Homework 1, Math. 562
Prof. Jens Lorenz, Instructor
Assigned: Jan. 27, 2023.   Due: Feb. 10, 2023

1) Let Γ denote the circle with parameterization $z(t) = 2e^{it}, 0 \le t \le 2\pi$, and let

$$A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix} .$$

a) Compute the projector

$$P = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1}\, dz .$$

Integrate the four matrix elements $((zI - A)^{-1})_{jk}$ along Γ to compute P.

b) Determine the range of P and the nullsapce of P.

c) Write P in the form

$$P = T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} .$$

Determine the matrix T.

2) Let W be a vector space and let $P : W \to W$ be a projector, i.e., P is linear and $P^2 = P$. Set

$$U = range(P), \quad V = nullspace(P) .$$

Prove that $W = U \oplus V$.

3) Let W be a vector space with subspaces U and V satisfying $W = U \oplus V$. Define $P : W \to W$ as follows:

If $w \in W$ and $w = u + v$ with $u \in U, v \in V$ then set $Pw = u$. Prove that P is a projector and

$$U = range(P), \quad V = nullspace(P) .$$

4) Let

$$A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}$$

denote the matrix of Problem 1. Compute a matrix B with $A = e^B$.

5) Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} .$$

Let Γ denote the circle with radius 1 centered at $z = 2$. Compute

$$B = \frac{1}{2\pi i} \int_\Gamma (\log z)(zI - A)^{-1}\, dz .$$

Check if the equation $A = e^B$ holds or not.