y) = \delta_0(y)$$

then one can give the following intuitive argument showing that the function

$$u(x) = \int \Phi(x-y)f(y)\,dy \tag{2.23}$$

satisfies

$$-\Delta u(x) = f(x)\ .$$

Apply the operator $-\Delta$ to (2.23) and assume that one can apply $-\Delta$ under the integral sign. Obtain:

$$\begin{aligned} -\Delta u(x) &= \int -\Delta\Phi(x-y)f(y)\,dy \\ &= \int \delta_0(x-y)f(y)\,dy \\ &= f(x) \end{aligned}$$

2.6.4 Remarks on Coulomb's Law for Point Charges

Let $Q_1, \ldots, Q_N$ denote point charges at the points $y^{(1)}, \ldots, y^{(N)} \in \mathbb{R}^3$. They generate an electric field $E(x)$ which has the potential

$$u(x) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^N \frac{Q_j}{|x-y^{(j)}|}\ .$$

In the sense of distributions, we have

$$-\Delta\Big(\frac{1}{4\pi}\frac{1}{|x-y^{(j)}|}\Big) = \delta_0(x-y^{(j)}) \ ,$$

thus

$$-\Delta u(x) = \sum_{j=1}^{N} \frac{Q_j}{\varepsilon_0}\, \delta_0(x-y^{(j)}) \ .$$

On the other hand, if $\rho(y)$ is a smooth charge distribution, we have argued that it generates an electric field with potential

$$u(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(y)}{|x-y|}\, dy$$

which satisfies

$$-\Delta u(x) = \frac{\rho(x)}{\varepsilon_0} \ .$$

This suggests that we should assign to the point Q_j at the point $y^{(j)}$ the charge distribution function

$$Q_j\, \delta_0(x-y^{(j)}) \ .$$

2.7 The Fundamental Solution $\Phi(x)$ for the Laplace Operator on $\mathbb{R}^n$

By Sect. 2.4 all radial functions which are harmonic in $\mathbb{R}^n \setminus \{0\}$ have the form

$$u(x) = \begin{cases} c_1 \ln|x| + c_2, & n=2 \\ c_1|x|^{2-n} + c_2, & n \neq 2 \end{cases}$$

Taking $c_2 = 0$ and choosing c_1 properly, one obtains the fundamental solution $\Phi(x)$ for $-\Delta$ in $\mathbb{R}^n$,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x|, & n=2 \\ \frac{1}{\omega_n(n-2)}\, |x|^{2-n}, & n \geq 3 \end{cases} \tag{2.24}$$

Then we have

$$-\Delta\Phi = \delta_0$$

in the sense of distributions. It is not difficult to generalize Theorem 2.3 from $n=3$ to general n: If $f \in C_c^2(\mathbb{R}^n)$ then the equation

$$-\Delta u = f$$

is solved by $u = \Phi * f$.

Formally, the result also holds for $n = 1, \omega_1 = 2$. As solution of

$$-u''(x) = f(x), \quad x \in \mathbb{R} \ ,$$

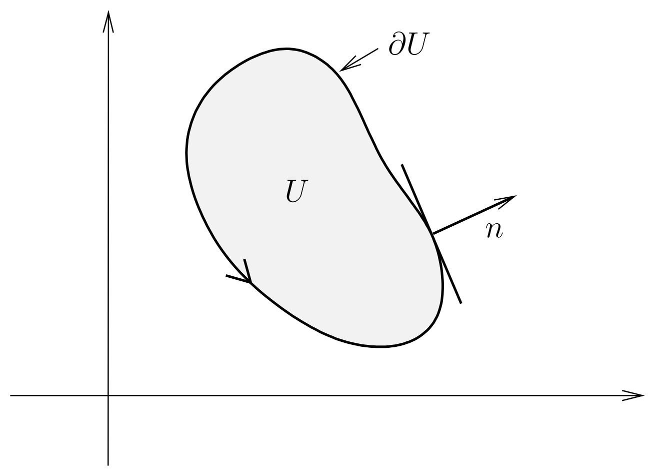

Figure 9: Set U and its outer normal

one obtains

$$u(x) = -\frac{1}{2}\int_{-\infty}^{\infty} |x-y| f(y)\, dy \ .$$

It is easy to check that $u(x)$ solves the equation $-u'' = f$ if $f \in C_c$.

Note that, for $n = 1$ and $n = 2$, the fundamental solution $\Phi(x)$ does not decay to zero as $|x| \to \infty$. Therefore, the solution $u = \Phi * f$ generally does not decay to zero as $|x| \to \infty$ for $n = 1$ or $n = 2$.

2.8 The Dirichlet Problem for Poisson's Equation in a Bounded Domain

Let $U \subset \mathbb{R}^3$ be a bounded open set with smooth boundary ∂U. Let $f : U \to \mathbb{R}$ and $g : \partial U \to \mathbb{R}$ be given smooth functions. We want to find a function $u \in C^2(U) \cap C(\bar{U})$ with

$$-\Delta u = f \quad \text{in} \quad U, \quad u = g \quad \text{on} \quad \partial U \ . \tag{2.25}$$

This is called Dirichlet's problem for Poisson's equation.

Let $n = (n_1, n_2, n_3), n_j = n_j(y), y \in \partial U$, denote the unit outward normal to the boundary surface ∂U at the point $y \in \partial U$.

One tries to construct a solution u of (2.25) in the form

$$u(x) = \int_U G(x,y) f(y)\, dy - \int_{\partial U} \frac{\partial G(x,y)}{\partial n(y)}\, g(y)\, dS(y) \ . \tag{2.26}$$

Why this is a good form is not at all obvious.

In the formula (2.26) the function $G(x,y)$ is the so–called Green's function for the Laplace operator, corresponding to Dirichlet boundary conditions and the domain U. The function

$G(x,y)$ does not depend on the right–hand sides f and g. Also,

$$\frac{\partial G(x,y)}{\partial n(y)} = \sum_{j=1}^{3} n_j(y) D_{y_j} G(x,y)$$

is the normal derivative of $G(x,y)$ for $y \in \partial U$.

We will motivate the formula (2.26) below. All considerations generalize rather easily from domains in $\mathbb{R}^3$ to domains in $\mathbb{R}^n$.

For some special domains U the Green's function $G(x,y)$ can be computed explicitly. In the next sections we discuss this for the half–space, $U = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \ : \ x_3 > 0\}$, and the open unit ball in $\mathbb{R}^3$, namely $U = B_1 = \{x \in \mathbb{R}^3 : |x| < 1\}$.

Next we list some general formulas, valid for smooth functions u, v on bounded domains U with smooth boundaries ∂U. As usual, dy is the volume element and $dS(y)$ is the surface element. Also, $D_j = \partial / \partial y_j$ and

$$\frac{\partial u}{\partial n} = \sum_j n_j D_j u = n \cdot \nabla u$$

denotes the directional derivative of u in direction of the unit outward normal.

The formulas have various names connected with Green, Gauss, and Stokes.

$$\int_U D_j u \, dy = \int_{\partial U} u n_j \, dS(y) \tag{2.27}$$

$$\int_U u D_j v \, dy = -\int_U (D_j u) v \, dy + \int_{\partial U} u v n_j \, dS(y) \tag{2.28}$$

$$\int_U \Delta u \, dy = \int_{\partial U} \frac{\partial u}{\partial n} \, dS(y) \tag{2.29}$$

$$\int_U (u \Delta v - v \Delta u) \, dy = \int_{\partial U} \Big(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \Big) \, dS(y) \tag{2.30}$$

First assume that u solves (2.25). Fix $x \in U$; let $\varepsilon > 0$ be so small that

$$B_\varepsilon(x) \subset U$$

and set

$$U_\varepsilon = U \setminus B_\varepsilon(x) \ .$$

Let

$$\Phi(x) = \frac{1}{4\pi |x|}$$

denote the fundamental solution for the Laplace operator on $\mathbb{R}^3$ and set

$$v(y) = \Phi(x - y), \quad y \neq x \ .$$

We want to explore implications of the formula (2.30) when U is replaced by U_ε and ε goes to 0.

Lemma 2.4 *Fix $x \in U$ and set $v(y) = \Phi(x-y)$ for $y \neq x$. If u solves the Dirichlet problem (2.25), then we have*

$$\int_U v(y)f(y)\,dy = u(x) + \int_{\partial U}\Big(g\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,dS \quad \textit{for all} \quad x \in U\ . \tag{2.31}$$

Proof: Use (2.30) with U replaced by U_ε and note that $\Delta v = 0, -\Delta u = f$ in U_ε. This yields

$$\int_{U_\varepsilon} vf\,dy = \int_{\partial U_\varepsilon}\Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,dS \tag{2.32}$$

As $\varepsilon \to 0$, the left–hand side converges to

$$\int_U vf\,dy\ .$$

The boundary ∂U_ε consists of two parts, ∂U and $\partial B_\varepsilon(x)$. Since $u = g$ on ∂U, we have

$$\int_{\partial U}\Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,dS = \int_{\partial U}\Big(g\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,dS\ .$$

It remains to consider

$$\int_{\partial B_\varepsilon(x)}\Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)\,dS\ .$$

Since $v = \mathcal{O}(\varepsilon^{-1})$ on $\partial B_\varepsilon(x)$ one obtains that

$$\int_{\partial B_\varepsilon(x)} v\frac{\partial u}{\partial n}\,dS \to 0 \quad \text{as} \quad \varepsilon \to 0\ .$$

Finally, for $y \in \partial B_\varepsilon(x)$,

$$D_{y_j}v(y) = \frac{1}{4\pi}\,\frac{x_j - y_j}{\varepsilon^3} \quad \text{and} \quad n_j(y) = \frac{x_j - y_j}{\varepsilon}\ .$$

Therefore,

$$\frac{\partial v}{\partial n}(y) = \frac{1}{4\pi\varepsilon^2}\ .$$

It follows that

$$\int_{\partial B_\varepsilon(x)} u\frac{\partial v}{\partial n}\,dS \to u(x) \quad \text{as} \quad \varepsilon \to 0\ .$$

This proves the lemma. $\diamond$

Recall that $v(y) = \Phi(x-y)$ in (2.31), thus

$$u(x) = \int_U \Phi(x-y)f(y)\,dy - \int_{\partial U}\Big(g(y)\frac{\partial\Phi(x-y)}{\partial n(y)} - \Phi(x-y)\frac{\partial u(y)}{\partial n}\Big)\,dS(y)\ . \tag{2.33}$$

One now tries to subtract a correction term, $H(x,y)$, from $\Phi(x-y)$ so that for the corrected function

$$G(x,y) = \Phi(x-y) - H(x,y)$$

the boundary term involving $\partial u/\partial n$ drops out. Thus, $H(x,y)$ should satisfy the following conditions:

1.

$$\Delta_y H(x,y) = 0 \quad \text{for all} \quad x,y \in U \ ;$$

2.

$$H(x,y) = \Phi(x-y) \quad \text{for all} \quad y \in \partial U, \quad x \in U \ .$$

3. The function $H(x,y)$ is not singular for $x \in U, y \in \bar{U}$.

If $H(x,y)$ satisfies these conditions, then

$$G(x,y) = \Phi(x-y) - H(x,y)$$

is the Green's function for the Laplacian w.r.t Dirichlet boundary conditions, and the formula (2.26) holds for every solution of the Dirichlet problem (2.25). To show this, repeat the proof of Lemma 2.4 with $v(y) = \Phi(x-y) - H(x,y)$.

Remark: Assume that $H(x,y)$ satisfies the three conditions. Fix $x \in U$ and set $h(y) = H(x,y)$. Then obtain from Green's formula (2.30) that

$$\int_U h(y) f(y)\, dy = \int_{\partial U} \Big(g \frac{\partial h}{\partial n} - h \frac{\partial u}{\partial n} \Big)\, dS(y) \ .$$

Subtracting this equation from (2.33) yields

$$u(x) = \int_U G(x,y) f(y)\, dy - \int_{\partial U} g(y)\, \frac{\partial G(x,y)}{\partial n(y)}\, dS(y) \tag{2.34}$$

with

$$G(x,y) = \Phi(x-y) - H(x,y) \ .$$

This is the formula (2.26).

So far, we have assumed that u solves the Dirichlet problem and have derived the representation (2.26) for $u(x)$. Conversely, one can prove that the formula (2.26) defines a solution $u \in C^2(U) \cap C(\bar{U})$ of (2.25) if f, g and ∂U are sufficiently regular. It suffices that U is a bounded open set, that the boundary ∂U is C^1, that $g \in C(\partial U)$ and that f is Hölder continuous in U.

Summary: Under suitable assumptions on $U, \partial U, f$, and g the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad U, \quad u = g \quad \text{on} \quad \partial U$$

can be solved as follows:

1) For every $x \in U$ solve the following Dirichlet problem for Laplace's equation:

$$\Delta h = 0 \quad \text{in} \quad U, \quad h(y) = \Phi(x-y) \quad \text{for} \quad y \in \partial U \ .$$

Call the solution

$$H(x,y) = h(y) \ .$$

2) Define the Green's function

$$G(x,y) = \Phi(x-y) - H(x,y) \ .$$

Then the solution of the Dirichlet problem is given by (2.26).

2.9 The Green's Function for a Half–Space

Let

$$U = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$$

denote a half–space with boundary

$$\partial U = \{x \in \mathbb{R}^3 : x_3 = 0\} \ .$$

We try to construct a solution of the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad U, \quad u = g \quad \text{on} \quad \partial U \ , \tag{2.35}$$

in the form

$$u(x) = \int_U G(x,y) f(y)\, dy + \int_{\partial U} K(x,y) g(y)\, dS(y) \ , \tag{2.36}$$

where

$$K(x,y) = -\frac{\partial G(x,y)}{\partial n(y)} \ .$$

Note that $n(y) = -e_3$, thus

$$K(x,y) = -\frac{\partial G(x,y)}{\partial n(y)} = D_{y_3} G(x,y) \ .$$

To construct $G(x,y)$ we follow the process of the previous section and write

$$G(x,y) = \Phi(x-y) - H(x,y) \ .$$

Here $\Phi(x-y) = \frac{1}{4\pi|x-y|}$ and $H(x,y)$ must satisfy

1. $$\Delta_y H(x,y) = 0 \quad \text{for all} \quad x,y \in U \ ;$$

2. $$H(x,y) = \Phi(x-y) \quad \text{for all} \quad y \in \partial U, \quad x \in U \ .$$

3. The function $H(x,y)$ is not singular for $x \in U, y \in \bar{U}$.

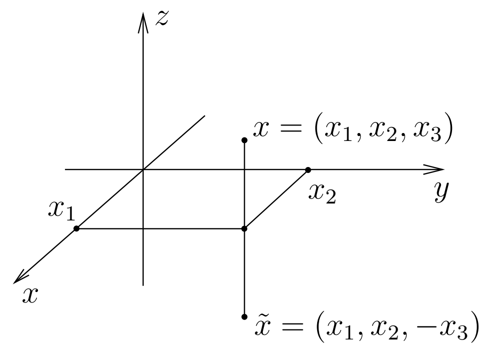

Figure 10: Reflection point $\tilde{x}$

For $x = (x_1, x_2, x_3) \in U$ denote the reflected point (w.r.t. the hyperplane ∂U) by

$$\tilde{x} = (x_1, x_2, -x_3) \ .$$

We claim that

$$H(x, y) = \Phi(\tilde{x} - y), \quad x \in U, \quad y \in \bar{U} \ ,$$

satisfies the above conditions. (Note that $H(x, y)$ does not become singular for $x \in U, y \in \bar{U}$.) The first condition, $\Delta_y H(x, y) = 0$, holds since $\Delta\Phi(z) = 0$ for $z \neq 0$. The second condition, $H(x, y) = \Phi(x - y)$ for $y \in \partial U$, holds since

$$|x - y| = |\tilde{x} - y| \quad \text{for} \quad y \in \partial U \ .$$

For $x \in U, y \in \partial U$ one obtains the following:

$$\begin{aligned} K(x, y) &= D_{y_3} G(x, y) \\ &= \frac{1}{4\pi}\Big(\frac{x_3 - y_3}{|x - y|^3} - \frac{\tilde{x}_3 - y_3}{|\tilde{x} - y|^3}\Big) \\ &= \frac{1}{2\pi} \frac{x_3}{|x - y|^3} \quad (\text{since } -\bar{x}_3 = x_3) \end{aligned}$$

The function $K(x, y)$ is called the Poisson kernel for the half–space $U = \{x \in \mathbb{R}^3 \ : \ x_3 > 0\}$.

If the functions $f : U \to \mathbb{R}$ and $g : \partial U \to \mathbb{R}$ are sufficiently regular, then one can prove that the Dirichlet problem (2.35) is solved by (2.36).

The function u given by (2.36) consists of two parts, $u = u_f + u_g$, where

$$\begin{aligned} u_f(x) &= \int_U G(x, y) f(y) \, dy \\ u_g(x) &= \int_{\partial U} K(x, y) g(y) \, dS(y) \end{aligned}$$

We consider the function u_f first.

Lemma 2.5 *Let $f \in C_c^2(U)$ and define*

$$u(x) = \int_U G(x,y) f(y)\, dy, \quad x \in \bar{U} \ .$$

Then $u(x) = 0$ for $x \in \partial U$ and $-\Delta u = f$ in U.

Proof: a) For $x \in \partial U$ we have $x = \tilde{x}$, thus $G(x,y) = 0$ for $x \in \partial U$ and $y \in U$. This yields that $u(x) = 0$ for $x \in \partial U$.

b) Extend f to be zero outside U. We have $u(x) = u_1(x) - u_2(x)$ with

$$\begin{aligned} u_1(x) &= \int \Phi(x-y) f(y)\, dy \\ u_2(x) &= \int \Phi(\tilde{x}-y) f(y)\, dy \\ u_2(\tilde{x}) &= u_1(x) \ . \end{aligned}$$

We have shown earlier that $-\Delta u_1(x) = f(x)$ in $\mathbb{R}^3$. Also,

$$-\Delta u_2(x) = -\Delta u_1(\tilde{x}) = f(\tilde{x}) = 0 \ .$$

This yields that $-\Delta u(x) = f(x)$ for $x \in U$. $\diamond$

In the next theorem, we consider only the part u_g generated by the boundary data g.

Theorem 2.8 *Recall that $U = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \ : \ x_3 > 0\}$ denotes the upper half–space. Let $g : \partial U \to \mathbb{R}$ be a bounded continuous function. Then the function*

$$u(x) = \int_{\partial U} K(x,y) g(y)\, dS(y), \quad x \in U \ , \tag{2.37}$$

satisfies:

a) $u \in C^\infty(U)$;

b) $\Delta u = 0$ in U;

c) if $x^0 \in \partial U$ then

$$\lim_{x \to x^0, x \in U} u(x) = g(x^0) \ . \tag{2.38}$$

Note that the formula (2.37) does not make sense for $x \in \partial U$ since

$$K(x,y) = \frac{1}{2\pi} \frac{x_3}{|x-y|^3}$$

is not defined for $x \in \partial U$, i.e., for $x_3 = 0$. It is incorrect to put $K(x,y) = 0$ for $x_3 = 0$ since the function

$$y \to \frac{1}{|x-y|^3}, \quad y \in \partial U \ ,$$

is not locally integrable if $x_3 = 0$. Therefore, the statement

$$u = g \quad \text{on} \quad \partial U$$

is made precise as in (2.38). In other words: The formula (2.37) defines a function $u(x)$ only for x in the open region U. Then one has to prove that this function $u(x), x \in U$, can be extended continuously to ∂U. The extended function is still denoted by u and satisfies $u = g$ on ∂U.

The difficulty that an integral representation of a solution is not valid on the boundary of a domain is typical for many PDE formulas.

Proof of Theorem 2.8: The kernel $K(x,y)$ does not have a singularity as x varies in U and y varies on ∂U. Therefore, it is not difficult to prove a) and to show that

$$D^\alpha u(x) = \int_{\partial U} D_x^\alpha K(x,y) g(y)\, dS(y), \quad x \in U \ .$$

For $x \neq y$ we have

$$\begin{aligned} \Delta_x \frac{1}{|x-y|} &= 0 \\ \frac{\partial}{\partial y_3} \frac{1}{|x-y|} &= \frac{x_3 - y_3}{|x-y|^3} \\ 0 &= \frac{\partial}{\partial y_3} \Delta_x \frac{1}{|x-y|} \\ &= \Delta_x \frac{\partial}{\partial y_3} \frac{1}{|x-y|} \\ &= \Delta_x \frac{x_3 - y_3}{|x-y|^3} \ . \end{aligned}$$

Since $2\pi K(x,y) = \frac{x_3 - y_3}{|x-y|^3}$ for $x \in U, y \in \partial U$, we obtain that $\Delta_x K(x,y) = 0$ for $x \in U, y \in \partial U$. This yields b).

To prove c) we first show the following.

Lemma 2.6 *For all $x \in U$ we have*

$$\int_{\partial U} K(x,y)\, dS(y) = 1 \ .$$

Remark: The result of the Lemma is plausible for the following reason: The Dirichlet problem

$$-\Delta u = 0 \quad \text{in} \quad U, \quad u = 1 \quad \text{on} \quad \partial U$$

has the solution $u \equiv 1$.

Proof of Lemma 2.6: We must prove that

$$I := \int_{\mathbb{R}^2} \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2 \Big)^{-3/2} dy = \frac{2\pi}{x_3} \ .$$

We have

$$\begin{aligned} I &= \int_{\mathbb{R}^2} (y_1^2 + y_2^2 + x_3^2)^{-3/2}\, dy \\ &= 2\pi \int_0^\infty r(r^2 + x_3^2)^{-3/2}\, dr \quad (\text{substitute } r = x_3\rho) \\ &= 2\pi x_3^2 \int_0^\infty \rho(x_3^2\rho^2 + x_3^2)^{-3/2}\, d\rho \\ &= \frac{2\pi}{x_3} \int_0^\infty \rho(\rho^2+1)^{-3/2}\, d\rho \end{aligned}$$

Using the substitution

$$\rho^2 + 1 = v, \quad dv = 2\rho d\rho \ ,$$

we obtain

$$\begin{aligned} \int_0^\infty \rho(\rho^2+1)^{-3/2}\, d\rho &= \frac{1}{2}\int_1^\infty v^{-3/2}\, dv \\ &= -v^{-1/2}\Big|_1^\infty \\ &= 1 \end{aligned}$$

This proves the lemma. $\diamond$

We continue the proof of Theorem 2.8. Let $x^0 \in \partial U$ be fixed. Using the lemma, we have for all $x \in U$,

$$|u(x) - g(x^0)| \le \int_{\partial U} K(x,y)|g(y) - g(x^0)|\, dS(y) =: Int \ .$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ with

$$|g(y) - g(x^0)| < \varepsilon \quad \text{if} \quad |y - x^0| < \delta, \quad y \in \partial U \ .$$

In the following, let $x \in U$ and let $|x - x^0| < \delta/2$. (A further restriction, $|x - x^0| < \delta_1$, will be required below.)

We split the integration domain of Int into

$$M_\delta = \{y \in \partial U : |y - x^0| < \delta\}$$

and

$$\partial U \setminus M_\delta = \{y \in \partial U : |y - x^0| \ge \delta\} \ .$$

Denote

$$\begin{aligned} Int_1 &= \int_{M_\delta} K(x,y)|g(y) - g(x^0)|\, dS(y) \\ Int_2 &= \int_{\partial U \setminus M_\delta} K(x,y)|g(y) - g(x^0)|\, dS(y) \end{aligned}$$

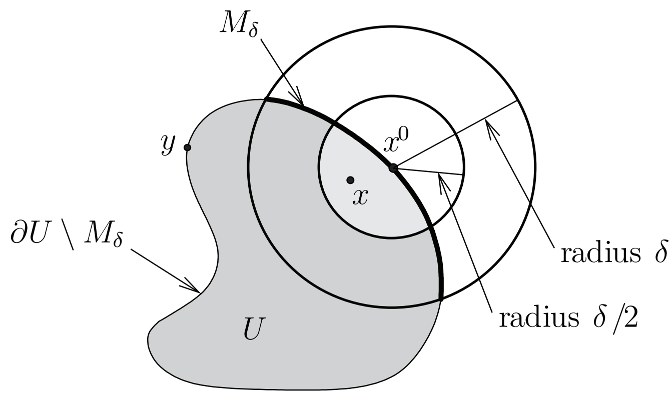

Figure 11: Region U from Theorem 2.8

For $y \in M_\delta$ we have $|g(y) - g(x^0)| < \varepsilon$, which yields

$$Int_1 < \varepsilon \ .$$

To estimate Int_2, we use the trivial bound $|g(y) - g(x^0)| \le 2|g|_\infty$, but are more careful when estimating the kernel $K(x,y)$.

Recall that $|x - x^0| < \frac{\delta}{2}$. For $y \in \partial U \setminus M_\delta$ we have

$$\begin{aligned} |y - x| &= |y - x^0 + x^0 - x| \\ &\ge |y - x^0| - |x^0 - x| \\ &\ge |y - x^0| - \frac{\delta}{2} \\ &\ge \frac{1}{2}\,|y - x^0| \end{aligned}$$

Therefore,

$$K(x,y) = \frac{x_3}{2\pi}\,\frac{1}{|x-y|^3} \le \frac{x_3}{2\pi}\,\frac{8}{|x^0 - y|^3} \ .$$

It follows that

$$Int_2 \le 2|g|_\infty \frac{x_3}{2\pi} \cdot 8 \cdot Int_3$$

with

$$\begin{aligned} Int_3 &= \int_{\partial U \setminus M_\delta} \frac{dy_1 dy_2}{\Big((x_1^0 - y_1)^2 + (x_2^0 - y_2)^2\Big)^{3/2}} \\ &= \int_{|z| \ge \delta} \frac{dz_1 dz_2}{|z|^3} \\ &= 2\pi \int_\delta^\infty r^{-2}\, dr \end{aligned}$$

$$= \frac{2\pi}{\delta}$$

This proves that

$$Int_2 \le \frac{16\,|g|_\infty\, x_3}{\delta} \ .$$

Together with the estimate for Int_1 we have shown that

$$Int \le \varepsilon + \frac{16\,|g|_\infty\, x_3}{\delta}$$

for all $x \in U$ with $|x - x^0| < \frac{\delta}{2}$. Here $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ are fixed. There exists $\delta_1 = \delta_1(\varepsilon) > 0$ with

$$\frac{16|g|_\infty \delta_1}{\delta} < \varepsilon \ .$$

Then, if $x \in U$ and

$$|x - x^0| < \min\{\delta_1, \delta/2\} \ ,$$

we have $0 < x_3 < \delta_1$ and therefore

$$|u(x) - g(x^0)| \le Int < 2\varepsilon \ .$$

This proves (2.38). $\diamond$

Decay of $u(x)$ as $|x| \to \infty$. If $g \equiv 1$, then the solution $u(x)$ given by (2.37) is $u \equiv 1$, a function that does not decay as $|x| \to \infty$. Now assume that $g \in C_c(\partial U)$, i.e., there exists $R > 0$ with

$$g(y) = 0 \quad \text{for} \quad |y| > R \ .$$

For $|y| < R$ and $|x| > 2R$ we have

$$|x - y| \ge |x| - |y| \ge \frac{1}{2}\,|x| \ ,$$

and therefore

$$K(x,y) \le \frac{x_3}{2\pi}\,\frac{8}{|x|^3} \le \frac{4}{\pi}\,\frac{1}{|x|^2} \ .$$

This implies the decay estimate

$$|u(x)| \le \frac{4}{\pi |x|^2} \int_{\partial U} |g(y)|\, dS(y) \quad \text{for} \quad |x| > 2R \ .$$

Uniqueness of decaying solutions. The homogeneous Dirichlet problem

$$\Delta w = 0 \quad \text{in} \quad U, \quad w = 0 \quad \text{on} \quad \partial U \ , \tag{2.39}$$

has many nontrivial solutions. For example,

$$w = x_3(x_1^2 - x_2^2) \quad \text{and} \quad w = x_3 e^{x_1} \cos x_2$$

are solutions. We claim, however, the following uniqueness result: If $w \in C^2(U) \cap C(\bar{U})$ solves (2.39) and

$$|w(x)| \to 0 \quad \text{as} \quad |x| \to \infty \ ,$$

then $w \equiv 0$.

Proof: Suppose that w attains a positive maximum at the point $x \in U$. (If $w \le 0$ then consider $-w$.) For $0 < R < x_3$ we have, by the mean–value theorem for harmonic functions,

$$w(x) = \frac{1}{area(\partial B_R(x))} \int_{\partial B_R(x)} w(y)\, dS(y) \ .$$

Since $w(y) \le w(x)$ for all y, this implies that $w(y) = w(x)$ for all $y \in \partial B_R(x)$. As $R \to x_3$ we obtain a contradiction to the boundary condition, $w(y) = 0$, for all $y \in \partial U$. This contradiction proves that $w \equiv 0$.

It is now easy to show the following result:

Lemma 2.7 *Recall that $U = \mathbb{R}^3_+$ denotes the upper half–space. Consider the Dirichlet problem*

$$\Delta u = 0 \quad in \quad U, \quad u = g \quad on \quad \partial U \ ,$$

where $g \in C_c(\partial U)$. The function

$$u(x) = \int_{\partial U} K(x,y) g(y)\, dS(y), \quad x \in U \ ,$$

solves this problem, and $u(x)$ is the only solution of the problem with $|u(x)| \to 0$ as $|x| \to \infty$.

2.10 The Green's Function for the Unit Ball

Let $U = B_1 = \{x \in \mathbb{R}^3 : |x| < 1\}$ denote the open unit ball in $\mathbb{R}^3$. We try to find a Green's function for the Laplace operator in the form

$$G(x,y) = \Phi(x-y) - H(x,y)$$

where $\Delta_y H(x,y) = 0$ for $x, y \in U$ and $H(x,y) = \Phi(x-y)$ for $x \in U, y \in \partial U$.

For $x \in U, x \ne 0$, define the reflected point

$$\tilde{x} = \frac{x}{|x|^2} \ .$$

Note that $|\tilde{x}| = \frac{1}{|x|} > 1$. We claim that we can take

$$H(x,y) = \Phi(|x|(\tilde{x} - y)), \quad x \in U, \quad y \in \bar{U} \ .$$

Since $\Delta_z \Phi(z) = 0$ for all $z \ne 0$, it is easy to see that $\Delta_y H(x,y) = 0$ for $y \ne \tilde{x}$.

Next we show the second condition, $H(x,y) = \Phi(x-y)$ for $x \in U$ and $y \in \partial U$.

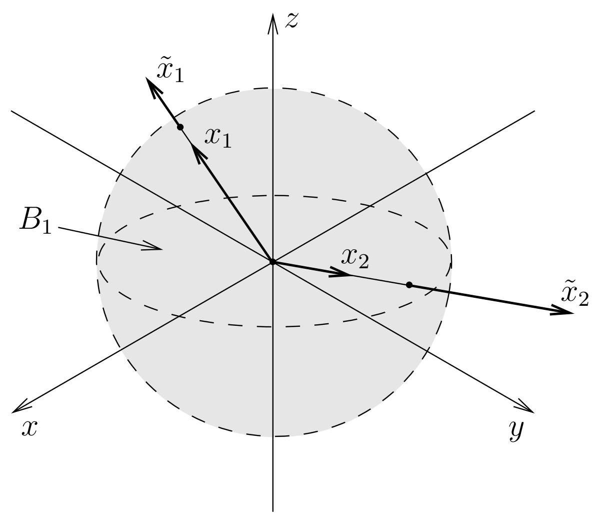

Figure 12: Reflected point $\tilde{x}$

Lemma 2.8 *If $|x| < 1$ and $|y| = 1$ then*

$$|x||\tilde{x} - y| = |x - y| \ , \tag{2.40}$$

and therefore $H(x,y) = \Phi(x-y)$ for $x \in U$ and $y \in \partial U$.

Proof: We have

$$\begin{aligned} |x|^2|\tilde{x} - y|^2 &= |x|^2\Big(|\tilde{x}|^2 - 2\tilde{x}\cdot y + 1\Big) \\ &= 1 - 2x\cdot y + |x|^2 \\ &= |x-y|^2 \end{aligned}$$

This proves the lemma. $\diamond$

Remark: For $x = 0$ the reflected point $\tilde{x} = x/|x|^2$ is not defined. Therefore, the function $H(x,y) = \Phi(|x|(\tilde{x} - y))$ is not defined for $x = 0$. However, this singular behavior of the expression $\Phi(|x|(\tilde{x}-y))$ at $x = 0$ is harmless. For $x \neq 0$ and $y \neq \tilde{x}$ we have

$$\begin{aligned} \Phi(|x|(\tilde{x}-y)) &= \frac{1}{4\pi}\,\frac{1}{|x|}\,\frac{1}{\left|\frac{x}{|x|^2} - y\right|} \\ &= \frac{1}{4\pi}\,\frac{1}{\left|\frac{x}{|x|} - |x|y\right|} \end{aligned}$$

As $x \to 0$ this expression converges to $\frac{1}{4\pi}$. We therefore define $H(0,y) = \frac{1}{4\pi}$ and note that the conditions $\Delta_y H(0,y) = 0$ and $H(0,y) = \Phi(0-y)$ for $|y| = 1$ are satisfied. In the following, the harmless singular behavior in the expression $H(x,y) = \Phi(|x|(\tilde{x}-y))$ at $x = 0$ will be ignored.

Computation of the Poisson kernel for $U = B_1$. We have

$$K(x,y) = -\frac{\partial G(x,y)}{\partial n(y)} \quad \text{for} \quad x \in U \quad \text{and} \quad y \in \partial U \ ,$$

and $n(y) = y$ for $|y| = 1$. Since

$$G(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|} - \frac{1}{4\pi} \frac{1}{|x|} \frac{1}{|\tilde{x}-y|}$$

we obtain

$$D_{y_j} G(x,y) = \frac{1}{4\pi} \Big(\frac{x_j - y_j}{|x-y|^3} - \frac{1}{|x|} \frac{\tilde{x}_j - y_j}{|\tilde{x}-y|^3} \Big) \ .$$

For all y with $|y| = 1$ we obtain

$$\begin{aligned} \frac{\partial G(x,y)}{\partial n(y)} &= y \cdot \nabla_y G(x,y) \\ &= \frac{1}{4\pi} \Big(\frac{x \cdot y - 1}{|x-y|^3} - \frac{1}{|x|} \frac{x \cdot y / |x|^2 - 1}{|\tilde{x}-y|^3} \Big) \\ &= \frac{1}{4\pi} \Big(\frac{x \cdot y - 1}{|x-y|^3} - \frac{x \cdot y - |x|^2}{|x|^3 |\tilde{x}-y|^3} \Big) \\ &= \frac{1}{4\pi} \frac{1}{|x-y|^3} \Big(x \cdot y - 1 - x \cdot y + |x|^2 \Big) \end{aligned}$$

In the last equation we have used (2.40).

From the above expression for $\partial G(x,y)/\partial n(y)$, we obtain the Poisson kernel,

$$K(x,y) = \frac{1 - |x|^2}{4\pi |x-y|^3} \quad \text{for} \quad |x| < 1 \quad \text{and} \quad |y| = 1 \ .$$

Similarly as in the half–space case, one can prove the following result:

Theorem 2.9 *Let $g \in C(\partial B_1)$. Then the function*

$$u(x) = \int_{\partial B_1} K(x,y) g(y) \, dS(y), \quad x \in B_1 \ ,$$

satisfies:

1. $u \in C^\infty(B_1)$ *;*
2. $\Delta u = 0$ *in* B_1*;*
3. *for all* $x^0 \in \partial B_1$ *we have*
$$\lim_{x \to x_0, x \in B^1} u(x) = g(x^0) \ .$$

Furthermore, if a function $v \in C^2(B_1) \cap C(\bar{B}_1)$ satisfies

$$\Delta v = 0 \quad in \quad B_1, \quad v = g \quad on \quad \partial B_1 \ ,$$

then $v = u$.

2.11 Symmetry of the Green's Function

We know from matrix theory that real symmetric matrices have many special properties. For example, their eigenvalues are real and can be characterized by variational properties. Also, the matrices can be diagonalized by an orthogonal transformation. Similar results hold for many integral operators with *symmetric* kernels, i.e., for operators T of the form

$$(Tf)(x) = \int_U G(x,y) f(y)\, dy \quad \text{with} \quad G(x,y) = G(y,x) \ .$$

In the two examples that we have considered it is easy to check directly (i.e., by using the expression derived) that the Green's function $G(x,y)$ is symmetric, i.e.,

$$G(x,y) = G(y,x) \quad \text{for} \quad x, y \in U, \quad x \neq y \ .$$

To check this, we only have to show that $H(x,y) = H(y,x)$.

1) For $U = \mathbb{R}^3_+$ we have

$$4\pi H(x,y) = \frac{1}{|\tilde{x} - y|} = \Big((x_1 - y_1)^2 + (x_2 - y_2)^2 + (-x_3 - y_3)^2 \Big)^{-1/2}$$

and

$$4\pi H(y,x) = \frac{1}{|\tilde{y} - x|} = \Big((y_1 - x_1)^2 + (y_2 - x_2)^2 + (-y_3 - x_3)^2 \Big)^{-1/2}$$

We see that $H(x,y) = H(y,x)$, and therefore $G(x,y) = G(y,x)$.

2) For $U = B_1$ we have

$$\begin{aligned} 4\pi H(x,y) &= \frac{1}{|x|} \frac{1}{\left| \frac{x}{|x|^2} - y \right|} \\ &= \frac{1}{\left| \frac{x}{|x|} - |x| y \right|} \\ &=: \frac{1}{|z_1|} \end{aligned}$$

and, similarly,

$$\begin{aligned} 4\pi H(y,x) &= \frac{1}{\left| \frac{y}{|y|} - |y| x \right|} \\ &=: \frac{1}{|z_2|} \end{aligned}$$

Here we have

$$\begin{aligned} |z_1|^2 &= \Big(\frac{x}{|x|} - |x| y \Big) \cdot \Big(\frac{x}{|x|} - |x| y \Big) \\ &= 1 - 2 x \cdot y + |x|^2 |y|^2 \end{aligned}$$

This expression is symmetric in x and y. Therefore, $|z_1| = |z_2|$, which implies $H(x,y) = H(y,x)$ and $G(x,y) = G(y,x)$.

Let us show that the symmetry of $G(x,y)$ in theses examples is not accidental, but is implied by the symmetry of the operator $Lu = -\Delta u$ under homogeneous Dirichlet boundary conditions.

For functions f, h defined on a domain U one defines the L^2 inner product by

$$(f,h) = \int f(x)h(x)\,dx\ .$$

The integration domain is always U.

Then, if u, v are smooth functions on U with $u = v = 0$ on ∂U, we obtain through integration by parts,

$$\begin{aligned}(-\Delta u, v) &= -\int (\Delta u)v\,dx \\ &= \sum_j \int (D_j u)(D_j v)\,dx \\ &= -\int u\Delta v\,dx \\ &= (u, -\Delta v)\end{aligned}$$

The equality $(-\Delta u, v) = (u, -\Delta v)$ expresses the symmetry of the operator $L = -\Delta$ on the space of functions satisfying homogeneous Dirichlet boundary conditions.

Let us show that this symmetry of $-\Delta$ implies symmetry of the Green's function. To this end, consider the two Dirichlet problems

$$-\Delta u = f \quad \text{in} \quad U, \quad u = 0 \quad \text{on} \quad \partial U\ ,$$

and

$$-\Delta v = h \quad \text{in} \quad U, \quad v = 0 \quad \text{on} \quad \partial U\ ,$$

where f and h are given functions. The solutions are

$$u(x) = \int G(x,y)f(y)\,dy$$

and

$$v(x) = \int G(x,y)h(y)\,dy\ .$$

We have $Lu = f$, and therefore

$$(Lu, v) = \int\int G(x,y)f(x)h(y)\,dydx\ .$$

Similarly, since $Lv = h$ we obtain that

$$\begin{aligned}(u, Lv) &= \int\int G(x,y)f(y)h(x)\,dydx \\ &= \int\int G(y,x)f(x)h(y)\,dxdy\end{aligned}$$

The last equation follows by renaming of the variables x and y. Since $(Lu, v) = (u, Lv)$ we have shown that

$$\int \int \Big(G(x,y) - G(y,x) \Big) f(x) h(y) \, dy dx = 0 \ .$$

Here f and h are arbitrary smooth functions on U. It follows that $G(x,y) - G(y,x) = 0$ at all points $(x,y) \in U \times U$ where $G(x,y)$ is continuous. Therefore, $G(x,y) = G(y,x)$ for $x \neq y$.

3 The Heat Equation and Other Evolution Equations With Constant Coefficients

3.1 The Cauchy Problem for the Heat Equation

3.1.1 Solution Using the Heat Kernel

In one space dimension, the Cauchy problem for the heat equation reads

$$u_t = u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0 \; , \tag{3.1}$$

$$u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R} \; . \tag{3.2}$$

Here $f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is a given function. We claim that a solution can be written in the form [4]

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t) f(y)\, dy, \quad x \in \mathbb{R}, \quad t > 0 \; , \tag{3.3}$$

where

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \quad x \in \mathbb{R}, \quad t > 0 \; , \tag{3.4}$$

is the so–called heat kernel. The function (3.3) is called the heat kernel solution of the problem (3.1), (3.2).

Remark: It is easy to generalize to n space dimensions. Then the heat equation reads $u_t = \Delta u$, and the heat kernel is

$$\Phi(x,t) = (4\pi t)^{-n/2}\, e^{-|x|^2/4t}, \quad x \in \mathbb{R}^n, \quad t > 0 \; . \tag{3.5}$$

For simplicity, we assume $n = 1$.

Lemma 3.1 *The heat kernel (3.4) has the following properties:*

1. $\Phi \in C^\infty(\mathbb{R} \times (0,\infty))$;
2. $\Phi_t = \Phi_{xx}$ *in* $\mathbb{R} \times (0,\infty)$;
3. $\int_{-\infty}^{\infty} \Phi(x,t)\, dx = 1$ *for all* $t > 0$.
4. $$\lim_{t \to 0+} \Phi(x,t) = \begin{cases} \infty & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0 \end{cases}$$

[4] The solution of the Cauchy problem is not unique unless one puts some growth restrictions on $u(x,t)$ as $|x| \to \infty$. We will only discuss the solution given in terms of the heat kernel.

Proof: Properties 1 and 4 are obvious. Property 2 is easy to check: Set $E = E(x,t) = e^{-x^2/4t}$ and set $q(x,t) = t^{-1/2}E(x,t)$. Obtain that

$$\begin{aligned} q_t &= -\frac{1}{2}t^{-3/2}E + t^{-1/2}\frac{x^2}{4t^2}\,E \\ q_x &= t^{-1/2}E\frac{-x}{2t} \\ q_{xx} &= -\frac{1}{2}t^{-3/2}E + t^{-1/2}E\Big(\frac{x}{2t}\Big)^2 \end{aligned}$$

The equation $q_t = q_{xx}$ follows. To check 3, substitute $y = x/\sqrt{4t}$,

$$\int_{-\infty}^{\infty} e^{-x^2/4t}\,dx = \sqrt{4t}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{4\pi t}\ .$$

$\diamond$

Theorem 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and let $u(x,t)$ be defined by (3.3). Then we have:*

1. *$u \in C^\infty(\mathbb{R}\times(0,\infty))$, and all derivatives of u can be computed by differentiating the formula (3.3) under the integral sign,*

$$D_t^i D_x^j u(x,t) = \int_{-\infty}^{\infty} D_t^i D_x^j \Phi(x-y,t) f(y)\,dy, \quad x \in \mathbb{R}, \quad t > 0\ .$$

2. *$u_t = u_{xx}$ for $x \in \mathbb{R}$ and $t > 0$;*

3. *for all $x^0 \in \mathbb{R}$,*

$$\lim_{(x,t)\to(x^0,0)} u(x,t) = f(x^0)\ . \tag{3.6}$$

Proof: 1. Suppose we want to show that $u_x(x,t)$ exists and

$$u_x(x,t) = \int_{-\infty}^{\infty} D_x\Phi(x-y,t) f(y)\,dy\ . \tag{3.7}$$

This is easy if $f \in C_c(\mathbb{R})$: Just note that

$$\Phi(x+h-y,t) = \Phi(x-y,t) + h\Phi_x(x-y,t) + \mathcal{O}(h^2)\ .$$

If one only assumes that $f \in C(\mathbb{R})$ is bounded, then one must use decay of the derivatives of $\Phi(x,t)$ as $|x| \to \infty$. We carry this out below.

Once 1. is shown, the equation $u_t = u_{xx}$ follows from $\Phi_t = \Phi_{xx}$.

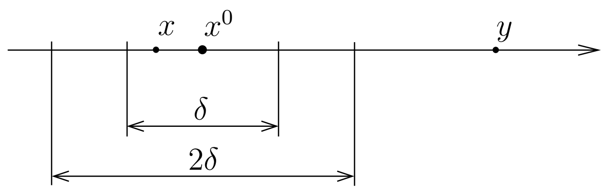

Figure 13: For proof of Theorem 3.1

Proof of (3.6): Let $\varepsilon > 0$ be given. There exists $\delta > 0$ with $|f(y)-f(x^0)| < \varepsilon$ for $|y-x^0| < \delta$. In the following, let $|x - x^0| < \delta/2$. We have

$$\begin{aligned} |u(x,t) - f(x^0)| &\leq \int \Phi(x-y,t)|f(y)-f(x^0)|\,dy \\ &= \int_{|y-x^0|<\delta} \ldots + \int_{|y-x^0|>\delta} \ldots \\ &=: I_1 + I_2 \end{aligned}$$

Here $I_1 \leq \varepsilon$, and it remains to estimate I_2. We have

$$I_2 \leq 2|f|_\infty \int_{|y-x^0|>\delta} \Phi(x-y,t)\,dy =: 2|f|_\infty J \ .$$

Since $|x - x^0| < \delta/2$ and $|y - x^0| > \delta$ we have

$$\begin{aligned} |y-x| &\geq |y-x^0| - |x-x^0| \\ &\geq |y-x^0| - \frac{\delta}{2} \\ &\geq \frac{1}{2}\,|y-x^0| \ . \end{aligned}$$

Therefore,

$$\begin{aligned} J &= (4\pi t)^{-1/2} \int_{|y-x^0|>\delta} e^{-(x-y)^2/4t}\,dy \\ &\leq (4\pi t)^{-1/2} \int_{|y-x^0|>\delta} e^{-(y-x^0)^2/16t}\,dy \\ &= (4\pi t)^{-1/2}\, 2 \int_\delta^\infty e^{-z^2/16t}\,dz \quad (\text{substitute } \xi = z/4\sqrt{t}) \\ &= C \int_{\delta/4\sqrt{t}}^\infty e^{-\xi^2}\,d\xi \end{aligned}$$

Note that $\int_r^\infty e^{-\xi^2}\,d\xi \to 0$ as $r \to \infty$. It follows that, for $|x - x^0| < \frac{\delta}{2}$,

$$|u(x,t) - f(x^0)| \leq I_1 + I_2$$

$$
\begin{aligned}
&\leq \varepsilon + 2|f|_\infty C \int_{\delta/4\sqrt{t}}^\infty e^{-\xi^2}\, d\xi \\
&\leq 2\varepsilon
\end{aligned}
$$

for $0 < t \leq \delta_1$. $\diamond$

Supplement to the proof of Theorem 3.1. Suppose we only assumes $f \in C \cap L^\infty$, but do not assume that f has compact support. To justify that one can differentiate the formula (3.3) under the integral sign, one must use that the derivatives of the heat kernel decay as $|x| \to \infty$. We carry out some details.

Lemma 3.2 *Let $G(y) = e^{-y^2}$. Then, for $n = 1, 2, \ldots$*

$$D^n G(y) = p_n(y) G(y) \quad \textit{with} \quad D = \frac{d}{dy}$$

where $p_n(y)$ is a polynomial of degree n.

Proof: Induction in n.

Lemma 3.3 *Let $G(y) = e^{-y^2}$ and let $\Phi(x,t) = (4\pi t)^{-1/2} G(x(4t)^{-1/2})$ denote the heat kernel. Then, for all $t_0 > 0$ and all $i = 0, 1, \ldots$ and all $j = 0, 1, \ldots$ there is a constant $C(i,j,t_0)$ so that*

$$|D_t^i D_x^j \Phi(x,t)| \leq C(i,j,t_0) e^{-x^2/8t} \quad \textit{for} \quad t \geq t_0 \quad \textit{and} \quad x \in \mathbb{R} \ .$$

Proof: Since $D_t \Phi = D_x^2 \Phi$ we have

$$D_t^i D_x^j \Phi = D_x^{j+2i} \Phi \ .$$

Let $n = j + 2i$. Using the previous lemma, we can write

$$D_x^n \Phi(x,t) = (4\pi t)^{-1/2} (4t)^{-n/2} p_n(y) G(y) \quad \text{with} \quad y = x(4t)^{-1/2} \ .$$

Since

$$|p_n(y) G(y)| \leq C_n (1 + |y|^n) e^{-y^2} \leq C_n' e^{-y^2/2}$$

the claim follows. $\diamond$

A generally useful mathematical tool is Taylor's formula with remainder in integral form, which can be obtain through integration by parts. We carry this out to lowest order only.

Lemma 3.4 *For $q \in C^2[0,1]$ we have*

$$q(1) = q(0) + q'(0) + \int_0^1 (1-s) q''(s)\, ds \ .$$

Proof: We have

$$\begin{aligned} q(1) - q(0) &= \int_0^1 q'(s)\, ds \\ &= q'(s)(s-1)\Big|_0^1 - \int_0^1 (s-1) q''(s)\, ds \\ &= q'(0) + \int_0^1 (1-s) q''(s)\, ds \end{aligned}$$

$\diamond$

If $\phi \in C^2[0,h]$ and one sets $q(s) = \phi(sh), 0 \le s \le 1$, then one obtains

$$\phi(h) = \phi(0) + h\phi'(0) + h^2 \int_0^1 (1-s)\phi''(sh)\, ds \ .$$

This is Taylor's formula with remainder in integral form.

If we apply the formula to

$$\phi(h) = \Phi(x+h-y,t) \ ,$$

then we obtain

$$\Phi(x+h-y,t) = \Phi(x-y,t) + hD_x\Phi(x-y,t) + h^2 \int_0^1 (1-s) D_x^2 \Phi(x+sh-y,t)\, ds \ .$$

Therefore,

$$u(x+h,t) - u(x,t) = h \int_{-\infty}^{\infty} D_x \Phi(x-y,t) f(y)\, dy + h^2 R(x,t,h) \tag{3.8}$$

with

$$R(x,t,h) = \int_{-\infty}^{\infty} \int_0^1 (1-s) D_x^2 \Phi(x+sh-y,t)\, ds f(y)\, dy \ .$$

Using the bound for $D_x^2\Phi$ of Lemma 3.3, we have

$$\begin{aligned} |R(x,t,h)| &\le |f|_\infty \int_0^1 \int_{-\infty}^{\infty} |D_x^2 \Phi(x+sh-y,t)|\, dxds \\ &\le |f|_\infty C(t) \int_0^1 \int_{-\infty}^{\infty} e^{-(x+sh-y)^2/8t}\, dxds \ . \end{aligned}$$

It is now clear that $|R(x,t,h)$ remains bounded as $h \to 0$. Therefore, if we divide formula (3.8) by h and let $h \to 0$, we obtain that $D_x u(x,t)$ exists and is given by

$$u_x(x,t) = \int_{-\infty}^{\infty} D_x \Phi(x-y,t) f(y)\, dy \ .$$

The higher derivatives of $u(x,t)$ are treated in the same way.

3.1.2 Decay of the Heat Kernel Solution as $|x| \to \infty$

Assume that $f \in C_c(\mathbb{R})$, i.e., there exists $R > 0$ with $f(y) = 0$ for $|y| > R$. Let $|x| > 2R$. Then we have

$$|x-y| \geq |x| - R \geq \frac{1}{2}|x| \ ,$$

and therefore,

$$e^{-(x-y)^2/4t} \leq e^{-x^2/16t} \ .$$

For $|x| > 2R$ we obtain the solution estimate

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \, e^{-x^2/16t} \, \|f\|_{L^1} \ .$$

Thus, the solution decays rapidly as $|x| \to \infty$.

In particular, one obtains the following:

Lemma 3.5 *Let $f : \mathbb{R} \to \mathbb{R}$ denote a continuous function with compact support and let $u(x,t)$ denote the heat–kernel solution of the Cauchy problem*

$$u_t = u_{xx}, \quad u(x,0) = f(x) \ .$$

Then, for every $\varepsilon > 0$ and every $T > 0$ there exists $R = R(\varepsilon, T) > 0$ with

$$|u(x,t)| \leq \varepsilon \quad \textit{for} \quad |x| \geq R \quad \textit{and} \quad 0 \leq t \leq T \ .$$

We claim that all derivatives of $u(x,t)$ also decay rapidly as $|x| \to \infty$. For example, we have

$$|u_x(x,t)| \leq \int_{-R}^{R} |\Phi_x(x,t)||f(y)| \, dy \ ,$$

where

$$\Phi_x(x,t) = \frac{1}{\sqrt{4\pi t}} \, \frac{x}{4t} \, e^{-x^2/4t} \ .$$

For $|x| > 2R$ we obtain, as above,

$$|u_x(x,t)| \leq \frac{1}{\sqrt{4\pi t}} \, \frac{|x|}{4t} \, e^{-x^2/16t} \ .$$

This proves that $|u_x(x,t)|$ tends to zero rapidly as $|x| \to \infty$.

3.1.3 Uniqueness of Decaying Solutions Via Maximum Principle

We consider the Cauchy problem

$$u_t = u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0; \quad u(x,0) = f(x) \quad \text{for} \quad x \in \mathbb{R} \ ,$$

where $f \in C_c(\mathbb{R})$. Let $u(x,t) = \int \Phi(x-y,t) f(y)\, dy$ denote the heat kernel solution and let $v(x,t)$ denote any solution which satisfies the decay property formulated in Lemma 3.5. We claim that $v = u$.

To show this, set $w = u - v$. Clearly,

$$w_t = w_{xx}, \quad w(x,0) = 0 \ ,$$

and w has the decay property formulated in Lemma 3.5. Set

$$q(x,t) = e^{-t} w(x,t) \ .$$

It is easy to show that

$$q_t = q_{xx} - q$$

and q has the decay property formulated in Lemma 3.5. Suppose that q is not identically zero. There exists $x_1 \in \mathbb{R}$ and $T > 0$ with $q(x_1, T) \neq 0$. There exists $R > 0$ with

$$|q(x,t)| < |q(x_1,T)| \quad \text{for} \quad |x| \geq R \quad \text{and} \quad 0 \leq t \leq T \ .$$

Therefore, in the region

$$\mathbb{R} \times [0,T]$$

the function $|q(x,t)|$ attains a maximum. Let the maximum be attained at (x_0, t_0) where $0 < t_0 \leq T$ and assume that

$$q(x_0, t_0) > 0 \ .$$

(If $q(x_0,t_0) < 0$ then consider $-q$.)

We have $q_{xx}(x_0,t_0) \leq 0$ and the equation

$$q_t = q_{xx} - q$$

yields that

$$q_t(x_0,t_0) < 0 \ .$$

However, this implies that

$$q(x_0, t - \varepsilon) > q(x_0, t_0) \quad \text{for} \quad 0 < \varepsilon << 1 \ .$$

This contradiction proves that $q \equiv 0$, i.e., $v = u$.

Remarks on Non–Uniqueness. Consider the Cauchy problems

$$u_t = u_{xx}, \quad u(x,0) = 0 \ .$$

If one does not imposes any growth restrictions on u as $|x| \to \infty$, then one cannot conclude that $u = 0$. An example is given by Fritz John:

Let $\alpha > 1$ and define the function

$$g(t) = \begin{cases} 0 & \text{for} \quad t \le 0 \\ \exp(-t^{-\alpha}) & \text{for} \quad t > 0 \end{cases}$$

One can show that $g \in C^\infty(\mathbb{R})$.

For $(x,t) \in \mathbb{R}^2$ set

$$\begin{aligned} u(x,t) &= \sum_{j=0}^\infty \frac{g^{(j)}(t)}{(2j)!}\, x^{2j} \\ &= g(t) + \frac{1}{2}\, g'(t)x^2 + \frac{1}{4!}\, g''(t)x^4 + \frac{1}{6!}\, g'''(t)x^6 + \ldots \end{aligned}$$

One can show that $u \in C^\infty(\mathbb{R}^2)$ and all derivatives can be applied term by term.

One obtains that

$$\begin{aligned} u_t &= g'(t) + \frac{1}{2}\, g''(t)x^2 + \frac{1}{4!}\, g'''(t)x^4 + \ldots \\ u_{xx} &= 0 + g'(t) + \frac{1}{2}\, g''(t)x^2 + \frac{1}{4!}\, g'''(t)x^4 + \ldots \end{aligned}$$

It follows that $u_t = u_{xx}$ and $u(x,t) = 0$ for $t \le 0$. Clearly, $u(0,t) = g(t) > 0$ for $t > 0$.

3.1.4 Derivation of the Heat Kernel Using Fourier Transformation

Let $\mathcal{S}$ denote the Schwartz space, i.e., $\mathcal{S}$ consists of all functions $f : \mathbb{R} \to \mathbb{C}$ which are in C^∞ and which satisfy

$$|x|^N |D^j f(x)| \le C_{N,j}, \quad x \in \mathbb{R} \ ,$$

for all $N = 1, 2, \ldots$ and all $j = 0, 1, \ldots$. The functions $f \in \mathcal{S}$ are smooth and all their derivatives are rapidly decaying.

For $f \in \mathcal{S}$ the Fourier transform is

$$\hat{f}(k) = c \int e^{-ikx} f(x)\, dx, \quad k \in \mathbb{R} \ ,$$

where

$$c = \frac{1}{\sqrt{2\pi}} \ .$$

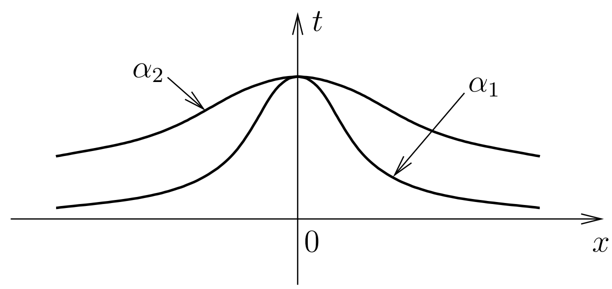

Figure 14: Gaussian $e^{-\alpha x^2}$ for $0 < \alpha_2 < \alpha_1$

In this section, the constant c always equals $c = 1/\sqrt{2\pi}$.

One can prove that $\hat{f} \in \mathcal{S}$ and

$$f(x) = c \int e^{ikx} \hat{f}(k)\, dk, \quad x \in \mathbb{R} \ . \tag{3.9}$$

The formula (3.9) is called the Fourier representation of f.

The next lemma says that the Fourier transform of a Gaussian is again a Gaussian.

Lemma 3.6 *Let $f(x) = e^{-\alpha x^2}$ with $\alpha > 0$. Then we have*

$$\hat{f}(k) = c\sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha} \ .$$

In particular, the Fourier transform of $e^{-x^2/2}$ is $e^{-k^2/2}$.

Proof: Let

$$g(k) = \hat{f}(k) = c \int e^{-ikx} e^{-\alpha x^2}\, dx \ ,$$

thus

$$g'(k) = -ic \int e^{-ikx} x e^{-\alpha x^2}\, dx \ .$$

Note that

$$f'(x) = -2\alpha x f(x), \quad x f(x) = -\frac{1}{2\alpha}\, f'(x) \ .$$

This yields

$$\begin{aligned} g'(k) &= \frac{ic}{2\alpha} \int e^{-ikx} f'(x)\, dx \\ &= -\frac{ic}{2\alpha}\, (-ik) \int e^{-ikx} f(x)\, dx \\ &= -\frac{k}{2\alpha}\, g(k) \end{aligned}$$

Also,

$$g(0) = c\sqrt{\frac{\pi}{\alpha}} \; .$$

Applying the ODE method of separation of variables to the differential equation $g'(k) = -\frac{k}{2\alpha}\, g(k)$, we obtain that

$$\log\Big(\frac{g(k)}{g(0)}\Big) = -\frac{k^2}{4\alpha}\, g(k) \; ,$$

thus

$$g(k) = c\sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha} \; .$$

◇

Lemma 3.7 *Let $\hat{f}(k) = e^{-k^2 t}$ where $t > 0$. Then we have*

$$f(x) = \frac{1}{c}\, \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t} \; .$$

Proof: Consider

$$f(x) = \beta e^{-\alpha x^2}, \quad \alpha > 0 \; .$$

By the previous lemma,

$$\hat{f}(k) = \beta c \sqrt{\frac{\pi}{\alpha}}\, e^{-k^2/4\alpha} \; .$$

We choose α and β so that

$$\beta c \sqrt{\frac{\pi}{\alpha}} = 1, \quad \frac{1}{4\alpha} = t \; .$$

This yields

$$\alpha = \frac{1}{4t}, \quad \beta = \frac{1}{c}\, \frac{1}{\sqrt{4\pi t}} \; .$$

◇

Consider the initial value problem

$$u_t = u_{xx}, \quad u(x,0) = f(x) \; .$$

We Fourier transform in the x–variable and obtain

$$\hat{u}_t(k,t) = -k^2 \hat{u}(k,t), \quad \hat{u}(k,0) = \hat{f}(k) \; .$$

Thus, we have obtained a simple ODE initial value problem for each wave number k. Clearly,

$$\hat{u}(k,t) = e^{-k^2 t} \hat{f}(k), \quad k \in \mathbb{R} \; .$$

Using the definition of the Fourier transform, and its inverse, we have

$$\begin{aligned} u(x,t) &= c\int e^{ikx-k^2t}\hat{f}(k)\,dk \\ &= c^2\int\int e^{ikx-k^2t-iky}f(y)\,dydk \\ &= c^2\int\int e^{ik(x-y)-k^2t}\,dkf(y)\,dy \\ &= \int\Phi(x-y,t)f(y)\,dy \end{aligned}$$

where

$$\begin{aligned} \Phi(z,t) &= c^2\int e^{ikz}e^{-k^2t}\,dk \\ &= c\Big(c\int e^{ikz}e^{-k^2t}\,dk\Big) \\ &= \frac{1}{\sqrt{4\pi t}}e^{-z^2/4t} \end{aligned}$$

In the last equation we have used the previous lemma. We have computed the heat kernel via Fourier transformation.

3.2 Evolution Equations with Constant Coefficients: Well–Posed and Ill–Posed Problems

We consider general constant coefficient evolution equations $u_t = Pu$ with initial condition $u(x,0) = f(x)$. We will define when the problem is well–posed and when it is ill–posed. For simplicity, we will assume that $f(x)$ is a 2π–periodic trigonometric polynomial. Then one can always solve the problem $u_t = Pu, u(x,0) = f(x)$, even if it is ill–posed.

The crucial question is if the solution operator $S_0(t)$ is bounded for $t \geq 0$ or not. In the bounded case, the problem is well–posed, in the unbounded case it is ill–posed.

3.2.1 Solution via Fourier Expansion

Consider an evolution equation

$$u_t = Pu$$

where $u = u(x,t)$ is an unknown function and P is a spatial differential operator of the general form

$$Pu = \sum_{j=0}^{m} a_j D^j u, \quad D = \partial/\partial x\ .$$

Here the a_j are constant numbers, which are real or complex.

Examples:

$$\begin{aligned} u_t &= u_{xx}, \quad P = D^2 \ , \\ u_t &= -u_{xx}, \quad P = -D^2 \ , \\ u_t &= iu_x + u_{xxx}, \quad P = iD + D^3 \ . \end{aligned}$$

We consider the equation $u_t = Pu$ together with an initial condition

$$u(x,0) = f(x) \ .$$

Let us first assume that

$$f(x) = e^{ikx} = \cos(kx) + i\sin(kx)$$

where k is an integer, the wave number of the function $f(x)$.

We try to obtain a solution in the form

$$u(x,t) = a(t)e^{ikx}$$

where $a(t)$ is an amplitude that needs to be determined. Note that

$$De^{ikx} = ike^{ikx}, \quad D^2e^{ikx} = (ik)^2 e^{ikx} \ ,$$

etc. This yields

$$Pe^{ikx} = \hat{P}(ik)e^{ikx}$$

where

$$\hat{P}(ik) = \sum_{j=0}^{m} a_j (ik)^j$$

is the so–called symbol of the differential operator P. The symbol $\hat{P}(ik)$ is obtained from P by formally replacing D with ik. Note that the function $k \to \hat{P}(ik)$ is a polynomial of degree m.

The function $u(x,t) = a(t)e^{ikx}$ solves the equation $u_t = Pu$ if and only if

$$a'(t) = \hat{P}(ik)a(t) \ .$$

Also, the initial condition $u(x,0) = e^{ikx}$ requires $a(0) = 1$. One obtains that

$$a(t) = e^{\hat{P}(ik)t}, \quad u(x,t) = e^{\hat{P}(ik)t}\, e^{ikx} \ .$$

If $f(x)$ is a trigonometric polynomial,

$$f(x) = \sum_{k=-M}^{M} b_k e^{ikx} \ , \tag{3.10}$$

then the solution of

$$u_t = Pu, \quad u(x,0) = f(x) \ ,$$

is obtained by superposition,

$$u(x,t) = \sum_{k=-M}^{M} b_k e^{\hat{P}(ik)t}\, e^{ikx} \ . \tag{3.11}$$

3.2.2 The Operator Norm of the Solution Operator

To measure the solution at any fixed time t, we introduce the L^2 inner product and norm of 2π–periodic functions by

$$(u,v) = \int_0^{2\pi} \bar{u}(x)v(x)\,dx, \quad \|u\| = (u,u)^{1/2}, \quad u,v \in L^2(0,2\pi) \ .$$

The sequence of functions

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx}, \quad k \in \mathbb{Z} \ ,$$

forms an orthonormal system in $L^2(0,2\pi)$.

Let $\mathcal{T}$ denote the space of all 2π–perioic trigonometric polynomials, i.e., $\mathcal{T}$ consists of all functions $f(x)$ that have the form (3.10) for some finite M.

If $f \in \mathcal{T}$ then we can write

$$f(x) = \sum_{k=-M}^{M} \hat{f}(k)\phi_k(x) \tag{3.12}$$

where

$$\hat{f}(k) = (\phi_k, f), \quad k \in \mathbb{Z} \ ,$$

are the Fourier coefficients of f. It is easy to show that Parseval's relation holds,

$$\|f\|^2 = \sum_k |\hat{f}(k)|^2 \ .$$

The solution formula (3.11) takes the form

$$u(x,t) = \sum_k \hat{u}(k,t)\phi_k(x) \tag{3.13}$$

$$= \sum_k e^{\hat{P}(ik)t}\hat{f}(k)\phi_k(x) \tag{3.14}$$

For every fixed real t (positive, negative, or zero) define the solution operator $S_0(t) : \mathcal{T} \to \mathcal{T}$ by

$$S_0(t)f = u(\cdot, t) \ .$$

Thus, $S_0(t)$ maps the initial function f to the solution at time t. Since

$$e^{\hat{P}(ik)(s+t)} = e^{\hat{P}(ik)s}e^{\hat{P}(ik)t} \quad \text{for all} \quad s,t \in \mathbb{R} \ ,$$

it is easy to see that

1. $S_0(0) = id$;
2. $S_0(s+t) = S_0(s)S_0(t)$ for all real s,t.

Definition 3.1 *Let $(X, \|\cdot\|)$ be a normed space. A linear operator $L : X \to X$ is called bounded if there is a constant $C \ge 0$ with*

$$\|Lu\| \le C\|u\| \quad \textit{for all} \quad u \in X \ . \tag{3.15}$$

If L is bounded then the smallest constant C with (3.15) is called the operator norm of L, denote by $\|L\|$. If L is unbounded we will write $\|L\| = \infty$.

We will apply the concept to the operators $S_0(t) : \mathcal{T} \to \mathcal{T}$, where the norm on $\mathcal{T}$ is the L^2 norm and where t is fixed.

Lemma 3.8 *We have*

$$\|S_0(t)\| = \sup_{k \in \mathbb{Z}} \left| e^{\hat{P}(ik)t} \right| \ . \tag{3.16}$$

In particular, the operator $S_0(t)$ is bounded if and only if the supremum on the right side of (3.16) is finite.

Proof:

a) Let Q denote the supremum. Let $f \in \mathcal{T}$ be given as in (3.12). Then we have

$$\begin{aligned} \|S_0(t)f\|^2 &= \sum_k |\hat{u}(k,t)|^2 \\ &= \sum_k \left| e^{\hat{P}(ik)t} \hat{f}(k) \right|^2 \\ &\le Q^2 \|f\|^2 \end{aligned}$$

Thus, if $Q < \infty$ then the operator $S_0(t)$ is bounded and $\|S_0(t)\| \le Q$.

b) Assume that $Q < \infty$ and let $\varepsilon > 0$ be arbitrary. There is a wave number k with

$$\left| e^{\hat{P}(ik)t} \right| \ge Q - \varepsilon \ .$$

Choosing $f = \phi_k(x)$ we obtain $\|S_0(t)f\| \ge (Q - \varepsilon)\|f\|$. Therefore, $\|S_0(t)\| \ge Q - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain that $\|S_0(t)\| \ge Q$. Together with a), the equality $\|S_0(t)\| = Q$ is shown.

c) Assume that $Q = \infty$. Given any positive integer n there is a wave number k with

$$\left| e^{\hat{P}(ik)t} \right| \ge n \ .$$

As in b) obtain that $\|S_0(t)f\| \ge n\|f\|$ for $f = \phi_k$. Since n is arbitray, we obtain that $\|S_0(t)\| = \infty$. $\diamond$

Examples:

1. Consider the heat equation, $u_t = u_{xx}$. Here
$$P = D^2, \quad \hat{P}(ik) = -k^2, \quad |e^{\hat{P}(ik)t}| = e^{-k^2 t} ,$$
thus
$$\|S_0(t)\| = 1 \quad \text{for all} \quad t \geq 0 .$$
2. Consider the backward heat equation, $u_t = -u_{xx}$. Here
$$P = -D^2, \quad \hat{P}(ik) = k^2, \quad |e^{\hat{P}(ik)t}| = e^{k^2 t} ,$$
thus
$$\|S_0(t)\| = \infty \quad \text{for all} \quad t > 0 .$$
3. Consider the first order equation $u_t = iu_x$. Here
$$P = iD, \quad \hat{P}(ik) = -k, \quad |e^{\hat{P}(ik)t}| = e^{-kt} ,$$
thus
$$\|S_0(t)\| = \infty \quad \text{for all} \quad t > 0 .$$
4. Consider the Schrödinger equation $u_t = iu_{xx}$. Here
$$P = iD^2, \quad \hat{P}(ik) = -ik^2, \quad |e^{\hat{P}(ik)t}| = 1 ,$$
thus
$$\|S_0(t)\| = 1 \quad \text{for all} \quad t \in \mathbb{R} .$$
5. Consider the heat equation with a lower order term, $u_t = u_{xx} + 10u$. Here
$$P = D^2 + 10, \quad \hat{P}(ik) = -k^2 + 10, \quad |e^{\hat{P}(ik)t}| = e^{(-k^2+10)t} ,$$
thus
$$\|S_0(t)\| = e^{10t} \quad \text{for all} \quad t \geq 0 .$$

3.3 Well–Posed and Ill–Posed Initial Value Problems for Constant Coefficient Operators

Definition 3.2 *The initial value problem*
$$u_t = Pu, \quad u(x,0) = f(x) ,$$
with 2π–periodic initial data $f(x)$ is called well–posed if there is a time $t_0 > 0$ and a constant $C \geq 1$ with
$$\|S_0(t)\| \leq C \quad \textit{for} \quad 0 \leq t \leq t_0 . \tag{3.17}$$
If such an estimate does not hold, then the initial value problem is called ill–posed.

Examples: We obtain form the previous computations of $\|S_0(t)\|$ that the 2π–periodic initial value problem is well pased for the equations

$$u_t = u_{xx}, \quad u_t = iu_{xx}, \quad u_t = u_{xx} + 10u \ ,$$

and ill–posed for the equations

$$u_t = -u_{xx}, \quad u_t = iu_x \ .$$

If the problem is well–posed, then the norm of $\|S_0(t)\|$ cannot grow faster than exponential for increasing t. We will show this next.

Lemma 3.9 *Assume that the 2π–periodic initial value problem for $u_t = Pu$ is well–posed. Then there are constants $\alpha \geq 0$ and $C \geq 1$ with*

$$\|S_0(t)\| \leq Ce^{\alpha t}, \quad t \geq 0 \ .$$

Proof: Let $C \geq 1$ and $t_0 > 0$ be determined as in (3.17). Given $t \geq 0$ we write

$$t = nt_0 + \tau \ ,$$

where $n = 0, 1, \ldots$ and $0 \leq \tau < t_0$. Obtain that

$$\begin{aligned} \|S_0(t)\| &\leq \|S_0(t_0)\|^n \|S_0(\tau)\| \\ &\leq C^n C \ . \end{aligned}$$

Here $n \leq t/t_0$, thus

$$C^n = e^{n \log C} \leq e^{\alpha t}$$

with

$$\alpha = \frac{\log C}{t_0} \ .$$

$\diamond$

Assume that the 2π–periodic initial value problem for $u_t = Pu$ is well–posed. If $f \in L^2(0, 2\pi)$ then define the finite Fourier sums

$$f_n(x) = \sum_{k=-n}^{n} \hat{f}(k)\phi_k(x), \quad \hat{f}(k) = (\phi_k, f) \ .$$

One can show that

$$\|f - f_n\| \to 0 \quad \text{as} \quad n \to \infty \ .$$

Therefore, for $t \geq 0$,

$$\|S_0(t)f_m - S_0(t)f_n\| \leq \|S_0(t)\| \|f_m - f_n\| \ .$$

Since $L^2(0, 2\pi)$ is a complete space w.r.t. the L^2–norm, the sequence $S_0(t)f_n$ converges. One calls

$$u(\cdot, t) = \lim_{n\to\infty} S_0(t)f_n =: S(t)f$$

the generalized solution of

$$u_t = Pu, \quad u(x, 0) = f(x) \ .$$

Thus, if the problem is well–posed, one obtains a generalized solution for initial data $f \in L^2(0, 2\pi)$. If the problem is ill–posed, then the sequence $S_0(t)f_n$ will generally not converge as $n \to \infty$.

3.4 First–Order Hyperbolic Systems in One Space Dimension

Let $u = u(x, t)$ denote a function depending on the space variable $x \in \mathbb{R}$ and the time variable t. We assume that u takes values in $\mathbb{C}^N$. If $A \in \mathbb{C}^{N\times N}$ is a given matrix, then

$$u_t + Au_x = 0$$

is a first–order system for u. Such a system is called strongly hyperbolic if all eigenvalues λ_j of A are real and there is a non–singular matrix $S \in \mathbb{C}^{N\times N}$ which diagonalizes A,

$$S^{-1}AS = \Lambda = diag(\lambda_j) \ .$$

If $u_t + Au_x = 0$ is a strongly hyperbolic system and if $S^{-1}AS = \Lambda$ is real and diagonal, then define new variables $v(x, t)$ by

$$u(x, t) = Sv(x, t) \ .$$

The system $u_t + Au_x = 0$ transforms to $v_t + \Lambda v_x = 0$, thus one obtains N decoupled scalar equations,

$$v_{jt} + \lambda_j v_{jx} = 0, \quad j = 1, \ldots, N \ .$$

An initial condition

$$u(x, 0) = f(x)$$

transforms to

$$v_j(x, 0) = g_j(x)$$

where

$$f(x) = Sg(x) = \sum_j S_j g_j(x) \ .$$

Here the S_j are the columns of S, which are the eigenvectors of A.

One obtains that

$$\begin{aligned} u(x, t) &= Sv(x, t) \\ &= \sum_j S_j v_j(x, t) \\ &= \sum_j S_j g_j(x - \lambda_j t) \end{aligned}$$

if

$$u(x,0) = \sum_j S_j g_j(x) \; .$$

Thus the eigenvalues of A are the propagation speeds for a strongly hyperbolic system $u_t + Au_x = 0$.

3.5 The 1D Wave Equation Written as a Symmetric Hyperbolic System

A first order system $u_t + Au_x = 0$ is called symmetric hyperbolic if $A = A^*$. Clearly, such a system is strongly hyperbolic. We will show here that one can write the 1D wave equation as a symmetric hyperbolic system. Then we apply the solution process of the previous section.

Consider the 1D wace equation

$$w_{tt} = c^2 w_{xx} \; .$$

Assume that $w(x,t)$ is a smooth solution and set

$$u_1 = w_t, \quad u_2 = cw_x \; .$$

Then we have

$$u_{1t} = w_{tt} = c^2 w_{xx} = cu_{2x}$$

and

$$u_{2t} = cw_{xt} = cw_{tx} = cu_{1x} \; .$$

We can write the two equations for u_1, u_2 in systems form,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x \; .$$

Thus the system has the form $u_t = Au_x$ where the matrix A equals

$$A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \; .$$

Thus A is real and symmetric with eigenvalues $\pm c$. We have

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$S^{-1}AS = diag(c, -c) \; .$$

3.5.1 Derivation of d'Alembert's Formula

We want to solve

$$w_{tt} = c^2 w_{xx}, \quad w(x,0) = \alpha(x), \quad w_t(x,0) = \beta(x) \; .$$

Introducing $u_1 = w_t, u_2 = cw_x$, as above, we obtain the symmetric hyperbolic system $u_t = Au_x$ and the initial condition

$$u(x,0) = f(x) \quad \text{with} \quad f_1(x) = \beta(x), \quad f_2(x) = c\alpha'(x) \; .$$

Then, following the general process of diagonalizing the hyperbolic system etc., we obtain

$$u(x,t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (f_1 + f_2)(x+ct) + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (f_1 - f_2)(x-ct) \; .$$

To obtain the solution of the wave equation, consider the second component, $u_2 = cw_x$. We have

$$\begin{aligned} cw_x(x,t) &= u_2(x,t) \\ &= \frac{1}{2}(f_1+f_2)(x+ct) - \frac{1}{2}(f_1-f_2)(x-ct) \\ &= \frac{1}{2}\beta(x+ct) + \frac{c}{2}\alpha'(x+ct) - \frac{1}{2}\beta(x-ct) + \frac{c}{2}\alpha'(x-ct) \end{aligned}$$

Therefore,

$$w_x(x,t) = \frac{1}{2}\Big(\alpha'(x+ct) + \alpha'(x-ct)\Big) + \frac{1}{2c}\Big(\beta(x+ct) - \beta(x-ct)\Big)$$

Integration yields

$$w(x,t) = \frac{1}{2}\Big(\alpha(x+ct) + \alpha(x-ct)\Big) + \frac{1}{2c}\Big(\int_0^{x+ct} \beta(y)dy - \int_0^{x-ct} \beta(y)dy\Big) + \phi(t) \; .$$

By requiring the initial conditions,

$$w(x,0) = \alpha(x), \quad w_t(x,0) = \beta(x)$$

it follows that $\phi(0) = \phi'(0) = 0$. Also, the wave equation $w_{tt} = c^2 w_{xx}$ requires that $\phi''(t) = 0$. It follows that $\phi \equiv 0$. We have derived d'Alembert's solution formula,

$$\begin{aligned} w(x,t) &= \frac{1}{2}\Big(\alpha(x+ct) + \alpha(x-ct)\Big) + \frac{1}{2c}\Big(\int_0^{x+ct} \beta(y)dy - \int_0^{x-ct} \beta(y)dy\Big) \\ &= \frac{1}{2}\Big(\alpha(x+ct) + \alpha(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(y)dy \; . \end{aligned}$$

Remark: There is a faster but less systematic way to derive the formula: One can write the wave equation as

$$(D_t + cD_x)(D_t - cD_x)w = 0 \; .$$

This implies that any smooth function of the form

$$w(x,t) = a(x+ct) + b(x-ct)$$

satisfies the wave equation. Requiring the initial conditions

$$\begin{aligned} \alpha(x) &= w(x,0) = a(x) + b(x) \\ \beta(x) &= w_t(x,0) = ca'(x) - cb'(x) \end{aligned}$$

one obtains

$$\begin{aligned} a(x) + b(x) &= \alpha(x) \\ a(x) - b(x) &= \frac{1}{c}\int_0^x \beta(y)dy + const \end{aligned}$$

Solving this linear system for $a(x)$ and $b(x)$ leads to d'Alembert's formula.

3.6 The Euler Equations for 1D Compressible Flow

We use the 1D Euler equations as an example to illustrate the important process of linearization. It is not difficult to generalize to 2D and 3D.

Let $u(x,t), \rho(x,t), p(x,t)$ denote velocity, density, and pressure of a gas, respectively. Here we assume, for simplicity, that u, ρ, and p depend on only one space variable, $x \in \mathbb{R}$. The scalar $u(x,t)$ represents the velocity in x–direction. The 1D Euler equations consist of the momentum equation, the continuity equation, and an equation of state. In their simplest form, the equations read

$$\begin{aligned} (\rho u)_t + (\rho u u)_x + p_x &= 0 \\ \rho_t + (\rho u)_x &= 0 \\ p &= p_0\Big(\frac{\rho}{\rho_0}\Big)^\gamma \end{aligned}$$

Here ρ_0 and p_0 are reference values for ρ and p. The exponent γ is the ratio of specific heats at constant temperature and constant volume. For air one has $\gamma = 1.4$.

Subtracting u times the continuity equation from the momentum equation, one obtains the equivalent system

$$\rho(u_t + uu_x) + p_x = 0 \tag{3.18}$$

$$\rho_t + (\rho u)_x = 0 \tag{3.19}$$

$$p = p_0\Big(\frac{\rho}{\rho_0}\Big)^\gamma \tag{3.20}$$

Note that any constant vector function

$$(u, \rho, p) = (U, \rho_0, p_0)$$

solves the system.

We want to derive the small disturbance equations. To this end, substitute

$$\begin{aligned} u(x,t) &= U + \varepsilon \tilde{u}(x,t) \\ \rho(x,t) &= \rho_0 + \varepsilon \tilde{\rho}(x,t) \\ p(x,t) &= p_0 + \varepsilon \tilde{p}(x,t) \end{aligned}$$

into the system (3.18) to (3.20). If one divides the resulting equations by ε and then neglects terms of order ε, one obtains the following linear system for $\tilde{u}, \tilde{\rho}$, and $\tilde{p}$,

$$\rho_0(\tilde{u}_t + U\tilde{u}_x) + \tilde{p}_x = 0 \tag{3.21}$$

$$\tilde{\rho}_t + \rho_0\tilde{u}_x + U\tilde{\rho}_x = 0 \tag{3.22}$$

$$\tilde{p} = \frac{\gamma p_0}{\rho_0}\,\tilde{\rho} \tag{3.23}$$

We can use the last equation to eliminate $\tilde{p}$ from the first equation. If we then drop the tilde notation, we obtain the equations

$$\begin{aligned} u_t + Uu_x + \frac{\gamma p_0}{\rho_0^2}\,\rho_x &= 0 \\ \rho_t + \rho_0 u_x + U\rho_x &= 0 \end{aligned}$$

In systems form, we have

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t + A \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0 \tag{3.24}$$

with

$$A = \begin{pmatrix} U & \gamma p_0 \rho_0^{-2} \\ \rho_0 & U \end{pmatrix} .$$

The eigenvalues of A are

$$\lambda_{1,2} = U \pm \sqrt{\frac{\gamma p_0}{\rho_0}} .$$

Since the eigenvalues of A are real and distince, the system (3.24) is strongly hyperbolic. The eigenvalues $\lambda_{1,2}$ are the propagation speeds of small disturbances in the Euler equations. The speed

$$a = \sqrt{\frac{\gamma p_0}{\rho_0}}$$

is the sound speed corresponding to the state (ρ_0, p_0).

The values

$$\begin{aligned} p_0 &= 101 kilopascals = 1.01 * 10^5 Newton\ meter^{-2} \\ \rho_0 &= 1.293 * 10^{-3} g\ cm^{-3} \\ \gamma &= 1.4 \end{aligned}$$

correspond to air at standard conditions. One obtains that

$$\begin{aligned} a^2 &= \frac{1.4 * 1.01}{1.293} * 10^8 * 10^3 * 10^{-2} \, \frac{cm^2}{sec^2} \\ &= \frac{1.4 * 1.01 * 10}{1.293} * 10^4 \, \frac{m^2}{sec^2} \end{aligned}$$

which yields

$$a = 330.7 \, \frac{m}{sec}$$

for the speed of sound.

3.7 General Constant Coefficient Systems, First Order in Time

In this section we let $x \in \mathbb{R}^n$ and consider a system

$$u_t = Pu, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \; , \tag{3.25}$$

where

$$P = \sum_{|\alpha| \le m} A_\alpha D^\alpha \; .$$

Here

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

is a multi–index of order

$$|\alpha| = \sum_j \alpha_j$$

and

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \quad D_j = \frac{\partial}{\partial x_j} \; .$$

The matrices $A_\alpha \in \mathbb{C}^{N \times N}$ are constant. We first consider the system $u_t = Pu$ with an initial condition

$$u(x,0) = e^{ik \cdot x} a_0, \quad x \in \mathbb{R}^n \; , \tag{3.26}$$

where $k \in \mathbb{R}^n$ is a fixed wave vector and $a_0 \in \mathbb{C}^N$ is a fixed vector of amplitudes.

To solve (3.25), (3.26), we make the ansatz

$$u(x,t) = e^{ik \cdot x} a(t) \; .$$

We obtain that the function u satisfies the system $u_t = Pu$ iff

$$a'(t) = \hat{P}(ik) a(t)$$

where

$$\hat{P}(ik) = \sum_\alpha (ik)^\alpha A_\alpha$$

is the so–called symbol of P and

$$(ik)^\alpha = (ik_1)^{\alpha_1} \cdots (ik_n)^{\alpha_n} \ .$$

Thus, (3.25), (3.26) is solved by

$$u(x,t) = e^{ik\cdot x} e^{\hat{P}(ik)t} a_0 \ .$$

Let $\mathcal{T} = \mathcal{T}(\mathbb{R}^n, \mathbb{C}^N)$ denote the space of all trigonometric polynomials in n variables, $x_1, \ldots, x_n$, taking values in $\mathbb{C}^N$. Thus, every $f \in \mathcal{T}$ has the form

$$f(x) = (2\pi)^{-n/2} \sum_k e^{ik\cdot x} \hat{f}(k) \tag{3.27}$$

where $\hat{f}(k) \in \mathbb{C}^N$ and where the sum is taken over finitely many wave vectors $k \in \mathbb{Z}^n$. The factor $(2\pi)^{-n/2}$ is introduced here in order to avoid a scaling factor (different from one) in Parseval's relation; see formula (3.28) below.

The system $u_t = Pu$ with the initial condition $u(x,0) = f(x)$ is solved by

$$u(x,t) = (2\pi)^{-n/2} \sum_k e^{ik\cdot x} e^{\hat{P}(ik)t} \hat{f}(k) \ .$$

For every $t \in \mathbb{R}$ we have obtained the solution operator $S_0(t) : \mathcal{T} \to \mathcal{T}$ which maps $f \in \mathcal{T}$ to the solution $u(\cdot, t)$.

In order to define well–posedness of the 2π–periodic initial value problem, we introduce a norm on the space $\mathcal{T}$:

For $f \in \mathcal{T}$ let

$$\|f\|^2 = \int_{Q_n} |f(x)|^2 \, dx \ .$$

Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^N$ and

$$Q_n = [0, 2\pi]^n \ .$$

If $k, l \in \mathbb{Z}^n$ are wave vectors with integer components, then

$$(e^{ik\cdot x}, e^{il\cdot x}) = \int_{Q_n} e^{i(l-k)\cdot x} \, dx = 0$$

for $k \neq l$. Using this, it is not difficult to show **Parseval's relation**:

$$\|f\|^2 = \sum_k |\hat{f}(k)|^2, \quad f \in \mathcal{T} \ . \tag{3.28}$$

Using this relation, the proof of the following result is easy.

Lemma 3.10 *If $S_0(t) : \mathcal{T} \to \mathcal{T}$ denotes the solution operator defined above, then we have*

$$\|S_0(t)\| = \sup_{k \in \mathbb{Z}^n} |e^{\hat{P}(ik)t}| \ .$$

In particular, the operator $S_0(t)$ is bounded if and only if the supremunm is finite.

Proof: Let $f \in \mathcal{T}$ be arbitray and let $u(\cdot, t) = S_0(t)f$. We have

$$\begin{aligned} \|u(\cdot,t)\|^2 &= \sum_k |e^{\hat{P}(ik)t}\hat{f}(k)|^2 \\ &\leq \sum_k |e^{\hat{P}(ik)t}|^2 \, |\hat{f}(k)|^2 \\ &\leq \Big(\sup_k |e^{\hat{P}(ik)t}|^2\Big) \sum_k |\hat{f}(k)|^2 \\ &= \Big(\sup_k |e^{\hat{P}(ik)t}|^2\Big) \, \|f\|^2 \end{aligned}$$

Taking square roots, this estimate proves that

$$\|S_0(t)\| \leq \sup_{k \in \mathbb{Z}^n} |e^{\hat{P}(ik)t}| \ .$$

To prove that equality holds, let $\varepsilon > 0$ be arbitray. There exists $k_0 \in \mathbb{Z}^n$ with

$$|e^{\hat{P}(ik_0)t}| \geq \sup_{k \in \mathbb{Z}^n} |e^{\hat{P}(ik)t}| - \varepsilon \ .$$

Furthermore, there exists a nonzero vector $a \in \mathbb{C}^N$ with

$$|e^{\hat{P}(ik_0)t}a| = |e^{\hat{P}(ik_0)t}| \, |a| \ .$$

If one chooses

$$f(x) = e^{ik_0 \cdot x}a$$

then one obtains that

$$\||S_0(t)f\| \geq \Big(\sup_{k \in \mathbb{Z}^n} |e^{\hat{P}(ik)t}| - \varepsilon\Big) \, \|f\| \ .$$

This completes the proof of the lemma. $\diamond$

Definition 3.3 *The 2π–periodic initial value problem for the equation $u_t = Pu$ is called well–posed, if there exist constants $t_0 > 0$ and $C \geq 1$ with*

$$\|S_0(t)\| \leq C \quad \textit{for} \quad 0 \leq t \leq t_0 \ .$$

Using the previous lemma, we can express the operator norm $\|S_0(t)\|$ by the matrix norms of

$$e^{\hat{P}(ik)t} .$$

One obtains the following result.

Theorem 3.2 *The 2π–periodic initial value problem for the equation $u_t = Pu$ is well–posed if and only if there exist constants $t_0 > 0$ and $C \ge 1$ with*

$$|e^{\hat{P}(ik)t}| \le C \quad \textit{for all} \quad k \in \mathbb{Z}^n \quad \textit{and} \quad 0 \le t \le t_0 .$$

3.8 Symmetric Hyperbolic Systems: Maxwell's Equations as an Example

Definition 3.4 *A first order system*

$$u_t = A_1 D_1 u + \cdots + A_n D_n u$$

is called symmetric hyperbolic if $A_j = A_j^*$ for $j = 1, \ldots, n$.

An important example is given by the (scaled) Maxwell system in vacuum. If $E = E(x,t)$ and $H = H(x,t)$ denote the electric and magnetic fields, then Maxwell's equations read

$$\begin{aligned} E_t - \frac{1}{\mu_0} \nabla \times H &= 0 , \\ H_t + \frac{1}{\varepsilon_0} \nabla \times E &= 0 . \end{aligned}$$

Introduce the scaled variables

$$\tilde{E} = \sqrt{\mu_0}\, E, \quad \tilde{H} = \sqrt{\varepsilon_0}\, H .$$

In these variables, the system becomes

$$\begin{aligned} \tilde{E}_t - \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \tilde{H} &= 0 , \\ \tilde{H}_t + \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \nabla \times \tilde{E} &= 0 . \end{aligned}$$

The quantity

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

turns out to be the speed of propagation for the system, which is the speed of light.

Dropping the tilde notation, we obtain the system

$$E_t - c \nabla \times H = 0 , \tag{3.29}$$
$$H_t + c \nabla \times E = 0 . \tag{3.30}$$

If $V:\mathbb{R}^3\to\mathbb{R}^3$ is any smooth vector function, then we have

$$\nabla\times V=\sum_{j=1}^{3}B_jD_jV \tag{3.31}$$

where the matrices $B_j\in\mathbb{R}^{3\times 3}$ are skew symmetric,

$$B_j^T=-B_j,\quad j=1,2,3\ .$$

For example, since

$$\nabla\times V=\begin{pmatrix}D_2V_3-D_3V_2\\ D_3V_1-D_1V_3\\ D_1V_2-D_2V_1\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}D_1V+\cdots$$

one obtains that

$$B_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\ .$$

With the notation (3.31), the (scaled) Maxwell system (3.29), (3.30) reads

$$\begin{pmatrix}E\\H\end{pmatrix}_t+\sum_j cA_jD_j\begin{pmatrix}E\\H\end{pmatrix}=0$$

where A_j is a 6×6 matrix that has the block form

$$A_j=\begin{pmatrix}0&-B_j\\B_j&0\end{pmatrix}\ .$$

Since $B_j^T=-B_j$ it follows that $A_j^T=A_j$. In other words, the (scaled) Maxwell system (3.29), (3.30) is symmetric hyperbolic.

Theorem 3.3 *The 2π periodic initial value problem for a symmetric hyperbolic system*

$$u_t=Pu\equiv\sum_j A_jD_ju,\quad A_j^*=A_j\ ,$$

is well–posed. The operator norm of the solution operator $S_0(t)$ satisfies

$$\|S_0(t)\|=1\quad\textit{for all}\quad t\ .$$

Proof: We have

$$\hat{P}(ik)=i\sum_j k_jA_j=:U$$

and note that $U^*=-U$, i.e., U is skew Hermitian. Therefore, the theorem follows from the following lemma.

Lemma 3.11 *If $U \in \mathbb{C}^{N \times N}$ satisfies $U^* = -U$, then we have*

$$|e^{Ut}| = 1 \quad \textit{for all} \quad t \ .$$

Proof: Let $a(t)$ denote the solution of the ODE system

$$a'(t) = Ua(t), \quad a(0) = a_0 \ .$$

We have

$$\begin{aligned} \frac{d}{dt}\,|a|^2 &= \langle a, a' \rangle + \langle a', a \rangle \\ &= \langle a, Ua \rangle + \langle Ua, a \rangle \\ &= 0 \ . \end{aligned}$$

Therefore, $|a(t)| = |a_0|$ for all t. Since $a(t) = e^{Ut} a_0$ we obtain that $|e^{Ut}| = 1$. $\diamond$

3.9 An Ill–Posed Problem

In the next example we will consider an ill–posed problem. **Example:** Consider the system

$$u_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u_x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u \ .$$

In this case,

$$\hat{P}(ik) = \begin{pmatrix} ik & ik \\ 1 & ik \end{pmatrix} \ .$$

The eigenvalues of $\hat{P}(ik)$ are

$$\lambda_{1,2} = ik \pm \sqrt{i}\sqrt{k} \quad \text{where} \quad \sqrt{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \ .$$

For $k > 0$ the matrix $\hat{P}(ik)$ has an eigenvalue with real part

$$Re\,\lambda_1 = \frac{1}{\sqrt{2}}\,\sqrt{k} \ .$$

It follows that, for $t > 0$,

$$|e^{\hat{P}(ik)t}| \geq e^{\sqrt{k/2}\,t} \ .$$

For $t > 0$, the right–hand side tends to infinity as $k \to \infty$. Therefore,

$$\sup_k |e^{\hat{P}(ik)t}| = \infty \ .$$

This implies that the solution operator $S_0(t)$ is unbounded for $t > 0$. Thus, the 2π periodic initial value for the system is ill–posed.

3.10 Strongly Parabolic Systems in 1D

Definition 3.5 *A 2nd order system*

$$u_t = Au_{xx} + Bu_x + Cu \tag{3.32}$$

is called strongly parabolic if $A + A^*$ is positive definite.

Theorem 3.4 *The 2π periodic initial value problem for a strongly parabolic system (3.32) is well–posed.*

Proof: We have

$$\hat{P}(ik) = -k^2 A + ikB + C \ .$$

Let $a(t)$ satisfy

$$a'(t) = \hat{P}(ik)a(t), \quad a(0) = a_0 \ .$$

Then we have, for $t \geq 0$ and some positive constants c_1 and δ,

$$\begin{aligned}
\frac{d}{dt}|a|^2 &= \langle a, a'\rangle + \langle a', a\rangle \\
&= \langle a, (-k^2 A + ikB + C)a\rangle + \langle(-k^2 A + ikB + C)A, a\rangle \\
&\leq -k^2\langle a, (A + A^*)a\rangle + c_1(1 + |k|)|a|^2 \\
&\leq (-k^2\delta + c_1(1 + |k|))|a|^2
\end{aligned}$$

Here the constant $\delta > 0$ can be chosen as the smallest eigenvalue of $A + A^*$; one then has the estimate

$$\langle a, (A + A^*)a\rangle \geq \delta|a|^2$$

for all $a \in \mathbb{C}^N$. The above estimate for $(d/dt)|a|^2$ implies that there exists a constant $\alpha > 0$, independent of k and a_0, so that

$$\frac{d}{dt}|a|^2 \leq 2\alpha|a|^2 \ .$$

It then follows that

$$|a(t)| \leq e^{\alpha t}|a_0| \ ,$$

and therefore,

$$|e^{\hat{P}(ik)t}| \leq e^{\alpha t} \quad \text{for all} \quad t \geq 0 \ .$$

$\diamond$

4 The Wave Equation and First Order Hyperbolic Systems

4.1 The Wave Equation in 1D: d'Alembert's Formula

Consider the Cauchy problem

$$u_{tt} = c^2 u_{xx}, \quad c > 0 \ , \tag{4.1}$$

with initial condition

$$u = g, \quad u_t = h \quad \text{at} \quad t = 0 \ . \tag{4.2}$$

The solution is

$$u(x,t) = \frac{1}{2}\Big(g(x+ct) + g(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy \ . \tag{4.3}$$

4.2 An Initial–Boundary–Value Problem for the 1D Wave Equation

The formula (4.5) given below will be used later for the 3D wave equation.

Consider

$$u_{tt} = c^2 u_{xx}, \quad x \geq 0, \quad t \geq 0 \ ,$$

with initial condition

$$u = g, \quad u_t = h \quad \text{at} \quad t = 0, \quad x \geq 0 \ ,$$

and boundary condition

$$u = 0 \quad \text{at} \quad x = 0, \quad t \geq 0 \ .$$

Assume the compatibility condition

$$g(0) = h(0) = 0 \ .$$

Extend g and h to $x < 0$ as odd functions; thus define

$$g(-x) = -g(x), \quad h(-x) = -h(x), \quad x > 0 \ .$$

Then solve the Cauchy problem. Note that the solution satisfies the boundary condition $u = 0$ at $x = 0$. Obtain the following solution of the IBV problem:

$$u(x,t) = \frac{1}{2}\Big(g(x+ct) + g(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy, \quad 0 \leq ct \leq x \ , \tag{4.4}$$

$$u(x,t) = \frac{1}{2}\Big(g(x+ct) - g(-x+ct)\Big) + \frac{1}{2c}\int_{-x+ct}^{x+ct} h(y)dy, \quad 0 \leq x \leq ct \ . \tag{4.5}$$

Formula (4.4) is just d'Alembert's formula. To obtain (4.5) note that

$$g(x-ct) = -g(-x+ct)$$

and

$$\int_{x-ct}^{-x+ct} h(y)\, dy = 0 \ .$$

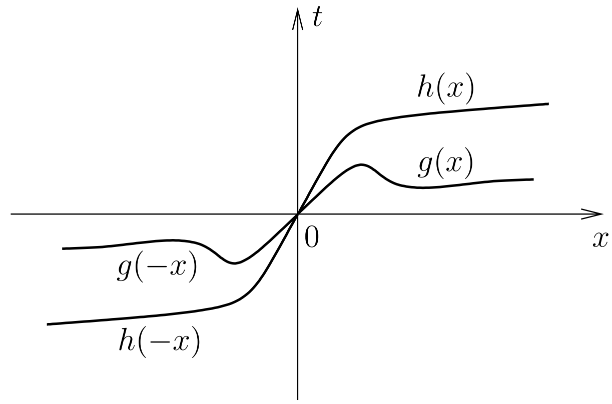

Figure 15: Extention of $g(x)$ and $g(x)$

4.3 The Wave Equation in 3D: Kirchhoff's Formula

Consider the Cauchy problem for the 3D wave equation,

$$u_{tt} = c^2 \Delta u, \quad x \in \mathbb{R}^3, \quad t \geq 0 \ ,$$

$$u = g, \quad u_t = h \quad \text{at} \quad t = 0 \ .$$

First assume that $u(x,t)$ solves the problem. For fixed x define the spherical means (for $r > 0$):

$$\begin{aligned} U(r,t) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} u(y,t)\, dS(y) \\ &= \frac{1}{4\pi} \int_{\partial B_1} u(x+rz,t)\, dS(z) \ , \\ G(r) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} g(y)\, dS(y) \ , \\ H(r) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} h(y)\, dS(y) \ . \end{aligned}$$

These means also depend on the fixed point x, but this is suppressed in our notation.

We will show below how to compute the function $U(r,t)$ by solving a 1D initial–boundary value problem. Then, since

$$\lim_{r \to 0} U(r,t) = u(x,t) \tag{4.6}$$

by the mean–value theorem, we can compute $u(x,t)$ from $U(r,t)$.

To compute $U(r,t)$, first note that

$$U(r,0) = G(r), \quad U_t(r,0) = H(r), \quad r > 0 \ ,$$

since $u = g$ and $u_t = h$ at $t = 0$.

In the next lemma we show that $U(r,t)$ satisfies a PDE, the so–called Euler–Poisson–Darboux equation.

Lemma 4.1 *The function $U(r,t)$ satisfies the Euler–Poisson–Darboux equation,*

$$U_{tt} = c^2 (U_{rr} + \frac{2}{r} U_r) \ . \tag{4.7}$$

Proof: We have, using the relation $y = x + rz$ for $y \in \partial B_r(x)$ and $z \in \partial B_1$,

$$\begin{aligned} U(r,t) &= \frac{1}{4\pi} \int_{\partial B_1} u(x+rz,t)\, dS(z) \\ U_r(r,t) &= \frac{1}{4\pi} \sum_j \int_{\partial B_1} (D_j u)(x+rz,t) z_j \, dS(z) \\ &= \frac{1}{4\pi r^2} \sum_j \int_{\partial B_r(x)} D_j u(y,t) \ \frac{y_j - x_j}{r} \, dS(y) \end{aligned}$$

where

$$\frac{y_j - x_j}{r} = n_j(y), \quad |n(y)| = 1 \ .$$

Therefore, by Green's theorem,

$$\begin{aligned} U_r(r,t) &= \frac{1}{4\pi r^2} \sum_j \int_{B_r(x)} D_j^2 u(y,t)\, dy \\ &= \frac{1}{4\pi r^2 c^2} \int_{B_r(x)} u_{tt}(y,t)\, dy \end{aligned}$$

Obtain that

$$\begin{aligned} r^2 U_r(r,t) &= \frac{1}{4\pi c^2} \int_{B_r(x)} u_{tt}(y,t)\, dy \\ &= \frac{1}{4\pi c^2} \int_0^r \int_{\partial B_\rho(x)} u_{tt}(y,t)\, dS(y)\, d\rho \ . \end{aligned}$$

Differentiation in r yields

$$\begin{aligned} (r^2 U_r(r,t))_r &= \frac{1}{4\pi c^2} \int_{\partial B_r(x)} u_{tt}(y,t)\, dS(y) \\ &= \frac{r^2}{c^2} U_{tt}(r,t) \end{aligned}$$

Thus we have shown that $U(r,t)$ satisfies the equation,

$$r^2 U_{rr} + 2r U_r = \frac{r^2}{c^2} U_{tt} \ .$$

Multiplication by c^2/r^2 proves the lemma. $\diamond$

Remark: If one generalizes from $\mathbb{R}^3$ to $\mathbb{R}^n$ and defines the spherical means by

$$U(r,t) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y,t)\, dS(y) \ ,$$

then one obtains the equation

$$U_{tt} = c^2 (U_{rr} + \frac{n-1}{r}\, U_r) \ .$$

Consider again the 3D case where the function $U(r,t)$ satisfies (4.7) and set

$$\tilde{U}(r,t) = rU(r,t), \quad \tilde{G}(r) = rG(r), \quad \tilde{H}(r) = rH(r) \ .$$

The function $\tilde{U}(r,t)$ satisfies the differential equation

$$\tilde{U}_{tt} = c^2 \tilde{U}_{rr} \ , \tag{4.8}$$

the initial condition

$$\tilde{U} = \tilde{G}, \quad \tilde{U}_t = \tilde{H} \quad \text{at} \quad t = 0 \ ,$$

and the boundary condition

$$\tilde{U} = 0 \quad \text{at} \quad r = 0, \quad t \geq 0 \ .$$

Proof of (4.8): We have

$$\begin{aligned} \tilde{U}_r &= (rU)_r = U + rU_r \\ \tilde{U}_{rr} &= 2U_r + rU_{rr} \\ \tilde{U}_{tt} &= rU_{tt} \\ &= rc^2 (U_{rr} + \frac{2}{r} U_r) \\ &= c^2 \tilde{U}_{rr} \end{aligned}$$

Our aim is to compute $u(x,t)$ as the limit, as $r \to 0$, of $U(r,t)$; see (4.6). Therefore, we have to compute $U(r,t) = \frac{1}{r}\, \tilde{U}(r,t)$ for small $r > 0$. Using (4.5) we have, for $t > 0$ and $0 < r \leq ct$,

$$U(r,t) = \frac{1}{2r}\Big(\tilde{G}(ct+r) - \tilde{G}(ct-r)\Big) + \frac{1}{2cr} \int_{ct-r}^{ct+r} \tilde{H}(y) dy \ .$$

Taking the limit as $r \to 0$ one obtains,

$$\begin{aligned} u(x,t) &= \tilde{G}'(ct) + \frac{1}{c}\, \tilde{H}(ct) &\quad (4.9) \\ &= \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} \Big(g(y) + (y-x)\cdot \nabla g(y) + t h(y) \Big)\, dS(y) \ . &\quad (4.10) \end{aligned}$$

The first formula follows from the mean value theorem. To derive the formula (4.10), note the following:

$$\begin{aligned}
\tilde{G}(r) &= rG(r) \\
\tilde{G}'(r) &= G(r) + rG'(r) \\
G(r) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} g(y)\, dS(y) \\
&= \frac{1}{4\pi} \int_{\partial B_1} g(x+rz)\, dS(z) \\
G'(r) &= \frac{1}{4\pi} \int_{\partial B_1} z \cdot (\nabla g)(x+rz)\, dS(z) \\
&= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} \frac{y-x}{r} \cdot \nabla g(y)\, dS(y) \\
rG'(r) &= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} (y-x) \cdot \nabla g(y)\, dS(y) \\
\tilde{H}(r) &= rH(r) \\
&= \frac{1}{4\pi r} \int_{\partial B_r(x)} h(y)\, dS(y)
\end{aligned}$$

The formula (4.10) follows by adding the expressions for $G(r), rG'(r)$, and $\frac{1}{c}\tilde{H}(r)$, evaluated at $r = ct$.

To summarize, we have shown the following solution formula for the Cauchy problem of the 3D wave equation,

$$u(x,t) = \tilde{G}'(r) + \frac{1}{c}\tilde{H}(r) \quad \text{with} \quad r = ct \;, \tag{4.11}$$

or, more explicitly,

$$u(x,t) = \frac{1}{4\pi r^2} \int_{\partial B_r(x)} \Big(g(y) + (y-x)\cdot \nabla g(y) + t h(y) \Big)\, dS(y) \quad \text{with} \quad r = ct \;. \tag{4.12}$$

These formulas are called Kirchhoff's formulas for the 3D wave equation.

Discussion: In the following, we assume $t > 0$. Note that $r = ct$ is the distance traveled by a signal in time t if the propagation speed is c. The formula (4.12) shows that the value $u(x,t)$ depends on $g, \nabla g$, and h on the *surface* of the ball $B_r(x)$ only. In other words, the initial data $g, \nabla g, h$ at some point $x^{(0)}$ influence the solution precisely at the surface of the so–called forward light cone of $x^{(0)}$, which consists of all world points $(x,t) \in \mathbb{R}^4$ with

$$|x - x^{(0)}| = ct, \quad x \in \mathbb{R}^3, \quad t \geq 0 \;. \tag{4.13}$$

(The points (x,t) with $|x - x^{(0)}| < ct$ lie strictly inside the light cone (4.13); the points (x,t) with $|x - x^{(0)}| > ct$ lie strictly outside the light cone (4.13).) In particular, the data $g, \nabla g, h$ at $x^{(0)}$ do not influence the solution $u(x,t)$ at points (x,t) that lie strictly *inside* the light cone

(4.13). Therefore, one says that for the 3D wave equation, $u_{tt} = c^2 \Delta u$, one has a *sharp* speed of propagation of signals. This speed is the coefficient c. If, for some other problem, the data at $x^{(0)}$ do not influence the solution outside the light cone, but do influence the solution inside the light cone, then c is called the *maximal* speed of propagation, but not the sharp speed of propagation. In the next section we discuss the 2D wave equation and will see that c is the maximal, but not the sharp speed of propagation.

Focussing of Singularities: Let $\phi : [0,\infty) \to \mathbb{R}$ denote a smooth function and consider the 3D wave equation with initial condition

$$u(x,0) = \phi(|x|), \quad u_t(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \; .$$

Let us consider $u(0,t)$, i.e., we let $x = 0$ in Kirchhoff's formula.

We have $g(x) = \phi(|x|)$ and

$$G(r) = \frac{1}{4\pi} \int_{\partial B_1} g(rz)\, dS(z) = \phi(r) \; .$$

Also,

$$\tilde{G}(r) = rG(r) = r\phi(r) \quad \text{and} \quad \tilde{G}'(r) = \phi(r) + r\phi'(r) \; .$$

The formula

$$u(0,t) = \tilde{G}'(r) \quad \text{with} \quad r = ct$$

yields that

$$u(0,t) = \phi(r) + r\phi'(r) \quad \text{with} \quad r = ct \; .$$

Consider the continuous function

$$\phi(r) = \begin{cases} \sqrt{1-r} & \text{for} \quad 0 \le r \le 1 \\ 0 & \text{for} \quad r > 1 \end{cases}$$

and its derivative

$$\phi'(r) = \begin{cases} -\frac{1}{2}\frac{1}{\sqrt{1-r}} & \text{for} \quad 0 \le r < 1 \\ 0 & \text{for} \quad r > 1 \end{cases}$$

For $0 \le ct < 1$ the solution $u(0,t)$ only depends on the initial data in the open ball $B_1(0)$ where the initial data are smooth. One obtains that

$$u(0,t) = \sqrt{1-ct} - \frac{1}{2}\frac{ct}{\sqrt{1-ct}} \quad \text{for} \quad 0 \le ct < 1 \; .$$

Therefore,

$$u(0,t) \to -\infty \quad \text{as} \quad t \to \frac{1}{c} \ .$$

Here the initial data are continuous, but not differentiable. The singular behavior of the derivative of the function $\phi(r)$ at $r = 1$ gets focussed at $x = 0$ and leads to a blow–up of $u(0,t)$.

4.4 The Wave Equation in 2D: The Method of Descent

A solution u of the 2D wave equation also solves the 3D wave equation if one assumes that u is independent of the third space dimension. We use the 3D formula (4.11) and then descent to 2D.

Let

$$x = (x_1, x_2), \quad \bar{x} = (x_1, x_2, x_3) \ .$$

Assume that $u(x,t)$ solves the Cauchy problem for the 2D wave equation,

$$u_{tt} = c^2 \Delta u, \quad x \in \mathbb{R}^2, \quad t \geq 0 \ ,$$

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in \mathbb{R}^2 \ .$$

Set

$$\begin{aligned} \bar{u}(x_1,x_2,x_3,t) &= u(x_1,x_2,t) \\ \bar{g}(x_1,x_2,x_3,t) &= g(x_1,x_2,t) \\ \bar{h}(x_1,x_2,x_3,t) &= h(x_1,x_2,t) \end{aligned}$$

Clearly, $\bar{u}(\bar{x},t)$ solves the 3D wave equation with Cauchy data $\bar{u} = \bar{g}, \bar{u}_t = \bar{h}$ at $t = 0$. From (4.11) obtain that

$$\bar{u}(\bar{x},t) = \tilde{G}'(r) + \frac{1}{c}\tilde{H}(r) \quad \text{with} \quad r = ct$$

where

$$\tilde{G}(r) = \frac{1}{4\pi r} \int_{\partial B_r(\bar{x})} \bar{g}(\bar{y}) \, dS(\bar{y}) \ , \tag{4.14}$$

and $\tilde{H}(r)$ is defined similarly. The integral in the formula for $\tilde{G}$ is

$$\begin{aligned} Int &= \int_{\partial B_r(\bar{x})} g(y_1,y_2) \, dS(\bar{y}) && (4.15) \\ &= 2 \int_{B_r(x)} g(y_1,y_2) \, \frac{r}{(r^2 - |x-y|^2)^{1/2}} \, dy && (4.16) \end{aligned}$$

Explanation 1: Let $\rho = |x-y|$ and $f(\rho) = (r^2 - \rho^2)^{1/2}$, thus

$$f'(\rho) = -\rho (r^2 - \rho^2)^{-1/2} \ .$$

One can parameterize the upper and lower half–spheres of $\partial B_r(\bar{x})(\subset \mathbb{R}^3)$ by the solid circle $B_r(x)(\subset \mathbb{R}^2)$ in an obvious way: For example, the upper half–sphere is parametrized by

$$R(y_1, y_2) = \Big(y_1, y_2, x_3 + \sqrt{r^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}\Big), \quad y = (y_1, y_2) \in B_r(x) \ .$$

Then, at distance ρ from the center of the circle $B_r(x)$, the (local) surface area of the sphere $\partial B_r(\bar{x})$ is larger than the corresponding (local) area of the circle $B_r(x)$ by a factor dl where

$$\begin{aligned} (dl)^2 &= (d\rho)^2 + (df)^2 \\ &= (d\rho)^2\Big(1 + (df/d\rho)^2\Big) \\ &= (d\rho)^2 r^2/(r^2 - \rho^2) \ , \end{aligned}$$

thus

$$dl = \frac{r}{(r^2 - \rho^2)^{1/2}} \, d\rho \ .$$

This leads to the factor $r/(r^2 - |x-y|^2)^{1/2}$ in the above integral (4.16). The factor 2 appears in (4.16) since the sphere $\partial B_r(\bar{x})$ consists of two half–spheres.

Explanation2: Let

$$R(y_1, y_2) = \Big(y_1, y_2, x_3 + \sqrt{r^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}\Big), \quad y = (y_1, y_2) \in B_r(x)$$

denote the parametrization of the upper half–sphere of $\partial B_r(\bar{x})(\subset \mathbb{R}^3)$, as above.

We have

$$\begin{aligned} D_1 R(y_1, y_2) &= \Big(1, 0, \frac{x_1 - y_1}{\sqrt{\cdots}}\Big) \\ D_2 R(y_1, y_2) &= \Big(0, 1, \frac{x_2 - y_2}{\sqrt{\cdots}}\Big) \end{aligned}$$

with

$$\sqrt{\cdots} = \sqrt{r^2 - |x-y|^2} \ .$$

It is elementary to compute the length of the cross product and obtain

$$|D_1 R(y) \times D_2 R(y)| = \frac{r}{\sqrt{\cdots}} \ .$$

The formulas (4.14) and (4.16) yield that

$$\tilde{G}(r) = \frac{1}{2\pi} \int_{B_r(x)} \frac{g(y)}{(r^2 - |x-y|^2)^{1/2}} \, dy \ , \tag{4.17}$$

and similarly

$$\tilde{H}(r) = \frac{1}{2\pi} \int_{B_r(x)} \frac{h(y)}{(r^2 - |x-y|^2)^{1/2}}\, dy \ .$$

It remains to compute $\tilde{G}'(r)$. To this end, we first transform the integral in (4.17) to the fixed domain B_1 using the substitution

$$y = x + rz, \quad z \in B_1, \quad dy = r^2 dz \ .$$

One obtains that

$$\tilde{G}(r) = \frac{r}{2\pi} \int_{B_1} \frac{g(x+rz)}{(1-|z|^2)^{1/2}}\ dz \ ,$$

thus

$$\tilde{G}'(r) = \frac{1}{2\pi} \int_{B_1} \frac{g(x+rz)}{(1-|z|^2)^{1/2}}\ dz + \frac{r}{2\pi} \int_{B_1} \frac{z \cdot (\nabla g)(x+rz)}{(1-|z|^2)^{1/2}}\ dz \ .$$

We transform back to the integration domain $B_r(x)$ using the substitution

$$x + rz = y, \quad dz = r^{-2} dy, \quad z = \frac{y-x}{r} \ .$$

Therefore,

$$\begin{aligned} \tilde{G}'(r) &= \frac{1}{2\pi r^2} \int_{B_r(x)} \frac{g(y)}{(1-|y-x|^2 r^{-2})^{1/2}}\, dy + \frac{r}{2\pi r^2} \int_{B_r(x)} \frac{\frac{y-x}{r} \cdot \nabla g(y)}{(1-|y-x|^2 r^{-2})^{1/2}}\, dy \\ &= \frac{r}{2}\, \frac{1}{\pi r^2} \int_{B_r(x)} \frac{g(y) + (y-x)\cdot \nabla g(y)}{(r^2 - |x-y|^2)^{1/2}}\, dy \ . \end{aligned}$$

Adding the corresponding contribution from $\frac{1}{c}\tilde{H}(r)$, one obtains the following Kirchhoff formula for the solution of the 2D wave equation,

$$u(x,t) = \frac{r}{2}\, \frac{1}{\pi r^2} \int_{B_r(x)} \frac{g(y) + (y-x)\cdot\nabla g(y) + t h(y)}{(r^2 - |x-y|^2)^{1/2}}\, dy \quad \text{with} \quad r = ct \ .$$

The formula shows that signals can propagate at most at speed c. To compute $u(x,t)$, one needs the initial data in the whole circle $B_{ct}(x)$, not just on the boundary of the circle. Conversely, let $x^{(0)}$ be some fixed point in $\mathbb{R}^2$ and consider the light cone emanating from $x^{(0)}$, i.e., consider all world points (x,t) with

$$|x - x^{(0)}| = ct, \quad x \in \mathbb{R}^2, \quad t \geq 0 \ .$$

The initial data at $x^{(0)}$ influence the solution $u(x,t)$ at all points (x,t) *inside and on* the light cone, i.e., at all points (x,t) with

$$|x - x^{(0)}| \leq ct, \quad x \in \mathbb{R}^2, \quad t \geq 0 \ .$$

Thus, for the 2D wave equation $u_{tt} = c^2 \Delta u$ the speed c is the maximal speed of propagation and there is no sharp speed of propagation.

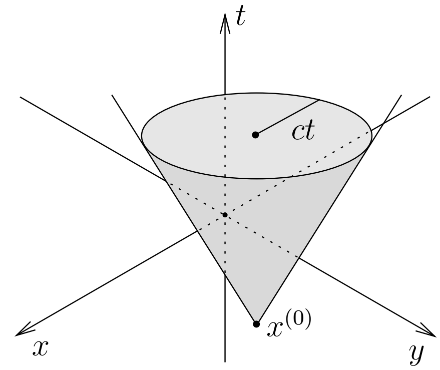

Figure 16: Cone $|x - x^{(0)}| \le ct$, $x \in \mathbb{R}^2$

4.5 The Inhomogeneous Wave Equation: Duhamel's Principle

As a motivation, first consider an ODE system

$$u'(t) = Au(t) + F(t), \quad u(0) = a \ , \tag{4.18}$$

where $A \in \mathbb{R}^{n \times n}$ is a constant matrix, $a \in \mathbb{R}^n$ is an initial value, and $F(t)$ is a continuous inhomogeneous term. If $F = 0$ then the solution is

$$u(t) = e^{At} a \ .$$

If the inhomogeneous term $F(t)$ is present in (4.18), then the solution is

$$u(t) = e^{At} a + \int_0^t e^{A(t-s)} F(s)\, ds \ .$$

Thus, the solution operator $S_0(t) = e^{At}$ of the homogeneous equation $u' = Au$ can be used to solve the inhomogeneous equation. This idea works in great generality. Though the wave equation is of second order in time, we have to modify the idea only slightly.

Consider the inhomogeneous wave equation,

$$u_{tt} = c^2 \Delta u + F(x,t), \quad x \in \mathbb{R}^3, \quad t \ge 0 \ ,$$

with homogeneous initial data

$$u = u_t = 0 \quad \text{at} \quad t = 0 \ .$$

For $0 \le s \le t$ let $v(x,t;s)$ denote the solution of the Cauchy problem

$$\begin{aligned} v_{tt} &= c^2 \Delta v \quad \text{for} \quad t \ge s \\ v(x,t) = 0, \quad v_t(x,t) &= F(x,t) \quad \text{at} \quad t = s \ . \end{aligned}$$

Lemma 4.2 *If $v(x,t;s)$ is defined as above, then the function*

$$u(x,t) = \int_0^t v(x,t;s)\,ds$$

solves the inhomogeneous wave equation, $u_{tt} = c^2 \Delta u + F$, with initial condition $u = u_t = 0$ at $t = 0$.

Proof: It is clear that $u = 0$ at $t = 0$. Also,

$$\begin{aligned} u_t(x,t) &= v(x,t;t) + \int_0^t v_t(x,t;s)\,ds \\ &= \int_0^t v_t(x,t;s)\,ds \ , \end{aligned}$$

thus $u_t = 0$ at $t = 0$. Differentiating again in t we obtain that

$$\begin{aligned} u_{tt}(x,t) &= v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,ds \\ &= F(x,t) + c^2 \int_0^t \Delta v(x,t;s)\,ds \\ &= F(x,t) + c^2 \Delta u(x,t) \ . \end{aligned}$$

This proves the lemma. $\diamond$

Remark: The above arguments were only formal; we did not check that we actually obtain a smooth function $u(x,t)$ and are allowed to differentiate under the integral sign etc. All this can be justified if $F(x,t)$ is sufficiently smooth.

Using Kirchhoff's formula (4.12) (with $g = 0$ and $h(y) = F(y,s)$) we have for $0 \le s < t$,

$$v(x,t;s) = \frac{1}{4\pi c^2 (t-s)} \int_{\partial B(x,c(t-s))} F(y,s)\,dS(y) \ ,$$

thus

$$u(x,t) = \frac{1}{4\pi c} \int_0^t \int_{\partial B(x,c(t-s))} \frac{F(y,s)}{c(t-s)}\,dS(y) ds \ .$$

Substitute

$$c(t-s) = r, \quad ds = -\frac{dr}{c}, \quad s = t - \frac{r}{c}$$

to obtain that

$$u(x,t) = \frac{1}{4\pi c^2} \int_0^{ct} \int_{\partial B(x,r)} \frac{F(y, t - \frac{r}{c})}{r}\,dS(y) dr \ .$$

This is an integral over the solid 3D ball $B(x,ct)$:

$$u(x,t) = \frac{1}{4\pi c^2} \int_{B(x,ct)} \frac{F(y, t - \frac{|x-y|}{c})}{|x-y|}\,dy \ . \tag{4.19}$$

Note that

$$\tau = \frac{|x-y|}{c}$$

is the time for a signal to travel from y to x at speed c. If we want to compute $u(x,t)$ we must integrate over $B(x,ct)$ and, for every $y \in B(x,ct)$, we must evaluate $F(y,\cdot)$ at the retarded time $t-\tau$. The integrand in the above formula (4.19) is called a retarded potential.

4.6 Energy Conservation for Symmetric Hyperbolic Systems and for the Wave Equation

Consider a symmetric hyperbolic system

$$u_t = \sum_{j=1}^{n} A_j D_j u, \quad A_j = A_j^*$$

where $u(x,t) \in \mathbb{C}^N, x \in \mathbb{R}^n$. The energy at time t is

$$\begin{aligned} E(t) &= \|u(\cdot,t)\|^2 \\ &= \int |u(x,t)|^2 \, dx \end{aligned}$$

If $|u(x,t)| \to 0$ sufficiently fast as $|x| \to \infty$ then one obtains through integration by parts,

$$\begin{aligned} \frac{d}{dt} E(t) &= \int \langle u_t, u\rangle + \langle u, u_t\rangle \, dx \\ &= \sum_j \int \langle A_j D_j u, u\rangle + \langle u, A_j D_j u\rangle \, dx \\ &= 0 \end{aligned}$$

Thus the energy is conserved during the evolution.

As an example, consider the 2D wave equation,

$$w_{tt} = c^2(w_{xx} + w_{yy}) \ .$$

If one sets

$$\begin{aligned} u_1 &= w_t \\ u_2 &= c w_x \\ u_3 &= c w_y \end{aligned}$$

then one obtains a symmetric hyperbolic system for $u(x,y,t)$. In this case the conserved energy is

$$E(t) = \int (w_t)^2 + c^2(w_x)^2 + c^2(w_y)^2 \, dxdy \ .$$

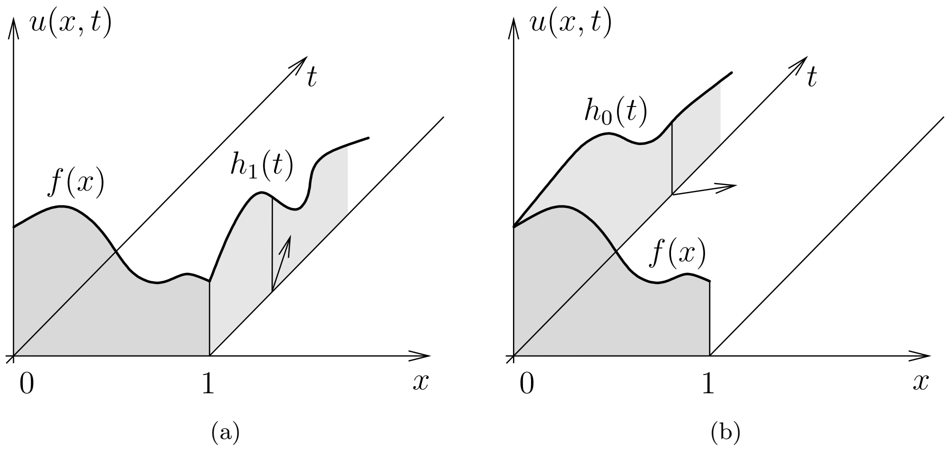

Figure 17: IBVP $u_t + au_x = 0$ for (a) $a < 0$, (b) $a > 0$

4.7 Initial–Boundary Value Problems for Strongly Hyperbolic Systems in 1D

First consider the scalar equation

$$u_t + au_x = 0, \quad a \in \mathbb{R} \ ,$$

in the strip

$$0 \le x \le 1, \quad t \ge 0 \ . \tag{4.20}$$

At $t = 0$ we give an initial condition

$$u(x, 0) = f(x) \ .$$

To determine a solution in the whole strip (4.20) one needs boundary conditions. The sign of the characteristic speed a is important.

Case 1: $a > 0$. Since u travels to the right, we give a boundary condition at the left boundary $x = 0$,

$$u(0, t) = h_0(t), \quad t \ge 0 \ .$$

Case 2: $a < 0$. Since u travels to the left, we give a boundary condition at the right boundary $x = 1$,

$$u(1, t) = h_1(t), \quad t \ge 0 \ .$$

Case 3: $a = 0$. In this case the solution is $u(x, t) = f(x)$, i.e., the solution is determined by the initial data. We do not need any boundary condition and are also not allowed to prescribe an boundary condition.

If the initial data $f(x)$ and the boundary data $h_0(t)$ or $h_1(t)$ are compatible, then the above problem has a unique solution.

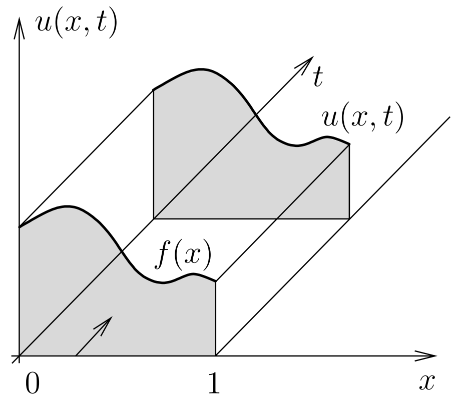

Figure 18: IBVP $u_t + au_x = 0$ for $a = 0$

Remark: Let $a > 0$, say. One can also determine a unique solution by prescribing $u(1,t) = h_1(t)$ for $t > 1/a$. However, in this case one needs future data, for $t > t_1$, to determine the solution at the time t_1. This is often unreasonable, and we will ignore such possibilities to determine a solution.

Second, consider a strongly hyperbolic system

$$u_t + Au_x = 0$$

in the strip (4.20) with initial condition $u(x,0) = f(x)$. By assumption, there is a transformation S with

$$S^{-1}AS = \Lambda = diag(\lambda_1, \ldots, \lambda_n), \quad \lambda_j \in \mathbb{R} \ .$$

Introduce a new unknown function $v(x,t)$ by

$$u = Sv$$

and obtain

$$v_t + \Lambda v_x = 0, \quad v(x,0) = g(x) \ ,$$

with $f(x) = Sg(x)$. The variables $v_j(x,t)$ are called characteristic variables. The system for v decouples into n independent scalar equations,

$$v_{jt} + \lambda_j v_{jx} = 0, \quad v_j(x,0) = g_j(x) \ .$$

The solution is

$$v_j(x,t) = g_j(x - \lambda_j t)$$

but we need boundary conditions to determine the solutions v_j in the whole strip. The guiding principle is the following: At each boundary, the boundary conditions must determine the ingoing characteristic variables in terms of the other characteristic variables.

As an example, take a case where $n = 2$ and

$$\lambda_1 > 0 > \lambda_2 \ .$$

Then, at $x = 0$ the variable v_1 is ingoing and v_2 is outgoing. Conversely, at $x = 1$ the variable v_2 is ingoing and v_1 is outgoing. Therefore, one can use the following kind of boundary conditions to determine a unique solution in the strip:

$$v_1(0,t) = \alpha v_2(0,t) + h_0(t) \tag{4.21}$$
$$v_2(1,t) = \beta v_1(1,t) + h_1(t) \tag{4.22}$$

As before, a smooth solution will exist if compatibility conditions are satisfied between initial and boundary data. If the boundary conditions are formulated directly in terms of the (physical) u variables, instead of the characteristic v variables, then one needs one boundary condition at $x = 0$ and one boundary condition at $x = 1$. In addition, in principle it must be possible to rewrite the conditions in the form (4.21), (4.22). If this can be done or not depends, in general, on the transformation $u = Sv$ between the variables.

Now assume that we have $n = 2$ and

$$\lambda_1 > 0, \quad \lambda_2 > 0 \ .$$

In this case both variables v_1 and v_2 are ingoing at $x = 0$ and outgoing at $x = 1$. We need to prescribe both variables at $x = 0$,

$$v_1(0,t) = h_{01}(t), \quad v_2(0,t) = h_{02}(t) \ .$$

Equivalently, we can prescribe the vector $u(0,t)$ at the boundary $x = 0$.

In general, let

$$\begin{aligned} \lambda_j > 0 \quad & \text{for} \quad j = 1, \ldots, l \ , \\ \lambda_j < 0 \quad & \text{for} \quad j = l+1, \ldots, l+r \ , \\ \lambda_j = 0 \quad & \text{for} \quad j = l+r+1, \ldots, n \ . \end{aligned}$$

We partition the vector v correspondingly,

$$v = \begin{pmatrix} v^I \\ v^{II} \\ v^{III} \end{pmatrix} \ .$$

In terms of the chracteristic variables the boundary conditions can take the following form:

At $x = 0$:

$$v^I(0,t) = R_0 \begin{pmatrix} v^{II}(0,t) \\ v^{III}(0,t) \end{pmatrix} + h_0(t) \ . \tag{4.23}$$

At $x = 1$:

$$v^{II}(1,t) = R_1 \begin{pmatrix} v^{I}(1,t) \\ v^{III}(1,t) \end{pmatrix} + h_1(t) \ .$$

Here the matrix R_0 is $l \times (n-l)$ and R_1 is $r \times (n-r)$

In terms of physical variables, the boundary condition at $x = 0$ can take the form

$$Q_0 u(0,t) = q_0(t) \tag{4.24}$$

where Q_0 is $l \times n$. If one is careful one should check that the condition (4.24) can be rewritten in the form (4.23). To this end, note that $u = Sv$; therefore, (4.24) reads

$$Q_0 S v(0,t) = q_0(t) \ .$$

The matrix $Q_0 S$ has the block form

$$Q_0 S = (Q_0 S^I, Q_0 S^{II}, Q_0 S^{III})$$

where S^I contains the first l columns of S, etc. If

$$det(Q_0 S^I) \neq 0$$

then the condition (4.24) can be written in the form (4.23) and therefore the condition (4.24) is an admissible boundary condition at $x = 0$.

Similar considerations apply to the boundary condition at $x = 1$: It can have the form $Q_1 u(1,t) = q_1(t)$ where Q_1 is $r \times n$, and one should check that $det(Q_1 S^{II}) \neq 0$.

As an example, we consider the linearized Euler equations. We recall that they have the form

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t + A \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0$$

with

$$A = \begin{pmatrix} U & \gamma p_0 \rho_0^{-2} \\ \rho_0 & U \end{pmatrix} \ .$$

Here U is the speed of the underlying base flow and p_0 and ρ_0 are the pressure and the density of the underlying flow. The number γ is the ratio of specific heats. (For air, $\gamma = 1.4$.) The sound speed corresponding to the underlying base flow is

$$a = \sqrt{\frac{\gamma p_0}{\rho_0}} \ .$$

It is easy to check that the matrix A has the eigenvalues

$$\lambda_{1,2} = U \pm a \ .$$

Let us check how many boundary conditions are needed in different cases.

1. $|U| < a$; the base flow is subsonic. We have $\lambda_1 > 0 > \lambda_2$ and need one boundary condition at $x = 0$ and one boundary condition at $x = 1$.

2. $|U| > a$; the base flow is supersonic.

   a) If $U > a > 0$ then we have supersonic inflow at $x = 0$ and need two boundary conditions at $x = 0$. In this case $\lambda_{1,2} > 0$.

   b) If $U < -a < 0$ then we have supersonic inflow at $x = 1$ and need two boundary conditions at $x = 1$. In this case $\lambda_{1,2} < 0$.

3. $|U| = a$; the base flow is sonic.

   a) If $U = a > 0$ then we have sonic inflow at $x = 0$ and need one boundary conditions at $x = 0$, no boundary condition at $x = 1$. In this case $\lambda_1 > 0 = \lambda_2$.

   b) If $U = -a < 0$ then we have sonic inflow at $x = 1$ and need one boundary conditions at $x = 1$, no boundary condition at $x = 0$. In this case $\lambda_1 = 0 > \lambda_2$.

Another simple example is given by the wave equation,

$$w_{tt} = c^2 w_{xx}, \quad c > 0 \ ,$$

written as a first order system. If

$$u_1 = w_t, \quad u_2 = c w_x$$

then one obtains

$$u_t + A u_x = 0$$

with

$$A = -c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{4.25}$$

The eigenvalues of A are

$$\lambda_{1,2} = \pm c \ .$$

Thus one needs one boundary condition at $x = 0$ and one boundary condition at $x = 1$. A possibility is to require

$$u_1 = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1 \ .$$

If

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

then

$$S^{-1} A S = diag(-c, c) \ .$$

The transformation $u = Sv$ reads

$$\begin{aligned} u_1 &= v_1 + v_2 \\ u_2 &= v_1 - v_2 \end{aligned}$$

Thus, in terms of characteristic variables, the boundary condition $u_1 = 0$ is

$$v_2 = -v_1 \quad \text{at} \quad x = 0 \ ,$$

$$v_1 = -v_2 \quad \text{at} \quad x = 1 \ .$$

Here we see that at each boundary the ingoing characteristic variable is determined in terms of the outgoing one. (At $x = 0$, the variable v_2 corresponding to $\lambda_2 = c > 0$ is ingoing, for example.)

Consider again the above system $u_t + Au_x = 0$ where A is given by (4.25) with $c > 0$. Consider the boundary conditions

$$u_1 + u_2 = 0 \quad \text{at} \quad x = 0, \quad u_1 = 0 \quad \text{at} \quad x = 1 \ .$$

Thus we have changed the boundary condition at $x = 0$. In terms of characteristic variables, the boundary condition $u_1 + u_2 = 0$ at $x = 0$ reads

$$v_1 = 0 \quad \text{at} \quad x = 0$$

since

$$v = S^{-1}u, \quad S^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

Thus, the boundary condition prescribes the outgoing characteristic variable v_1 at $x = 0$. (The variable v_1 is outgoing since it corresponds to the eigenvalue $\lambda_1 = -c < 0$ of A.) In general, the resulting initial–boundary value problem will not have a solution since, at least for some time, the values of v_1 at $x = 0$ are determined by the initial data, and the boundary condition $v_1 = 0$ will typically contradict these values.

5 Parabolic Systems with Variable Coefficients: Solution by Iteration

Consider the initial value problem,

$$u_t = Au_{xx} + B(x,t)u_x + C(x,t)u + F(x,t), \quad u(x,0) = f(x) \tag{5.1}$$

for $x \in \mathbb{R}, 0 \le t \le T$. We make the following assumptions:

1. $A, B(x,t), C(x,t) \in \mathbb{R}^{N\times N}, F(x,t), f(x) \in \mathbb{R}^N$ for all $x \in \mathbb{R}, 0 \le t \le T$.
2. A is constant and
$$A = A^T \ge \delta I, \quad \delta > 0 \ .$$
3. The functions B, C, F, f are C^∞ and are 2π–periodic in x for each time t with $0 \le t \le T$. For example,
$$F(x+2\pi, t) \equiv F(x,t) \ .$$

Because of the assumption $A = A^T > 0$, one calls the system symmetric parabolic.

We want to prove that the initial value problem (5.1) has a unique solution $u(x,t)$ which is C^∞ and 2π–periodic in x. The main difficulty is to prove the existence of the solution.

To this end, consider the sequence of functions $u^n(x,t), n = 0, 1, \ldots$ defined by the iteration

$$u^{n+1}_t = Au^{n+1}_{xx} + B(x,t)u^n_x + C(x,t)u^n + F(x,t), \quad u^{n+1}(x,0) = f(x) \ , \tag{5.2}$$

starting with $u^0(x,t) \equiv f(x)$. Each function $v = u^{n+1}$ solves a problem

$$v_t = Av_{xx} + G(x,t), \quad v(x,0) = f(x) \ , \tag{5.3}$$

where $G \in C^\infty$ is 2π–periodic in x. Using Fourier expansion, we will construct a solution $v(x,t)$ of (5.3).

First proceeding formally, we write

$$v(x,t) = \sum_{k=-\infty}^{\infty} \phi_k(x)\hat{v}(k,t) \tag{5.4}$$

where

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \ .$$

For each wave number $k \in \mathbb{Z}$, one obtains an ODE initial value problem for the function $\hat{v}(k,t)$,

$$\hat{v}_t(k,t) = -k^2 A\hat{v}(k,t) + \hat{G}(k,t), \quad \hat{v}(k,0) = \hat{f}(k) \ . \tag{5.5}$$

We know that this IVP has the unique solution

$$\hat{v}(k,t) = e^{-k^2 tA}\hat{f}(k) + \int_0^t e^{-k^2(t-s)A}\,\hat{G}(k,s)\,ds \ . \tag{5.6}$$

One can prove the following: If $\hat{v}(k,t)$ is defined by (5.6), then the series (5.4) defines a C^∞ solution of (5.3). We will prove this in Section 5.4. To summarize:

Theorem 5.1 *Let $G(x,t)$ and $f(x)$ be C^∞ functions that are 2π–periodic in x. Then the problem (5.3) has a unique classical solution $v(x,t)$ that is 2π–periodic in x. This solution v is C^∞ smooth and is given by (5.4) with $\hat{v}(k,t)$ given by (5.5).*

Using this theorem, it is clear that the sequence $u^n(x,t), n = 0, 1, \ldots$ is well–defined, and each function $u^n(x,t)$ is C^∞ and is 2π–periodic in x. Our aim is to prove that the sequence $u^n(x,t)$ converges to a C^∞ function $u(x,t)$, which then is the unique solution of (5.1).

This is done in several steps. In a first step, we show that the functions $u^n(x,t)$ are uniformly smooth. This means that all derivative satisfy bounds with constants independent of n.

5.1 Uniform Smoothness of the Functions $u^n(x,t)$

We begin with the basic energy estimate. For any matrix or vector function $M(x,t)$, which is assumed to be smooth and 2π–periodic in x, we use the notation

$$|M|_\infty = \max\Big\{|M(x,t)| : 0 \le x \le 2\pi, 0 \le t \le T\Big\} .$$

Also, the L^2–inner product of vector functions $u(x), v(x)$ taking values in $\mathbb{R}^N$ is

$$(u,v) = \int_0^{2\pi} \langle u(x), v(x)\rangle \, dx$$

and $\|u\| = (u,u)^{1/2}$ is the corresponding norm. If $u = u(x,t)$ is a function of (x,t) we often write for brevity,

$$\|u(\cdot,t)\| = \|u\| .$$

Lemma 5.1 *There is a constant $c_1 > 0$, depending only on $\delta, |B|_\infty, |C|_\infty, |B_x^T|_\infty$, with*

$$\frac{d}{dt}\,\|u^{n+1}\|^2 \le c_1\|u^{n+1}\|^2 + c_1\|u^n\|^2 + \|F\|^2, \quad 0 \le t \le T . \tag{5.7}$$

Proof: We have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\,\|u^{n+1}\|^2 &= (u^{n+1}, u_t^{n+1}) \\
&= (u^{n+1}, Au_{xx}^{n+1}) + (u^{n+1}, Bu_x^n) + (u^{n+1}, Cu^n) + (u^{n+1}, F) \\
&\le -\delta\|u^{n+1}\|^2 - (B_x^T u^{n+1}, u^n) - (Bu_x^{n+1}, u^n) + (u^{n+1}, Cu^n) + (u^{n+1}, F) \\
&\le -\delta\|u^{n+1}\|^2 + \Big(|B_x^T|_\infty + |C|_\infty\Big)\|u^n\|\|u^{n+1}\| + |B|_\infty\|u^n\|\|u_x^{n+1}\| + \|u^{n+1}\|\|F\|
\end{aligned}$$

Using the inequality

$$ab = (\varepsilon a)(\frac{1}{\varepsilon}\, b) \le \frac{\varepsilon^2}{2}\, a^2 + \frac{1}{2\varepsilon^2}\, b^2$$

to estimate the term

$$|B|_\infty\|u^n\|\|u_x^{n+1}\| ,$$

the claim follows. ⋄

Next we prove a simple result about a differential inequality. We will then apply the result to (5.7).

Lemma 5.2 *Let* $\phi \in C^1[0,T], g \in C[0,T]$ *and assume that*

$$\phi'(t) \le c\phi(t) + g(t), \quad 0 \le t \le T \ ,$$

where $c \ge 0$*. Then* $\phi(t)$ *satisfies the bound*

$$\phi(t) \le e^{ct}\phi(0) + \int_0^t e^{c(t-s)} g(s)\, ds, \quad 0 \le t \le T \ .$$

Proof: Set $r(t) = e^{-ct}\phi(t)$. It is easy to show that

$$r'(t) \le e^{-ct} g(t)$$

and therefore

$$r(t) \le r(0) + \int_0^t e^{-cs} g(s)\, ds \ .$$

Since $\phi(t) = e^{ct} r(t)$ the claim follows. ⋄

Using the previous two lemmas, it follows that there exist positive constants a and b, independent of n, with

$$\|u^{n+1}(\cdot,t)\|^2 \le a + b \int_0^t \|u^n(\cdot,s)\|^2\, ds, \quad 0 \le t \le T \ ,$$

for $n = 0, 1, \ldots$. This says that we can recursively estimate u^{n+1} in terms of u^n.

In the next lemma, called **Picard's lemma**, we show how such recursive estimates can be used to obtain absolute bounds for u^n, which are independent of n.

Lemma 5.3 *For* $n = 0, 1, \ldots$ *let* $\phi_n \in C[0,T]$ *denote a nonnegative function. Let* a, b *denote two nonnegative real numbers. Assume the recursive estimate,*

$$\phi_{n+1}(t) \le a + b \int_0^t \phi_n(s)\, ds, \quad 0 \le t \le T \ ,$$

for $n = 0, 1, \ldots$*. Then we have*

$$\phi_n(t) \le a \sum_{j=0}^{n-1} \frac{b^j t^j}{j!} + \frac{b^n t^n}{n!} M_t, \quad 0 \le t \le T \ , \tag{5.8}$$

with

$$M_t = \max_{0 \le s \le t} \phi_0(s) \ .$$

Consequently,

$$\phi_n(t) \le (a + |\phi_0|_\infty) e^{bt}, \quad 0 \le t \le T \ . \tag{5.9}$$

Proof: The bound (5.8) follows by induction in n. Then (5.9) follows since $e^{bt} = \sum \frac{b^j t^j}{j!}$. $\diamond$

Using the previous three lemmas, one obtains the following basic estimate for the sequence of functions $u^n(x,t)$:

Lemma 5.4 *There is a constant $K_0 > 0$, independent of n, so that*

$$\|u^n(\cdot,t)\| \le K_0, \quad 0 \le t \le T \; . \tag{5.10}$$

We now show that we can also bound all derivatives of u^n in a similar way. The basic reason is that the derivatives of u^n satisfy equations that are quite similar to those satisfied by u^n.

If we differentiate the equation (5.2) in x, we obtain

$$u^{n+1}_{xt} = A u^{n+1}_{xxx} + B u^n_{xx} + (B_x + C) u^n + G^n$$

with

$$G^n = F_x + C_x u^n \; .$$

Thus, for the sequence u^n_x one obtains a similar iteration as for u^n. One should note that $\|G^n(\cdot,t)\|$ is already estimated if one uses the previous lemma. In this way one obtains a bound

$$\|u^n_x(\cdot,t)\| \le K_1, \quad 0 \le t \le T \; .$$

The process can be repeated and one obtains bounds for u^n_{xx} etc. Then time derivatives can be expressed by space derivatives. One obtains the following result:

Theorem 5.2 *For every $i = 0,1,\ldots$ and every $j = 0,1,\ldots$ there is a constant K_{ij}, independent of n, with*

$$\|D^i_t D^j_x u^n(\cdot,t)\| \le K_{ij}, \quad 0 \le t \le T \; . \tag{5.11}$$

We want to show that we can bound $D^i_t D^j_x u^n$ also in maximum norm. To this end, we use the following simple example of a Sobolev inequality.

Theorem 5.3 *Let $u \in C^1[a,b], u : [a,b] \to \mathbb{R}$. Then the following estimate holds:*

$$|u|^2_\infty \le \Big(1 + \frac{1}{b-a}\Big)\|u\|^2 + \|Du\|^2, \quad Du(x) = u'(x) \; . \tag{5.12}$$

Proof: Let

$$M^2 = \max_x u^2(x) = u^2(x_1)$$

and

$$m^2 = \min_x u^2(x) = u^2(x_0) \; .$$

Suppose that $x_0 < x_1$, for definiteness. We have

$$\begin{aligned} M^2 - m^2 &= \int_{x_0}^{x_1} (u^2)'(x)\,dx \\ &= 2\int_{x_0}^{x_1} u(x)u'(x)\,dx \end{aligned}$$

Therefore,

$$M^2 \le m^2 + 2\|u\|\|Du\| \ .$$

Since

$$\int_a^b u^2(x)\,dx \ge (b-a)m^2$$

we have

$$M^2 \le \frac{1}{b-a}\,\|u\|^2 + 2\|u\|\|Du\| \ ,$$

and the claim follows. $\diamond$

It is easy to generalize the Sobolev inequality to vector valued functions, $u : [a,b] \to \mathbb{R}^N$, by applying the previous lemma to every component of u. Therefore, a simple implication of the previous two theorems is the following. The result says that the sequence of functions $u^n(x,t)$ is uniformly smooth.

Theorem 5.4 *For every $i = 0, 1, \ldots$ and every $j = 0, 1, \ldots$ there is a constant K_{ij}, independent of n, with*

$$|D_t^i D_x^j u^n(x,t)| \le K_{ij}, \quad x \in \mathbb{R}, \quad 0 \le t \le T \ . \tag{5.13}$$

We cite a theorem of analysis, the **Arzela–Ascoli theorem**. The theorem will be applied with

$$\Omega = [0, 2\pi] \times [0, T] \ .$$

Theorem 5.5 *Let $\Omega \subset \mathbb{R}^s$ be a closed and bounded set. For every $n = 0, 1, \ldots$ let $u^n : \Omega \to \mathbb{R}^N$ be a continuous function.*

Assume:

1) For all $\varepsilon > 0$ there exists $\delta > 0$, independent of n, so that

$$|u^n(y) - u^n(z)| \le \varepsilon \quad \textit{if} \quad |y - z| \le \delta \ .$$

(One says that the sequence u^n is uniformly equicontinuous.)

2) There is a constant K, independent of n, so that

$$|u^n(y)| \le K \quad \textit{for all} \quad y \in \Omega \quad \textit{and} \quad n = 0, 1, \ldots$$

(One says that the sequence u^n is uniformly bounded.)

Under these assumptions there exists $u \in C(\Omega, \mathbb{R}^N)$ and a subsequence u^{n_k} with

$$|u^{n_k} - u|_\infty \to 0 \quad \textit{as} \quad n_k \to \infty \ .$$

We can apply Theorem 5.5 to every derivative, $D_t^i D_x^j u^n$. However, since the indices n and $n+1$ appear in the iteration (5.2), this does not suffice to obtain a solution u of the given equation (5.1) in the limit as $n \to \infty$. One has to ensure convergence of the whole sequence $D_t^i D_x^j u^n$ as $n \to \infty$, not just of a subsequence. A key result, proved next, says that the whole sequence u^n converges in some weak norm. This will then imply *uniqueness* of all possible limits of subsequences of u^n, and will in fact imply convergence as $n \to \infty$ of u^n. The details of the arguments are given below.

5.2 Convergence of u^n in the L_2–Norm

We will show that $u^n(\cdot, t)$ converges w.r.t. the L_2–norm for $0 \le t \le t_0$ where $t_0 > 0$ is sufficiently small.

Set

$$v^n = u^{n+1} - u^n, \quad n = 0, 1, \ldots$$

Define

$$\|v^n\|_{t_0} = \max_{0 \le t \le t_0} \|v^n(\cdot, t)\| \ .$$

Lemma 5.5 *If $t_0 > 0$ is small enough, then*

$$\|v^{n+1}\|_{t_0} \le \frac{1}{2} \|v^n\|_{t_0}, \quad n = 0, 1, \ldots \tag{5.14}$$

Proof: We have

$$v_t^{n+1} = A v_{xx}^{n+1} + B v_x^n + C v^n, \quad v^{n+1}(x, 0) = 0 \ .$$

Proceeding as in the energy estimate, we have:

$$\frac{d}{dt} \|v^{n+1}\|^2 \le c_1 \Big(\|v^{n+1}\|^2 + \|v^n\|^2 \Big)$$

where $c_1 > 0$ depends only on

$$\delta, \quad |B_x^T|_\infty, \quad |B|_\infty, \quad |C|_\infty \ .$$

Therefore,

$$\|v^{n+1}(t)\|^2 \le c_1 \int_0^t e^{c_1(t-s)} \|v^n(s)\|^2 \, ds, \quad 0 \le t \le T \ .$$

For $0 \le t \le t_0$:

$$\|v^{n+1}(t)\|^2 \le \|v^n\|_{t_0}^2 \, c_1 \int_0^t e^{c_1(t-s)} \, ds \ .$$

Evaluating the integral, we obtain

$$\|v^{n+1}(t)\|^2 \le \|v^n\|_{t_0}^2 (e^{c_1 t_0} - 1) \quad \text{for} \quad 0 \le t \le t_0 \ .$$

If we choose $t_0 > 0$ so small that

$$e^{c_1 t_0} - 1 \leq \frac{1}{4}$$

then the bound (5.14) holds. $\diamond$

A consequence of the previous lemma is the following: If $0 \leq t \leq t_0$, then

$$u^n(\cdot, t)$$

is a Cauchy sequence in $L_2(0, 2\pi)$. Thus, there is a function $U(\cdot, t) \in L_2(0, 2\pi)$ with

$$\|u^n(\cdot, t) - U(\cdot, t)\| \to 0 \quad \text{as} \quad n \to \infty \tag{5.15}$$

for $0 \leq t \leq t_0$.

5.3 Existence of a Solution of (5.1)

We begin with a simple convergence criterion of analysis.

Theorem 5.6 *Let $b_n \in \mathbb{R}$ denote a sequence of real numbers. Assume:*
1) Every subsequence of b_n has a convergent subsequence;
2) All convergent subsequences of b_n converge to the same limit, called b.
Then, $b_n \to b$.

Proof: If b_n does not converge to b, then there exists an $\varepsilon > 0$ and a subsequence b_{n_k} with

$$|b_{n_k} - b| \geq \varepsilon \quad \text{for all} \quad n_k \ . \tag{5.16}$$

By assumption, the subsequence b_{n_k} has a convergent subsequence with limit b, which contradicts (5.16). $\diamond$

It is clear that the result and its proof generalize to sequences in normed spaces:

Theorem 5.7 *Let $b_n \in X$ denote a sequence in a normed space X. Assume:*
1) Every subsequence of b_n has a convergent subsequence;
2) All convergent subsequences of b_n converge to the same limit, called b.
Then, $b_n \to b$.

Let

$$\Omega = [0, 2\pi] \times [0, t_0]$$

and let X denote the space of all continuous functions

$$v : \Omega \to \mathbb{R}^N$$

with norm

$$|v|_\infty = \max_{(x,t) \in \Omega} |v(x, t)| \ .$$

We regard $u^n(x,t)$ as a sequence in X. First, by the Arzela–Ascoli theorem, every subsequence of u^n has a convergent subsequence with limit in X. Second, let $u \in X$ and assume

$$|u^{n_k} - u|_\infty \to 0 \quad \text{as} \quad n_k \to \infty \ .$$

This implies that

$$\|u^{n_k}(\cdot,t) - u(\cdot,t)\| \to 0$$

for $0 \le t \le T$. Therefore, $u = U$, where U is the limit constructed in (5.15). This shows that $U \in X$ and that *all* convergent subsequences of u^n (with limits and convergence in X) converge to the same limit, namely U. By Theorem 5.7 we have

$$|u^n - U|_\infty \to 0 \quad \text{as} \quad n \to \infty \ .$$

We now want to prove that $U \in C^\infty$ and that all derivatives of u^n converge to the corresponding derivatives of U w.r.t. $|\cdot|_\infty$.

Set

$$v^n = u^n_x \in X \ .$$

By the Arzela–Ascoli theorem, every subsequence of v^n has a convergent subsequence. Suppose that $V \in X$ and that

$$|v^{n_k} - V|_\infty \to 0 \quad \text{as} \quad n_k \to \infty \ .$$

We have, for any fixed t with $0 \le t \le t_0$,

$$u^{n_k}(x,t) - u^{n_k}(0,t) = \int_0^x v^{n_k}(\xi,t)\, d\xi \ .$$

In the limit as $n_k \to \infty$,

$$U(x,t) - U(0,t) = \int_0^x V(\xi,t)\, d\xi \ .$$

This proves that U can be differentiated w.r.t. x and that

$$U_x = V \ .$$

In particular, every limit $V \in X$ (w.r.t. $|\cdot|_\infty$) of any convergent subsequence of $v^n = u^n_x$ is unique, because it equals $V = U_x$. By Theorem 5.7 we obtain that

$$|u^n_x - U_x|_\infty \to 0 \quad \text{as} \quad n \to \infty \ .$$

This argument can be repeated for u^n_{xx}, u^n_t etc. One obtains:

Theorem 5.8 *Let $U = U(x,t)$ denote the limit of the sequence $u^n(x,t)$ constructed in (5.15). Then $U \in C^\infty$ is 2π–periodic in x and*

$$|D_t^i D_x^j u^n - D_t^i D_x^j U|_\infty \to 0 \quad \textit{as} \quad n \to \infty \ .$$

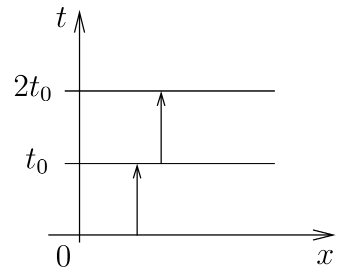

Figure 19: Restart process

It is now clear from (5.2) that the limit function $U(x,t)$ solves (5.1) for $0 \le t \le t_0$. At $t = t_0$ we can restart the argument and obtain a solution in $t_0 \le t \le 2t_0$. The interval length, the number t_0, is the same as before since t_0 only depends on $\delta, |B_x^T|_\infty, |B|_\infty, |C|_\infty$. After finitely many steps we obtain a solution in the whole time interval, $0 \le t \le T$. It is easy to see that the solution is unique: If there are two solutions, u_1 and u_2, subtract them and make an energy estimate for $v = u_1 - u_2$.

We summarize the result:

Theorem 5.9 *Under the assumptions listed above, the initial value problem (5.1) has a unique classical solution that is 2π–periodic in x. This solution is C^∞ smooth.*

5.4 Results on Fourier Expansion

The smoothness of a 2π–periodic function, $u(x)$, can be characterized by the decay rate of its Fourier coefficients, $\hat{u}(k)$, as $|k| \to \infty$. The following result says, basically, that $u \in C^\infty$ if and only if $|\hat{u}(k)|$ decays faster than any power, $|k|^{-m}$, as $|k| \to \infty$.

Lemma 5.6 *a) Let $u : \mathbb{R} \to \mathbb{C}$ denote a C^∞ function with $u(x+2\pi) \equiv u(x)$. Let*

$$\hat{u}(k) = (\phi_k, u) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ikx} u(x)\, dx, \quad k \in \mathbb{Z} \ ,$$

denote the Fourier coefficients of u. Then, for every $m = 1, 2, \ldots$

$$|\hat{u}(k)| \le c \int_0^{2\pi} |D^m u(x)|\, dx\, |k|^{-m}, \quad k \ne 0 \ .$$

Here $c = 1/\sqrt{2\pi}$.

b) Let $\alpha_k \in \mathbb{C}, k \in \mathbb{Z}$, denote a sequence of numbers. Assume that for every $m = 1, 2, \ldots$ there is a constant C_m with

$$|\alpha_k| \le C_m\, |k|^{-m}, \quad k \ne 0 \ .$$

Then the formula

$$u(x) = \sum_k \alpha_k \phi_k(x)$$

defines a 2π*–periodic* C^∞ *function and* $\hat{u}(k) = \alpha_k$.

Proof:

a) Through integration by parts,

$$\begin{aligned} \hat{u}(k) &= \frac{c}{ik} \int_0^{2\pi} e^{-ikx} Du(x)\, dx \\ &= \frac{c}{(ik)^2} \int_0^{2\pi} e^{-ikx} D^2 u(x)\, dx \end{aligned}$$

etc. This proves the decay of the Fourier coefficients.

b) Set

$$S_n(x) = \sum_{k=-n}^{n} \alpha_k \phi_k(x) \ .$$

For $n > j$ we have

$$\begin{aligned} |S_n - S_j|_\infty &\leq \sum_{j<|k|\leq n} |\alpha_k| \\ &\leq 2C_2 \sum_{j<|k|\leq n} \frac{1}{k^2} \ . \end{aligned}$$

Since the series $\sum_{k=1}^\infty k^{-2}$ converges, it follows that the sequence of functions $S_n(x)$ converges in maximum norm. The limit is a continuous, 2π–periodic function that we call $u(x)$. Thus, $u \in C_{per}$.

Fix any derivative order, m. We have

$$D^m S_n(x) = \sum_{k=-n}^{n} \alpha_k (ik)^m \phi_k(x) \ .$$

Using the same argument as above, we find that the sequence of functions $D^m S_n(x)$ converges in maximum norm to a continuous, 2π–periodic function that we call $U^{(m)}$.

Take $m = 1$, for example. We have

$$S_n(x) - S_n(0) = \int_0^x DS_n(y)\, dy \ .$$

In the limit as $n \to \infty$ we obtain that

$$u(x) - u(0) = \int_0^x U^{(1)}(y)\, dy \ .$$

This proves that $u \in C^1$ and $Du = U^{(1)}$. In the same way, $U^{(1)} \in C^1$ and $DU^{(1)} = U^{(2)}$. Thus, $u \in C^2$ and $D^2 u = U^{(2)}$. Clearly, this process can be continued. One obtains that $u \in C^\infty$. $\diamond$

5.4.1 Application to the System (5.3)

First proceeding formally, one obtains from (5.3) the ODE initial value problems

$$\hat{v}_t(k,t) = -k^2 A\hat{v}(k,t) + \hat{G}(k,t), \quad \hat{v}(k,0) = \hat{f}(k) \ , \tag{5.17}$$

for $k \in \mathbb{Z}$. Applying Lemma 5.6 a) we obtain that for every $m = 1, 2, \ldots$ there is a constant C_m with

$$|\hat{G}(k,t)| + |\hat{f}(k)| \le C_m |k|^{-m} \quad \text{for} \quad 0 \le t \le T \quad \text{and} \quad k \ne 0 \ . \tag{5.18}$$

For the solution $\hat{v}(k,t)$ of (5.17) we have

$$\begin{aligned} \frac{d}{dt}\,|\hat{v}(k,t)|^2 &= \langle \hat{v}, \hat{v}_t\rangle + \langle \hat{v}_t, \hat{v}\rangle \\ &\le -2\delta k^2 |\hat{v}|^2 + 2|\hat{v}||\hat{G}| \\ &\le |\hat{v}|^2 + |\hat{G}|^2 \end{aligned}$$

Thus, for $0 \le t \le T$, we obtain the (crude) estimate

$$\begin{aligned} |\hat{v}(k,t)|^2 &\le e^t |\hat{f}(k)|^2 + \int_0^t e^{t-s} |\hat{G}(k,s)|^2 \, ds \\ &\le K\Big(|\hat{f}(k)|^2 + \int_0^t |\hat{G}(k,s)|^2 \, ds \Big) \end{aligned}$$

with $K = e^T$. Using the bounds (5.18) it follows that for every $m = 1, 2, \ldots$ there is a constant C'_m with

$$|\hat{v}(k,t)| \le C'_m |k|^{-m} \quad \text{for} \quad 0 \le t \le T \quad \text{and} \quad k \ne 0 \ . \tag{5.19}$$

Consider the sequence of partial sums of the Fourier series (5.4),

$$S_n(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} e^{ikx} \hat{v}(k,t) \ .$$

It is clear that $S_n \in C^\infty$ and that S_n is 2π–periodic in x for each fixed n. Using the bounds (5.19) it is not difficult to show that the functions $S_n(x,t)$ are *uniformly* smooth, i.e., for all $i = 0, 1, \ldots$ and all $j = 0, 1, \ldots$ there is a constant K_{ij}, independent of n, with

$$|D_t^i D_x^j S_n(x,t)| \le K_{ij} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad 0 \le t \le T \ .$$

(To see this, first bound all derivatives of S_n in the L_2–norm, using (5.19) and Parseval's relation; time derivatives can be expressed using the differential equation (5.17). Then use Sobolev's inequality to bound the maximum norm of all derivatives.)

Also, for $n_1 > n_2$ we have

$$\begin{aligned} \|S_{n_1}(\cdot,t) - S_{n_2}(\cdot,t)\|^2 &\le \sum_{n_2 < |k| \le n_1} |\hat{v}(k,t)|^2 \\ &\le C'_1 \sum_{n_2 < |k| \le n_1} |k|^{-2} \end{aligned}$$

if we use the bound (5.19) with $m = 1$. This implies that $S_n(\cdot, t)$ is a Cauchy sequence in L_2.

The same arguments as in Section 5.3 can be used to prove that the sequence $S_n(x,t)$, along with all derivatives, converges in maximum norm to a C^∞ function $v(x,t)$,

$$\max_{x,t} |D_t^i D_x^j (S_n(x,t) - v(x,t))| \to 0 \quad \text{as} \quad n \to \infty \ .$$

The limit function $v(x,t)$ then solves (5.3).

6 Exact Boundary Conditions and Their Approximations

In many physical problems one introduces artificial boundaries for computational purposes and then has to formulate boundary conditions at these boundaries. A typical process to arrive at boundary conditions involves the following steps:

1. Linearize the equations about a current approximation;
2. Freeze coefficients in a variable coefficient problem;
3. Choose boundary conditions based on the analysis of the resulting constant coefficient problems.

To illustrate the last step, we consider a model problem.

6.1 A Model Problem: Derivation of Exact Boundary Conditions

Consider the model problem

$$u_t + u_x = \nu u_{xx}, \quad u(x,0) = f(x) \ , \tag{6.1}$$

where f is a smooth function supported in $-L < x < L$.

Assume $\nu > 0$. Let $u = u_{CP}(x,t)$ denote the solution of the Cauchy problem. We first try to find boundary conditions, imposed at $x = \pm L$, such that the corresponding IBVP is well–posed and its solution, u_{IBVP}, agrees with the solution u_{CP} of the Cauchy problem for $|x| \le L$. Such boundary conditions are called exact, because they reproduce the solution of the Cauchy problem exactly. It turns out that exact boundary conditions are not local in time. Therefore, one cannot easily implement them in a numerical process. A second aim, then, is to find approximations to the exact boundary conditions so that (1) the resulting IBVP is well–posed; (2) the boundary conditions are easy to implement; (3) the error between the solution of the IBVP and the desired solution of the Cauchy problem is small.

Let $u(x,t)$ denote the solution of the Cauchy problem (6.1). Use Laplace transformation in t. With

$$\tilde{u}(x,s) = \int_0^\infty u(x,t)e^{-st}\,dt$$

we denote the Laplace transform of $u(x,t)$ in time. Since $u(x,t)$ is bounded, its Laplace transform is defined for all s with $\operatorname{Re} s > 0$.

Recall a general rule of Laplace transformation: If $v(t)$ has the Laplace transform $\mathcal{L}(v(t))(s)$, then we have, for $\operatorname{Re} s$ sufficiently large,

$$\begin{aligned}
\mathcal{L}(v(t))(s) &= \int_0^\infty v(t)e^{-st}\,dt \\
&= \int_0^\infty v(t)e^{-st}\,dt \\
&= v(t)\frac{1}{-s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty v'(t)e^{-st}\,dt \\
&= \frac{1}{s}v(0) + \frac{1}{s}\int_0^\infty v'(t)e^{-st}\,dt
\end{aligned}$$

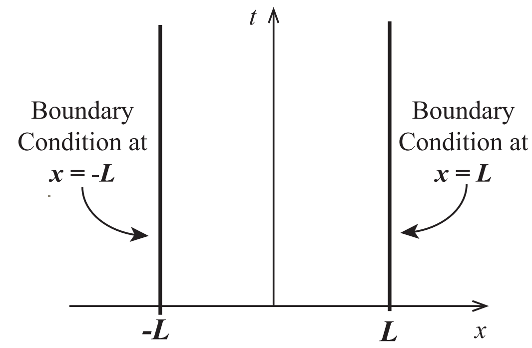

Figure 20: Domain for an initial–boundary value poblem

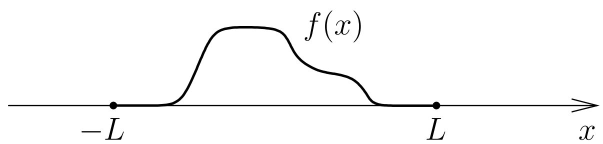

Figure 21: Function with compact support

Thus,

$$\mathcal{L}(v'(t))(s) = s\mathcal{L}(v(t))(s) - v(0) \ .$$

We apply this rule to $\tilde{u}(x,s)$, for each fixed x, and obtain the following family of ODEs:

$$s\tilde{u}(x,s) - f(x) + \tilde{u}_x(x,s) = \nu\tilde{u}_{xx}(x,s), \quad x \in \mathbb{R} \ . \tag{6.2}$$

Here s is a complex parameter, $\operatorname{Re} s > 0$. For each fixed s, the general solution of (6.2) contains two free constants. These are determined by the behavior of $\tilde{u}(x,s)$ for $|x| \to \infty$.

It is not difficult to prove that $u(x,t) \to 0$ as $|x| \to \infty$, for each fixed t. This makes it plausible that

$$|\tilde{u}(x,s)| \to 0 \quad \text{as} \quad |x| \to \infty, \quad \operatorname{Re} s > 0 \ . \tag{6.3}$$

We now solve the ODE (6.2) with the boundary conditions (6.3).

Write the equation for $\tilde{u}$ as a 1st order system,

$$U_x = M(s)U + F(x), \quad x \in \mathbb{R} \ ,$$

with

$$U = \begin{pmatrix} \tilde{u} \\ \tilde{u}_x \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ -f(x)/\nu \end{pmatrix}, \quad M(s) = \begin{pmatrix} 0 & 1 \\ s/\nu & 1/\nu \end{pmatrix} \ .$$

The eigenvalues $\lambda_{1,2} = \lambda_{1,2}(s)$ of $M(s)$ satisfy the characteristic equation

$$\lambda^2 - \frac{1}{\nu}\lambda - \frac{s}{\nu} = 0 \ . \tag{6.4}$$

The eigenvalues of $M(s)$ are (for $\mathrm{Re}\, s > 0$),

$$\begin{aligned} \lambda_1 &= \frac{1}{2\nu}\Big(1 - \sqrt{1+4\nu s}\Big), \quad \mathrm{Re}\,\lambda_1 < 0 \ , \\ \lambda_2 &= \frac{1}{2\nu}\Big(1 + \sqrt{1+4\nu s}\Big), \quad \mathrm{Re}\,\lambda_2 > 0 \ . \end{aligned}$$

The signs of $\mathrm{Re}\,\lambda_j$ are very important. In the above case one can discuss these signs directly by studying the root. Sometimes it is easier to use the following simple lemma.

Lemma 6.1 *Let $\lambda_{1,2}$ denote the solutions of a quadratic equation*

$$\lambda^2 + a\lambda + b = 0$$

where

$$\mathrm{Re}\, b < 0 \ .$$

Then one can order λ_1 and λ_2 so that

$$\mathrm{Re}\,\lambda_1 < 0 < \mathrm{Re}\,\lambda_2 \ .$$

Proof: This follows directly from $\lambda_1\lambda_2 = b$ and $\mathrm{Re}\, b < 0$. $\diamond$

One can transform $M(s)$ to diagonal form,

$$S^{-1}(s)M(s)S(s) = diag(\lambda_1, \lambda_2) = \Lambda$$

with

$$S(s) = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad S^{-1}(s) = \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \ .$$

Define the variables $V = S^{-1}U$ and obtain the diagonal system

$$V_x = \Lambda V + G(x), \quad G(x) = S^{-1}F(x) \ .$$

Here $G(x)$ is supported in $-L < x < L$. This system consists of two scalar equations:

$$v_{jx} = \lambda_j v_j + g_j(x), \quad j = 1, 2 \ .$$

Since

$$\mathrm{Re}\,\lambda_1 < 0 < \mathrm{Re}\,\lambda_2$$

the solution $V = (v_1, v_2)$ corresponding to u_{CP} satisfies the following boundary conditions:

$$\begin{aligned} v_1(x,s) &= 0 \quad \text{at} \quad x = -L \ , \\ v_2(x,s) &= 0 \quad \text{at} \quad x = L \ . \end{aligned}$$

Since $V = S^{-1}U$ these conditions read

$$\begin{aligned} \lambda_2 \tilde{u} - \tilde{u}_x &= 0 \quad \text{at} \quad x = -L \ , \\ \lambda_1 \tilde{u} - \tilde{u}_x &= 0 \quad \text{at} \quad x = L \ . \end{aligned}$$

These are the exact boundary conditions, formulated in terms of the Laplace transform $\tilde{u}(x, s)$ of the solution of the Cauchy problem.

In principle, by taking the inverse Laplace transform, one can write these conditions as boundary conditions for $u(x, t)$. The resulting conditions are not local in t, however. For computations, this is very inconvenient.

6.2 Approximation of the Exact Boundary Conditions

The idea is to approximate the functions $\lambda_j(s)$ by rational functions of s. Suppose, for example, that

$$\lambda_1(s) \approx \frac{p(s)}{q(s)}$$

with polynomials $p(s)$ and $q(s)$. This suggests to replace the exact outflow condition

$$\lambda_1 \tilde{u} - \tilde{u}_x = 0 \quad \text{at} \quad x = L$$

by

$$p(s)\tilde{u} - q(s)\tilde{u}_x = 0 \quad \text{at} \quad x = L \ .$$

Taking the inverse Laplace transform, one must replace s with $\partial/\partial t$. (Note that $u = 0$ at $t = 0, x = L$. Also, $u_t = 0$ at $t = 0, x = L$ by the differential equation, etc.) In this way one obtains conditions formulated directly in terms of the physical variable $u(x, t)$.

A possibility is to approximate $\lambda_1(s)$ accurately near $s = 0$. (Note that small s correspond to large t; good approximation near $s = 0$ correspond to good approximation near steady state.)

Using Taylor expansion about $\nu s = 0$, we obtain

$$\lambda_1 = -s + \nu s^2 + \mathcal{O}(\nu^2 s^3) \ .$$

Using the approximation

$$\lambda_1 \approx -s$$

one obtains the boundary condition

$$u_t + u_x = 0 \quad \text{at} \quad x = L \ ,$$

which is reasonable since it corresponds to neglecting the term νu_{xx} at the boundary. The error which the exact solution produces in the boundary condition $u_t + u_x = 0$ is of order ν. This might be satisfactory if ν is very small. However, one might try to obtain more accurate boundary conditions. For example, one can try to obtain boundary conditions that are satisfied by the exact solution up to order ν^2, say.

If one uses the approximation

$$\lambda_1 \approx -s + \nu s^2$$

one obtains the boundary condition

$$u_t - \nu u_{tt} + u_x = 0 \quad \text{at} \quad x = L \ .$$

We will show in the next section that the resulting $IBVP$ is badly unstable, however. (Basically this can be understood as follows. Write the boundary condition in Laplace space,

$$(-s + \nu s^2)\tilde{u} - \tilde{u}_x = 0 \quad \text{at} \quad x = L \ .$$

The real part of the coefficient $-s + \nu s^2$ has the wrong sign for large s.)

One can use instead a rational approximation of $\lambda_1(s)$,

$$\lambda_1 \approx -s(1 - \nu s) \approx \frac{-s}{1 + \nu s}$$

which results in the boundary condition

$$u_t + u_x + \nu u_{xt} = 0 \quad \text{at} \quad x = L \ .$$

One can show that the resulting IBVP is well–posed and stable.

The approximation of the exact boundary condition at inflow is easier. The exact condition is

$$(1 + \sqrt{1 + 4\nu s})\tilde{u} - 2\nu \tilde{u}_x = 0 \quad \text{at} \quad x = -L \ .$$

A reasonable approximation for $0 < \nu << 1$, which is valid uniformly for $\text{Re}\, s \geq 0$, is $\tilde{u} = 0$ at $x = -L$. This results in the inflow condition

$$u = 0 \quad \text{at} \quad x = -L \ .$$

6.3 Introduction to Normal Mode Analysis

Consider the differential equation (where $\nu > 0$)

$$u_t + u_x = \nu u_{xx}, \quad -L \leq x \leq L, \quad t \geq 0 \ ,$$

with the initial condition

$$u(x, 0) = f(x), \quad -L < x < L$$

and inflow boundary condition

$$u(-L, t) = 0, \quad t \geq 0 \ .$$

We consider the following three outflow boundary conditions at $x = L$:

$$\begin{aligned} u_t + u_x &= 0 && (6.5) \\ u_t - \nu u_{tt} + u_x &= 0 && (6.6) \\ u_t + \nu u_{xt} + u_x &= 0 && (6.7) \end{aligned}$$

We try to understand why the boundary condition (6.6) leads to bad instability whereas the other two conditions do not.

First ignore boundary and initial conditions and try to determine solutions of the differential equation of the form

$$u = e^{st+\kappa x}$$

where s and κ are complex parameters. These are so–called normal modes. Such a normal mode satisfies the differential equation if and only if

$$s + \kappa = \nu\kappa^2 \ . \qquad (6.8)$$

Equation (6.8) is called the dispersion relation of the differential equation since it relates a spatial wave number, κ, and a temporal frequency, s. One should note, however, that s and κ may both be complex.

Remark: The dispersion relation (6.8) and the characteristic equation (6.4) are the same equations if one identifies λ and κ. This is not accidental. When one solves the characteristic equation for λ, one determines spatial modes of the form $e^{\lambda x}$ for the homogeneous equation (6.2).

Modes $u = e^{st+\kappa x}$ with $\mathrm{Re}\, s > 0$ are growing in t and therefore describe unstable behavior. These modes are not allowed in the solution if one wants to have a stable IBVP.

If $\mathrm{Re}\, s > 0$ then the dispersion relation (6.8) has two solutions, $\kappa_{1,2}$, where

$$\mathrm{Re}\, \kappa_1 < 0 < \mathrm{Re}\, \kappa_2, \quad \kappa_j = \kappa_j(s) \ .$$

Consider an unstable mode of the form

$$u = e^{st+\kappa_2 x}, \quad \mathrm{Re}\, \kappa_2 > 0 \ . \qquad (6.9)$$

Such a mode grows in space as x increases. It should be eliminated by the boundary condition at $x = L$.

Remark: If one considers the Cauchy problem, then modes $e^{st+\kappa x}$ with $\mathrm{Re}\, \kappa \neq 0$ are not present in the solution since one *assumes* the solution to be bounded as $|x| \to \infty$. If one considers a problem on a finite interval, $-L < x < L$, then such modes are potentially present, however, and may lead to instabilities or even ill–posedness of the IBVP.

We now consider the boundary conditions (6.5) to (6.7) separately. For the mode (6.9) the boundary condition (6.5) requires

$$s + \kappa_2 = 0 \ .$$

However, since $\mathrm{Re}\ s > 0$ and $\mathrm{Re}\, \kappa_2 > 0$, this equation has no solution and we conclude that (6.5) does not allow an unstable mode of the form (6.9).

Similarly, the boundary condition (6.7) requires

$$s + \nu\kappa_2 s + \kappa_2 = 0 \ .$$

Since Re $s > 0$ and Re $\kappa_2 > 0$, this equation has no solution and we conclude that (6.7) does not allow an unstable mode of the form (6.9).

The boundary condition (6.6) requires

$$s - \nu s^2 + \kappa_2 = 0 \ .$$

Together with the dispersion relation one obtains the equations

$$s + \kappa_2 = \nu s^2 = \nu \kappa_2^2 \ .$$

It is easy to see that these equations are solved by

$$s = \kappa_2 = \frac{2}{\nu} \ .$$

For small ν one obtains the badly unstable normal mode

$$u = \exp(2(t + x)/\nu) \ .$$

Typically, it will be present if one uses the boundary condition (6.6) and will lead to a useless approximation of the solution of the Cauchy problem.

6.4 Exact Solution of the Model Problem

For $\nu = 0$ the exact solution is

$$u(x, t) = f(x - t) \ .$$

In this case

$$u(\xi + t, t) = f(\xi), \quad \xi \in \mathbb{R} \ .$$

Therefore, even if $\nu > 0$, one may expect that the following transformation simplifies the equation: Set

$$v(\xi, t) := u(\xi + t, t) \ .$$

(One can say that we have changed to a moving coordinate system.) We have

$$\begin{aligned} v_t(\xi, t) &= (u_t + u_x)(\xi + t, t) \\ v_{\xi\xi}(\xi, t) &= u_{xx}(\xi + t, t) \end{aligned}$$

Thus, for $v(\xi, t)$ we obtain the heat equation,

$$v_t(\xi, t) = \nu v_{\xi\xi}(\xi, t), \quad v(\xi, 0) = f(\xi) \ .$$

To transform the coefficient ν to one, we rescale time. Set

$$w(\xi,\tau) = v(\xi,\tau/\nu) \; .$$

Obtain:

$$\begin{aligned} w_\tau(\xi,\tau) &= \frac{1}{\nu}\, v_t(\xi,\tau/\nu) \\ w_{\xi\xi}(\xi,\tau) &= v_{\xi\xi}(\xi,\tau/\nu) \end{aligned}$$

The equation for w is

$$w_\tau = w_{\xi\xi}, \quad w(\xi,0) = f(\xi) \; .$$

Therefore,

$$w(\xi,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{-(\xi-y)^2/4\tau}\, f(y)\, dy \; .$$

Since $v(\xi,t) = w(\xi,\nu t)$ we have

$$v(\xi,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-(\xi-y)^2/4\nu t}\, f(y)\, dy \; .$$

Since $u(x,t) = v(x-t,t)$ we have

$$u(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{-(x-t-y)^2/4\nu t}\, f(y)\, dy \; .$$

Thus we have obtained an explicit solution of the Cauchy problem (6.1). It is not easy, however, to discuss the solution formula. For example, try to show that one obtains $u(x,t) = f(x-t)$ in the limit as $\nu \to 0$.

7 Strongly Hyperbolic Systems in 1D: The Cauchy problem and IBVPs

Consider

$$u_t + Au_x = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}, \quad t \geq 0 \; .$$

We assume that the Cauchy problem is well–posed, i.e., all eigenvalues of A are real and A can be diagonalized. Let

$$S^{-1}AS = \Lambda = diag(\lambda_1, \ldots, \lambda_n) \; .$$

Define new variables $v(x,t)$, so–called characteristic variables, by

$$u(x,t) = Sv(x,t) = \sum_j v_j(x,t) S_j \; .$$

Here S_j is the j–th column of S. Define transformed initial data $g(x)$ by $f(x) = Sg(x)$ and obtain

$$v_{jt} + \lambda_j v_{jx} = 0, \quad v_j(x,0) = g_j(x) \; .$$

Thus,

$$v_j(x,t) = g_j(x - \lambda_j t) \ .$$

The solution $u(x,t)$ of the Cauchy problem is

$$u(x,t) = \sum_j g_j(x - \lambda_j t) S_j \ .$$

The formula shows that the eigenvalues λ_j of A are the propagation speeds.

Now consider the problem in a strip,

$$u_t + Au_x = 0, \quad u(x,0) = f(x), \quad 0 \leq x \leq 1, \quad t \geq 0 \ .$$

We must add boundary conditions. Since we can transform to characteristic variables, we first consider the scalar problem

$$v_t + \lambda v_x = 0, \quad v(x,0) = g(x) \ ,$$

in the strip

$$0 \leq x \leq 1, \quad t \geq 0 \ .$$

A boundary condition must provide the solution at inflow. For $\lambda > 0$ inflow occurs at $x = 0$, and we can use

$$v(0,t) = g_0(t), \quad \lambda > 0 \ .$$

For $\lambda < 0$ inflow occurs at $x = 1$, and we can use

$$v(1,t) = g_1(t), \quad \lambda < 0 \ .$$

For $\lambda = 0$ a boundary condition is neither required nor allowed.

Now consider a strongly hyperbolic system $u_t + Au_x = 0$. For simplicity, assume that zero is not an eigenvalue of A. Let

$$S^{-1}AS = diag(\Lambda^I, \Lambda^{II}), \quad \Lambda^I < 0 < \Lambda^{II} \ ,$$

and partition v accordingly. A well–posed IBVP is obtained if at each boundary the ingoing characteristic variables are determined in terms of given data and the outgoing characteristic variables.

As an example, consider the linearized Euler equations

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t + A \begin{pmatrix} u \\ \rho \end{pmatrix}_x = 0$$

with

$$A = \begin{pmatrix} U & c^2/\rho_0 \\ \rho_0 & U \end{pmatrix} \ .$$

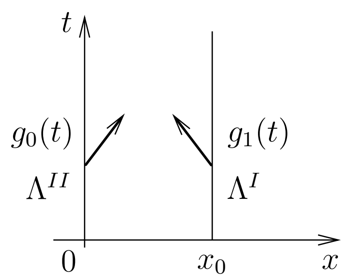

Figure 22: Hyperbolic systems in 1D

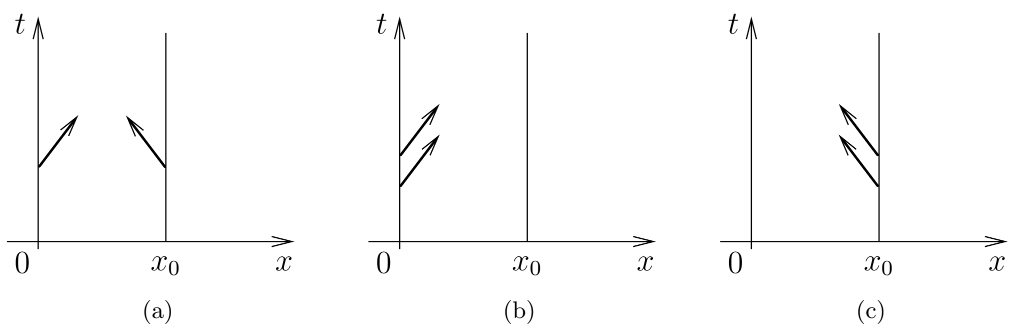

Figure 23: Base flow: (a) subsonic, (b) supersonic in positive x–direction, (c) supersonic in negative x–direction

Here $c > 0$ is the speed of sound, U is the velocity in x direction of the underlying flow, and $\rho_0 > 0$ is the density of the underlying flow. The eigenvalues of A are

$$\lambda_1 = U - c, \quad \lambda_2 = U + c \; .$$

Since $c > 0$ the eigenvalues are real and distinct, i.e., the given system is strictly hyperbolic. The correct number of boundary conditions is determined by the signs of the eigenvalues λ_j.

In the subsonic case, $|U| < c$, we have $\lambda_1 < 0 < \lambda_2$ and need one boundary condition at $x = 0$ and one boundary condition at $x = 1$. If $U > c$ the underlying flow is supersonic in positive x–direction. We have $0 < \lambda_1 < \lambda_2$ and need two boundary conditions at $x = 0$. If $U < -c$ the flow is supersonic in negative x–direction. We have $\lambda_1 < \lambda_2 < 0$ and need two boundary conditions at $x = 1$.

8 Extensions

8.1 Weak Solutions of Burgers' Equation

Consider Burgers' equation

$$u_t + \frac{1}{2}\,(u^2)_x = 0, \quad u(x,0) = f(x)$$

and first assume that $u(x,t)$ is a classical solution. Let

$$\phi : \mathbb{R}^2 \to \mathbb{R}$$

denote a test function, i.e., $\phi \in C_0^\infty(\mathbb{R}^2)$. Multiply the differential equation by $\phi(x,t)$ and integrate by parts to obtain

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + \frac{1}{2}\,u^2\phi_x)(x,t)\,dxdt + \int_{-\infty}^\infty f(x)\phi(x,0)\,dx = 0\ . \tag{8.1}$$

Assume now that $f \in L^1(\mathbb{R})$. If $u \in L^2_{loc}(\mathbb{R} \times [0,\infty)$ then the above integral is well defined and we can make the following definition:

Definition: A function $u \in L^2_{loc}(\mathbb{R} \times [0,\infty)$ is called a weak solution of Burgers' equation with initial data $f(x)$ if (8.1) holds for all $\phi \in C_0^\infty(\mathbb{R}^2)$.

8.2 Piecewise Smooth Weak Solutions of Burgers' Equation

We want to understand weak solutions $u(x,t)$ which are piecewise smooth and may have jump discontinuities.

a) Let $u(x,t)$ denote a weak solution which is smooth in an open set

$$\Omega \subset \mathbb{R} \times (0,\infty)\ .$$

If ϕ is a test function with support in Ω then

$$\begin{aligned} 0 &= \int_\Omega (u\phi_t + \frac{1}{2}u^2\phi_x)\,dxdt \\ &= -\int_\Omega (u_t + \frac{1}{2}(u^2)_x)\phi\,dxdt \end{aligned}$$

Since ϕ is an arbitray test function with support in Ω, it follows that the differential equation

$$u_t(x,t) + \frac{1}{2}(u^2)_x(x,t) = 0$$

holds classically in Ω.

b) We now want to understand a weak solution $u(x,t)$ which is smooth to the left and to the right of a curve

$$\Gamma \ : \ (\xi(t), t), \quad a \le t \le b \ ,$$

and which has a jump discontinuity across Γ.

For later reference we note that the vector

$$(\xi'(t), 1)$$

is tangent to Γ at the point

$$P = (\xi(t), t)$$

and the vector

$$\mathbf{n}(P) = \frac{1}{\sqrt{1 + (\xi'(t))^2}} \, (1, -\xi'(t))$$

is the unit normal to Γ at P pointing to the right. Clearly,

$$n_2(P) = -\xi'(t) n_1(P) \quad \text{for} \quad P = (\xi(t), t) \tag{8.2}$$

Let

$$\Omega \subset \mathbb{R} \times (0, \infty)$$

denote an open bounded set and let

$$\begin{aligned} \Omega_l &= \{(x,t) \in \Omega \ : \ x < \xi(t)\} \\ \Omega_r &= \{(x,t) \in \Omega \ : \ x > \xi(t)\} \end{aligned}$$

We assume that $u(x,t)$ is a weak solution which is smooth in Ω_l and Ω_r and has a jump discontinuity across $\Gamma \cap \Omega$.

For $P = (\xi(t), t) \in \Gamma \cap \Omega$ let

$$\begin{aligned} u_l(P) &= \lim_{x \to \xi(t)-} u(x,t) \\ u_r(P) &= \lim_{x \to \xi(t)+} u(x,t) \end{aligned}$$

Let ϕ denote a test function with support in Ω. We then have

$$0 = \int_\Omega (u\phi_t + \frac{1}{2} u^2 \phi_x) \, dxdt = \int_{\Omega_l} \ldots + \int_{\Omega_r} \ldots \ .$$

Here

$$\int_{\Omega_l} u^2 \phi_x dxdt = - \int_{\Omega_l} (u^2)_x \phi dxdt + \int_\Gamma u_l^2 \phi n_1 \, dP$$

and

$$\int_{\Omega_l} u \phi_t dxdt = - \int_{\Omega_l} u_t \phi dxdt + \int_\Gamma u_l \phi n_2 \, dP$$

If we now use that u is a classical solution in Ω_l and use (8.2), then we obtain

$$\int_{\Omega_l} (u\phi_t + \frac{1}{2} u^2 \phi_x) \, dxdt = \int_\Gamma \Big(\frac{1}{2} u_l^2 - u_l \xi'(t) \Big) n_1 \phi \, dP \ .$$

Similarly,

$$\int_{\Omega_r} (u\phi_t + \frac{1}{2} u^2 \phi_x) \, dxdt = - \int_\Gamma \Big(\frac{1}{2} u_r^2 - u_r \xi'(t) \Big) n_1 \phi \, dP \ .$$

Therefore,

$$\int_\Gamma \Big(\frac{1}{2} u_l^2 - u_l \xi'(t) \Big) n_1 \phi \, dP = \int_\Gamma \Big(\frac{1}{2} u_r^2 - u_r \xi'(t) \Big) n_1 \phi \, dP \ .$$

It follows that

$$\int_\Gamma \Big(\frac{1}{2} u_l^2 - \frac{1}{2} u_r^2 - (u_l - u_2) \xi'(t) \Big) n_1 \phi \, dP = 0 \ .$$

Since ϕ is an arbitray test function with support in Ω and since $n_1(P) \neq 0$ one obtains that

$$\frac{1}{2}(u_l^2 - u_r^2)(\xi(t), t) = \xi'(t)(u_l - u_r)(\xi(t), t) \quad \text{for all} \quad (\xi(t), t) \in \Gamma \cap \Omega \ .$$

We have proved the Rankine–Hugoniot jump condition:

$$\xi'(t) = \frac{1}{2}(u_l + u_r)(\xi(t), t) \ .$$

The condition relates the shock speed $\xi'(t)$ to the average between u_l and u_r at the point $(\xi(t), t)$.

One can prove that a piecewise smooth function $u(x, t)$, which has only jump discontinuities, is a weak solution of Burgers' equation if and only if the function solves Burgers' equation classically in the regions where it is smooth and satisfies the Rankine–Hugoniot condition along jump discontinuities.

Remark: If one generalizes Burgers' equation to

$$u_t + (F(u))_x = 0$$

then the Rankine–Hugoniot condition becomes

$$\xi'(t)(u_l - u_r)(\xi(t), t) = F(u_l(\xi(t), t)) - F(u_r(\xi(t), t)) \ .$$

Burgers equation is the special case where the flux function is

$$F(u) = \frac{1}{2}u^2 \ .$$

8.3 Examples

We consider Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \quad u(x,0) = f(x) \ ,$$

with different initial functions.

Example 1:

$$f(x) = \begin{cases} 1 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases}$$

The shock at $x = 0$ propages with speed $\frac{1}{2}$. The shock line is

$$(\xi(t), t) = (\frac{t}{2}, t) \ .$$

A weak solution is

$$u(x,t) = \begin{cases} 1 & \text{for} \quad x < \frac{t}{2} \\ 0 & \text{for} \quad x > \frac{t}{2} \end{cases} \ .$$

Example 2:

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

The function

$$u(x,t) = \begin{cases} 0 & \text{for} \quad x < \frac{t}{2} \\ 1 & \text{for} \quad x > \frac{t}{2} \end{cases}$$

is a weak solution, but it is *physically unreasonable* since characteristics leave the shock.

Instead, a physically reasonable weak solution is the rarefaction wave

$$u(x,t) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x/t & \text{for} \quad 0 < x < t \\ 1 & \text{for} \quad x > t \end{cases}$$

Remark: It is easy to check that the function $u(x,t) = x/t$ is a classical solution of Burgers' equation.

Example 3:

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Here a shock forms at $x = 1$ and a rarefaction wave forms at $x = 0$. The shock line is

$$(1 + \frac{t}{2}, t)$$

and the rarefaction wave x/t is valid for $0 < x < t$. The equation

$$t = 1 + \frac{t}{2}$$

holds for $t = 2$. At time $t = 2$ the rarefaction wave catches up with the shock wave. This happens at

$$(x, t) = (2, 2) \ .$$

After time $t = 2$ the shock line

$$(\xi(t), t)$$

is determined by the conditions

$$\xi'(t) = \frac{1}{2} \frac{\xi(t)}{t}, \quad \xi(2) = 2 \ .$$

One obtains that

$$\xi(t) = \sqrt{2t} \quad \text{for} \quad t \geq 2 \ .$$

The solution $u(x, t)$ is
for $0 < t \leq 2$:

$$u(x,t) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x/t & \text{for} \quad 0 < x < t \\ 1 & \text{for} \quad t < x < 1 + \frac{t}{2} \\ 0 & \text{for} \quad x > 1 + \frac{t}{2} \end{cases}$$

for $t \geq 2$:

$$u(x,t) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x/t & \text{for} \quad 0 < x < \sqrt{2t} \\ 0 & \text{for} \quad x > \sqrt{2t} \end{cases}$$

9 Existence of Nonlinear PDEs via Iteration

9.1 Setup

Let

$$f : \mathbb{R} \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$$

denote a C^∞–function and let $u_0 : \mathbb{R} \to \mathbb{R}$ be C^∞.

We consider the initial value problem

$$\begin{aligned} u_t &= u_{xx} + f(x, t, u, u_x), \quad x \in \mathbb{R}, \quad t \geq 0 \;, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R} \;. \end{aligned}$$

We assume that $u_0(x)$ and $f(x, t, u, u_x)$ are 1–periodic in x and we want to determine a solution $u(x, t)$ which is also 1–peripdic in x for each $0 \leq t \leq T$.

We will assume that f and all its derivatives are bounded.

We try to obtain a solution $u(x, t)$ as the limit of a sequence of functions $u^n(x, t)$, which is determined by the iteration (for $n = 0, 1, \ldots$):

$$\begin{aligned} u_t^{n+1} &= u_{xx}^{n+1} + f(x, t, u^n, u_x^n), \quad x \in \mathbb{R}, \quad t \geq 0 \;, \\ u^{n+1}(x, 0) &= u_0(x), \quad x \in \mathbb{R} \;, \end{aligned}$$

starting with

$$u^0(x, t) = u_0(x) \;.$$

We must first address the solution of linear problems:

$$\begin{aligned} u_t &= u_{xx} + G(x, t), \quad x \in \mathbb{R}, \quad 0 \leq t \leq T \;, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R} \;, \end{aligned}$$

were $G(x, t)$ is a C^∞ function which is 1–periodic in x for each $0 \leq t \leq T$.

The linear problem can be solved by Fourier expansion in x and Duhamel's principle.

One obtains that the sequence of functions

$$u^n(x, t), \quad n = 0, 1, \ldots$$

exists for $x \in \mathbb{R}, 0 \leq t \leq T$, and each function $u^n(x, t)$ has period 1 in x for each t.

Our aim is to show that the functions $u^n(x, t)$ converge with all their derivatives to a limit $u(x, t)$, which is the unique solution of the nonlinear problem.

For simplicity of presentation, we will assume that the nonlinear function f depends on u_x only, i.e., the PDE reads

$$u_t = u_{xx} + f(u_x) \ .$$

The existence of a C^∞ solution $u(x,t)$ will be proved in four steps:

1. We will prove a priori estimates for $u(x,t)$ and its derivatives in any finite time interval $0 \le t \le T$.

2. In a similar way, we will show that the functions $u^n(x,t)$ are uniformly smooth in any finite time interval $0 \le t \le T$. Precisely: For $j = 0, 1, \ldots$ and $k = 0, 1, \ldots$ and for any $T > 0$ there exists a constant $C = C(j,k,T)$ independent of n so that

$$\max \Big\{ \Big| \Big(\frac{\partial}{\partial x} \Big)^j \Big(\frac{\partial}{\partial t} \Big)^k u^n(x,t) \Big| \ : \ x \in \mathbb{R}, 0 \le t \le T \Big\} \le C(j,k,T \ .$$

3. Using the Arzela–Ascoli Theorem, we will show that there exists a function $u \in C^\infty(\mathbb{R} \times [0,T])$ and a subsequence of the sequence $u^n(x,t)$ which, along with all derivatives, converges to $u(x,t)$ in maximum norm.

4. We then use a contraction argument in a small time interval $0 \le t \le t_0$ to show that the whole sequence $u^n(x,t)$ (not just a subsequence) converges (with its derivatives) to $u(x,t)$ in $0 \le t \le t_0$. This implies that $u(x,t)$ solves the PDE in $0 \le t \le t_0$. One can restart the argument at $t = t_0$ and, in finitely many steps obtain a solution in $0 \le t \le T$. Since $T > 0$ is arbitrary, the argument proves existence of a solution for all $t \ge 0$.

9.2 Solution of Linear Problems via Fourier Expansion

9.3 Auxiliary Results

9.3.1 A Sobolev Inequality

Let $u : [a,b] \to \mathbb{R}, u \in C^1$. Define the norms

$$|u|_\infty = \max_{a \le x \le b} |u(x)|$$

and

$$\|u\|_{H^1}^2 = \int_a^b \Big(|u(x)|^2 + |u'(x)|^2 \Big) \, dx \ .$$

Theorem 9.1 *There exists a constant $C > 0$, depending on $b - a$ but independent of u, so that*

$$|u|_\infty \le C \|u\|_{H^1} \quad \textit{for all} \quad u \in C^1[a,b] \ .$$

Proof: Let

$$\begin{aligned} m : &= \min |u(x)| = |u(x_0)| \\ M : &= \max |u(x)| = |u(x_1)| = |u|_\infty \end{aligned}$$

We may assume that $a \le x_0 < x_1 \le b$. Then we have

$$u(x_1) = u(x_0) + \int_{x_0}^{x_1} u'(x)\, dx \ ,$$

thus

$$\begin{aligned} M &\le m + \int_a^b |u'(x)|\, dx \\ &\le m + \sqrt{b-a}\, \|u'\| \end{aligned}$$

Also,

$$\|u\|^2 \ge \int_a^b m^2\, dx = m^2(b-a) \ ,$$

thus

$$m \le \frac{\|u\|}{\sqrt{b-a}} \ .$$

Set $C = \max(\sqrt{b-a}, \frac{1}{\sqrt{b-a}})$ and obtain:

$$\begin{aligned} |u|_\infty &\le \frac{\|u\|}{\sqrt{b-a}} + \sqrt{b-a}\|u'\| \\ &\le C(\|u\| + \|u'\|) \\ &\le \sqrt{2}C\|u\|_{H^1} \end{aligned}$$

$\diamond$

9.3.2 Picard's Lemma

(from recursive estimates to an absolute estimate)

Lemma 9.1 *Let $\phi_n \in C[0,T]$ denote a sequence of nonnegative functions satisfying the recursive estimates*

$$\phi_{n+1}(t) \le a + b\int_0^t \phi_n(s)\, ds \quad \textit{for} \quad 0 \le t \le T \quad \textit{and} \quad n = 0, 1, \ldots$$

where $a \ge 0$ and $b > 0$. Then we have

$$\phi_n(t) \le ae^{bt} + \frac{b^n t^n}{n!}\, |\phi_0|_\infty \quad \textit{for} \quad 0 \le t \le T \quad \textit{and} \quad n = 0, 1, \ldots \tag{9.1}$$

In particular,

$$\phi_n(t) \le e^{bT}(a + |\phi_0|_\infty) \quad \textit{for} \quad 0 \le t \le T \quad \textit{and} \quad n = 0, 1, \ldots$$

Proof: The estimate holds for $n = 0$. Assuming that it holds for $\phi_n(t)$ we have

$$\begin{aligned} \phi_{n+1}(t) &\leq a + b\int_0^t \phi_n(s)\,ds \\ &\leq a + b\int_0^t ae^{bs}\,ds + b|\phi_0|_\infty \int_0^t \frac{b^n s^n}{n!}\,ds \\ &= a + ab\,\frac{1}{b}(e^{bt} - 1) + \frac{b^{n+1}t^{n+1}}{(n+1)!}\,|\phi_0|_\infty \\ &= ae^{bt} + \frac{b^{n+1}t^{n+1}}{(n+1)!}\,|\phi_0|_\infty \end{aligned}$$

This proves the lemma. $\diamond$

We next prove two simple results about sequences in a normed space. These results will be used together with the Arzela–Ascoli Theorem.

Theorem 9.2 *Let $(U, \|\cdot\|)$ denote a normed space and let b_n denote a sequence in U with the following properties:*

1) Every subsequence of b_n has a convergent subsequence.

2) All convergent subsequences of b_n converge to the same limit b.

Under these assumption we have $b_n \to b$.

Proof: Suppose that b_n does not converge to b. Then there exists $\varepsilon > 0$ with

$$\|b - b_{n_j}\| \geq \varepsilon$$

for a subsequence b_{n_j}. However, this subsequence does not have a subsequence converging to b, which contradicts the assumptions. $\diamond$

Theorem 9.3 *Let $(U, \|\cdot\|)$ denote a normed space and let b_n denote a Cauchy sequence in U. Assume that b_n has a subsequence converging to $b \in U$.*

Under these assumption we have $b_n \to b$.

Proof: Given $\varepsilon > 0$ there exists $N(\varepsilon)$ with

$$\|b_n - b_m\| \leq \frac{\varepsilon}{2} \quad \text{for} \quad n, m \geq N(\varepsilon)\ .$$

Also, there exists an index $n_j \geq N(\varepsilon)$ with

$$\|b - b_{n_j}\| \leq \frac{\varepsilon}{2}\ .$$

It follows that

$$\|b_n - b\| \leq \varepsilon \quad \text{for} \quad n \geq N(\varepsilon)\ .$$

$\diamond$

Theorem 9.4 *(Mean Value Theorem in Integral Form) Let $f : \mathbb{R} \to \mathbb{R}$ denote a C^1–function and let $p, q \in \mathbb{R}$. Then we have*

$$f(q) - f(p) = \Big(\int_0^1 f'(p + t(q-p))\, dt \Big) (q-p) \ .$$

Proof: Set

$$\begin{aligned} \alpha(t) &= p + t(q-p) \\ g(t) &= f(\alpha(t)) \end{aligned}$$

We have

$$\begin{aligned} f(q) - f(p) &= f(\alpha(1)) - f(\alpha(0)) \\ &= g(1) - g(0) \\ &= \int_0^1 g'(t)\, dt \\ &= \int_0^1 f'(\alpha(t))\alpha'(t)\, dt \\ &= \Big(\int_0^1 f'(p + t(q-p))\, dt \Big) (q-p) \end{aligned}$$

$\diamond$

The mean value theorem is often stated as

$$f(q) - f(p) = f'(\xi)(q-p)$$

for some ξ between p and q.

An advantage of the formula in Theorem 9.4 is that the term

$$\int_0^1 f'(p + t(q-p))\, dt$$

depends smoothly and p and q if f' is smooth. Also, it is easy to generalize Theorem 9.4 to C^1–functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

9.4 A Priori Estimates

Consider the PDE

$$u_t = u_{xx} + f(u_x) \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R} \; .$$

Assume that

$$u_0 \in C^\infty, \quad u_0(x+1) \equiv u_0(x)$$

and assume that $u(x,t)$ is a C^∞ solution with

$$u(x+1,t) \equiv u(x,t) \quad \text{for} \quad 0 \le t \le T \; .$$

Assume that $f \in C^\infty$ and that f and all its derivatives are bounded:

$$|f^{(j)}(v)| \le M_j \quad \text{for all} \quad v \in \mathbb{R} \quad \text{and} \quad j = 0, 1, \ldots$$

For simplicity, we assume that all functions are real valued.

We want to derive bounds for u and all its derivatives for $0 \le t \le T$.

We first estimate all space derivatives of u.

We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 &= (u, u_t) \\ &= -\|u_x\|^2 + (u, f(u_x)) \\ &\le \int_0^1 |u(x,t)| M_0 \, dx \\ &\le \|u\| M_0 \\ &\le \frac{1}{2}\|u\|^2 + \frac{1}{2}M_0^2 \end{aligned}$$

Thus we obtain a bound of the form

$$\max_{0 \le t \le T} \|u(\cdot,t)| \le C(T, M_0, \|u_0\|) \; .$$

Let $D = \partial/\partial x$. We have

$$u_{xt} = u_{xxx} + Df(u_x) \; .$$

Therefore,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_x(\cdot,t)\|^2 &= (u_x, u_{xt}) \\ &= -\|u_{xx}\|^2 - (u_{xx}, f(u_x)) \\ &\le -\|u_{xx}\|^2 + \|u_{xx}\| M_0 \\ &\le \frac{1}{2}M_0^2 \end{aligned}$$

Thus we obtain a bound of the form

$$\max_{0\le t\le T} \|u_x(\cdot,t)| \le C(T, M_0, \|u_{0x}\|) \ .$$

We have

$$u_{xxt} = u_{xxxx} + D^2 f(u_x) \ .$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}\,\|u_{xx}(\cdot,t)\|^2 &= (u_{xx}, u_{xxt}) \\
&= -\|u_{xxx}\|^2 - (u_{xxx}, Df(u_x)) \\
&= -\|u_{xxx}\|^2 - (u_{xxx}, f'(u_x)u_{xx}) \\
&\le -\|u_{xxx}\|^2 + M_1\|u_{xxx}\|\|u_{xx}\| \\
&\le \frac{1}{2}M_1^2\|u_{xx}\|^2
\end{aligned}$$

We obtain a bound of the form

$$\max_{0\le t\le T} \|u_{xx}(\cdot,t)| \le C(T, M_1, \|u_{0xx}\|) \ .$$

We have

$$D^3 u_t = D^5 u + D^3 f(u_x) \ .$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}\,\|D^3u(\cdot,t)\|^2 &= (D^3u, D^3u_t) \\
&= -\|D^4u\|^2 - (D^4u, D^2 f(u_x)) \\
&\le -\|D^4u\|^2 + \|D^4u\|\|D^2 f(u_x)\| \\
&\le \frac{1}{2}\|D^2 f(u_x)\|^2
\end{aligned}$$

Here

$$D^2 f(u_x) = f''(u_x)Du_xDu_x + f'(u_x)D^2u_x \ .$$

It follows that

$$\|D^2 f(u_x)\| \le C(|D^2u|_\infty\|D^2u\| + \|D^3u\|) \ .$$

Using Sobolev's inequality and the bound for $\max_t \|D^2u(\cdot,t)|$, we obtain that

$$\max_{0 \le t \le T} \|u_{xxx}(\cdot,t)\| \le C(T, M_0, M_1, M_2, \|u_0\|_{H^3}) \ .$$

We now make an induction argument. Let $j \ge 3$ and assume that we have shown bounds

$$\max_{0 \le t \le T} \|D^l u(\cdot,t)\| \le C(l,T) \quad \text{for} \quad 0 \le l \le j$$

where the constant $C(l,T)$ may also depend on the constants $M_0, \ldots, M_l$ and $\|u_0\|_{H^l}$. We want to bound

$$\max_{0 \le t \le T} \|D^{j+1} u(\cdot,t)\| \ .$$

We have

$$D^{j+1} u_t = D^{j+3} u + D^{j+1} f(u_x)$$

and obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|D^{j+1}u(\cdot,t)\|^2 &= (D^{j+1}u, D^{j+1}u_t) \\ &= -\|D^{j+2}u\|^2 - (D^{j+2}u, D^j f(u_x)) \\ &\le -\|D^{j+2}u\|^2 + \|D^{j+2}u\|\|D^j f(u_x)\| \\ &\le \frac{1}{2}\|D^j f(u_x)\|^2 \end{aligned}$$

We have

$$\begin{aligned} Df(u_x) &= f'(u_x)Du_x \\ D^2 f(u_x) &= f''(u_x)Du_x Du_x + f'(u_x)D^2 u_x \\ D^3 f(u_x) &= f'''(u_x)Du_x Du_x Du_x + 3f''(u_x)Du_x D^2 u_x + f'(u_x)D^3 u_x \end{aligned}$$

Using induction we obtain that

$$D^j f(u_x) = \sum_\alpha c(\alpha) f^{(l(\alpha))}(u_x)\, D^{\alpha_1}u_x \ldots D^{\alpha_k}u_x$$

where the sum extends over all multi–indices α with

$$\alpha_1 \ge \ldots \ge \alpha_k \ge 1 \quad \text{and} \quad \alpha_1 + \ldots + \alpha_k = j \ .$$

Also,

$$1 \le l(\alpha) \le j \ .$$

Consider a term

$$P_\alpha = D^{\alpha_1} u_x \dots D^{\alpha_k} u_x \ .$$

If

$$\alpha_1 = j$$

then

$$\|P_\alpha\| = \|D^{j+1} u\| \ .$$

If

$$\alpha_1 = j - 1$$

then $\alpha_2 = 1$. The product P_α is

$$P_\alpha = D^j u D^2 u$$

with

$$\|P_\alpha\| \le \|D^j u\| |D^2 u|_\infty \le C \|D^j u\| \|u\|_{H^3} \ .$$

The right-hand side is already estimated.

If $\alpha_1 \le j - 2$ then all terms in the product

$$P_\alpha = D^{\alpha_1} u_x \dots D^{\alpha_k} u_x$$

are already estimated in maximum norm. One obtains that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|D^{j+1} u(\cdot,t)\|^2 &\le \frac{1}{2} \|D^j f(u_x)\|^2 \\ &\le C_1 + C_2 \|D^{j+1} u(\cdot,t)\|^2 \end{aligned}$$

This completes the induction in j.

Using Sobolev's inequality, we have shown bounds

$$\max_{0 \le t \le T} |D^j u(\cdot,t)|_\infty \le C(j,T) \ .$$

We can now use the differential equation

$$u_t = u_{xx} + f(u_x)$$

and

$$D^j u_t = D^{j+2} u + D^j f(u_x)$$

to obtain bounds

$$\max_{0\le t\le T} |D^j u_t(\cdot,t)|_\infty \le \tilde{C}(j,T) \ .$$

Then differentiate the differential equation in t to obtain

$$u_{tt} = u_{txx} + f'(u_x)u_{tx}$$

etc.

To summarize, we have shown:

Theorem 9.5 *Let $u(x,t)$ denote a C^∞–solution of*

$$u_t = u_{xx} + f(u_x), \quad u(x,0) = U_0(x)$$

for

$$(x,t) \in \mathbb{R} \times [0,T]$$

which is 1–periodic in x for every $0 \le t \le T$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ function with

$$|f^{(j)}(v)| \le M_j \quad \textit{for} \quad v \in \mathbb{R} \quad \textit{and} \quad j = 0,1,\ldots$$

Then there are constants $C(k,l)$, depending only on

$$T \quad \textit{and} \quad M_0, M_1, \ldots \quad \textit{and} \quad \|D^j u_0\| \quad \textit{for} \quad j = 0,1,\ldots$$

with

$$\max\Big\{ \Big| \frac{\partial^{k+l}}{\partial x^k \partial t^l} u(x,t) \Big| \ : \ x \in \mathbb{R}, 0 \le t \le T \Big\} \le C(k,l) \ .$$

This holds for all $k = 0,1,\ldots$ and all $l = 0,1,\ldots$

9.5 Uniform Smoothness of the Iterates

We make the same assumptions on f and u_0 as above and consider the sequence of functions

$$u^n \in C^\infty(\mathbb{R} \times [0,T])$$

defined in Section 9.2. Our aim is to prove that estimates as in the previous theorem hold for the sequence u^n with constants $C(k,l)$ independent of n. I.e.,

$$\max\Big\{ \Big| \frac{\partial^{k+l}}{\partial x^k \partial t^l} u^n(x,t) \Big| \ : \ x \in \mathbb{R}, 0 \le t \le T \Big\} \le C(k,l) \quad \text{for} \quad n = 0,1\ldots$$

for all $k = 0,1,\ldots$ and all $l = 0,1,\ldots$

It is easy to estimate

$$\max_{0\le t\le T}\|D^j u^{n+1}(\cdot,t)\|$$

for $j=0$ and $j=1$.

Consider the case $j=2$. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|D^2u^{n+1}\|^2 &= -\|D^3u^{n+1}\|^2-(D^3u^{n+1},Df(u^n_x))\\ &\le \|Df(u^n_x)\|^2\\ &\le M_1^2\|D^2u^n\|^2\end{aligned}$$

Integration in t yields the estimate

$$\|D^2u^{n+1}(\cdot,t)\|^2\le\|D^2u_0\|^2+C\int_0^t\|D^2u^n(\cdot,s)\|^2\,ds\ .$$

Applying Picard's Lemma, we obtain the bound

$$\max_{0\le t\le T}\|D^2u^n(\cdot,t)\|\le C_2\quad\text{for}\quad n=0,1,\ldots$$

with C_2 independent of n. Proceeding as for the a priori estimates and applying Picard's Lemma, one obtains bounds

$$\max_{0\le t\le T}\|D^ju^n(\cdot,t)\|\le C_j\quad\text{for}\quad n=0,1,\ldots\quad\text{and}\quad j=0,1,\ldots$$

with constants C_j independent of n. These bounds imply bounds for all space derivatives of the $u^n(x,t)$ in maximum norm, with constants independent of n. Then the differential equations imply bounds for time derivatives etc.

9.6 Application of Arzela–Ascoli

The following theorem will be applied to the sequence $u^n(x,t)$ for (x,t) in the compact set

$$\Omega=[0,1]\times[0,T]$$

and to the derivatives of u^n.

Theorem 9.6 *(Arzela–Ascoli) Let Ω denote a compact subset of $\mathbb{R}^s$ and let $u_n\in C(\Omega)$ denote a sequence of functions with the following two properties:*

1) For every $\varepsilon>0$ there is $\delta>0$, independent of n, so that

$$|u_n(x)-u_n(y)|\le\varepsilon\quad\textit{for all}\quad x,y\in\Omega\quad\textit{with}\quad|x-y|\le\delta\ .$$

2) There is a constant K, independent of n, so that

$$\max_{x\in\Omega} |u_n(x)| \le K \ .$$

Then there exists $u \in C(\Omega)$ *and a subsequence* u_{n_j} *with*

$$|u - u_{n_j}|_\infty \to 0 \quad \textit{as} \quad n_j \to \infty \ .$$

In the following, let $\Omega = [0,1] \times [0,T]$. Clearly, the Theorem of Arzela–Ascoli applies to u^n and to every derivative of u^n. Thus, there exists $u \in C(\Omega)$ and a subsequence

$$u^n, \quad n \in \mathbb{N}_1$$

with

$$|u - u^n|_\infty \to 0, \quad n \in \mathbb{N}_1 \ .$$

Now apply the Theorem of Arzela–Ascoli to the sequence

$$u^n_x, \quad n \in \mathbb{N}_1$$

and obtain: There exists $v \in C(\Omega)$ and a subsequence $\mathbb{N}_2 \subset \mathbb{N}_1$ so that

$$|v - u^n_x|_\infty \to 0, \quad n \in \mathbb{N}_2 \ .$$

In the identity

$$u^n(x,t) - u^n(0,t) = \int_0^x u^n_x(\xi,t)\, d\xi$$

we let $n \to \infty, n \in \mathbb{N}_2$, and obtain that

$$u(x,t) - u(0,t) = \int_0^x v(\xi,t) \ .$$

This yields existence of u_x and

$$v = u_x \ .$$

We now apply Theorem 9.2 to the sequence

$$b_n = u^n_x \in C(\Omega), \quad n \in \mathbb{N}_1 \ ,$$

where we use the maximum norm on $C(\Omega$. We obtain that

$$|u_x - u^n_x|_\infty \to 0, \quad n \in \mathbb{N}_1 \ .$$

This argument can be repeated for all derivatives. We have proved that $u \in C^\infty(\Omega)$, and every derivative of u^n converges to the corresponding derivative of u in maximum norm over Ω. Here we let $n \to \infty$ and $n \in \mathbb{N}_1$.

9.7 Convergence of the Whole Sequence

We will show that there is an interval $0 \le t \le t_0$ with $t_0 > 0$ so that the *whole* sequence u^n (not just a subsequence) converges to u in maximum norm on

$$[0,1] \times [0, t_0] \; .$$

The arguments given above then show that every derivative of u^n also converges to the corresponding derivative of u as $n \to \infty$. This will allow us to let $n \to \infty$ in the equation

$$u_t^{n+1} = u_{xx}^{n+1} + f(u_x^n)$$

to obtain that

$$u_t = u_{xx} + f(u_x) \quad \text{for} \quad 0 \le t \le t_0 \; .$$

Consider two consecutive iterates

$$\begin{aligned} u_t^n &= u_{xx}^n + f(u_x^{n-1}) \\ u_t^{n+1} &= u_{xx}^{n+1} + f(u_x^n) \end{aligned}$$

Set

$$v = u^{n+1} - u^n, \quad w = u^n - u^{n-1}$$

and obtain that

$$v_t = v_{xx} + f(u_x^n) - f(u_x^{n-1}) \; .$$

By the Mean Value Theorem we can write

$$f(u_x^n(x,t)) - f(u_x^{n-1}(x,t)) = \alpha(x,t,n) w_x(x,t)$$

where $\alpha(x,t,n)$ and all derivatives of these functions are bounded independently of n. The equation

$$v_t = v_{xx} + \alpha w_x$$

yields that

$$\frac{1}{2} \frac{d}{dt} \|v(\cdot,t)\|^2 = -\|v_x\|^2 + (v, \alpha w_x) \; .$$

Here

$$(v, \alpha w_x) = (\alpha v w_x) = (-\alpha v_x, w) - (\alpha_x, v, w) \; ,$$

and, therefore,

$$(v, \alpha w_x) \le C_1(\|v\| + \|v_x\|)\|w\| \ .$$

One obtains that there is a constant K, independent of n, with

$$\frac{d}{dt}\,\|v(\cdot,t)\|^2 \le K(\|v\|^2 + \|w\|^2) \quad \text{for} \quad 0 \le t \le T \ .$$

Since $v(x,0) = 0$ obtain that

$$\|v(\cdot,t)\|^2 \le K \int_0^t e^{K(t-s)} \|w(\cdot,s)\|^2 \, ds \ .$$

Therefore, if $t_0 >$ is sufficiently small:

$$\max_{0 \le t \le t_0} \|v(\cdot,t)\| \le \frac{1}{2} \max_{0 \le t \le t_0} \|w(\cdot,t)\| \ ,$$

where t_0 depends only on K. We have shown that

$$\max_{0 \le t \le t_0} \|u^{n+1}(\cdot,t) - u^n(\cdot,t)\| \le \frac{1}{2} \max_{0 \le t \le t_0} \|u^n(\cdot,t) - u^{n-1}(\cdot,t)\| \ ,$$

and, therefore,

$$\max_{0 \le t \le t_0} \|u^{n+1}(\cdot,t) - u^n(\cdot,t)\| \le 2^{-n} C \ .$$

Clearly, this implies that the sequence of functions

$$u^n(\cdot,t) \in C[0,1]$$

is a Cauchy sequence w.r.t. $\| \cdot \|$ for every fixed $0 \le t \le t_0$. Since $u^{n_j}(\cdot,t)$ is a convergent subsequnce (converging to $u(\cdot,t)$) it follows from Theorem 9.3 that

$$\|u(\cdot,t) - u^n(\cdot,t)\| \to 0 \quad \text{as} \quad n \to \infty \ .$$

We can now apply Theorem 9.2 to the sequence

$$u^n(\cdot,t) \in C[0,1]$$

w.r.t. $| \cdot |_\infty$ and obtain that

$$|u(\cdot,t) - u^n(\cdot,t)|_\infty \to 0 \quad \text{as} \quad n \to \infty \ .$$

10 Appendix 1: Fourier Expansion

Fourier Expansion of 2π–peridoic functions. For $f, g \in L_2(-\pi, \pi)$ introduce the innerproduct and norm by

$$(f,g) = \int_{-\pi}^{\pi} \bar{f}(x) g(x)\, dx, \quad \|f\|^2 = (f,f) \ .$$

The sequence of functions

$$\phi_k(x) = (2\pi)^{-1/2}\, e^{ikx}, \quad k \in \mathbb{Z} \ .$$

is orthonormal in this space,

$$(\phi_j, \phi_k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-j)x}\, dx = \delta_{jk} \ .$$

Let $f \in L_2(-\pi, \pi)$ and assume, tentatively, that we can write f as a series,

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \phi_x(x) \ .$$

If we take the innerproduct with ϕ_j and use the orthonormality of the ϕ_k, we obtain that

$$(\phi_j, f) = a_j \ .$$

One defines the sequence of Fourier coefficients of f by

$$\hat{f}(k) = (\phi_k, f) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} e^{-iky} f(y)\, dx$$

and calls

$$\sum_k \hat{f}(k) \phi_k(x)$$

the Fourier series of f. One can prove that the series converges to f in the mean.

11 Appendix 2: Fourier Transformation

11.1 Fourier Transform on the Schwartz Space

The Schwartz Space $\mathcal{S}$: The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ consists of all functions $f : \mathbb{R}^N \to \mathbb{C}$ where $f \in C^\infty$ and

$$\sup_x |x|^j |D^\alpha f(x)| =: p(j, \alpha, f) < \infty$$

for all $j = 0, 1, \ldots$ and all multi–indices α. One says that f and all its derivatives are rapidly decaying.

Each function $f \to p(j, \alpha, f)$ is a seminorm on $\mathcal{S}$. Convergence in $\mathcal{S}$ is defined as follows: If $f_n, f \in \mathcal{S}$ then

$$f_n \to f \quad \text{in} \quad \mathcal{S}$$

means that for all $j = 0, 1, \ldots$ and all multi–indices α it holds that

$$p(j, \alpha, f_n - f) \to 0 \quad \text{as} \quad n \to \infty \; .$$

The Fourier Transform and Its Inverse on $\mathcal{S}$: For $f \in \mathcal{S}$ the Fourier transform is defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^{N/2}} \int e^{-ik\cdot x} f(x)\, dx \quad \text{for} \quad k \in \mathbb{R}^N \; .$$

Here, and below, the integral extends over $\mathbb{R}^N$ and

$$k \cdot x = \sum_{j=1}^N k_j x_j$$

denotes the usual scalar product in $\mathbb{R}^N$.

We also use the notation

$$\hat{f}(k) = (\mathcal{F} f)(k)$$

and call $\mathcal{F}$ the Fourier transform operator on $\mathcal{S}$.

One can prove that $\hat{f} \in \mathcal{S}$ and the following Fourier representation holds for all $f \in \mathcal{S}$:

$$f(x) = \frac{1}{(2\pi)^{N/2}} \int e^{ik\cdot x} \hat{f}(k)\, dk \quad \text{for} \quad x \in \mathbb{R}^N \; .$$

This implies that the operator

$$\mathcal{F} : \mathcal{S} \to \mathcal{S}$$

is a bijection and its inverse is given by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^{N/2}} \int e^{ik\cdot x}\, g(k)\, dk \quad \text{for} \quad x \in \mathbb{R}^N \ .$$

Complex Conjugates: Let $B : \mathcal{S} \to \mathcal{S}$ denote the operator given by $Bf = \bar{f}$ where $\bar{f}(x)$ denotes the complex conjugate of $f(x)$. We have

$$\begin{aligned} B\hat{f}(k) &= \frac{1}{(2\pi)^{N/2}} \int e^{ik\cdot x}\, Bf(x)\, dx \\ &= (\mathcal{F}^{-1}Bf)(k) \end{aligned}$$

This says that

$$B\mathcal{F} = \mathcal{F}^{-1}B \ . \tag{11.1}$$

Fourier Transform and the L_2–Inner Product: Let $u, v \in \mathcal{S}$. We have

$$\begin{aligned} \int (\mathcal{F}u)(x)v(x)\, dx &= \frac{1}{(2\pi)^{N/2}} \int\int e^{-ix\cdot y}\, u(y)\, dy\, v(x)\, dx \\ &= \frac{1}{(2\pi)^{N/2}} \int u(y) \int e^{-ix\cdot y} v(x)\, dx dy \\ &= \int u(y)(\mathcal{F}v)(y)\, dy \end{aligned}$$

This shows the important simple rule:

$$\int (\mathcal{F}u)v\, dx = \int u(\mathcal{F}v)\, dx \ , \tag{11.2}$$

i.e., in the above integrals we may move the operator $\mathcal{F}$ from one factor to the other.

The L_2–inner product on $L_2(\mathbb{R}^N)$ is defined by

$$(u, v) = (u, v)_{L_2} = \int \bar{u}(x)v(x)\, dx$$

and

$$\|u\| = \sqrt{(u, u)}$$

denotes the L_2–norm.

For $u, v \in \mathcal{S}$ we have

$$\begin{aligned} (\mathcal{F}u, \mathcal{F}v) &= \int (B\mathcal{F}u)(\mathcal{F}v)\, dx \\ &= \int (\mathcal{F}B\mathcal{F}u)v\, dx \end{aligned}$$

By (11.1) we have

$$\mathcal{F}B\mathcal{F} = B$$

and obtain that

$$(\mathcal{F}u, \mathcal{F}v) = (u, v) \ .$$

We have proved Parseval's relation:

Lemma 11.1 *For all $u, v \in \mathcal{S}$ we have*

$$(\mathcal{F}u, \mathcal{F}v) = (u, v)$$

and

$$\|\mathcal{F}u\| = \|u\| \ .$$

Convolution: For $u, v \in \mathcal{S}(\mathbb{R}^N)$ define the convolution by

$$(u * v)(x) = \int u(x-y)v(y)\, dy, \quad x \in \mathbb{R}^N \ .$$

One can show that $u * v \in \mathcal{S}$. We have

$$\begin{aligned} (\mathcal{F}(u*v))(k) &= \frac{1}{(2\pi)^{N/2}} \int e^{-ik\cdot x}\, (u*v)(x)\, dx \\ &= \frac{1}{(2\pi)^{N/2}} \int\int e^{-ik\cdot(x-y+y)}\, u(x-y)v(y)\, dydx \\ &= \int \Big(\frac{1}{(2\pi)^{N/2}} \int e^{-ik\cdot(x-y)}\, u(x-y)\, dx \Big)\, e^{-ik\cdot y}\, v(y)\, dy \\ &= \hat{u}(k) \int e^{-ik\cdot y}\, v(y)\, dy \\ &= (2\pi)^{N/2}\, \hat{u}(k)\hat{v}(k) \end{aligned}$$

This proves:

Lemma 11.2 *If $u, v \in \mathcal{S}(\mathbb{R}^N)$ then*

$$(\mathcal{F}(u*v))(k) = (2\pi)^{N/2}\, (\mathcal{F}u)(k)(\mathcal{F}v)(k), \quad k \in \mathbb{R}^N \ ,$$

or

$$(u*v)\hat{}\,(k) = (2\pi)^{N/2}\, \hat{u}(k)\hat{v}(k), \quad k \in \mathbb{R}^N \ .$$

11.2 Tempered Distributions

Definition: A linear functional

$$g : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$$

is called continuous if

$$f_n \to f \quad \text{in} \quad \mathcal{S} \quad \text{implies} \quad g(f_n) \to g(f) \quad \text{in} \quad \mathbb{C} \ .$$

A continuous linear functional $g : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ is called a tempered distribution on $\mathbb{R}^N$. By definition, the space

$$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^N)$$

consists of all continuous linear functionals $g : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$. We use the notation

$$\langle g, f \rangle = g(f) \quad \text{for} \quad g \in \mathcal{S}', f \in \mathcal{S} \ .$$

Clearly, $\mathcal{S}'$ is a linear vector space over $\mathbb{C}$, called the dual of the Schwartz space $\mathcal{S}$. One uses the following convergence concept in $\mathcal{S}'$: If $g_n, g \in \mathcal{S}'$ then

$$g_n \to g \quad \text{in} \quad \mathcal{S}' \quad \text{means that} \quad \langle g_n, f \rangle \to \langle g, f \rangle \quad \text{for all} \quad f \in \mathcal{S} \ .$$

12 Appendix 3: Fundamental Solution of Poisson's Equation via Fourier Transform

We have shown that the equation

$$-\Delta\Phi = \delta_0$$

has the solution (for $n \geq 3$):

$$\Phi(x) = \frac{c_n}{|x|^{n-2}}$$

with

$$c_n = \frac{1}{(n-2)\omega_n}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \ ,$$

thus

$$\begin{aligned} c_n &= \pi^{-n/2}\frac{\Gamma(n/2)}{2(n-2)} \\ &= \frac{1}{4}\pi^{-n/2}\Gamma(\frac{n}{2}-1) \end{aligned}$$

In this Appendix we want to derive the formula for $\Phi(x)$ via Fourier transformation.
The Fourier transform of the equation

$$-\Delta\Phi = \delta_0$$

reads

$$|k|^2\hat{\Phi} = (2\pi)^{-n/2} \ ,$$

thus

$$\hat{\Phi}(k) = (2\pi)^{-n/2}|k|^{-2} \ .$$

We have to compute the inverse Fourier transform of the locally integrable function $|k|^{-2}$.
Recall:

$$\begin{aligned} \mathcal{F}(e^{-\alpha x^2})(k) &= (2\alpha)^{-1/2}e^{-k^2/4\alpha} \quad (\text{in } 1D) \\ \mathcal{F}(e^{-\alpha|x|^2})(k) &= (2\alpha)^{-n/2}e^{-|k|^2/4\alpha} \quad (\text{in } nD) \\ \mathcal{F}^{-1}(e^{-|k|^2/4\alpha})(x) &= (2\alpha)^{n/2}e^{-\alpha|x|^2} \quad (\text{in } nD) \end{aligned}$$

Setting

$$s = \frac{1}{4\alpha}, \quad 2\alpha = (2s)^{-1}$$

the last formula reads

$$\mathcal{F}^{-1}(e^{-s|k|^2})(x) = (2s)^{-n/2} e^{-|x|^2/4s} \quad (\text{in } nD) \ .$$

We now write

$$|k|^{-2} = \int_0^\infty e^{-s|k|^2}\, ds \ .$$

We now assume that we may apply $\mathcal{F}^{-1}$ under the integral sign and obtain that

$$\mathcal{F}^{-1}(|k|^{-2})(x) = \int_0^\infty (2s)^{-n/2}\, e^{-|x|^2/4s}\, ds =: Int \ .$$

In the integral we substitute

$$\frac{|x|^2}{4s} = q, \quad s = \frac{|x|^2}{4}\, q^{-1}, \quad ds = -\frac{|x|^2}{4}\, q^{-2} dq \ .$$

This yields that

$$\begin{aligned} Int &= 2^{-n/2} \Big(\frac{|x|^2}{4} \Big)^{-n/2} \frac{|x|^2}{4} \int_0^\infty q^{\frac{n}{2}-2}\, e^{-q}\, dq \\ &= \frac{1}{|x|^{n-2}}\, 2^{n/2}\, \frac{1}{4}\, \Gamma(\frac{n}{2}-1) \end{aligned}$$

and

$$\begin{aligned} \Phi(x) &= (2\pi)^{-n/2} \mathcal{F}^{-1}(|k|^{-2})(x) \\ &= (2\pi)^{-n/2} \frac{1}{|x|^{n-2}}\, 2^{n/2}\, \frac{1}{4}\, \Gamma(\frac{n}{2}-1) \\ &= \frac{1}{4}\, \pi^{-n/2} \Gamma(\frac{n}{2}-1)\, \frac{1}{|x|^{n-2}} \end{aligned}$$

This confirms our previous result for the fundamental solution $\Phi(x)$ of Poisson's equation.

13 Appendix 4: The Ray Equation

13.1 Derivation

Let $\psi(x,y)$ denote a solution of the eikonal equation,

$$(\psi_x)^2 + (\psi_y)^2 = n^2(x,y) \; . \tag{13.1}$$

If the amplitude $A(x,y)$ solves

$$2A_x\psi_x + 2A_y\psi_y + A\Delta\psi = 0$$

then

$$v(x,y,t) = A(x,y)e^{i(k\psi(x,y)-\omega t)} \tag{13.2}$$

approximately solves the wave equation,

$$v_{tt} = c^2(x,y)\Delta v, \quad c(x,y) = \frac{\omega/k}{n(x,y)} \; . \tag{13.3}$$

Therefore, the (x,y)–curves

$$\psi(x,y) = const \tag{13.4}$$

are called wave fronts and the parametrized lines

$$\mathbf{R}(s) = (x(s), y(s)) \; , \tag{13.5}$$

which are orthogonal to the wave fronts, are called rays. Intuitively, the wave (13.2) propagates along the rays. We want to determine an ODE system for the rays $\mathbf{R}(s)$.

Recall the characteristic system for the eikonal equation,

$$\begin{aligned}
\frac{dx}{ds} &= 2\psi_x(x,y) \\
\frac{dy}{ds} &= 2\psi_y(x,y) \\
\frac{d\psi}{ds} &= 2n^2(x,y) \\
\frac{d\psi_x}{ds} &= 2n_x(x,y) \\
\frac{d\psi_y}{ds} &= 2n_y(x,y)
\end{aligned}$$

Using the notation (13.5) we obtain that

$$\begin{aligned}\frac{d}{ds}\mathbf{R}(s) &= 2\nabla\psi(\mathbf{R}(s)) \qquad (13.6)\\ \frac{d}{ds}\nabla\psi(s) &= 2\nabla n(\mathbf{R}(s)) \qquad (13.7)\end{aligned}$$

Here $\psi(s)$ and $\psi(\mathbf{R}(s))$ are identical.

Equation (13.6) implies that any characteristic curve (13.5) is in fact orthogonal to any wave front (13.4), i.e., any characteristic curve is a ray.

From (13.6) and (13.7) one obtains that

$$\frac{d^2}{ds^2}\mathbf{R}(s) = 4\nabla n(\mathbf{R}(s)) \ .$$

These second order equations for $\mathbf{R}(s)$, which can also be written as

$$\mathbf{R}''(s) = 2\nabla n^2(\mathbf{R}(s)) \ , \qquad (13.8)$$

are called the ray equations.

If the index of refraction $n(x, y)$ is a known function of the spatial variable (x, y) then the equations (13.8) determine the rays $\mathbf{R}(s)$ of wave propagation. Recall, however, that the function (13.2) only approximately solves the wave equation (13.3). The ray equations yield the geometrical optics approximation for wave propagation.

13.2 Snell's Law

Assume that the index of refraction $n(x, y)$ has a jump discontinuity at $x = 0$, but is constant for $x < 0$ and $x > 0$:

$$n(x, y) = \begin{cases} n_{left} = n_l & \text{for} \quad x < 0 \\ n_{right} = n_r & \text{for} \quad x > 0 \end{cases}$$

The eikonal equation for the phase function $\psi(x, y)$ becomes

$$\begin{aligned}(\psi_{lx})^2 + (\psi_{ly})^2 &= n_l^2 \quad \text{for} \quad x < 0\\ (\psi_{rx})^2 + (\psi_{ry})^2 &= n_r^2 \quad \text{for} \quad x > 0\end{aligned}$$

and at $x = 0$ we impose the continuity condition

$$\psi_l(0, y) = \psi_r(0, y) \quad \text{for} \quad y \in \mathbb{R} \ . \qquad (13.9)$$

Since the rays are straight lines in the regions $x < 0$ and $x > 0$ and since the wave fronts described by $\psi(x, y) = const$ are orthogonal to the rays, the functions $\psi_l(x, y)$ (for $x < 0$) and $\psi_r(x, y)$ (for $x > 0$) are linear functions of (x, y). Then, solving the eikonal equation, we may assume that

$$
\begin{aligned}
\psi_l(x,y) &= n_l(x\cos\alpha + y\sin\alpha) \quad \text{for} \quad x<0 \\
\psi_r(x,y) &= n_r(x\cos\beta + y\sin\beta) \quad \text{for} \quad x>0
\end{aligned}
$$

The continuity requirement (13.9) yields that

$$n_l \sin\alpha = n_r \sin\beta \; . \tag{13.10}$$

The angles α and β are the angles between that ray directions and the normal to the surface $x = 0$. Equation (13.10) is called Snell's law.

13.3 Electromagnetic Waves in the Atmosphere

The index of refraction for electromagnetic waves in the atmosphere is approximately given by the formula

$$n^2(x,y) = n_0^2(1+ay)$$

where

$$1.00025 \le n_0 \le 1.0004 \quad \text{and} \quad a = -\frac{0.039}{meter} \; .$$

Here $y \ge 0$ is the height above sea level. The ray equations become

$$
\begin{aligned}
x''(s) &= 0 \\
y''(s) &= \frac{n_0^2 a}{2}
\end{aligned}
$$

For a ray $\mathbf{R}(s) = (x(s), y(s))$ passing through the origin one obtains that

$$
\begin{aligned}
x(s) &= s\cos\alpha \\
y(s) &= s\sin\alpha + \frac{n_0^2 a}{4}\, s^2
\end{aligned}
$$

where α is the angle between the ray and the horizontal line $y = 0$. If one eliminates the parameter s and expresses y as a function of x, one obtains the parabola

$$y(x) = x\tan\alpha + \frac{n_0^2 a}{4\cos^2\alpha}\, x^2 \; .$$

14 Appendix 5: The Wave Equation via Fourier Transform

14.1 General Remarks on the Cauchy Problem

Consider the Cauchy problem

$$u_{tt} = c^2 \Delta u, \quad u(x,0) = 0, \quad u_t(x,0) = h(x) \ . \tag{14.1}$$

If $u(x,t)$ denotes its solution and we set

$$v(x,t) = u_t(x,t)$$

then the function $v(x,t)$ satisfies

$$v_{tt} = c^2 \Delta v, \quad v(x,0) = h(x), \quad v_t(x,0) = 0 \ . \tag{14.2}$$

Therefore, it suffices to solve the wave equation with initial data of the form

$$u(x,0) = 0, \quad u_t(x,0) = h(x) \ .$$

The 1D Case: The problem

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = h(x)$$

has the solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy$$

and

$$u_t(x,t) = \frac{1}{2}\Big(h(x+ct) + h(x-ct) \Big) \ .$$

This yields that the solution of

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = g(x), \quad u_t(x,0) = h(x)$$

is

$$u(x,t) = \frac{1}{2}\Big(g(x+ct) + g(x-ct) \Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy \ .$$

which is d'Alembert's formula.

The 3D Case: The problem

$$u_{tt} = c^2 \Delta u, \quad u(x,0) = 0, \quad u_t(x,0) = h(x)$$

has the solution

$$\begin{aligned} u(x,t) &= \frac{t}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} h(y)\, dS(y) \\ &= \frac{t}{4\pi} \int_{\partial B_1} h(x+ctz)\, dS(z) \end{aligned}$$

One obtains that

$$\begin{aligned} u_t(x,t) &= \frac{1}{4\pi} \int_{\partial B_1} \Big(h(x+ctz) + ctz \cdot (\nabla h)(x+ctz) \Big)\, dS(z) \\ &= \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} \Big(h(y) + (y-x) \cdot \nabla h(y) \Big)\, dS(y) \end{aligned}$$

One obtains Kirchhoff's formula for the solution of

$$u_{tt} = c^2 \Delta u, \quad u(x,0) = g(x), \quad u_t(x,0) = h(x) \ .$$

The solution is

$$u(x,t) = \frac{1}{4\pi c^2 t^2} \int_{\partial B_{ct}(x)} \Big(g(y) + (y-x) \cdot \nabla g(y) + t h(y) \Big)\, dS(y) \ .$$

14.2 The 1D Case via Fourier Transform

Consider the equation

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = h(x) \tag{14.3}$$

with solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy \ .$$

To solve the problem via Fourier transform, we first consider

$$h(x) = e^{ikx}$$

and determine a solution of the form

$$u(x,t) = a(t) e^{ikx} \ .$$

One obtains the amplitude equation

$$a''(t) + c^2 k^2 a(t) = 0$$

and the initial conditions

$$a(0) = 0, \quad a'(0) = 1$$

yield

$$a(t) = \frac{\sin(ckt)}{ck} \; .$$

Therefore,

$$u(x,t) = \frac{\sin(ckt)}{ck}\, e^{ikx} \; .$$

Next let $h \in \mathcal{S}$. Then $h(x)$ has the Fourier representation

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) e^{ikx}\, dk \; .$$

For the solution of (14.3) one obtains that

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) \frac{\sin(ckt)}{ck}\, e^{ikx}\, dk \; .$$

From this formula it is not obvious that there is a finite speed c of propagation.

First derivation of d'Alembert's formula: We have

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_{-\infty}^{\infty} \hat{h}(k) \frac{1}{ck}\, (e^{ikct} - e^{-ikct}) e^{ikx}\, dk \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2c} \int_{-\infty}^{\infty} \hat{h}(k) \frac{1}{ik}\, (e^{ik(x+ct)} - e^{ik(x-ct)})\, dk \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2c} \int_{-\infty}^{\infty} \hat{h}(k) \int_{x-ct}^{x+ct} e^{iky}\, dy\, dk \\
&= \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) e^{iky}\, dk\, dy \\
&= \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy
\end{aligned}$$

This derivation was tricky and did not follow the usual path of solution by Fourier transformation.

We now follow the standard path where one expresses $\hat{h}(k)$ via $h(y)$.

We have:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) \frac{\sin(ckt)}{ck}\, e^{ikx}\, dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(y) \frac{\sin(ckt)}{ck}\, e^{ik(x-y)}\, dy\, dk
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(ckt)}{ck}\,e^{ik(x-y)}\,dk h(y)\,dy$$

We have proceeded formally by exchanging the orders of integration. However, the function

$$k \to \frac{\sin(ckt)}{ck}\,e^{ik(x-y)}$$

decays too slowly and is not in L_1. We ignore will ignore this.

To arrive at d'Alembert's formula, we must show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(ckt)}{ck}\,e^{ik(x-y)} = \begin{cases} \frac{1}{2c} & \text{for} \quad x-ct < y < x+ct \\ 0 & \text{for} \quad y \notin [x-ct, x+ct] \end{cases}$$

Equivalently, we must show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ckt}{k}\,e^{ik\xi}\,dk = \begin{cases} 1 & \text{for} \quad |\xi| < ct \\ 0 & \text{for} \quad |\xi| > ct \end{cases}$$

Set

$$ctk = \kappa, \quad \frac{dk}{k} = \frac{d\kappa}{\kappa}, \quad \frac{\xi}{ct} = q\ .$$

We must show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\kappa}{\kappa}\,e^{i\kappa q}\,d\kappa = \begin{cases} 1 & \text{for} \quad |q| < 1 \\ 0 & \text{for} \quad |q| > 1 \end{cases} \tag{14.4}$$

Define the function

$$H(q) = \begin{cases} 1 & \text{for} \quad |q| < 1 \\ 0 & \text{for} \quad |q| > 1 \end{cases}$$

Its Fourier transform is

$$\begin{aligned} \hat{H}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ikq}\,dq \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{-ik}(e^{-ik} - e^{-k}) \\ &= \frac{1}{\sqrt{2\pi}}\frac{2}{k}\frac{e^{-ik} - e^{-k}}{2i} \\ &= \sqrt{\frac{2}{\pi}}\frac{\sin k}{k} \end{aligned}$$

The Fourier inversion formula yields that

$$H(q) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin k}{k}e^{ikq}\,dk\ .$$

This agrees with formula (14.4).

14.3 The 3D Case via Fourier Transform

By Kirchhoff's formula the solution of

$$u_{tt} = c^2 \Delta u, \quad u(x,0) = 0, \quad u_t(x,0) = h(x)$$

is

$$u(x,t) = \frac{t}{4\pi} \int_{\partial B_1} h(x + ctz)\, dS(z) \ .$$

For $x = 0$ and $c = t = 1$ one obtains that

$$u(0,1) = \frac{1}{4\pi} \int_{\partial B_1} h(y)\, dS(y) \ .$$

We want to derive this formula via Fourier transformation.

We first solve the Cauchy problem with

$$h(x) = e^{ik\cdot x} \ .$$

If $u(x,t) = a(t)e^{ik\cdot x}$ then one obtains that

$$a''(t) + c^2|k|^2 a(t) = 0, \quad a(0) = 0, \quad a'(0) = 1 \ .$$

Therefore,

$$a(t) = \frac{\sin(c|k|t)}{c|k|}$$

and

$$u(x,t) = \frac{\sin(c|k|t)}{c|k|}\, e^{ik\cdot x} \ .$$

If $h \in \mathcal{S}$ then

$$h(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{h}(k) e^{ik\cdot x}\, dk$$

and

$$u(x,t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{h}(k)\, \frac{\sin(c|k|t)}{c|k|}\, e^{ik\cdot x}\, dk \ .$$

For $x = 0$ and $c = t = 1$ one obtains that

$$u(0,1) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \hat{h}(k) \frac{\sin|k|}{|k|}\, dk \ .$$

Here

$$\hat{h}(k) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} h(y) e^{-ik\cdot y}\, dy \ .$$

This yields, formally,

$$u(0,1) = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sin(|k|}{|k|}\, e^{-ik\cdot y}\, dk\, h(y)\, dy \ .$$

We set

$$G(y) = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\sin |k|}{|k|}\, e^{-ik\cdot y}\, dk$$

and obtain that

$$u(0,1) = \int_{\mathbb{R}^3} G(y) h(y)\, dy \ .$$

This is the formal result via Fourier transformation.

By Kirchhoff:

$$u(0,1) = \frac{1}{4\pi} \int_{\partial B_1} h(y)\, dS(y) \ .$$

To prove that the two formulas agree we must show that

$$G(y) dy = \frac{1}{4\pi}\, dS(y) \tag{14.5}$$

where $dS(y)/4\pi$ is scaled surface measure on the unit sphere ∂B_1.

Summary: Let $u(x,t)$ denote the solution of (14.1) where $h \in \mathcal{S}$ and $c = 1$. Then Kirchhoff's formula yields

$$u(0,1) = \frac{1}{4\pi} \int_{\partial B_1} h(y)\, dS(y)$$

and solution by Fourier transformation yields formally

$$u(0,1) = \int_{\mathbb{R}^3} G(y) h(y)\, dy$$

with

$$G(y) = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\sin |k|}{|k|}\, e^{-ik\cdot y}\, dk \ .$$

We will prove that

$$G(y) dy = \frac{1}{4\pi}\, dS(y)$$

by showing that both sides of the above equation have the same Fourier transform. Note that

$$G(y) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} (2\pi)^{-3/2} \frac{\sin|k|}{|k|} e^{ik\cdot y}\, dk \ ,$$

which yields that

$$\hat{G}(k) = (2\pi)^{-3/2} \frac{\sin|k|}{|k|} \ .$$

To compute the Fourier transform of the scaled surface measure, we will approximate the measure by an ordinary function below.

We first show the following auxiliary result:

Theorem 14.1 *The Fourier transform of a radial function is radial.*

Proof: Let $f : \mathbb{R}^n \to \mathbb{C}$ be a radial function, i.e.,

$$f(Rx) = f(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{if} \quad R \in \mathbb{R}^{n\times n} \quad \text{and} \quad R^T R = I \ .$$

If R is an orthogonal matrix, then R^T is also orthogonal, Therefore,

$$\begin{aligned} \hat{f}(Rk) &= c \int f(x) e^{-i\langle Rk,x\rangle}\, dx \\ &= c \int f(x) e^{-i\langle k,R^T x\rangle}\, dx \\ &= c \int f(R^T x) e^{-i\langle k,R^T x\rangle}\, dx \\ &= c \int f(y) e^{-i\langle k,y\rangle}\, dy \\ &= \hat{f}(k) \end{aligned}$$

$\diamond$

For $\varepsilon > 0$ set

$$V_\varepsilon = \{x \in \mathbb{R}^3 \ : \ 1 \le x \le 1+\varepsilon\}$$

and define the function

$$H_\varepsilon(x) = \begin{cases} 1/vol(V_\varepsilon) & \text{for} \quad x \in V_\varepsilon \\ 0 & \text{for} \quad x \notin V_\varepsilon \end{cases}$$

For all $h \in \mathcal{S}$ we have

$$\int_{\mathbb{R}^3} H_\varepsilon(y) h(y)\, dy \to \frac{1}{4\pi} \int_{\partial B_1} h(y)\, dS(y) \quad \text{as} \quad \varepsilon \to 0 \ .$$

Formally,

$$H_\varepsilon(y)dy \to \frac{1}{4\pi}dS(y) \quad \text{as} \quad \varepsilon \to 0 \ .$$

On the Fourier side:

$$\hat{H}_\varepsilon(k) = (2\pi)^{-3/2}\frac{1}{vol(V_\varepsilon)}\int_{V_\varepsilon} e^{-ik\cdot x}\,dx \ .$$

As $\varepsilon \to 0$ one obtains that

$$\hat{H}_\varepsilon(k) \to (2\pi)^{-3/2}\,\frac{1}{4\pi}\int_{\partial B_1} e^{-ik\cdot x}\,dS(x) =: \hat{H}_0(k) \ .$$

Since $H_\varepsilon(x)$ is a radial function, we may assume that

$$k = (0,0,k_3), \quad k_3 = |k| \ .$$

Using spherical coordinates, we have

$$x_3 = \cos\theta, \quad dA = \sin\theta\,d\phi d\theta$$

and obtain that (with $-\cos\theta = q, dq = \sin\theta d\theta$):

$$\begin{aligned}
\hat{H}_0(k) &= (2\pi)^{-3/2}\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} e^{-ik_3\cos\theta}\,\sin\theta\,d\phi d\theta \\
&= (2\pi)^{-3/2}\frac{1}{2}\int_0^\pi e^{-ik_3\cos\theta}\,\sin\theta\,d\theta \\
&= (2\pi)^{-3/2}\frac{1}{2}\int_{-1}^1 e^{ik_3q}\,dq \\
&= (2\pi)^{-3/2}\frac{1}{2}\frac{1}{ik_3}(e^{ik_3} - e^{-ik_3}) \\
&= (2\pi)^{-3/2}\frac{\sin k_3}{k_3} \\
&= (2\pi)^{-3/2}\frac{\sin|k|}{|k|}
\end{aligned}$$

Since

$$H_\varepsilon(y)\,dy \to \frac{1}{4\pi}\,dS(y)$$

and since

$$\hat{H}_\varepsilon(k) \to \hat{H}_0(k) = (2\pi)^{-3/2}\frac{\sin|k|}{|k|}$$

we obtain that

$$\mathcal{F}\Big(\frac{1}{4\pi}\, dS(y)\Big) = (2\pi)^{-3/2}\frac{\sin|k|}{|k|} \ .$$

We have already shown that

$$\hat{G}(k) = (2\pi)^{-3/2}\frac{\sin|k|}{|k|}$$

and conclude that

$$G(y)\, dy = \frac{1}{4\pi}\, dS(y) \ .$$