Functions of a Complex Variable I
Math 561, Fall 2022

Jens Lorenz

January 18, 2023

Department of Mathematics and Statistics,
UNM, Albuquerque, NM 87131

Contents

1 The Field $\mathbb{C}$ of Complex Numbers; Some Simple Concepts

Summary: The set of all complex numbers $z = x+iy$ forms a commutative field, denoted by $\mathbb{C}$, were addition and multiplication are defined. The mapping of complex conjugation, $z = x + iy \to \bar{z} = z - iy$, commutes with addition and multiplication. With the distance function $d(z_1, z_2) = |z_1 - z_2|$ the set $\mathbb{C}$ becomes a complete metric space.

Important analytical concepts are convergence of sequences z_n and series $\sum_{j=0}^{\infty} a_j$; continuity and complex differentiability of functions $f : U \to \mathbb{C}$ where U denotes an open subset of $\mathbb{C}$.

1.1 The Field $\mathbb{C}$ of Complex Numbers and the Euclidean Plane

Let

$$\mathbb{R}^2 = \{(x, y) \ : \ x, y \in \mathbb{R}\}$$

denote the Euclidean plane, consisting of all ordered pairs of real numbers x, y. One defines addition in $\mathbb{R}^2$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \ .$$

It is remarkable that one can define multiplication in $\mathbb{R}^2$ which, together with the above addition, turns $\mathbb{R}^2$ into a commutative field.

To motivate the definition of multiplication, let us identify the pair $(x, 0)$ with $x \in \mathbb{R}$ and set $(0, 1) =: i$. Then

$$(x, y) = (x, 0) + (0, y) = x + iy \ .$$

If one now postulates that $i^2 = -1$ and also postulates distributive laws, one obtains

$$\begin{aligned} (x_1, y_1) \cdot (x_2, y_2) &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2) \\ &= (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2) \end{aligned}$$

This motivates to define multiplication in $\mathbb{R}^2$ by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2) \ .$$

It is tedious, but not difficult to prove:

Theorem 1.1 *The set $\mathbb{R}^2$, together with addition and multiplication defined above, is a commutative field.*

Partial Proof: The zero–element in $\mathbb{R}^2$ is $(0, 0) = 0$ and the one–element is $(1, 0) = 1$. We want to check that every element $(x, y) \neq (0, 0)$ has a multiplicative inverse. Motivation for the formula for the inverse: Let

$$z = (x, y) = x + iy \ .$$

Then we have

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} \; .$$

This motivates to set

$$(a,b) = \Big(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\Big) \; .$$

Then we calculate

$$\begin{aligned} (x,y)\cdot(a,b) &= \Big(\frac{x^2}{x^2+y^2} - \frac{y(-y)}{x^2+y^2}, \frac{yx}{x^2+y^2} + \frac{x(-y)}{x^2+y^2}\Big) \\ &= (1,0) \\ &= 1 \end{aligned}$$

This shows that (a,b) is indeed a multiplicative inverse of (x,y).

It is, of course, also important to check that

$$\begin{aligned} i^2 &= i\cdot i \\ &= (0,1)\cdot(0,1) \\ &= (-1,0) \\ &= -1 \end{aligned}$$

As usual, we will identify $(x,0)$ with $x \in \mathbb{R}$ and write $i = (0,1)$,

$$(x,y) = x + iy \; .$$

With these notations and the above definitions of addition and multiplication, one writes $\mathbb{C}$ for $\mathbb{R}^2$. It is convenient to think of $\mathbb{R}$ as a subfield of $\mathbb{C}$, i.e. $\mathbb{R} \subset \mathbb{C}$.

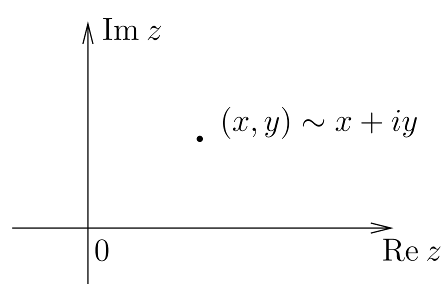

Figure 1.1: Identification of $\mathbb{R}^2$ and $\mathbb{C}$

Summary: The Euclidean plane $\mathbb{R}^2$ and the field of complex numbers $\mathbb{C}$ can be identified via the mapping

$$\mathbb{R}^2 \longleftrightarrow \mathbb{C}, \quad (x,y) \longleftrightarrow z = x + iy \; .$$

Addition and multiplication in $\mathbb{C}$ are defined by

$$
\begin{aligned}
(x_1 + iy_1) + (x_2 + iy_2) &= x_1 + x_2 + i(y_1 + y_2) \\
(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)
\end{aligned}
$$

1.2 Some Simple Concepts

Complex Conjugation. If $z = x + iy$ with real x, y, then $\bar{z} = x - iy$ is called the complex conjugate of z. One easily checks the rules:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

and

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \ .$$

Furthermore, $z = \bar{z}$ if and only if z is real.

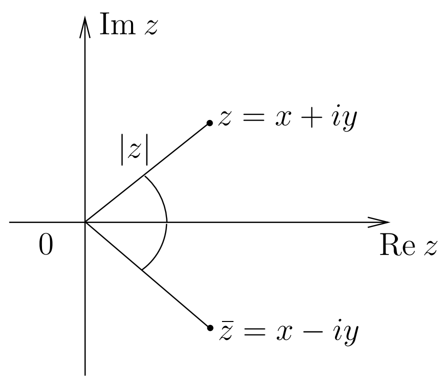

Figure 1.2: The complex conjugate

Exercise: Prove: If $z \neq 0$ then $\overline{(1/z)} = 1/\bar{z}$.

A simple consequence of the rules for taking complex conjugates is the following:

Lemma 1.1 *Let $p(z) = a_0 + a_1 z + \ldots + a_k z^k$ be a polynomial with real coefficients, $a_j \in \mathbb{R}$. If $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then $p(\bar{z}_0) = 0$. In other words, the non–real roots of a polynomial with real coefficients come in pairs of complex conjugate numbers. Further implication: The non–real eigenvalues of a real matrix $A \in \mathbb{R}^{n \times n}$ come in complex conjugate pairs.*

Absolute Value. If $z = x + iy$ with real x, y, then

$$|z| = \sqrt{x^2 + y^2}$$

is the Euclidean distance of z from 0. We have the triangle inequality,

$$|z + w| \leq |z| + |w| \ ,$$

and the multiplication rule:

$$|zw| = |z||w| \; .$$

Distance of Two Complex Numbers and Convergence of Sequences. If $z_1, z_2 \in \mathbb{C}$ are two complex numbers then their Euclidean distance is

$$|z_1 - z_2| \; .$$

This distance concept leads, as usual, to a concept of convergence for sequences: If z_n is a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$, then z_n converges to z (for short: $z_n \to z$) if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with

$$|z_n - z| < \varepsilon \quad \text{for} \quad n \geq N \; .$$

A sequence z_n of complex numbers is called a Cauchy sequence in $\mathbb{C}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $|z_m - z_n| < \varepsilon$ for $m, n \geq N$. It is easy to check that every convergent sequence is a Cauchy sequence. An important result of analysis says that every Cauchy sequence in $\mathbb{C}$ has a limit in $\mathbb{C}$. In other words, the metric space $\mathbb{C}$ with distance $d(z_1, z_2) = |z_1 - z_2|$ is complete.

A Result from Real Analysis: Let $\alpha_n \in \mathbb{R}$ denote a sequence of real numbers. Assume that there exists $\gamma \in \mathbb{R}$ so that

$$\gamma \leq \alpha_{n+1} \leq \alpha_n \quad \text{for all} \quad n \in \mathbb{N} \; .$$

Then the sequence α_n converges to some $\alpha \geq \gamma$.

To prove this result one uses that the sequence α_n is a Cauchy sequence in $\mathbb{R}$.

Lemma 1.2 *Let $z \in \mathbb{C}, |z| < 1$. Then $z^n \to 0$ as $n \to \infty$.*

Proof: If $\alpha_n := |z^n| = |z|^n$ then

$$0 \leq \alpha_{n+1} \leq \alpha_n \quad \text{for all} \quad n \; .$$

Convergence $|z^n| = |z|^n \to \alpha \geq 0$ follows. We have to show that $\alpha = 0$ and may assume that $0 < |z| < 1$. We have

$$|z|^{n+1} = |z||z|^n \; .$$

Convergence $|z|^n \to \alpha$ implies that

$$\alpha = |z|\alpha \; .$$

Therefore, $\alpha = 0$. $\diamond$

Convergence of Series. Similar as in real analysis, we will consider series, which are expressions of the form

$$\sum_{j=0}^{\infty} a_j$$

where $a_j \in \mathbb{C}$. The sequence

$$s_n = \sum_{j=0}^{n} a_j$$

is the corresponding sequence of partial sums. The series $\sum_{j=0}^\infty a_j$ is called convergent if the sequence s_n of partial sums converges. If $s_n \to s$ then one writes

$$\sum_{j=0}^{\infty} a_j = s \ .$$

In other words, the symbol $\sum_j a_j$ may denote just an expression, but it also may denote the complex number

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_j \ .$$

This double meaning of $\sum_j a_j$, though sometimes confusing, turns out to be very convenient.

The series $\sum_j a_j$ is called absolutely convergent if the series $\sum_j |a_j|$ converges.

Exercise: Prove: If the series $\sum_j |a_j|$ converges, then the series $\sum_j a_j$ also converges, i.e., absolute convergence implies convergence. (The proof uses completeness of $\mathbb{C}$.)

If the series $\sum_j a_j$ is convergent, but not absolutely convergent, then one calls it conditionally convergent. The standard example of a conditionally convergent series is

$$\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

In the theory of complex variables one typically works with absolutely convergent series.

Example 1.1: The Geometric Series: Consider the series

$$\sum_{j=0}^{\infty} z^j \quad \text{for} \quad |z| < 1 \ .$$

We have

$$(1 + z + \ldots + z^n)(1 - z) = 1 - z^{n+1} \ ,$$

thus

$$1 + z + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad \text{for} \quad z \neq 1 \ .$$

Since $z^{n+1} \to 0$ for $|z| < 1$ one obtains that

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1 - z} \quad \text{for} \quad |z| < 1 \ .$$

Since the series

$$\sum_{j=0}^{\infty} |z^j|$$

converges, the convergence of the geometric series is absolute.

Two simple results on convergent series are the comparison and the quotient criteria.

Theorem 1.2 *(Comparison Criterion) Assume that* $|a_j| \le |b_j|$ *for all* j*. If* $\sum_j |b_j|$ *converges, then* $\sum_j |a_j|$ *converges, too.*

This result follows, essentially, from the convergence of Cauchy sequences in $\mathbb{R}$.

Theorem 1.3 *(Quotient Criterion) Assume that there exists* $J \in \mathbb{N}$ *and* $0 \le q < 1$ *so that*

$$\Big| \frac{a_{j+1}}{a_j} \Big| \le q < 1 \quad \textit{for} \quad j \ge J \ .$$

Then the series $\sum_j a_j$ *converges absolutely.*

Proof: The proof uses convergence of the geometric series,

$$\sum_{j=0}^{\infty} q^j = \frac{1}{1-q}, \quad |q| < 1 \ ,$$

and the Comparison Criterion: For $k \ge 0$ we have

$$|a_{J+k}| \le q^k |a_J| \ .$$

Therefore,

$$\sum_{k=0}^{n} |a_{J+k}| \le |a_J| \sum_{k=0}^{n} q^k \le |a_J| \frac{1}{1-q} \ .$$

This implies that the series

$$\sum_{k=0}^{\infty} |a_{J+k}|$$

converges. $\diamond$

Example 1.2: The Quotient Criterion can be used to prove absolute convergence of the series defining the exponential function,

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C} \ .$$

In this case, $a_j = z^j/j!$ and

$$|a_{j+1}/a_j| = |z|/(j+1) \le \frac{1}{2} \quad \text{for} \quad j+1 \ge 2|z| \ .$$

Pointwise and Uniform Convergence Let $U \subset \mathbb{C}$ and let $s_n : U \to \mathbb{C}$ denote a sequence of functions, $s_n = s_n(z)$. Let $s : U \to \mathbb{C}$ denote a function. The sequence of functions $s_n(z)$ converges pointwise on U to the function $s(z)$ if for all $z \in U$ and all $\varepsilon > 0$ there exists $N(\varepsilon, z)$ with

$$|s_n(z) - s(z)| \le \varepsilon \quad \text{for} \quad n \ge N(\varepsilon, z) \ .$$

The sequence of functions $s_n(z)$ converges uniformly on U to the function $s(z)$ if for all $\varepsilon > 0$ there exists $N(\varepsilon)$ with

$$|s_n(z) - s(z)| \le \varepsilon \quad \text{for} \quad n \ge N(\varepsilon) \quad \text{for all} \quad z \in U \ .$$

The concepts of pointwise and uniform convergence are often used for series. Let $f_j : U \to \mathbb{C}$ denote a sequence of functions and let $f : U \to \mathbb{C}$ denote a function. Then the series

$$\sum_{j=0}^{\infty} f_j(z)$$

converges pointwise on U to $f(z)$ if for all $z \in U$ and all $\varepsilon > 0$ there exists $N(\varepsilon, z)$ with

$$|\sum_{j=0}^{n} f_j(z) - f(z)| \le \varepsilon \quad \text{for} \quad n \ge N(\varepsilon, z) \ .$$

The series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on U to $f(z)$ if for all $\varepsilon > 0$ there exists $N(\varepsilon)$ with

$$|\sum_{j=0}^{n} f_j(z) - f(z)| \le \varepsilon \quad \text{for} \quad n \ge N(\varepsilon) \quad \text{for all} \quad z \in U \ .$$

Example: Let $0 \le r < 1$. We claim that

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$$

where the convergence is uniform for $|z| \le r$. This follows from

$$\Big|\sum_{j=0}^{n} z^j - \frac{1}{1-z}\Big| = \frac{|z|^{n+1}}{|1-z|} \le \frac{r^{n+1}}{1-r} \quad \text{for} \quad |z| \le r < 1 \ .$$

We claim that

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$$

does **not** hold with uniform convergence for $|z| < 1$.

Proof: Consider $z = x$ for $0 \le x < 1$. We have

$$\Big|\sum_{j=0}^{n} x^j - \frac{1}{1-x}\Big| = \frac{x^{n+1}}{1-x} \quad \text{for} \quad 0 \le x < 1 \ .$$

Here, for every $n \in \mathbb{N}$,

$$\frac{x^{n+1}}{1-x} \to \infty \quad \text{as} \quad x \to 1- \ .$$

Therefore, if $\varepsilon = 1$ for example, then n with

$$\Big|\sum_{j=0}^{n} x^j - \frac{1}{1-x}\Big| \le \varepsilon = 1 \quad \text{for} \quad 0 \le x < 1$$

does not exist. $\diamond$

Example: The series which defines the exponential function,

$$\sum_{j=0}^{\infty} \frac{z^j}{j!} = \exp(z) \ ,$$

converges pointwise in $\mathbb{C}$, but not uniformly on $\mathbb{C}$. If $0 < r < \infty$ is fixed, then the convergence is uniform for all $z \in \mathbb{C}$ with $|z| \le r$.

Continuity of a Function: Let $U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$ denote a function. Let $z_0 \in U$. The function f is called continuous at z_0 if for all $\varepsilon > 0$ there is $\delta > 0$ so that

$$|f(z_0) - f(z)| < \varepsilon$$

for all $z \in U$ with $|z_0 - z| < \delta$.

1.3 Complex Differentiability

Notation and Definitions: Let $z_0 \in \mathbb{C}$ and let $r > 0$. The set

$$D(z_0, r) = \{z \in \mathbb{C} \ : \ |z_0 - z| < r\}$$

is the open disk of radius r centered at z_0.

A set $U \subset \mathbb{C}$ is called *open* if for every $z_0 \in U$ there exists $\varepsilon > 0$ with $D(z_0, \varepsilon) \subset U$. (Show that the set $D(z_0, r)$ is open.)

A set $V \subset \mathbb{C}$ is called *closed* if the following holds: If z_n is a sequence in V which converges,

$$z_n \to z \quad \text{as} \quad n \to \infty \ ,$$

then the limit z is an element of the set V, i.e, $z_n \in V$ and $z_n \to z$ implies $z \in V$.

It is not difficult to prove that a set $V \subset \mathbb{C}$ is closed if and only if its complement, $V^c = \mathbb{C} \setminus V$, is open.

An important concept of complex variables is *complex differentiability* of a function. Here the field structure of $\mathbb{C}$ is used in an essential way since in the formula (1.1) below *division* by the complex number h occurs.

Definition 1.1: *Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$. The function f is called complex differentiable at z_0 if*

$$\lim_{h \to 0} \frac{1}{h}(f(z_0 + h) - f(z_0)) \tag{1.1}$$

exists.[1] *If the limit exists, it is denoted by*

$$f'(z_0) = \frac{df}{dz}(z_0) \ .$$

The number $f'(z_0)$ is called the complex derivative of f at z_0. The function $f : U \to \mathbb{C}$ is called complex differentiable in U if it is complex differentiable at every point z_0 in U. We then write $f \in H(U)$ and call f a holomorphic function in U.

[1] The limit exists and equals the complex number a if for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|\frac{1}{h}(f(z_0 + h) - f(z_0)) - a| < \varepsilon$ for all complex numbers h with $0 < |h| < \delta$.

Example 1.3: Let $U = \mathbb{C}$ and let $f(z) = z^n$ where n is a positive integer. We have

$$\begin{aligned} f(z+h) - f(z) &= (z+h)^n - z^n \\ &= \Big(z^n + nhz^{n-1} + R(h)\Big) - z^n \\ &= nhz^{n-1} + R(h) \end{aligned}$$

where $|R(h)| \leq C|h|^2$ for $|h| \leq 1$. It follows that

$$\lim_{h\to 0} \frac{1}{h}(f(z+h) - f(z)) = nz^{n-1} \ ,$$

thus the function $f(z) = z^n$ is complex differentiable with derivative

$$(z^n)' = nz^{n-1} \ .$$

Example 1.4 The function $f(z) = x$ where $z = x + iy$ with real x, y is nowhere complex differentiable. To show this, take first $h = h_1, h_1 \in \mathbb{R}, h_1 \neq 0$ and obtain

$$\frac{1}{h}(f(z+h) - f(z)) = \frac{h_1}{h_1} = 1 \ .$$

Second, let $h = ih_2, h_2 \in \mathbb{R}, h_2 \neq 0$. In this case

$$\frac{1}{h}(f(z+h) - f(z)) = \frac{0}{ih_2} = 0 \ .$$

Therefore, the limit

$$\lim_{h\to 0} \frac{1}{h}(f(z+h) - f(z))$$

does not exist.

The theory of complex variables is the study of functions $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is an open set and where f is complex differentiable in U.

Any complex function $f : U \to \mathbb{C}$ can be written as

$$f(z) = u(x,y) + iv(x,y) \quad \text{with} \quad z = x + iy$$

where $u(x,y)$ and $v(x,y)$ are real valued. It is important to understand the relation between complex differentiability of f and real differentiability of the functions $u(x,y)$ and $v(x,y)$. As Example 1.3 shows, complex differentiability is more than just smoothness of the functions $u(x,y)$ and $v(x,y)$.

Roughly speaking, differentiation corresponds to approximation by a linear map. We can consider $\mathbb{R}^2 \simeq \mathbb{C}$ as a 2–dimensional **real** vector space or as a 1–dimensional **complex** vector space. If we have a map

$$L : \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{R}^2 \simeq \mathbb{C}$$

we then must distinguish between real and complex linearity of L. This distinction is of an algebraic nature.

Therefore, as we will explain in Chapter 3, the difference between real and complex differentiability is of an algebraic nature. The main issue is addressed by the following question: Which real–linear maps $L : \mathbb{R}^2 \to \mathbb{R}^2$ correspond to complex–linear maps from $\mathbb{C}$ to $\mathbb{C}$?

1.4 Alternating Series

The following result is often useful to show convergence of real series whose terms have alternating signs.

Theorem 1.4 *Let $a_n, n = 0, 1, \ldots$ denote a monotonically decreasing sequence of positive real numbers converging to zero,*

$$a_0 \geq a_1 \ldots \geq a_n \geq a_{n+1} \geq \ldots > 0, \quad a_n \to 0 \ .$$

The series

$$\sum_{j=0}^{\infty} (-1)^j a_j$$

converges.

Proof: Consider the partial sums

$$A_n = \sum_{j=0}^{n} (-1)^j a_j \ .$$

We have

$$\begin{aligned}
A_0 &= a_0 \\
A_2 &= a_0 - (a_1 - a_2) \\
&\leq A_0 \\
A_4 &= A_2 - (a_3 - a_4) \\
&\leq A_2 \\
A_{2n+2} &= A_{2n} - (a_{2n+1} - a_{2n+2}) \\
&\leq A_{2n}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
A_1 &= a_0 - a_1 \\
A_3 &= a_0 - a_1 + (a_2 - a_3) \\
&\geq A_1 \\
A_{2n+1} &= A_{2n-1} + (a_{2n} - a_{2n+1}) \\
&\geq A_{2n-1}
\end{aligned}$$

We also have that

$$A_{2n+1} = A_{2n} - a_{2n+1} \leq A_{2n} \ .$$

Therefore,

$$A_1 \leq A_3 \leq A_5 \leq \ldots \leq A_4 \leq A_2 \leq A_0 \ .$$

It follows that the limits

$$\lim_{n\to\infty} A_{2n+1} = A \quad \text{and} \quad \lim_{n\to\infty} A_{2n} = B$$

exist. Furthermore, the assumption $a_n \to 0$ implies that $A = B$. Convergence

$$A_n \to A = B$$

follows. $\diamond$

1.5 History

Euclid (Mid 4th century BC – Mid 3rd century BC), Greek

Franciscus Vieta (Francois Viète) (1540–1603), French

Jacob Bernoulli (1655–1705), Swiss
Jacob Ricatti (1676–1754), from Venice

Leonhard Euler (1701–1783), Swiss
Augustine–Jean Fresnel (1788–1827), French
Augustin–Louis Cauchy (1789–1857), French

Niels Henrik Abel (1802–1829), Norwegian
Joseph Liouville (1809–1882), French
Karl Theodor Wilhelm Weierstrass (1815–1897), German
Arthur Cayley (1821–1895), British
Charles Hermite (1822–1901), French
Bernhard Riemann (1826–1866), German
Felice Casorati (1835–1890), Italian
Edouard Goursat (1858–1936), French
Giacinto Morera (1856–1909), Italian
Émile Picard (1856–1941), French
Jacques Hadamard (1865–1963), French

Laurent Schwarz (1915–2002), French
Roger Apéry (1916–1994), French

In 1978 Apéry proved that the number $\zeta(3)$ is irrational.

2 The Cauchy Product of Two Series: Proof of the Addition Theorem for the Exponential Function

Summary: The Cauchy product of two series will be introduced and will be used to prove the Addition Theorem for the exponential function,

$$\exp(a+b) = \exp(a)\exp(b), \quad a, b \in \mathbb{C} ,$$

where

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \quad z \in \mathbb{C} .$$

The exponential function maps the open strip

$$S_\pi = \{z = x + iy \ : \ x \in \mathbb{R}, -\pi < y < \pi\}$$

bijectively onto the slit plane

$$\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$$

and, by definition, the inverse function from $\mathbb{C}^-$ onto S_π is the main branch of the complex logarithm, which we denote by $\log w$. We have

$$\log(\exp(z)) = z \quad \text{for all} \quad z \in S_\pi$$

and

$$\exp(\log(w)) = w \quad \text{for all} \quad w \in \mathbb{C}^- .$$

2.1 The Cauchy Product of Two Series

Let

$$\sum_{j=0}^{\infty} a_j \quad \text{and} \quad \sum_{j=0}^{\infty} b_j \tag{2.1}$$

denote two series of complex numbers. Proceeding formally, we obtain for their product

$$\begin{aligned}
(a_0 + a_1 + a_2 + \ldots) \cdot (b_0 + b_1 + b_2 + \ldots) \quad = \quad & a_0b_0 + a_0b_1 + a_0b_2 + \ldots \\
& + a_1b_0 + a_1b_1 + a_1b_2 + \ldots \\
& + a_2b_0 + a_2b_1 + a_2b_2 + \ldots \\
& + \ldots \\
= \quad & c_0 + c_1 + c_2 + \ldots
\end{aligned}$$

with

$$c_0 = a_0b_0, \quad c_1 = a_0b_1 + a_1b_0, \quad c_2 = a_0b_2 + a_1b_1 + a_2b_0, \quad \text{etc.}$$

In general, set

$$c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = \sum_{j=0}^{n} a_j b_{n-j} \; . \tag{2.2}$$

Then the series

$$\sum_{n=0}^{\infty} c_n$$

is called the Cauchy product of the two series (2.1).

Theorem 2.1 *a) Assume that both series (2.1) converge, and at least one of them converges absolutely. Then their Cauchy product also converges, and for the values of the series we have*

$$\Big(\sum_{j=0}^{\infty} a_j\Big) \cdot \Big(\sum_{j=0}^{\infty} b_j\Big) = \sum_{n=0}^{\infty} c_n \; . \tag{2.3}$$

b) If both series (2.1) converge absolutely, then their Cauchy product also converges absolutely.

Proof: a) Let

$$\begin{aligned} A_n &:= \sum_{j=0}^{n} a_j \to A \\ B_n &:= \sum_{j=0}^{n} b_j \to B \\ C_n &:= \sum_{j=0}^{n} c_j \end{aligned}$$

We must show that $C_n \to AB$ if at least one of the series (2.1) converges absolutely.

Assume that $\sum a_j$ converges absolutely and let

$$\alpha := \sum_{j=0}^{\infty} |a_j| \; .$$

Set

$$\beta_n := B_n - B = - \sum_{k=n+1}^{\infty} b_k \; .$$

Then we have $B_n = B + \beta_n$. Since $\beta_n \to 0$ as $n \to \infty$ there exists a constant $\beta_{max} > 0$ with

$$|\beta_n| \le \beta_{max} \quad \text{for all} \quad n \; .$$

We now rewrite C_n:

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + \ldots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \ldots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \ldots + a_n (B + \beta_0) \\ &= A_n B + \gamma_n \end{aligned}$$

with

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0 = \sum_{j=0}^n a_j\beta_{n-j} \ .$$

Since $A_n \to A$ we have to show that $\gamma_n \to 0$.

Let $\varepsilon > 0$ be given. Since $\beta_n \to 0$ there exists $N = N_1(\varepsilon)$ with

$$|\beta_n| \le \varepsilon \quad \text{for all} \quad n \ge N+1 \ . \tag{2.4}$$

In the following, N is fixed with (2.4). Using the absolute convergence of the series $\sum_j a_j$ and $\alpha = \sum_{j=1}^\infty |a_j|$ we have for all $n \ge N$:

$$\begin{aligned} |\gamma_n| &\le |\beta_0 a_n| + \ldots + |\beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1}| + \ldots + |\beta_n a_0| \\ &\le |\beta_0 a_n| + \ldots + |\beta_N a_{n-N}| + \varepsilon\alpha \\ &\le \beta_{max}\Big(|a_n| + \ldots + |a_{n-N}|\Big) + \varepsilon\alpha \end{aligned}$$

Here the bracket contains $N+1$ terms. Since $a_n \to 0$ there exists $N_2(\varepsilon) = N_2(\varepsilon, N)$ so that

$$|a_n| \le \frac{\varepsilon}{N+1}, \ldots, |a_{n-N}| \le \frac{\varepsilon}{N+1}$$

for $n \ge N_2(\varepsilon)$. It follows that

$$|\gamma_n| \le \varepsilon(\beta_{max} + \alpha) \quad \text{for} \quad n \ge N_2(\varepsilon) \ .$$

Here the constants β_{max} and α are independent of ε, and $\varepsilon > 0$ is arbitrary. This proves that $\gamma_n \to 0$.

b) Assume that both series (2.1) converge absolutely. We have

$$|c_n| \le |a_0||b_n| + \ldots + |a_n||b_0| =: d_n \ .$$

Here $\sum d_n$ is the Cauchy product of the series $\sum |a_j|$ and $\sum |b_j|$. By part a), the series $\sum d_n$ converges and, therefore, $\sum |c_n|$ also converges. $\diamond$

Remark: Assume that both series (2.1) converge, but none of them converges absolutely. Can one still conclude that the Cauchy product of the two series converges? The answer is *No*, in general. To give an example, let $a_j = b_j = \frac{(-1)^j}{\sqrt{j+1}}$ for $j = 0, 1, \ldots$ Then, by Theorem 1.4, the series (2.1) converge, but the convergence is not absolute. Here the general term of the Cauchy product is

$$c_n = (-1)^n \sum_{j=0}^n \frac{1}{\sqrt{j+1}\sqrt{n+1-j}}$$

and

$$\frac{1}{\sqrt{j+1}\sqrt{n+1-j}} \ge \frac{1}{\sqrt{n+1}\sqrt{n+1}} = \frac{1}{n+1}, \quad 0 \le j \le n \ .$$

It follows that $|c_n| \ge 1$; the Cauchy product of $\sum a_j$ and $\sum b_j$ diverges.

2.2 The Addition Theorem for the Exponential Function

For all $z \in \mathbb{C}$ the series

$$\exp(z) := \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

converges absolutely by the quotient criterion (Theorem 1.3). We use the previous theorem to prove the fundamental Addition Theorem for the exponential function.

Theorem 2.2

$$\exp(a+b) = \exp(a)\exp(b) \quad \textit{for all} \quad a, b \in \mathbb{C} \; . \tag{2.5}$$

Proof: Note that

$$\exp(a) = \sum_{j=0}^{\infty} a_j \quad \text{with} \quad a_j = \frac{a^j}{j!}$$

and

$$\exp(b) = \sum_{j=0}^{\infty} b_j \quad \text{with} \quad b_j = \frac{b^j}{j!} \; .$$

If $\sum_{n=0}^{\infty} c_n$ denotes the Cauchy product of the series $\exp(a)$ and $\exp(b)$ then

$$c_n = \sum_{j=0}^{n} \frac{a^j}{j!} \, \frac{b^{n-j}}{(n-j)!} \; .$$

Also,

$$\exp(a+b) = \sum_{n=0}^{\infty} \frac{1}{n!} (a+b)^n$$

where

$$(a+b)^n = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} a^j \, b^{n-j} \quad \text{with} \quad \begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{j!(n-j)!} \; .$$

It follows that

$$\frac{1}{n!} (a+b)^n = \sum_{j=0}^{n} \frac{a^j}{j!} \, \frac{b^{n-j}}{(n-j)!} = c_n \; .$$

Therefore,

$$\exp(a+b) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \frac{1}{j!} a^j \frac{1}{(n-j)!} b^{n-j} \Big) \; .$$

This is precisely the Cauchy product of the series for $\exp(a)$ and $\exp(b)$. The claim follows from Theorem 2.1. $\diamond$

Let us give a second proof of the Addition Theorem. It uses tools, however, which we will only justify later. The function

$$f(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

is entire and $f'(z) = f(z), f(0) = 1$. Fix $a \in \mathbb{C}$ and consider the function $g(z) = f(a+z)$. Then $g'(z) = g(z)$ and $g(0) = f(a)$. The function $h(z) = f(a)f(z)$ also satisfies $h'(z) = h(z), h(0) = f(a)$. Therefore, the functions $g(z)$ and $h(z)$ are both solutions of the initial–value problem

$$u'(z) = u(z), \quad u(0) = f(a) \ .$$

Uniqueness of the solution of this initial–value problem implies that $g(z) = h(z)$, i.e., $f(a+z) = f(a)f(z)$.

2.3 Powers of e

2.3.1 Integer Powers

One sets

$$e := \exp(1) = \sum_{j=0}^{\infty} \frac{1}{j!} = 2.71828\,18284\,59046\ldots \ ,$$

a notation due to Euler.[2] Then, by (2.5),

$$\begin{aligned} \exp(2) &= \exp(1)\exp(1) = e \cdot e = e^2 \\ \exp(3) &= \exp(1)\exp(2) = e \cdot e^2 = e^3 \end{aligned}$$

etc.

Also, since

$$\exp(1)\exp(-1) = \exp(0) = 1 \ ,$$

we obtain

$$\exp(-1) = \frac{1}{e} = e^{-1} \ .$$

In the same way as above,

$$\exp(-2) = \exp(-1)\exp(-1) = \frac{1}{e}\frac{1}{e} = e^{-2} \ .$$

etc. The arguments show that

$$\exp(n) = e^n \quad \text{for all} \quad n \in \mathbb{Z}$$

where, by definition,

[2] In 1873, Charles Hermite proved that the number e is transcendental; i.e., e is not a zero of any non–trivial polynomial with integer coefficients.

$$\exp(n) = \sum_{j=0}^{\infty} \frac{1}{j!} n^j$$

and where the standard definition of e^n is used.

2.3.2 Rational Powers

By definition,

$$\alpha := e^{1/2} = \sqrt{e}$$

is the positive real number with $\alpha^2 = e$. If we set

$$\beta := \exp(1/2)$$

then we have $\beta > 0$ and

$$\beta^2 = \exp(1/2)\exp(1/2) = \exp(1) = e \ .$$

Therefore, $\alpha = \beta$, i.e.,

$$e^{1/2} = \exp(1/2) \ .$$

More generally:

Lemma 2.1 *Let $q = m/n$ denote a positive rational number where $m, n \in \mathbb{N}$. If*

$$\alpha := e^{m/n} = e^q = \sqrt[n]{e^m}$$

denotes the positive n–th root of e^m, then

$$\exp(q) = \alpha \ .$$

In other words,

$$\exp(q) = e^q$$

for all positive rational numbers $q = m/n$.

Proof: Set $\beta := \exp(q)$. Then $\beta > 0$ and, using the addition theorem,

$$\begin{aligned} \beta^n &= \exp(\frac{m}{n}) \cdot \ldots \cdot \exp(\frac{m}{n}) \quad (n \text{ factors}) \\ &= \exp(m) \\ &= e^m \end{aligned}$$

Also, $\alpha^n = e^m$ and, therefore, $\alpha^n = \beta^n$. Since $\alpha > 0$ and $\beta > 0$ we conclude that $\alpha = \beta$, i.e.,

$$\sqrt[n]{e^m} = e^{m/n} = \exp(m/n) \ .$$

$\diamond$

With similar arguments, it follows that the equation

$$\exp(q) = e^q$$

also holds for negative rationals q. This justifies the standard notation

$$e^z = \exp(z), \quad z \in \mathbb{C} ,$$

where the exponential function is defined by the exponential series:

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!} .$$

2.4 Euler's Identity and Implications

Define

$$\begin{aligned} \sin z &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \\ \cos z &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k} \end{aligned}$$

For $z = x \in \mathbb{R}$ the above series are the Taylor series of the functions $\sin x$ and $\cos x$, centered at $x_0 = 0$. The series converge absolutely for every $z \in \mathbb{C}$. Using the definitions by the series, it is not difficult to prove Euler's identity,

Lemma 2.2

$$e^{iz} = \cos z + i \sin z \quad \textit{for all} \quad z \in \mathbb{C} .$$

Proof: We have

$$\begin{aligned} e^{iz} &= \sum_{j=0}^{\infty} \frac{(iz)^j}{j!} \\ &= \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \cos z + i \sin z \end{aligned}$$

$\diamond$

Lemma 2.3 *For all $z \in \mathbb{C}$:*

$$\cos^2 z + \sin^2 z = 1 .$$

Proof: We have

$$e^{iz} = \cos z + i \sin z \quad \text{and} \quad e^{-iz} = \cos z - i \sin z \quad \text{for all} \quad z \in \mathbb{C} \ .$$

Therefore,

$$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \ , \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \ . \end{aligned}$$

Using (2.5) one obtains that $(e^{iz})^2 = e^{2iz}$, thus

$$\cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1 \ .$$

$\diamond$

Lemma 2.4 *For all* $\theta \in \mathbb{R}$*:*

$$|e^{i\theta}| = 1 \ .$$

Proof: By Euler's identity:

$$e^{i\theta} = \cos\theta + i \sin\theta \ .$$

For real number θ, the values of $\cos\theta$ and $\sin\theta$ are real. Therefore,

$$|e^{i\theta}|^2 = \cos^2\theta + \sin^2\theta = 1 \ .$$

$\diamond$

2.5 The Polar Representation of a Complex Number

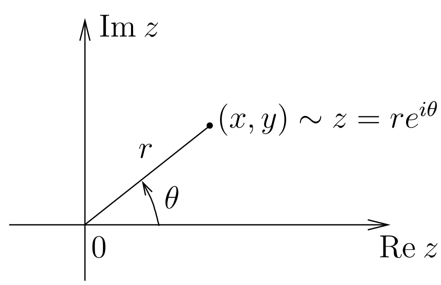

Figure 2.1: Polar representation

Let $z \in \mathbb{C}, z \neq 0$. Then $\zeta := z/|z| = x + iy$ satisfies $|\zeta| = |z|/|z| = 1$, thus

$$|\zeta|^2 = x^2 + y^2 = 1 \ .$$

From trigonometry (or calculus) we know the following result:

Lemma 2.5 *Given any two real numbers x, y with $x^2 + y^2 = 1$ there is a unique real number θ with $-\pi < \theta \le \pi$ and*

$$x = \cos\theta, \quad y = \sin\theta \ .$$

Remark: It is not at all obvious how to prove this result using the series representations of $\cos\theta$ and $\sin\theta$. In particular, one has to introduce the number π. One can define $\pi/2$ as the smallest positive zero of the cosine–function. One can prove that the functions $c(\theta) = \cos\theta$ and $s(\theta) = \sin\theta$ satisfy $c' = -s, s' = c$, thus $c'' + c = s'' + s = 0$. A proof of the lemma can be based on properties of the solutions of the differential equation $u'' + u = 0$.

Using the lemma we can write the number $\zeta = z/|z| = x + iy$ in the form

$$\zeta = x + iy = \cos\theta + i\sin\theta = e^{i\theta} \ .$$

The representation

$$z = re^{i\theta} \quad \text{with} \quad r = |z| > 0, \quad \theta = arg(z) \in (-\pi, \pi] \ ,$$

is called the polar representation of z. It is very useful if one wants to visualize complex multiplication geometrically since

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}$$

implies

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \ .$$

Regarding the real exponential function $x \to e^x$, we know from calculus:

Lemma 2.6 *a) The function $x \to e^x$ defined for $x \in \mathbb{R}$ is strictly increasing and maps the real line $\mathbb{R}$ onto the interval $(0, \infty)$ of positive real numbers.*

b) If one defines the real logarithm by

$$\ln r = \int_1^r \frac{ds}{s} \quad \textit{for} \quad r > 0$$

then

$$e^{\ln r} = r \quad \textit{for all} \quad r > 0$$

and

$$ln\Big(e^x\Big) = x \quad \textit{for all} \quad x \in \mathbb{R} \ .$$

2.6 Further Properties of the Exponential Function

In the following, let $z = x + iy$ with real x, y. We want to understand the map

$$z \to e^z$$

from $\mathbb{C}$ into itself. We make the following observations:

1) $e^z \neq 0$ for all $z \in \mathbb{C}$. This follows from $e^z e^{-z} = e^0 = 1$.

2) $|e^z| = |e^x e^{iy}| = e^x > 0$ since $|e^{iy}| = 1$.

3) For any fixed $y \in \mathbb{R}$, the horizontal line

$$H_y = \{z = x + iy \ : \ x \in \mathbb{R}\}$$

is mapped to the half–line

$$e^x(\cos y + i \sin y), \quad 0 < e^x < \infty \ .$$

4) For any fixed $x \in \mathbb{R}$, the vertical line

$$V_x = \{z = x + iy \ : \ y \in \mathbb{R}\}$$

is mapped (infinitely often) to the circle of radius e^x,

$$e^x(\cos y + i \sin y), \quad -\infty < y < \infty \ .$$

We note that the family of lines H_y is orthogonal to the family of lines V_x. Orthogonality also holds for the corresponding image lines: The radial line

$$x \to e^x(\cos y + i \sin y) \quad (y \text{ fixed})$$

is orthogonal to the circular line

$$y \to e^x(\cos y + i \sin y) \quad (x \text{ fixed}) \ .$$

We will see below that this is not accidental, but preservation of angles holds generally for holomorphic maps $f(z)$ with $f'(z) \neq 0$.

Roughly, the map

$$z \to e^z = e^x e^{iy}$$

is oscillatory in y and has real exponential behavior in x. For $x << -1$, the complex number e^z is very small in absolute value; for $x >> 1$, the complex number e^z is very large in absolute value. This follows simply from

$$|e^z| = e^x \ .$$

2.7 The Main Branch of the Complex Logarithm

Consider the open horizontal strip

$$S_\pi = \{z = x + iy \ : \ -\pi < y < \pi, \ x \in \mathbb{R}\}$$

and the slit plane

$$\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0] \ .$$

If $z = x + iy \in S_\pi$ then $-\pi < y < \pi$, thus

$$e^z = e^x e^{iy} \in \mathbb{C}^- \ .$$

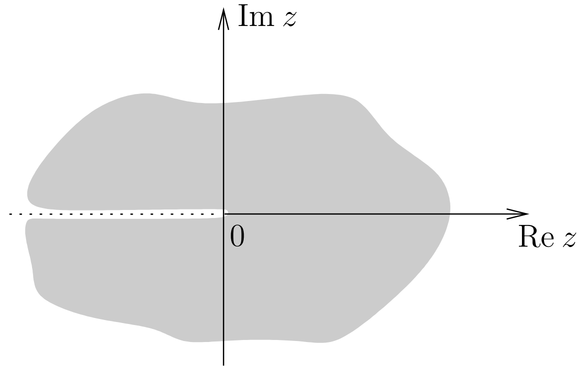

Figure 2.2: The slit plane $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$

Lemma 2.7 *The map*

$$\exp : \left\{ \begin{array}{ccc} S_\pi & \mapsto & \mathbb{C}^- \\ z & \to & e^z \end{array} \right. \tag{2.6}$$

is one–to–one and onto.

Proof: a) Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in S_\pi$ and let $e^{z_1} = e^{z_2}$. It follows that $e^{x_1} = e^{x_2}$ and $e^{iy_1} = e^{iy_2}$. Therefore, $x_1 = x_2$ is clear. The uniqueness statement of Lemma 2.5 yields that $y_1 = y_2$. (Here it is important that we assume $-\pi < y_j < \pi$ for $j = 1, 2$.)

b) Let $w = re^{i\theta} \in \mathbb{C}^-$ be given. Then we have $r > 0$ and may assume that $-\pi < \theta < \pi$. Let $x = \ln r$ and set $z = x + i\theta$. We have $z \in S_\pi$ and

$$e^z = e^x e^{i\theta} = re^{i\theta} = w \ .$$

$\diamond$

By definition, the inverse function of (2.6) is the main branch of the complex logarithm:

$$\log : \left\{ \begin{array}{ccc} \mathbb{C}^- & \mapsto & S_\pi \\ w & \to & \log w \end{array} \right. \tag{2.7}$$

with

$$\exp(\log w) = w \quad \text{for all} \quad w \in \mathbb{C}^- \ .$$

This log–function extends the real function

$$\ln : \left\{ \begin{array}{ccc} (0, \infty) & \mapsto & (-\infty, \infty) \\ r & \to & \ln r \end{array} \right. \tag{2.8}$$

from the positive real axis into the slit plane $\mathbb{C}^-$.

Given any $w \in \mathbb{C}^-$, write

$$w = re^{i\theta} \quad \text{with} \quad r > 0 \quad \text{and} \quad -\pi < \theta < \pi \ .$$

Here the numbers $r > 0$ and θ with $-\pi < \theta < \pi$ are unique.

It holds that

$$w = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta} \ ,$$

thus

$$\log w = \ln r + i\theta \ .$$

If $w = w_1 + iw_2 \in \mathbb{C}^-$ with $w_1, w_2 \in \mathbb{R}$, then

$$r = (w_1^2 + w_2^2)^{1/2}, \quad \theta = \arctan(w_2/w_1) \ ,$$

thus

$$\log(w_1 + iw_2) = \frac{1}{2} \ln(w_1^2 + w_2^2) + i \arctan(w_2/w_1) \ .$$

Here one has to choose the correct branch of arctan.

Example: Since $e^{i\pi/2} = i$ and $\frac{i\pi}{2} \in S_\pi$ we have

$$\log i = \frac{i\pi}{2} \ .$$

General Powers; Main Branch. Let $b \in \mathbb{C}$ and let $a \in \mathbb{C}^-$. One defines the main branch of a^b by

$$a^b = e^{b \log a} \ .$$

Example: We have

$$e^{\pi i/2} = i \ ,$$

thus

$$\log i = \pi i/2 \ .$$

Therefore,

$$\begin{aligned} i^i &= e^{i(\pi i/2)} \\ &= e^{-\pi/2} \\ &= 0.2078... \end{aligned}$$

Somewhat surprisingly, the number i^i is real. Euler discovered this result in 1746.

2.8 Remarks on the Multivalued Logarithm

If $z \in S_\pi$ and $w = e^z$ then $w \in \mathbb{C}^-$ and

$$\log w = z \ .$$

Here $\log w$ is defined above.

If $n \in \mathbb{Z}$ then

$$e^{z + 2\pi i n} = e^z = w \ .$$

A possible view is to say that

$$\log w = z + 2\pi i n$$

where n can take on any integer value and then call log a multivalued function. However, this view is not satisfying since it does not agree with the general notion of a function.

One can proceed as follows: Instead of defining $w \to \log w$ as a multivalued function on $\mathbb{C}^-$ or on $\mathbb{C} \setminus \{0\}$ one introduces an appropriate Riemann surface $\mathcal{S}$. On this surface the logarithm function will become single valued.

To get an intuitive idea of the Riemann surface $\mathcal{S}$, first consider the point $w = -1$ which lies outside the slit plane $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$.

Set

$$w_\varepsilon = e^{i(\pi-\varepsilon)} \quad \text{and} \quad \tilde{w}_\varepsilon = e^{i(-\pi+\varepsilon)} \quad \text{for} \quad 0 < \varepsilon << 1 \ .$$

Clearly,

$$w_\varepsilon \to -1 \quad \text{and} \quad \tilde{w}_\varepsilon \to -1 \quad \text{as} \quad \varepsilon \to 0 \ .$$

Let log denote the main branch of the complex logarithm as defined above. We have

$$\log w_\varepsilon = i(\pi - \varepsilon) \quad \text{and} \quad \log \tilde{w}_\varepsilon = i(-\pi + \varepsilon) \ .$$

Therefore,

$$\log w_\varepsilon \to i\pi \quad \text{and} \quad \log \tilde{w}_\varepsilon \to -i\pi \quad \text{as} \quad \varepsilon \to 0 \ .$$

This shows that one cannot continue the function log from $\mathbb{C}^-$ to $\mathbb{C} \setminus \{0\}$ as a continuous function. Therefore, cut the set $\mathbb{C} \setminus \{0\}$ along the line $(-\infty, 0)$ and then bend the part above the line upwards, the part below the line downwards. Then extend the resulting surface appropriately. The function $w \to \log w$ can be extended continuously to the extended surface. The process can be repeated an infinity of times. It leads to the Riemann surface for the function $\log w$. On this surface the function $\log w$ is single valued and smooth, except at $w = 0$. The point $w = 0$ is a so–called branch point of the Riemann surface.

3 Complex Differentiability and the Cauchy–Riemann Equations

3.1 Outline and Notations

We identify $\mathbb{R}^2$ and $\mathbb{C}$ using the correspondence

$$(x, y) \quad \longleftrightarrow \quad z = x + iy \; .$$

Let $U \subset \mathbb{C}$ denote an open set and let $f : U \to \mathbb{C}$ be a map. We then define two real–valued functions, $u, v : U \to \mathbb{R}$, by

$$f(x + iy) = u(x, y) + iv(x, y) \quad \text{for} \quad z = x + iy \in U \; .$$

Then the complex–valued map $f : U \to \mathbb{C}$ corresponds to the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = F(x, y) \tag{3.1}$$

from $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$.

Loosely speaking, a map is differentiable at a point P if it can be approximated at P by a linear map. In the present context, we must distinguish clearly between $\mathbb{R}$–linearity and $\mathbb{C}$–linearity. Therefore, in the next section, we consider $\mathbb{R}$–linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ and ask under what assumptions an $\mathbb{R}$–linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ corresponds to a $\mathbb{C}$–linear map $f : \mathbb{C} \to \mathbb{C}$. The condition is of an algebraic nature.

In Section 3.3 we will use this to discuss the relationship between real and complex differentiability. Complex differentiability leads to the Cauchy–Riemann equations for the functions $u(x, y)$ and $v(x, y)$.

3.2 $\mathbb{R}$–Linear and $\mathbb{C}$–Linear Maps from $\mathbb{R}^2 \simeq \mathbb{C}$ into Itself

If V is a vector space over a field $\mathbb{K}$ then a map $f : V \to V$ is called $\mathbb{K}$–linear (or simply linear if the field $\mathbb{K}$ is unambiguous) if

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2) \quad \text{for all} \quad v_1, v_2 \in V \quad \text{and} \qquad \text{for all} \quad \alpha, \beta \in \mathbb{K} \; .$$

The space $\mathbb{R}^2$ is a two–dimensional vector space over the field $\mathbb{R}$. The general $\mathbb{R}$–linear map from $\mathbb{R}^2$ into itself has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \tag{3.2}$$

where $a, b, c, d \in \mathbb{R}$.

The space $\mathbb{C}$ is a one–dimensional vector space over the field $\mathbb{C}$ and the general $\mathbb{C}$–linear map from $\mathbb{C}$ into itself has the form

$$z \to wz =: f(z)$$

where $w \in \mathbb{C}$.

Let $w = \alpha + i\beta$ and $z = x + iy$ where α, β, x and y are real. Then we have

$$\begin{aligned} f(z) &= wz \\ &= (\alpha + i\beta)(x + iy) \\ &= (\alpha x - \beta y) + i(\beta x + \alpha y) \ . \end{aligned}$$

The map $z \to f(z) = wz$ is a $\mathbb{C}$–linear map from $\mathbb{C}$ into itself. We obtain that $z \to f(z) = wz$ corresponds to the $\mathbb{R}$–linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ . \tag{3.3}$$

This is the map (3.2) with

$$a = d = \alpha, \quad -b = c = \beta \ .$$

In other words, an $\mathbb{R}$–linear map (3.2) corresponds to a $\mathbb{C}$–linear map iff

$$a = d \quad \text{and} \quad -b = c \ . \tag{3.4}$$

If we write the $\mathbb{R}$–linear map (3.2) in the form (3.1), then

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \tag{3.5}$$

and one obtains that

$$a = u_x, \quad b = u_y, \quad c = v_x, \quad d = v_y \ .$$

The condition (3.4) becomes

$$u_x = v_y, \quad -u_y = v_x \ .$$

In a more general setting, these are the Cauchy–Riemann equations. They require precisely that the (real) Jacobian of the map (3.1) corresponds to a $\mathbb{C}$–linear map.

To summarize:

Theorem 3.1 *The $\mathbb{R}$–linear map (3.2) corresponds to the $\mathbb{C}$–linear map*

$$z \to (\alpha + i\beta)z$$

if and only if

$$a = d = \alpha, \quad -b = c = \beta \ .$$

In other words, the $\mathbb{R}$–linear map (3.2) corresponds to the $\mathbb{C}$–linear map $z \to (\alpha + i\beta)z$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \ .$$

3.3 The Polar Representation of a Complex Number and the Corresponding Matrix

This section can be skipped.

Let

$$w = \alpha + i\beta = re^{i\theta} = r\cos\theta + ir\sin\theta, \quad w \neq 0 \; .$$

The $\mathbb{C}$–linear map $z \to wz$ corresponds to the $\mathbb{R}$–linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: F(x,y) \tag{3.6}$$

where

$$\alpha = r\cos\theta \quad \text{and} \quad \beta = r\sin\theta \; .$$

Therefore, system matrix is

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \sqrt{\alpha^2+\beta^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.7}$$

In terms of real variables, the complex map $z \to wz$ is the map of rotation by the angle θ, counterclockwise, followed by stretching by the factor

$$r = |w| = \sqrt{\alpha^2+\beta^2} \; .$$

Remark: The determinant of the matrix in (3.7) is

$$det\, F'(x,y) = \alpha^2 + \beta^2 = |w|^2 = r^2 \; .$$

The map $z \to wz$ stretches *lengths* by $|w|$. The determinant of the Jacobian matrix $F'(x,y)$ describes the stretching of *area*, which is described by the factor $|w|^2 = r^2$.

3.4 Real and Complex Differentiability

In the following, $\psi(h)$ denotes a function with $\psi(h) \to 0$ as $h \to 0$.

Let $a < c < b$ be real numbers and let $f : (a,b) \to \mathbb{R}$ be a real function. The function f is **real–differentiable** at c if the limit

$$\lim_{h\to 0} \frac{1}{h}\Big(f(c+h) - f(c)\Big) =: w \tag{3.8}$$

exists. Equivalently, f is real differentiable at c if there exists $w \in \mathbb{R}$ with

$$f(c+h) = f(c) + wh + h\psi(h) \quad \text{and} \quad \lim_{h\to 0}\psi(h) = 0 \; . \tag{3.9}$$

If $w \in \mathbb{R}$ with (3.9) exists then w is unique since (3.9) implies (3.8). One writes $w = f'(c)$.

Let $U \subset \mathbb{R}^m$ be an open set and let $f : U \to \mathbb{R}^n$ be a function. Let $c \in U$. The function f is **real–differentiable** at the vector c if there exists a matrix $A \in \mathbb{R}^{n\times m}$ with

$$f(c+h) = f(c) + Ah + \|h\|\psi(h) \quad \text{and} \quad \lim_{h\to 0}\psi(h) = 0 \; . \tag{3.10}$$

If a matrix $A \in \mathbb{R}^{m \times n}$ with (3.10) exists, then it is unique and one write $A = f'(c)$. This matrix is called the Jacobian of f at the point c.

The entries of A agree with the partial derivatives of the components of f,

$$a_{jk} = \frac{\partial f_j}{\partial x_k}(c) \ .$$

Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$. Then f is **complex differentiable** at z_0 if the limit

$$\lim_{h \to 0} \frac{1}{h}\Big(f(z_0 + h) - f(z_0) \Big) =: w \tag{3.11}$$

exists. Equivalently, f is **complex differentiable** at z_0 if there exists $w \in \mathbb{C}$ with

$$f(z_0 + h) = f(z_0) + wh + h\psi(h) \quad \text{and} \quad \lim_{h \to 0} \psi(h) = 0 \ . \tag{3.12}$$

If $w \in \mathbb{C}$ with (3.12) exists then w is unique since (3.12) implies (3.11). One writes $w = f'(z_0)$.

A function $f : U \to \mathbb{C}$ which is complex differentiable at every point $z \in \mathbb{C}$ is called a holomorphic function on U. We then write $f \in H(U)$. A function which is holomorphic on $U = \mathbb{C}$ is called an entire function.

Example 1: Let $n \in \mathbb{N}$. We claim that $f(z) = z^n$ is complex differentiable with $f'(z) = nz^{n-1}$.

Proof: Use the binomial formula

$$(a+b)^n = \sum_{j=0}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) a^{n-j} b^j$$

with

$$\left(\begin{array}{c} n \\ j \end{array} \right) = \frac{n!}{j!(n-j)!}$$

to obtain that

$$f(z+h) - f(z) = (z+h)^n - z^n = nz^{n-1}h + \mathcal{O}(h^2) \ .$$

Therefore,

$$\lim_{h \to 0} \frac{1}{h}\Big(f(z+h) - f(z) \Big) = nz^{n-1} \ .$$

Example 2: We claim that $f(z) = \bar{z}$ is not complex differentiable at any point $z \in \mathbb{C}$.

Proof: For any $z, h \in \mathbb{C}$ we have

$$f(z+h) - f(z) = \bar{z} + \bar{h} - \bar{z} = \bar{h} \ .$$

Therefore, for $h \neq 0$:

$$\frac{1}{h}(f(z+h) - f(z)) = \frac{\bar{h}}{h} \ .$$

If $h = h_1 \in \mathbb{R}, h_1 \neq 0$ then

$$\frac{\bar{h}}{h} = \frac{h_1}{h_1} = 1 \ .$$

If $h = ih_2 \neq 0, h_2 \in \mathbb{R}$, then

$$\frac{\bar{h}}{h} = \frac{-ih_2}{ih_2} = -1 \ .$$

Therefore, the limit

$$\lim_{h \to 0} \frac{\bar{h}}{h}$$

does not exist.

In the following, let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. We write

$$f(x+iy) = u(x,y) + iv(x,y)$$

and identify f with the function

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} =: F(x,y)$$

from U into $\mathbb{R}^2$.

Theorem 3.2 *Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$. Then the following two conditions are equivalent:*
1) f is complex differentiable at z_0.
2) F is real differentiable at (x_0, y_0) and the real matrix

$$A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} (x_0, y_0)$$

determines a $\mathbb{C}$–linear map, i.e.,

$$u_x = v_y, \quad -u_y = v_x \quad at \quad (x_0, y_0) \ .$$

Proof: First let f be complex differentiable and let

$$f(z_0 + h) = f(z_0) + wh + h\psi(h) \quad \text{and} \quad \lim_{h \to 0} \psi(h) = 0 \ .$$

Let $w = \alpha + i\beta$ and $h = h_1 + ih_2$. We have

$$\begin{aligned} wh &= (\alpha + i\beta)(h_1 + ih_2) \\ &= \alpha h_1 - \beta h_2 + i(\beta h_1 + \alpha h_2) \end{aligned}$$

Therefore, the complex number wh corresponds to the vector

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2 \ .$$

It follows that $F : U \to \mathbb{R}^2$ is real differentiable at the point (x_0, y_0) with Jacobian

$$F'(x_0, y_0) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} .$$

The converse follows similarly. $\diamond$

3.5 The Complex Logarithm as an Example

We have for $z = x + iy \in \mathbb{C}^-$:

$$f(z) = \log z = \frac{1}{2} \ln(x^2 + y^2) + i \arctan(y/x) ,$$

thus

$$\begin{aligned} u &= \frac{1}{2} \ln(x^2 + y^2) , \\ v &= \arctan(y/x) . \end{aligned}$$

The partial derivatives are

$$\begin{aligned} u_x &= \frac{x}{x^2 + y^2} \\ u_y &= \frac{y}{x^2 + y^2} \\ v_x &= -\frac{y}{x^2} \cdot \frac{1}{1 + (y/x)^2} \\ &= -\frac{y}{x^2 + y^2} \\ v_y &= \frac{1}{x} \cdot \frac{1}{1 + (y/x)^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

We obtain that

$$u_x = v_y, \quad u_y = -v_x .$$

Since the Cauchy–Riemann equations are satisfied, the function $f(z) = \log z$ is complex–differentiable in $\mathbb{C}^-$. We compute its complex derivative:

$$\begin{aligned} f'(z) &= f_x \\ &= u_x + iv_x \\ &= \frac{x - iy}{x^2 + y^2} \\ &= \frac{x - iy}{(x + iy)(x - iy)} \\ &= \frac{1}{x + iy} \\ &= \frac{1}{z} \end{aligned}$$

This, of course, is not unexpected since the derivative of $\ln x$ is $\frac{1}{x}$. We will obtain later that the functions

$$f(z) = \log z$$

and

$$f'(z) = \frac{1}{z}$$

are the only holomorphic extensions of the functions $\ln x$ and $1/x$, defined for $x > 0$, into the slit plane $\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]$.

3.6 Complex Conjugates

Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$. We write

$$f(x + iy) = u(x, y) + iv(x, y)$$

where u and v are real valued. We will assume that $u, v \in C^2(U)$. The function $f(z)$ is holomorphic on U if and only if

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{in} \quad U \ .$$

If the Cauchy–Riemann Equations hold then $\Delta u = \Delta v = 0$. The function v is called a harmonic conjugate of u.

Computing a Harmonic Conjugate: Let $u = xy$. We have $\Delta u = 0$ and want to compute a harmonic conjugate v of u.

We have

$$v_x = -u_y = -x \ ,$$

thus

$$v(x, y) = -\frac{x^2}{2} + \phi(y) \ .$$

Also,

$$v_y = \phi'(y) = u_x = y \ ,$$

thus

$$\phi(y) = \frac{y^2}{2} + const \ .$$

The function

$$v = \frac{1}{2}(y^2 - x^2)$$

is a harmonic conjugate of $u = xy$. The function

$$f(x + iy) = xy + \frac{i}{2}\,(y^2 - x^2)$$

is holomorphic on $\mathbb{C}$. We have

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= y - ix \\ &= -i(x+iy) \\ &= -iz \end{aligned}$$

and

$$\begin{aligned} f(z) &= -\frac{i}{2}\,z^2 \\ &= -\frac{i}{2}\,(x^2 + 2ixy - y^2) \\ &= xy + \frac{i}{2}\,(y^2 - x^2) \end{aligned}$$

Remark on Harmonic Conjugates: Let u and v be harmonic conjugates on U. For the inner product of the gradients we have

$$(u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y = 0$$

since $v_x = -u_y$ and $v_y = u_x$. The orthogonality of the gradients implies that the lines given by $u(x,y) = c_1 = const_1$ are orthogonal to the lines given by $v(x,y) = c_2 = const_2$.

3.7 The Operators $\partial/\partial z$ and $\partial/\partial \bar{z}$

This section can be skipped.

Let $\lambda, \mu \in \mathbb{C}$. Then the map

$$z \to f(z) = \lambda z + \mu \bar{z}$$

from $\mathbb{C}$ to $\mathbb{C}$ is $\mathbb{R}$–linear since the maps $z \to \lambda z$ and $z \to \bar{z}$ are $\mathbb{R}$–linear.

We can write $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$ (with $\lambda_j, \mu_j \in \mathbb{R}$) and obtain that the map $f(z)$ depends on four real parameters, $\lambda_1, \lambda_2, \mu_1, \mu_2$. We also can start with formula (3.2) and obtain that the general $\mathbb{R}$–linear map from $\mathbb{C}$ into $\mathbb{C}$ depends on four real parameters a, b, c, d.

Let us derive the relations between the parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$ and a, b, c, d. To do this, recall that

$$z = x + iy, \quad \bar{z} = x - iy$$

and

$$x = \frac{1}{2}(z + \bar{z}), \quad iy = \frac{1}{2}(z - \bar{z})\ .$$

Therefore, if we start from the general form (3.2), then we have

$$
\begin{aligned}
f(x+iy) &= (ax+by)+i(cx+dy) \\
&= (a+ic)x+(b+id)y \\
&= \frac{1}{2}(a+ic)(z+\bar{z})-\frac{i}{2}(b+id)(z-\bar{z}) \\
&= \lambda z+\mu\bar{z}
\end{aligned}
$$

with

$$
\begin{aligned}
\lambda &= \frac{1}{2}(a+ic)-\frac{i}{2}(b+id) \\
\mu &= \frac{1}{2}(a+ic)+\frac{i}{2}(b+id)
\end{aligned}
$$

This shows how to obtain the representation $z \to \lambda z+\mu\bar{z}$ from (3.2).

Conversely, if we start from the general form

$$
f(z)=\lambda z+\mu\bar{z}, \quad \lambda=\lambda_1+i\lambda_2, \quad \mu=\mu_1+i\mu_2 \ ,
$$

then we have

$$
\begin{aligned}
f(z) &= \lambda z+\mu\bar{z} \\
&= (\lambda_1+i\lambda_2)(x+iy)+(\mu_1+i\mu_2)(x-iy) \\
&= (\lambda_1+\mu_1)x+(\mu_2-\lambda_2)y+i\Big((\lambda_2+\mu_2)x+(\lambda_1-\mu_1)y\Big)
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
a &= \lambda_1+\mu_1 \\
b &= \mu_2-\lambda_2 \\
c &= \lambda_2+\mu_2 \\
d &= \lambda_1-\mu_1
\end{aligned}
$$

We obtain that the Cauchy–Riemann equations,

$$
a=d \quad \text{and} \quad -b=c \ ,
$$

are equivalent to the condition

$$
\mu=0 \ .
$$

Lemma 3.1 *The map*

$$
f(z)=\lambda z+\mu\bar{z}
$$

is complex differentiable if and only if $\mu=0$.

Proof: This is clear since $z \to \lambda z$ is complex differentiable and $z \to \bar{z}$ is not complex differentiable. Another proof follows from the Cauchy–Riemann equations. $\diamond$

Let

$$\begin{aligned} f(z) &= f(x+iy) \\ &= \lambda z + \mu \bar{z} \\ &= \lambda(x+iy) + \mu(x-iy) \end{aligned}$$

We have

$$f_x = \lambda + \mu \quad \text{and} \quad f_y = i\lambda - i\mu \ .$$

It follows that

$$\lambda = \frac{1}{2} f_x - \frac{i}{2} f_y \quad \text{and} \quad \mu = \frac{1}{2} f_x + \frac{i}{2} f_y \ . \tag{3.13}$$

Since

$$f(z) = \lambda z + \mu \bar{z}$$

it makes sense to write

$$f_z = \lambda \quad \text{and} \quad f_{\bar{z}} = \mu \ .$$

Then (3.13) yields that

$$f_z = \frac{1}{2} f_x - \frac{i}{2} f_y \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2} f_x + \frac{i}{2} f_y \ .$$

This motivates to define the operators

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y} \end{aligned}$$

Lemma 3.2 *Let $f : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$–linear function, i.e.,*

$$\begin{aligned} f(x+iy) &= (ax+by) + i(cx+dy) \\ &= (a+ic)x - i(b+id)iy \\ &= \frac{1}{2}(a+ic)(z+\bar{z}) - \frac{i}{2}(b+id)(z-\bar{z}) \\ &= \lambda z + \mu \bar{z} \end{aligned}$$

with

$$\begin{aligned} \lambda &= \frac{1}{2}(a+ic) - \frac{i}{2}(b+id) \\ \mu &= \frac{1}{2}(a+ic) + \frac{i}{2}(b+id) \end{aligned}$$

Then f is complex differentiable if and only if $\mu = f_{\bar{z}} = 0$.

Let $f : \mathbb{C} \to \mathbb{C}$ be $\mathbb{R}$–linear and complex differentiable. Then we have

$$f(x+iy) = (a+ic)x - i(b+id)iy = \lambda z$$

with

$$a = d = \lambda_1, \quad -b = c = \lambda_2 \ .$$

Therefore,

$$\begin{aligned} f' &= f_z \\ &= \lambda \\ &= \lambda_1 + i\lambda_2 \\ &= a + ic \\ &= -i(b+id) \\ &= f_x \\ &= -if_y \end{aligned}$$

One can show that for a complex differentiable function $f(z)$ the following holds:

$$f' = \frac{df}{dz} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \ .$$

4 Complex Line Integrals and Cauchy's Theorems

Summary: We first introduce parameterized curves and line integrals of continuous functions along such curves. Then we come to a central result of complex function theory, **Goursat's Lemma**. The lemma is a special case of **Cauchy's Integral Theorem**. The proof of Goursat's Lemma is remarkably clean. Based on the lemma, we construct a primitive of a holomorphic function in a disk and then prove Cauchy's Integral Theorem and Integral Formulas for holomorphic functions in a disk.

4.1 Curves

Let $\gamma : [a,b] \to \mathbb{C}$ denote a C^1–map. This means the following: If we write

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t), \quad a \le t \le b \ ,$$

then the two functions $\gamma_1, \gamma_2 : [a,b] \to \mathbb{R}$ are differentiable and their derivatives are continuous. Intuitively, we think of the image set

$$\{\gamma(t) \ : \ a \le t \le b\}$$

as a curve in $\mathbb{C}$, parameterized by the parameter t varying in $a \le t \le b$.

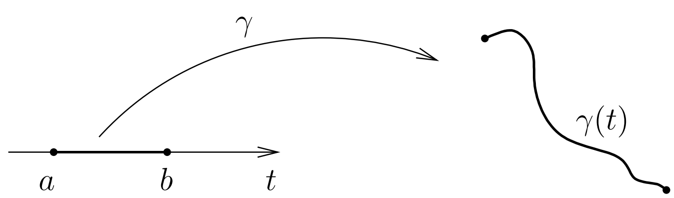

Figure 4.1: Parameterization of a curve

Every curve has many different parameterizations. For example, the mappings

$$\gamma(t) = e^{it}, \quad 0 \le t \le 2\pi \ ,$$

and

$$\delta(s) = e^{2is}, \quad 0 \le s \le \pi \ ,$$

both parameterize the circle $\mathcal{C}_1$ of radius one, centered at the origin. The map

$$\varepsilon(t) = e^{-it}, \quad 0 \le t \le 2\pi \ ,$$

has the same image set as γ but parameterizes $\mathcal{C}_1$ in opposite direction. We say that γ and δ both parameterize $\mathcal{C}_1$ whereas the map ε parameterizes $-\mathcal{C}_1$.

It is not trivial to define the notion of a curve precisely. One can proceed as follows.

Definition: Let $\mathcal{P}$ denote the set of all pairs (γ, I) where $I \subset \mathbb{R}$ is a finite closed interval and $\gamma : I \to \mathbb{C}$ is a C^1–map. Call $(\gamma, I_1), (\delta, I_2) \in \mathcal{P}$ equivalent if there exists a C^1-map (a parameter transformation)

$$\phi : I_1 \to I_2 \quad \text{with} \quad \phi'(t) > 0 \quad \text{for all} \quad t \in I_1$$

which is one-to-one and onto and satisfies

$$\delta(\phi(t)) = \gamma(t) \quad \text{for} \quad t \in I_1 \ .$$

A C^1-curve is an equivalence class in $\mathcal{P}$. If Γ is a C^1-curve and $(\gamma, I) \in \Gamma$, then (γ, I) is called a parameterization of the curve Γ.

Assume the curve Γ has the parameterization (γ, I). It is often convenient to identify the curve Γ with the set

$$\{\gamma(t) \ : \ t \in I\} \ ,$$

but one should at least assign a direction to the above set.

Furthermore, it is convenient to work with curves that are only piecewise C^1 and with parameterizations $\gamma(t)$ where t varies in an unbounded interval. A curve which is piecewise C^1 has a continuous parameterization which is piecewise C^1.

Length of a C^1–curve: Let Γ denote a C^1–curve with parameterization $\gamma(t), a \le t \le b$. Using real analysis, one obtains that

$$\begin{aligned} length(\Gamma) &= \int_a^b \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}\, dt \\ &= \int_a^b |\gamma'(t)|\, dt \ . \end{aligned}$$

The formula for the length of the curve Γ is plausible since $\gamma(t + \Delta t) - \gamma(t) \sim \gamma'(t)\Delta t$ for small $\Delta t > 0$, thus

$$|\gamma(t + \Delta t) - \gamma(t)| \sim |\gamma'(t)|\Delta t \ .$$

Example 4.1: Let

$$\gamma(t) = re^{it}, \quad 0 \le t \le 2\pi \ ,$$

denote a parameterization of the circle $\mathcal{C}_r$ of radius r centered at the origin. One obtains that $|\gamma'(t)| = r$ and $length(\mathcal{C}_r) = 2\pi r$.

4.2 Definition and Simple Properties of Line Integrals

Let $\gamma : [a, b] \to \mathbb{C}$ denote a C^1–map parameterizing the curve

$$\Gamma = \{\gamma(t) \ : \ a \le t \le b\}$$

and let

$$f : \Gamma \to \mathbb{C}$$

denote a continuous function. We want to define the line integral of f along Γ, which we denote by

$$\int_\Gamma f(z)\,dz \quad \text{or} \quad \int_\gamma f(z)\,dz \ .$$

This line integral can be defined as a limit of Riemann sums as follows: Let

$$a = t_0 < t_1 < \ldots < t_n = b$$

denote a partition of the parameterization interval $[a,b]$ and let $t_{j-1} \le s_j \le t_j$. The points $z_j = \gamma(t_j)$ and $w_j = \gamma(s_j)$ line up along Γ. We have:

$$\begin{aligned} \int_\Gamma f(z)\,dz &\approx \sum_{j=1}^n f(w_j)(z_j - z_{j-1}) \\ &= \sum_{j=1}^n f(\gamma(s_j))(\gamma(t_j) - \gamma(t_{j-1})) \\ &\approx \sum_{j=1}^n f(\gamma(s_j))\gamma'(s_j)(t_j - t_{j-1}) \\ &\approx \int_a^b f(\gamma(t))\gamma'(t)\,dt \end{aligned}$$

As the partition is refined, the sums converge. One obtains:

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt \ . \tag{4.1}$$

We will use equation (4.1) as the definition of the line integral $\int_\Gamma f(z)\,dz$. This is justified since the right–hand side is independent of the parameterization γ of the curve Γ. To obtain this, use the rule of substitution.

Note on Computation: To compute the integral on the right–hand side of (4.1), note the following: If $\psi : [a,b] \to \mathbb{C}$ is a continuous complex–valued function,

$$\psi(t) = \psi_1(t) + i\psi_2(t) \ ,$$

then

$$\int_a^b \psi(t)\,dt = \int_a^b \psi_1(t)\,dt + i\int_a^b \psi_2(t)\,dt \ . \tag{4.2}$$

Using (4.1) and (4.2) with $\psi(t) = f(\gamma(t))\gamma'(t)$ we obtain that, in principle, the evaluation of line integrals is standard calculus.

Example 4.2: Let Γ denote a curve in $\mathbb{C}$ from P to Q and let $f(z) = c = const$. Applying the Riemann sum definition one obtains that

$$\int_\Gamma c\,dz = c(Q-P) \ .$$

Another view: If $\gamma(t), a \le t \le b$, parameterizes Γ, then

$$\int_\Gamma c\,dz = c\int_a^b \gamma'(t)\,dt = c\Big(\gamma(b) - \gamma(a)\Big) = c(Q-P) \ .$$

Example 4.3: Using the parameterization

$$\gamma(t) = e^{it}, \quad 0 \le t \le 2\pi \ ,$$

of the unit circle $\mathcal{C}_1$, obtain for any integer n:

$$\int_{\mathcal{C}_1} z^n \, dz = 0 \quad \text{for} \quad n \neq -1 \quad \text{and} \quad \int_{\mathcal{C}_1} \frac{dz}{z} = 2\pi i \ .$$

If one integrates along $\mathcal{C}_r$ with parameterization

$$\gamma(t) = re^{it}, \quad 0 \le t \le 2\pi \ ,$$

one obtains the same results.

Details: If $f(z) = z^n, \gamma(t) = re^{it}, \gamma'(t) = ire^{it}$ then

$$\begin{aligned}\int_{\mathcal{C}_r} f(z) \, dz &= \int_0^{2\pi} r^n e^{int} ire^{it} \, dt \\ &= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt\end{aligned}$$

and the claim follows.

Example 4.4:

$$\int_{\mathcal{C}_r} \bar{z} \, dz = 2\pi i r^2 \ .$$

Note that $z\bar{z} = |z|^2$, thus $\bar{z} = \frac{r^2}{z}$ for $z \in \mathcal{C}_r$. The claim follows since

$$\int_{\mathcal{C}_r} \frac{dz}{z} = 2\pi i \ .$$

Another computation using the parameterization $z(t) = re^{it}, 0 \le t \le 2\pi$, of $\mathcal{C}_r$:

$$\begin{aligned}\int_{\mathcal{C}_r} \bar{z} \, dz &= \int_0^{2\pi} re^{-it} \, rie^{it} \, dt \\ &= r^2 i \int_0^{2\pi} dt \\ &= 2\pi i r^2\end{aligned}$$

A Simple Estimate: The estimate

$$\left| \int_\Gamma f(z) \, dz \right| \le \max_{z \in \tilde{\gamma}} |f(z)| \ length(\Gamma)$$

can be obtained using Riemann sums. It also follows from (4.1).

To practically evaluate line integrals, the following result, which is analogous to the fundamental theorem of calculus, is very useful:

Theorem 4.1 *Let $U \subset \mathbb{C}$ denote an open set and let $f : U \to \mathbb{C}$ be a continuous function. Suppose that $g : U \to \mathbb{C}$ is complex differentiable and $g' = f$ in U. If $\gamma : [a,b] \to U$ parameterizes a C^1–curve Γ, then*

$$\int_\Gamma f(z)\,dz = g(\gamma(b)) - g(\gamma(a)) = g(Q) - g(P)\ .$$

Here Γ goes from $P = \gamma(a)$ to $Q = \gamma(b)$. If Γ is continuous and piecewise C^1, the same result holds.

Proof: We have

$$\begin{aligned}\int_\Gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b g'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \frac{d}{dt}(g(\gamma(t))\,dt \\ &= g(\gamma(b)) - g(\gamma(a)) \\ &= g(Q) - g(P)\end{aligned}$$

$\diamond$

Definition: If $g \in H(U)$ and $g' = f$ in U, then g is called an *anti-derivative* or a *primitive* of f in U.

The previous theorem says that we can evaluate line integrals of f easily if we have an anti-derivative g of f. We will also obtain below that, conversely, line integrals can be used to construct an anti-derivative of f if f is complex differentiable.

Example 4.5: Let $f(z) = z^n$ where n is an integer, $n \neq -1$. If $n \geq 0$ then we can take $U = \mathbb{C}$ and $g(z) = \frac{1}{n+1}\,z^{n+1}$. If $n \leq -2$ we can take $U = \mathbb{C} \setminus \{0\}$ and again $g(z) = \frac{1}{n+1}\,z^{n+1}$. In both cases we have $g'(z) = f(z) = z^n$ in U. It follows that

$$\int_\Gamma z^n\,dz = 0$$

for any closed curve Γ in U.

Example 4.6: Consider $f(z) = 1/z$ in $U = \mathbb{C} \setminus \{0\}$. Since

$$\int_{\mathcal{C}_1} \frac{dz}{z} = 2\pi i \neq 0 \tag{4.3}$$

one obtains the following: There is no complex differentiable function $g : U \to \mathbb{C}$ with $g'(z) = 1/z$ in U. We have obtained in Section 3.5 that we can extend the real function $g(x) = \ln x$ (defined for $0 < x < \infty$) into the open slit plane

$$\mathbb{C}^- = \mathbb{C} \setminus (-\infty, 0]\ .$$

The extended function is the main branch of the complex logarithm, $g(z) = \log z$. One can show that $g(z) = \log z$ is holomorphic in $\mathbb{C}^-$ and $g'(z) = \frac{1}{z}$ in $\mathbb{C}^-$. However, because of (4.3), one cannot extend $g(z) = \log z$ holomorphically into $U = \mathbb{C} \setminus \{0\}$.

Example 4.7: Let Γ be a curve from $\gamma(a) = z_0$ to $\gamma(b) = z_1$. Then

$$\int_\gamma z^3\,dz = \frac{1}{4}z_1^4 - \frac{1}{4}z_0^4 \;.$$

4.3 Goursat's Lemma

A curve Γ from P to Q is called closed if $P = Q$. A closed curve is called simply closed if it does not intersect itself. Thus, if $\gamma(t), a \le t \le b$, parameterizes the simply closed curve Γ, then $\gamma(t_1) \ne \gamma(t_2)$ for $a \le t_1 < t_2 \le b$ unless $t_1 = a$ and $t_2 = b$.

Cauchy's Integral Theorem can be stated, somewhat loosely, as follows:

Theorem 4.2 *Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be complex differentiable in U. Let Γ denote a simply closed continuous curve in U which is piecewise C^1. Assume that the interior of Γ lies in U, i.e., Γ does not surround any holes of U. Then*

$$\int_\Gamma f(z)\,dz = 0 \;.$$

It is not easy to make precise what the interior of a closed curve is. (A possibility is to use the Jordan curve theorem, a result of topology that is notoriously difficult to prove.)

We prove Cauchy's theorem first for the case that the curve Γ is the boundary of a triangle Δ in U. The corresponding result is known as Goursat's Lemma.

Note that $f \in H(U)$ implies that f is continuous in U. Therefore, $\int_\Gamma f(z)\,dz$ is defined for any continuous curve Γ in U which is piecewise C^1.

Theorem 4.3 *(Goursat's Lemma) Let $U \subset \mathbb{C}$ denote an open set and let $f : U \to \mathbb{C}$ be complex differentiable. Let Δ be a closed triangle, $\Delta \subset U$, with boundary curve $\partial\Delta$. Then we have*

$$\int_{\partial\Delta} f(z)\,dz = 0 \;.$$

Proof: All the triangles below are assumed to be closed. Also, if $P \in \mathbb{C}$ and $\delta > 0$, then

$$D(P,\delta) = \{z \in \mathbb{C} \;:\; |z - P| < \delta\}$$

denotes the open disk of radius δ centered at P.

Some simple observations:

1) If Δ is any triangle then

$$w, z \in \Delta \quad \text{implies} \quad |w - z| \le \mathit{length}(\partial\Delta) \;. \tag{4.4}$$

2) If Δ is a triangle we subdivide it into four similar triangles by connecting the midpoints of the sides of Δ. Then, if Δ' is any of the four sub-triangles, we have

$$\mathit{length}(\partial\Delta') = \frac{1}{2}\,\mathit{length}(\partial\Delta) \;. \tag{4.5}$$

3) We use the abbreviation

$$a(\Delta) = \int_{\partial\Delta} f(z)\,dz \;.$$

If $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the four sub-triangles of Δ obtained by the subdivision, then

$$a(\Delta) = \sum_{j=1}^{4} a(\Delta_j) \ .$$

4) Choose $\Delta' \in \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ with

$$|a(\Delta')| = \max_{1 \leq j \leq 4} |a(\Delta_j)| \ .$$

Then we have

$$|a(\Delta)| \leq \sum_{j=1}^{4} |a(\Delta_j)| \leq 4|a(\Delta')| \ . \tag{4.6}$$

By subdividing Δ' etc. we obtain a sequence of triangles Δ^n with

$$\Delta^{n+1} \subset \Delta^n \subset \ldots \subset \Delta$$

and

$$length(\partial\Delta^n) = \frac{1}{2^n}\, length(\partial\Delta)$$

and

$$\begin{aligned} |a(\Delta)| &\leq 4\,|a(\Delta^1)| \\ &\leq 4^2|a(\Delta^2)| \\ &\leq 4^n|a(\Delta^n)| \end{aligned}$$

One can show that there is a unique point $P \in \Delta \subset U$ with

$$\bigcap_{n=1}^{\infty} \Delta^n = \{P\} \ .$$

Details: *Uniqueness of P:* Suppose P and Q are elements of $\bigcap_{n=1}^{\infty} \Delta^n$. Since $P, Q \in \Delta^n$ for all n it follows that

$$|P - Q| \leq length(\partial\Delta^n) \to 0 \quad \text{as} \quad n \to \infty \ ,$$

thus $P = Q$.

Existence of P: Let $P_n \in \Delta^n$. For $n > m \geq N$ we have

$$|P_n - P_m| \leq length(\partial\Delta^N) \leq \frac{1}{2^N}\, length(\partial\Delta) \ .$$

Therefore, P_n is a Cauchy sequence. Let $P_j \to P$. Since $P_j \in \Delta^n$ for $j \geq n$ and since Δ^n is closed, it follows that $P \in \Delta^n$. Here $n \in \mathbb{N}$ is arbitrary. Therefore, $P \in \bigcap_{n=1}^{\infty} \Delta^n$.

We now use complex differentiability of f at the point P and write

$$f(z) = f(P) + f'(P)(z - P) + R(z), \quad z \in U \ ,$$

where

$$R(z) = (z - P)\phi(z), \quad \phi \in C(U), \quad \phi(P) = 0 \ .$$

It is easy to show that

$$\int_{\partial \Delta^n} f(z)\, dz = \int_{\partial \Delta^n} R(z)\, dz$$

since a function of the form $l(z) = a + bz$, with constants $a, b \in \mathbb{C}$, has an antiderivative, thus

$$\int_{\partial \Delta^n} (a + bz)\, dz = 0 \ .$$

One obtains

$$\begin{aligned}
|a(\Delta)| &\leq 4^n |a(\Delta^n)| \\
&= 4^n \Big| \int_{\partial \Delta^n} f(z) dz \Big| \\
&= 4^n \Big| \int_{\partial \Delta^n} R(z) dz \Big| \\
&\leq 4^n \, length(\partial \Delta^n) \cdot \max\{|R(z)| \ : \ z \in \partial \Delta^n\} \\
&\leq 4^n \, length(\partial \Delta^n) \cdot length(\partial \Delta^n) \cdot \max\{|\phi(z)| \ : \ z \in \partial \Delta^n\} \\
&= length(\partial \Delta) \cdot length(\partial \Delta) \cdot \max\{|\phi(z)| \ : \ z \in \partial \Delta^n\}
\end{aligned}$$

Thus, we have shown that

$$|\int_{\partial \Delta} f(z) dz| = |a(\Delta)| \leq (length(\partial \Delta))^2 \cdot \max\{|\phi(z)| \ : \ z \in \partial \Delta^n\} \ .$$

Given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$|\phi(z)| \leq \varepsilon \quad \text{if} \quad |z - P| < \delta \ .$$

Also, if n is large enough, then

$$\Delta^n \subset D(P, \delta) \ .$$

Therefore, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with

$$\max\{|\phi(z)| \ : \ z \in \partial \Delta^n\} \leq \varepsilon \ .$$

Combining this bound with the above bound for $|\int_{\partial \Delta} f(z) dz|$ we obtain that

$$|\int_{\partial \Delta} f(z) dz| \leq (length(\partial \Delta))^2 \cdot \varepsilon \ .$$

Since $\varepsilon > 0$ is arbitrary the integral is zero. $\diamond$

4.4 Construction of a Primitive in a Disk

We now use Goursat's Lemma to construct an anti-derivative of a given function $f \in H(U)$ where U is an open disk.

Theorem 4.4 *Let $U = D(P, r) = \{z \in \mathbb{C} \ : \ |z - P| < r\}$ denote an open disk. If $f \in H(U)$ then there exists $g \in H(U)$ with $g' = f$.*

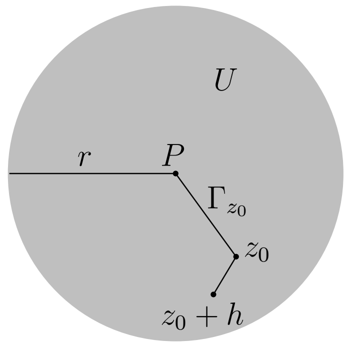

Figure 4.2: Construction of a Primitive

Proof: For any $z_0 \in U$ let Γ_{z_0} denote the straight line from P to z_0 and define

$$g(z_0) = \int_{\Gamma_{z_0}} f(z)\, dz \ .$$

We claim that $g \in H(U)$ and $g'(z_0) = f(z_0)$ for every $z_0 \in U$.

Fix $z_0 \in U$ and let $\varepsilon = r - |P - z_0|$, thus $\varepsilon > 0$. If $|h| < \varepsilon$ then

$$|P - (z_0 + h)| < |P - z_0| + \varepsilon = r \ ,$$

thus $z_0 + h \in U$. Also,

$$g(z_0 + h) = \int_{\Gamma_{z_0+h}} f(z)\, dz \ .$$

Let $\mathcal{C}_h$ denote the straight line from z_0 to $z_0 + h$. We have, by Goursat's Lemma:

$$\int_{\Gamma_{z_0+h}} f(z)\, dz = \int_{\Gamma_{z_0}} f(z)\, dz + \int_{\mathcal{C}_h} f(z)\, dz \ ,$$

thus

$$g(z_0 + h) = g(z_0) + \int_{\mathcal{C}_h} f(z)\, dz \ .$$

Since $\mathcal{C}_h$ has the parameterization

$$\gamma(t) = z_0 + th, \quad 0 \le t \le 1 \ ,$$

one obtains

$$g(z_0+h)-g(z_0)=\int_0^1 f(z_0+th)h\,dt\ .$$

Therefore, for $0<|h|<\varepsilon$:

$$\frac{1}{h}(g(z_0+h)-g(z_0))=\int_0^1 f(z_0+th)\,dt=:Int(h)\ .$$

We write

$$f(z_0+th)=f(z_0)+\Big(f(z_0+th)-f(z_0)\Big)\ .$$

Therefore,

$$Int(h)=f(z_0)+R(h)$$

with

$$|R(h)|\le\max_{0\le t\le 1}|f(z_0+th)-f(z_0)|\ .$$

Continuity of f in z_0 implies that $|R(h)|\to 0$ as $h\to 0$. This shows that $g'(z_0)=f(z_0)$. $\diamond$

Remark: A set $U\subset\mathbb{C}$ is called star–shaped if there exists a point $P\in U$ so that for every $Q\in U$ the straight line from P to Q lies in U. One then says that U is star–shaped w.r.t. P. For example, the set $\mathbb{C}^-=\mathbb{C}\setminus(-\infty,0]$ is star–shaped w.r.t. $P=1$. The set $\mathbb{C}\setminus\{0\}$ is not star–shaped.

If $U\subset\mathbb{C}$ is an open set that is star–shaped with respect to $P\in U$ and if $f\in H(U)$, then the same method as above can be used to construct a primitive g of f in U.

4.5 Cauchy's Theorem in a Disk

Theorem 4.5 *Let $U=D(P,r)$ and let $f\in H(U)$. If Γ is a closed curve in U, then*

$$\int_\Gamma f(z)\,dz=0\ .$$

Proof: Using the previous theorem, there exists $g\in H(U)$ with $g'=f$. Then, if Γ is a curve in U from P to Q we have

$$\int_\Gamma f(z)\,dz=g(Q)-g(P)\ .$$

If Γ is closed then $Q=P$, and the integral is zero. $\diamond$

4.6 Extensions

If $U\subset\mathbb{C}$ is any open set and $f\in H(U)$, will it hold that

$$\int_\Gamma f(z)\,dz=0 \tag{4.7}$$

whenever Γ is a closed curve in U? The example

$$U=\mathbb{C}\setminus\{0\},\quad f(z)=\frac{1}{z},\quad \Gamma=\partial D(0,1)\ ,$$

shows that the answer is *no*, in general, since $\int_\Gamma dz/z = 2\pi i$.

Definition: An open set $U \subset \mathbb{C}$ is called *connected* if for any two points $P, Q \in U$ there is a curve in U from P to Q. An open connected set is called a *region*.

Definition: Let Γ_0 and Γ_1 be two C^1–curves in U from P to Q parameterized by $\gamma_0(t)$ and $\gamma_1(t), a \le t \le b$. The curve Γ_0 is called *homotopic to* Γ_1 *in* U *with fixed endpoints* if there exists a continuous function

$$\gamma : [0,1] \times [a,b] \to U$$

with:

$$\begin{aligned}
\gamma(0,t) &= \gamma_0(t) \quad \text{for} \quad a \le t \le b \\
\gamma(1,t) &= \gamma_1(t) \quad \text{for} \quad a \le t \le b \\
\gamma(s,a) &= P \quad \text{for} \quad 0 \le s \le 1 \\
\gamma(s,b) &= Q \quad \text{for} \quad 0 \le s \le 1 \\
\gamma(s,\cdot) &\in C^1[a,b] \quad \text{for} \quad 0 \le s \le 1 \; .
\end{aligned}$$

Definition: A region U in $\mathbb{C}$ is called *simply connected* if every closed curve Γ in U, which goes from a point $P \in U$ to itself, is homotopic in U with fixed endpoints to the constant curve P.

If U is *simply connected*, then (4.7) holds whenever $f \in H(U)$ and Γ is a closed curve in U. We will explain this below.

Theorem 4.6 *Let U be a region in $\mathbb{C}$ and let Γ_0 and Γ_1 be two C^1 curves in U which are homotopic in U with fixed endpoints. If $f \in H(U)$ then*

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz \; .$$

Proof: Consider two curves Γ_{s_1} and Γ_{s_2} parameterized by $t \to \gamma(s_j, t)$ for $j = 1, 2$. If s_2 is sufficiently close to s_1 then one can use Theorem 4.5 to show that

$$\int_{\Gamma_{s_1}} f(z)\, dz = \int_{\Gamma_{s_2}} f(z)\, dz \; .$$

$\diamond$

The previous theorem is often used for closed curves as follows:

Theorem 4.7 *Let $U \subset \mathbb{C}$ be an open set. Let Γ denote a closed curve in U from P to P which is homotopic in U to the curve P. If $f \in H(U)$ then*

$$\int_\Gamma f(z)\, dz = 0 \; .$$

Fresnel Integrals. The Fresnel integrals

$$C(r) = \int_0^r \cos(x^2)\, dx \quad \text{and} \quad S(r) = \int_0^r \sin(x^2)\, dx$$

are used in optics. There is no simple expression for these integrals. But one can use Cauchy's Theorem to prove that

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}\ .$$

First recall that

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\ .$$

This is shown as follows: If

$$J := \int_{-\infty}^\infty e^{-x^2}\,dx$$

then

$$\begin{aligned} J^2 &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\,dxdy \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r\,drd\phi \\ &= \pi \int_0^\infty e^{-r^2} 2r\,dr \\ &= \pi \int_0^\infty e^{-q}\,dq \\ &= \pi \end{aligned}$$

Let $r > 0$ and consider the lines Γ_j with parameterizations

$$\begin{aligned} \gamma_1(t) &= t \quad \text{for} \quad 0 \le t \le r \\ \gamma_2(t) &= r + it \quad \text{for} \quad 0 \le t \le r \\ \gamma_3(t) &= (1+i)t \quad \text{for} \quad 0 \le t \le r \end{aligned}$$

Let $f(z) = e^{-z^2}$. We have

$$\int_{\Gamma_3} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz\ .$$

Here

$$Int_1(r) = \int_{\Gamma_1} f(z)\,dz = \int_0^r e^{-x^2}\,dx \to \frac{1}{2}\sqrt{\pi} \quad \text{as} \quad r \to \infty\ .$$

Also,

$$Int_2(r) = \int_{\Gamma_2} f(z)\,dz = \int_0^r e^{-(r+it)^2} i\,dt\ ,$$

thus

$$
\begin{aligned}
|Int_2(r)| &\le \int_0^r e^{-r^2+t^2}\, dt \\
&\le e^{-r^2} \int_0^r e^{rt}\, dt \\
&\le \frac{1}{r}
\end{aligned}
$$

Therefore,

$$Int_2(r) \to 0 \quad \text{as} \quad r \to \infty\ .$$

One obtains that

$$\int_{\Gamma_3} e^{-z^2}\, dz = (1+i) \int_0^r e^{-2it^2} \to \frac{1}{2}\sqrt{\pi} \quad \text{as} \quad r \to \infty\ .$$

Therefore,

$$\int_0^\infty e^{-2it^2}\, dt = \frac{1}{2}\sqrt{\pi}\, \frac{1}{1+i} = \frac{\sqrt{\pi}}{4}\,(1-i)\ .$$

Since

$$e^{-2it^2} = \cos(2t^2) - i\sin(2t^2)$$

one obtains that

$$\int_0^\infty \cos(2t^2)\, dt = \int_0^\infty \sin(2t^2)\, dt = \frac{\sqrt{\pi}}{4}\ .$$

With

$$2t^2 = x^2, \quad dt = dx/\sqrt{2}$$

this yields that

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\pi/2}\ .$$

Theorem 4.8 *Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. Then f has an anti-derivative in U if and only if*

$$\int_\Gamma f(z)\, dz = 0 \tag{4.8}$$

for every closed curve Γ in U.

Proof: a) If $g' = f$ in U then (4.8) holds by Theorem 4.1.

b) Assume that (4.8) holds for every closed curve in U. Fix a point $P \in U$ and, for every $z_0 \in U$, let Γ_{z_0} denote a curve in U from P to z_0. Define

$$g(z_0) = \int_{\Gamma_{z_0}} f(z)\, dz\ .$$

Because of (4.8) the value of $g(z_0)$ does not depend on the choice of the curve Γ_{z_0}. As in the proof of Theorem 4.4 it follows that $g'(z_0) = f(z_0)$. $\diamond$

If U is a simply connected region and $f \in H(U)$ then

$$\int_\Gamma f(z)\,dz = 0$$

for every closed curve Γ in U. This holds since Γ can be deformed continuously to a point in U where all deformed curves lie in U. The next result follows from the previous theorem.

Theorem 4.9 *Let U be a simply connected region in $\mathbb{C}$ and let $f \in H(U)$. Then f has an antiderivative in U, i.e., there exists $g \in H(U)$ with $g'(z) = f(z)$ for all $z \in U$.*

Example 4.8: Let

$$f(z) = \frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}\ , \quad z \in V := \mathbb{C} \setminus \{0,1\}\ .$$

We claim that f does not have an antiderivative in V. Let $\Gamma = \partial D(0, 1/2)$. Then

$$\int_\Gamma f(z)\,dz = \int_\Gamma \frac{dz}{z} = 2\pi i \neq 0\ ,$$

and Theorem 4.8 implies that an antiderivative of f in V does not exists.

We claim that f has an anti-derivative in

$$U = \mathbb{C} \setminus [0,1]\ .$$

First let Γ denote a simply closed curve in U which goes around $[0,1]$ once in the positive sense. Let $0 < \varepsilon < \frac{1}{2}$. Define the parameterizations

$$\gamma_1(t) = \varepsilon e^{it} \quad \text{and} \quad \gamma_2(t) = 1 + \varepsilon e^{it} \quad \text{for} \quad 0 \le t \le 2\pi$$

and let Γ_1 and Γ_2 denote the corresponding curves. We can deform Γ continuously in $\mathbb{C} \setminus \{0,1\}$ and obtain that

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz\ .$$

We have

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_1} \frac{dz}{z} = 2\pi i$$

and, similarly,

$$\int_{\Gamma_2} f(z)\,dz = \int_{\Gamma_2} \frac{dz}{1-z} = -2\pi i\ .$$

The equation

$$\int_\Gamma f(z)\,dz = 0$$

follows. One can then construct an anti-derivative of f in U using line–integrals as described in the proof of Theorem 4.8.

A more practical approach: First proceeding formally, we try

$$g(z) = \log z - \log(1-z) = \log \frac{z}{1-z} \ .$$

But we have to make precise how the log–function is defined.

We claim: If $z \in U := \mathbb{C} \setminus [0,1]$ then

$$\frac{z}{1-z} \in W := \mathbb{C} \setminus [0, \infty) \ .$$

Proof: Suppose that

$$\frac{z}{1-z} =: \alpha \geq 0 \ .$$

Then $z = \alpha - \alpha z$, thus

$$z = \frac{\alpha}{1+\alpha} \ ,$$

thus $0 \leq z < 1$. This shows that $z \in U = \mathbb{C} \setminus [0,1]$ impies

$$\frac{z}{1-z} \in W = \mathbb{C} \setminus [0, \infty) \ .$$

We have to define a log–function on W. If $w \in W$ then

$$w = |w| e^{i\phi} = e^{\ln |w| + i\phi} \quad \text{where} \quad 0 < \phi < 2\pi \ .$$

We set

$$\log_W(w) = \ln |w| + i\phi$$

and obtain

$$e^{\log_W(w)} = w, \quad w \log'_W(w) = 1 \ .$$

Therefore,

$$\log'_W(w) = \frac{1}{w} \quad \text{for} \quad w \in W \ .$$

The function

$$g(z) = \log_W \Big(\frac{z}{1-z} \Big) \ , \quad z \in U = \mathbb{C} \setminus [0,1] \ ,$$

is holomorphic on U and

$$g'(z) = \frac{1-z}{z} \cdot \frac{1}{(1-z)^2} = \frac{1}{z(1-z)} = f(z) \quad \text{for} \quad z \in U \ .$$

A second approach: Try

$$g(z) = \log z - \log(z-1) = \log \frac{z}{z-1} \quad \text{for} \quad z \in U = \mathbb{C} \setminus [0,1] \ .$$

We claim: If $z \in U$ then $z/(z-1) \notin (-\infty, 0]$. Suppose that

$$\frac{z}{z-1} = \beta \le 0 \ .$$

Then $z = \beta z - \beta$, thus

$$z = \frac{\beta}{\beta - 1} \ .$$

Sketching the function $\beta/(\beta-1)$ for $\beta \le 0$ we obtain that $z \in [0,1)$. Therefore, if $z \in U$ then $z/(z-1) \in \mathbb{C} \setminus (-\infty, 0]$. The formula

$$g(z) = \log \frac{z}{z-1}, \quad z \in U \ ,$$

gives an antiderivative of $f(z)$ where $\log w$ denotes the main branch of the logarithm.

4.7 Cauchy's Integral Formula in a Disk

Notations: Let $P \in \mathbb{C}$ and let $r > 0$. We set

$$\begin{aligned} D = D(P,r) &= \{z \ : \ |z-P| < r\} \\ \bar{D} = \bar{D}(P,r) &= \{z \ : \ |z-P| \le r\} \\ \partial D = \partial D(P,r) &= \{z \ : \ |z-P| = r\} \end{aligned}$$

With

$$\gamma(t) = \gamma(t,P,r) = P + re^{it}, \quad 0 \le t \le 2\pi \ ,$$

we denote the standard parameterization of the boundary curve of $D(P,r)$.

Theorem 4.10 *(Cauchy's integral formula) Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Let $\bar{D} = \bar{D}(P,r) \subset U$ and let ∂D denote the boundary curve of $D(P,r)$. Then we have for all $z_0 \in D(P,r)$:*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} \, dz \ . \tag{4.9}$$

Proof: Deform ∂D to a small curve Γ_ε about z_0 with parameterization

$$\gamma_\varepsilon(t) = z_0 + \varepsilon e^{it}, \quad 0 \le t \le 2\pi \ .$$

Write

$$f(z) = f(z) - f(z_0) + f(z_0)$$

and

$$\frac{f(z)}{z-z_0} = \frac{f(z)-f(z_0)}{z-z_0} + \frac{f(z_0)}{z-z_0}, \quad z \ne z_0 \ .$$

Integrate over Γ_ε to obtain

$$\int_{\partial D} \frac{f(z)}{z-z_0} \, dz = \int_{\Gamma_\varepsilon} \frac{f(z)-f(z_0)}{z-z_0} \, dz + 2\pi i \, f(z_0) \ .$$

Use that

$$\Big|\frac{f(z)-f(z_0)}{z-z_0}\Big| \le C \quad \text{for} \quad 0<|z-z_0|\le \varepsilon_0 \ .$$

Here C is a constant depending on z_0, but not on z. Then obtain for $\varepsilon \to 0$:

$$\int_{\partial D} \frac{f(z)}{z-z_0} = 2\pi i\, f(z_0) \ .$$

The formula (4.9) follows. $\diamond$

With a change of notation, the formula (4.9) is also written as

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in D \ . \tag{4.10}$$

Remark: The assumption that $z \in D$ is very important for the above formula. If z lies on the boundary of D, then the integral may not exist since the function $\zeta \to 1/(\zeta - z)$ is singular at $\zeta = z$. If z lies outside of D then the integral is zero.

One can use formula (4.10) to show that a holomorphic function $f(z)$ has complex derivatives of all orders. Differentiation of (4.10) with respect to z under the integral sign can be justified. One obtains:

Theorem 4.11 *(Cauchy's integral formula for derivatives) Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Let $\bar{D}(P,r) \subset U$ and let Γ denote the positively oriented boundary curve of $D(P,r)$. Then we have for all $z \in D(P,r)$:*

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{j+1}}\, d\zeta, \quad j = 0,1,2,\ldots \tag{4.11}$$

Proof: We may assume that $P=0$. We will prove the equation (4.11) for $j=1$. Using induction, one can prove (4.11) for $j = 1,2,3,\ldots$ with similar arguments.

Let $z \in D(0,r)$ and let

$$|z| =: r_1 < r \ .$$

Let $h \in \mathbb{C}$ with

$$0 < |h| \le \frac{1}{2}\,(r - r_1) \ .$$

We have

$$|z+h| \le r_1 + |h| \le \frac{1}{2}\,(r + r_1) < r \ .$$

By (4.10) obtain:

$$\frac{1}{h}\Big(f(z+h) - f(z)\Big) = \frac{1}{2\pi i}\,\frac{1}{h} \int_\Gamma f(\zeta)\Big(\frac{1}{\zeta - (z+h)} - \frac{1}{\zeta - z}\Big)\, d\zeta \ .$$

For fixed $\zeta \in \Gamma$ set

$$g(z) := \frac{1}{\zeta - z} \quad \text{for} \quad |z| < r \ .$$

We have

$$g^{(j)}(z) = \frac{j!}{(\zeta - z)^{j+1}} \quad \text{for} \quad j = 0, 1, 2, \ldots \tag{4.12}$$

Set

$$q(z) := \frac{1}{2\pi i} \int_\Gamma f(\zeta) g'(z)\, dz = \frac{1}{2\pi i} \int_\Gamma f(\zeta)(\zeta - z)^{-2}\, d\zeta \ .$$

We must show that

$$\Big| \frac{1}{h} \Big(f(z+h) - f(z) \Big) - q(z) \Big| \to 0 \quad \text{as} \quad h \to 0 \ . \tag{4.13}$$

Set

$$M := \max_{\zeta \in \Gamma} |f(\zeta)|$$

and obtain:

$$\Big| \frac{1}{h} \Big(f(z+h) - f(z) \Big) - q(z) \Big| \le M r \max_{|\zeta| = r} \Big| \frac{1}{h} \Big(g(z+h) - g(z) \Big) - g'(z) \Big| \tag{4.14}$$

where

$$|z| = r_1, \quad 0 < |h| \le \frac{1}{2}(r - r_1) \ .$$

By Taylor expansion,

$$g(z+h) = \sum_{j=0}^\infty \frac{h^j}{j!}\, g^{(j)}(z) \ ,$$

thus

$$g(z+h) = g(z) + h g'(z) + R(h)$$

where

$$\begin{aligned} |R(h)| &= \Big| \sum_{j=2}^\infty \frac{h^j}{j!}\, g^{(j)}(z) \Big| \\ &\le \sum_{j=2}^\infty \frac{|h|^j}{|\zeta - z|^{j+1}} \\ &\le \frac{|h|^2}{|\zeta - z|^3} \sum_{j=0}^\infty \frac{|h|^j}{|\zeta - z|^j} \end{aligned}$$

Here

$$|\zeta| = r, \quad |z| = r_1 < r, \quad |\zeta - z| \ge r - r_1$$

and $|h| \le \frac{1}{2}(r - r_1)$, thus

$$\frac{|h|}{|\zeta - z|} \le \frac{1}{2} \; .$$

It follows that

$$|R(h)| \le \frac{2|h|^2}{|\zeta - z|^3} \; .$$

The constant

$$C = \frac{2}{(r - r_1)^3}$$

is independent of h and independent of $\zeta \in \Gamma$; the estimate $|R(h)| \le C|h|^2$ holds. The convergence (4.13) follows from (4.14). $\diamond$

5 Holomorphic Functions Written As Power Series

Summary: If $U \subset \mathbb{C}$ is an open set then a function $f : U \to \mathbb{C}$ is called complex differentiable (or holomorphic) in U if the limit

$$\lim_{h \to 0} \frac{1}{h}(f(z+h) - f(z)) =: f'(z)$$

exists for every $z \in U$. (See Section 1.3.) In this chapter we will prove that a holomorphic function can locally be written as a power series,

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad z \in D(z_0, \rho) \subset U \ .$$

As we will prove in the next chapter, this implies that a complex differentiable function is always infinitely often differentiable. (This result also follows from Cauchy's integral formula for derivatives, Theorem 4.11.)

Clearly, this shows that there is a major difference between complex and real differentiability of functions.

The important concept of *uniform convergence* of a sequence of functions will be used.

5.1 Main Result

Definition: Let $U \subset \mathbb{C}$ and let $f_n : U \to \mathbb{C}$ denote a sequence of functions. Also, let $f : U \to \mathbb{C}$ denote a function. Let $K \subset U$. The sequence f_n converges to f uniformly on K if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ so that

$$|f(z) - f_n(z)| < \varepsilon \quad \text{for} \quad n \geq N(\varepsilon) \quad \text{for all} \quad z \in K \ .$$

It is important that the integer $N(\varepsilon)$ does not depend on the point $z \in K$: As n gets large, the difference $|f(z) - f_n(z)|$ goes to zero, uniformly on K.

Theorem 5.1 *Let U denote an open subset of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $z_0 \in U$ be arbitrary and assume*

$$D(z_0, \rho) \subset U, \quad \rho > 0 \ .$$

Then there exist unique complex numbers numbers $a_0, a_1, \ldots$ so that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < \rho \ . \tag{5.1}$$

The series converges absolutely for every $z \in D(z_0, \rho)$ and the convergence is uniform for $|z - z_0| \leq r$ if $0 < r < \rho$ is fixed.

We will show that the power series representation (5.1) follows rather easily from Cauchy's Integral Formula and convergence of the geometric series. Also, we will prove in the next chapter that the coefficients a_j of the power series (5.1) are uniquely determined.

The theorem says that any holomorphic function f can locally be written as a power series. Furthermore, the power series expansion is valid in any open disk $D(z_0, \rho)$ which lies completely in the open set U where f is holomorphic. The convergence of the power series is uniform on every closed disk $\bar{D}(z_0, r)$ if $0 < r < \rho$ and $D(z_0, \rho) \subset U$.

5.2 The Geometric Series

The power series

$$\sum_{j=0}^{\infty} w^j \tag{5.2}$$

is called the geometric series. Its partial sums

$$s_n(w) = \sum_{j=0}^{n} w^j, \quad w \in \mathbb{C} ,$$

satisfy

$$s_n(w)(1-w) = (1 + w + \ldots + w^n)(1-w) = 1 - w^{n+1} ,$$

thus

$$s_n(w) = \frac{1-w^{n+1}}{1-w} \quad \text{for} \quad w \neq 1 .$$

If $|w| < 1$ then $w^{n+1} \to 0$ as $n \to \infty$. Therefore, for $|w| < 1$:

$$\frac{1}{1-w} - s_n(w) = \frac{w^{n+1}}{1-w} \to 0 \quad \text{as} \quad n \to \infty ,$$

thus

$$\sum_{j=0}^{\infty} w^j = \frac{1}{1-w} \quad \text{for} \quad |w| < 1 . \tag{5.3}$$

Fix $0 < r < 1$ and consider the difference

$$\frac{1}{1-w} - s_n(w) = \frac{w^{n+1}}{1-w}$$

for $|w| \leq r < 1$. Obtain that

$$\Big| \frac{1}{1-w} - s_n(w) \Big| = \frac{|w|^{n+1}}{|1-w|} \leq \frac{r^{n+1}}{1-r} \to 0 \quad \text{as} \quad n \to \infty .$$

This shows that the convergence in formula (5.3) is uniform for $|w| \leq r$ if $0 < r < 1$ is fixed.

Remark on Exchange of Limits: For $0 < r < 1$ we have

$$\max_{|w| \leq r} \Big| \frac{1}{1-w} - \sum_{j=0}^{n} w^j \Big| = \frac{r^{n+1}}{1-r} .$$

Here

$$\lim_{r \to 1-} \Big(\lim_{n \to \infty} \frac{r^{n+1}}{1-r} \Big) = 0$$

and

$$\lim_{n\to\infty}\Big(\lim_{r\to 1-}\frac{r^{n+1}}{1-r}\Big)=\infty\ .$$

Clearly, exchanging the order of the two limit processes leads to different results. The first limit process expresses the uniform convergence of the geometric series in $\bar{D}(0,r)$ for each $0<r<1$. The second limit process yields that uniform convergence does not hold for $w\in D(0,1)$.

5.3 Power Series Expansion Using the Geometric Series

Let $U\subset\mathbb{C}$ be an open set and let $f:U\to\mathbb{C}$ be holomorphic. Let $D=D(z_0,r)$ and assume that $\bar{D}\subset U$. Let ∂D denote the positively oriented boundary curve of D.

By Cauchy's Integral Formula (Theorem 4.10) we have, for all $z\in D$,

$$f(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}\,d\zeta\ .$$

Let us first assume that $z_0=0$. Then we have for $z\in D$ and $\zeta\in\partial D$

$$|z|<|\zeta|=r$$

and can write

$$\zeta-z=\zeta\Big(1-\frac{z}{\zeta}\Big)\quad\text{with}\quad\Big|\frac{z}{\zeta}\Big|=\frac{|z|}{r}<1\ ,$$

thus

$$\begin{aligned}\frac{1}{\zeta-z}&=\frac{1}{\zeta}\cdot\frac{1}{1-\frac{z}{\zeta}}\\&=\frac{1}{\zeta}\sum_{j=0}^{\infty}\frac{z^j}{\zeta^j}\end{aligned}$$

For fixed $z\in D$ the convergence of the series is uniform for $\zeta\in\partial D$. Therefore, we may exchange the order of integration and summation to obtain

$$\begin{aligned}f(z)&=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z}\,d\zeta\\&=\frac{1}{2\pi i}\sum_{j=0}^{\infty}z^j\int_{\partial D}\frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta\\&=\sum_{j=0}^{\infty}a_jz^j\end{aligned}$$

with

$$a_j=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta\ .$$

Clearly, the numbers a_j do not depend on $z\in D$. We have written $f(z)$ as a convergent power series in z for $z\in D$.

In the general case, where z_0 is not assumed to be $z_0 = 0$, we write

$$\zeta - z = (\zeta - z_0) - (z - z_0) = (\zeta - z_0)\Big(1 - \frac{z - z_0}{\zeta - z_0}\Big)$$

and obtain that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{j=0}^{\infty} \frac{(z - z_0)^j}{(\zeta - z_0)^j} \ .$$

In the same way as for $z_0 = 0$ one obtains that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

with

$$a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \ .$$

The numbers a_j do not depend on $z \in D$.

We have shown:

Theorem 5.2 *Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $D = D(z_0, r)$ and assume that $\bar{D} \subset U$. With ∂D we denote the boundary curve of D. We have, for all $z \in D$,*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

with

$$a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \ .$$

This shows that any function $f \in H(U)$ can *locally* be written as a power series. If $\bar{D}(z_0, r) \subset U$ then the power series with expansion point z_0 converges to $f(z)$ at least in $D(z_0, r)$.

We now make a further fine point. Let $f \in H(U)$ and consider an open disk $D(z_0, \rho)$. Assume

$$D(z_0, \rho) \subset U, \quad \rho > 0 \ .$$

Fix $0 < r < \rho$. Set

$$a_j = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \ .$$

Our previous considerations show that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < r \ .$$

It is clear, by Cauchy's Integral Theorem, that the coefficients a_j are independent of r. Therefore, since the number r with $0 < r < \rho$ was arbitrary, one obtains that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < \rho$$

if the open disk $D(z_0, \rho)$ is a subset of U.

To complete the proof of Theorem 5.1 it remains to prove the following:

a) The coefficients a_j with (5.1) are unique.
b) The series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ converges absolutely for $z \in D(z_0, \rho)$ if $D(z_0, \rho) \subset U$.
c) The series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ converges uniformly to $f(z)$ for $z \in \bar{D}(z_0, r)$ if $0 < r < \rho$ and $D(z_0, \rho) \subset U$.

We will prove this in the next chapter where we consider general power series.

6 Functions Defined by Power Series

Summary: An expression of the form

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{6.1}$$

is called a power series centered at z_0 with coefficients $a_0, a_1, \ldots$ It is good to think of z in the expression (6.1) as a complex variable. As we will show, for any power series (6.1) there is a unique value r (with $0 \le r \le \infty$) so that:

a) If $0 < r < \infty$ then (6.1) converges for all z with $|z - z_0| < r$ and diverges for $|z - z_0| > r$.
b) If $r = 0$ then (6.1) converges only for $z = z_0$.
c) If $r = \infty$ then (6.1) converges for all $z \in \mathbb{C}$.

The number r (possibly $r = \infty$) is called the **radius of convergence** of the power series.
For many properties of power series it is convenient to assume $z_0 = 0$. Extensions to general z_0 are typically trivial.
The next theorem is an important result, which we prove in this chapter:

Theorem 6.1 *Assume that the power series $\sum a_j z^j$ has radius of convergence r where $0 < r \le \infty$. Then the function*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| < r \ , \tag{6.2}$$

is holomorphic in $D(0, r)$. Furthermore,

$$f'(z) = \sum_{j=1}^{\infty} j a_j z^{j-1}, \quad |z| < r \ , \tag{6.3}$$

and the series $\sum j a_j z^{j-1}$ also has radius of convergence equal to r.

The theorem says that the power series representation (6.2) of $f(z)$ can be differentiated term by term to give the power series representation (6.3) of $f'(z)$. In other words, two limit processes, differentiation and summation, can be exchanged for power series.

Hadamard's Formula for the Radius of Convergence of a Power Series. If $s_0, s_1, \ldots$ denotes any sequence of real numbers then one defines

$$\limsup_{j\to\infty} s_j := \lim_{n\to\infty} \Big(\sup_{j \ge n} s_j \Big) \quad \text{and} \quad \liminf_{j\to\infty} s_j := \lim_{n\to\infty} \Big(\inf_{j\ge n} s_j \Big) \ .$$

After reviewing some properties of these real analysis concepts, we will prove in Section 6.4 Hadamard's formula for the radius of convergence of the power series (6.1):

$$r = \frac{1}{\limsup_{j\to\infty} |a_j|^{1/j}} \ .$$

Here, by convention, $1/0 = \infty$ and $1/\infty = 0$.

6.1 Remarks on the Exchange of Limits

Let us recall the basic concept of a uniformly convergent sequence of functions $s_n : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{R}^m$ is a nonempty set. Let $s : \Omega \to \mathbb{C}$ be a function. The sequence $s_n = s_n(z)$ converges uniformly on Ω to $s = s(z)$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ with

$$|s_n(z) - s(z)| < \varepsilon \quad \text{for} \quad n \geq N \quad \text{and} \qquad \text{for all} \quad z \in \Omega \; .$$

We know from real analysis that the uniform limit of a sequence of continuous functions is continuous:

Theorem 6.2 *If $s_n \in C(\Omega)$ for all n and if s_n converges to s uniformly on Ω, then $s \in C(\Omega)$.*

Proof: Fix any $z_0 \in \Omega$, and let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ so that

$$\sup_{z \in \Omega} |s_N(z) - s(z)| < \varepsilon/3 \; .$$

Use the continuity of s_N: There is $\delta > 0$ so that $|s_N(z) - s_N(z_0)| < \varepsilon/3$ if $z \in \Omega$ and $|z - z_0| < \delta$. Then, using the triangle inequality,

$$\begin{aligned} |s(z) - s(z_0)| &\leq |s(z) - s_N(z)| + |s_N(z) - s_N(z_0)| + |s_N(z_0) - s(z_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

for $z \in \Omega$ with $|z - z_0| < \delta$. $\diamond$

Under the assumptions of the above theorem, let $x_k, x_0 \in \Omega$ and let $x_k \to x_0$ as $k \to \infty$. Consider the values

$$s_n(x_k) \in \mathbb{C} \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \qquad \text{for} \quad k = 1, 2 \ldots$$

and consider the following diagram

$$\begin{array}{ccccc} s_n(x_k) & \to & s(x_k) & \text{as} & n \to \infty \\ \downarrow & & \downarrow & \text{as} & k \to \infty \\ s_n(x_0) & \to & s(x_0) & \text{as} & n \to \infty \end{array}$$

The convergences

$$s_n(x_k) \to s(x_k) \quad \text{for all} \quad k \quad \text{and} \quad s_n(x_0) \to s(x_0) \quad (\text{as} \;\; n \to \infty)$$

express the pointwise convergence of the functions $s_n(x)$ to the function $s(x)$. We can also first fix n and let $k \to \infty$. The convergences

$$s_n(x_k) \to s_n(x_0) \quad \text{for all} \quad n \quad \text{and} \quad s(x_k) \to s(x_0) \quad (\text{as} \;\; k \to \infty)$$

express the continuity of the functions $s_n(x)$ and $s(x)$ at the point x_0.

Since the limit processes $n \to \infty$ and $k \to \infty$ lead to the same result, namely $s(x_0)$, one says that the above diagram commutes. Here the continuity of the limit function $s(x)$ is essential.

If continuous functions $s_n(x)$ only converge pointwise, but not uniformly, to a limit $s(x)$, then $s(x)$ may be discontinuous, and the two limit processes $n \to \infty$ and $k \to \infty$ may not commute.

Example: Consider $s_n(x) = x^n$ for $0 \le x \le 1$. We have for $n \to \infty$:

$$x^n \to \begin{cases} 0 & \text{for} \quad 0 \le x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases} \quad .$$

If

$$x_k = 1 - \frac{1}{k} \quad \text{for} \quad k = 1, 2, \ldots$$

then

$$\lim_{k\to\infty} \lim_{n\to\infty} s_n(x_k) = \lim_{k\to\infty} \lim_{n\to\infty} \Big(1 - \frac{1}{k}\Big)^n = 0$$

and

$$\lim_{n\to\infty} \lim_{k\to\infty} s_n(x_k) = \lim_{n\to\infty} \lim_{k\to\infty} \Big(1 - \frac{1}{k}\Big)^n = 1 \; .$$

A reasonable question is: Assume that s_n and s are smooth functions, for example infinitely often differentiable real functions. Is it allowed to exchange differentiation and taking the limit $n \to \infty$? In real analysis, the answer is *No*, in general. The sequence $s_n(x,y) = \frac{1}{n}\cos(n^2(x+y))$ gives a simple example. Clearly, s_n converges uniformly on $\mathbb{R}^2$ to $s(x,y) \equiv 0$, but the derivatives of s_n do not converge to the derivatives of s as $n \to \infty$.

It is, therefore, remarkable and important that for functions defined by power series, $f(z) = \sum_j a_j(z-z_0)^j$, one *can* differentiate term by term within the open disk of convergence. We will prove this in Section 6.6.

6.2 The Disk of Convergence of a Power Series

An expression

$$\sum_{j=0}^{\infty} a_j(z-z_0)^j$$

is called a power series centered at z_0. We often take $z_0 = 0$ for convenience.

The following simple result is important.

Lemma 6.1 *(Abel) Assume that the power series*

$$\sum_{j=0}^{\infty} a_j z^j$$

converges for some $z \ne 0$. If $|w| < |z|$, then the series

$$\sum_{j=0}^{\infty} a_j w^j$$

converges absolutely. If a number r with $0 < r < |z|$ is fixed, then the convergence is uniform for all w with $|w| \le r$.

Proof: Since $|a_j||z|^j \to 0$ as $j \to \infty$ there exists $M > 0$ so that

$$|a_j||z^j| \le M \quad \text{for all} \quad j = 0, 1, \ldots$$

Also,

$$q := \frac{|w|}{|z|} < 1 \quad \text{if} \quad |w| < |z| \ .$$

Therefore,

$$|a_j||w^j| = |a_j||z^j|\Big(\frac{|w|}{|z|}\Big)^j \le Mq^j \quad \text{if} \quad |w| < |z| \ .$$

Since $\sum q^j$ converges, the claim follows from the Comparison Criterion, Theorem 1.2. $\diamond$

Definition 6.1 For any given power series,

$$\sum_{j=0}^{\infty} a_j z^j \tag{6.4}$$

define the radius r of convergence as follows:

$$r := \sup\Big\{|z| \ : \ \sum_{j=0}^{\infty} a_j z^j \text{ converges}\Big\} \ .$$

Clearly, we have

$$0 \le r \le \infty \ .$$

There are three cases:

a) $r = \infty$: In this case, by the previous lemma, the series converges absolutely for every z. We will prove that the series (6.4) defines an entire function.

b) $r = 0$: In this case the series converges only for $z = 0$.

c) $0 < r < \infty$: In this case, the series converges absolutely for $|z| < r$ and diverges for $|z| > r$. We will prove that the series (6.4) defines a function which is holomorphic in the open disk $D(0, r)$.

In many cases, one can obtain the radius r of convergence as follows:

Theorem 6.3 *Let $\sum_{j=0}^{\infty} a_j z^j$ denote a power series and assume $a_j \ne 0$ for all large j. If*

$$\Big|\frac{a_{j+1}}{a_j}\Big| \to q \quad \textit{as} \quad j \to \infty$$

with $0 \le q \le \infty$, then the radius of convergence is

$$r = \frac{1}{q} \ .$$

Here one uses the conventions $1/\infty = 0$ and $1/0 = \infty$.

Proof: Let $\alpha_j = a_j z^j, z \neq 0$. We have

$$\left|\frac{\alpha_{j+1}}{\alpha_j}\right| \to q|z| \quad \text{as} \quad j \to \infty \ .$$

By the Quotient Criterion (Theorem 1.3) the power series $\sum a_j z^j$ converges absolutely if $q|z| < 1$ and diverges if $q|z| > 1$. This implies that the radius of convergence is $r = 1/q$. $\diamond$

Example 6.1: For $\sum_{j=0}^{\infty} j! z^j$ the radius of convergence is $r = 0$ by Theorem 6.3.

Example 6.2: For $\sum_{j=0}^{\infty} \frac{1}{j!} z^j$ the radius of convergence is $r = \infty$ by Theorem 6.3. We have

$$\sum_{j=0}^{\infty} \frac{1}{j!} z^j = e^z, \quad z \in \mathbb{C} \ .$$

Example 6.3: For $\sum_{j=0}^{\infty} z^j$ the radius of convergence is $r = 1$ by Theorem 6.3. We have

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad |z| < 1 \ .$$

Example 6.4: For $\sum_{j=1}^{\infty} j z^j$ the radius of convergence is $r = 1$ by Theorem 6.3. We have for $|z| < 1$:

$$\begin{aligned}
\sum_{j=1}^{\infty} j z^j &= z \sum_{j=0}^{\infty} \frac{d}{dz} z^j \\
&= z \frac{d}{dz} \sum_{j=0}^{\infty} z^j \\
&= z \frac{d}{dz} \frac{1}{1-z} \\
&= \frac{z}{(1-z)^2}
\end{aligned}$$

The fact that we can take d/dz out of the infinite sum will be justified below.

Example 6.5: Taylor expansion of the real function

$$f(x) = \ln(1+x), \quad x > -1 \ ,$$

about $x = 0$ leads to the series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j \ .$$

This follows from $f(0) = 0$ and

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{j=0}^{\infty} (-1)^j x \quad \text{for} \quad |x| < 1 \ .$$

The radius of convergence of the corresponding complex series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \, z^j$$

is $r = 1$ by Theorem 6.3. This suggest that

$$\log(1+z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \, z^j, \quad |z| < 1 \ ,$$

where log denotes the main branch of the complex logarithm. In other words, if $w = 1 + z$,

$$\log w = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \, (w-1)^j, \quad |w-1| < 1 \ .$$

We will show below that this expansion is valid, indeed.

Example 6.6: The Taylor expansion of the real function

$$f(x) = \frac{1}{1+x^2}$$

about $x = 0$ can be obtained using the geometric series: With $\varepsilon = -x^2$ we have for $|x| < 1$:

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-\varepsilon} \\ &= \sum_{j=0}^{\infty} \varepsilon^j \\ &= \sum_{j=0}^{\infty} (-1)^j x^{2j} \\ &= 1 - x^2 + x^4 \ldots \end{aligned}$$

The corresponding complex series

$$\sum_{j=0}^{\infty} (-1)^j z^{2j} = \frac{1}{1+z^2}$$

has the radius of convergence equal to 1.

6.3 Remarks on lim sup and lim inf

Definition 6.2: Let s_j denote a sequence of real numbers. One defines

$$\limsup_{j\to\infty} s_j := \lim_{n\to\infty} \Big(\sup_{j\ge n} s_j \Big) =: L \tag{6.5}$$

and

$$\liminf_{j\to\infty} s_j := \lim_{n\to\infty} \Big(\inf_{j\ge n} s_j \Big) \ . \tag{6.6}$$

Let us first show that the limit (6.5) always exists as an element of the extended real line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. (The limit (6.6) can be treated similarly.) Set

$$L_n := \sup_{j \geq n} s_j \ .$$

Case 1: The sequence s_j is not bounded from above. In this case $L_n = \infty$ for all n and, therefore, $L = \infty$.

Case 2: Assume $s_j \to -\infty$. In this case $L_n \to -\infty$, thus $L = -\infty$.

Case 3: In all other cases, the numbers L_n form a monotonically decreasing sequence of real numbers which is bounded from below. It therefore converges to some real number $\tilde{L}$,

$$\tilde{L} \leq \ldots \leq L_{n+1} \leq L_n \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad L_n \to L \ .$$

This shows that

$$L = \limsup_{j \to \infty} s_j = \lim_{n \to \infty} L_n$$

always exists as an element of $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Lemma 6.2 *Let* $s_j \in \mathbb{R}$ *and let*

$$L := \limsup_{j \to \infty} s_j = \lim_{n \to \infty} L_n \quad \textit{with} \quad L_n = \sup_{j \geq n} s_j \ .$$

Assume that $L \in \mathbb{R}$.

a) For any $\varepsilon > 0$ *there exist infinitely many* $j \in \mathbb{N}$ *with* $s_j \geq L - \varepsilon$.

b) For any $\varepsilon > 0$ *there exists* $J(\varepsilon) \in \mathbb{N}$ *with*

$$s_j \leq L + \varepsilon \quad \textit{for} \quad j \geq J(\varepsilon) \ .$$

Proof: a) Since $L_{n+1} \leq L_n$ for all $n \in \mathbb{N}$ we have $L_n > L - \varepsilon$ for all $n \in \mathbb{N}$. Set $n_1 = 1$ and consider

$$L - \varepsilon < L_1 = \sup_{j \geq 1} s_j \ .$$

There exists $j_1 \geq 1$ with

$$s_{j_1} \geq L - \varepsilon \ .$$

Set

$$n_2 = j_1 + 5 \ .$$

(The number 5 can be replaced by any positive integer.) We have

$$L - \varepsilon < L_{n_2} = \sup_{j \geq n_2} s_j \ .$$

There exists $j_2 \geq n_2 = j_1 + 5$ with

$$s_{j_2} \geq L - \varepsilon \ .$$

The process can be continued and one obtains a sequence of positive integers

$$j_1 < j_2 < j_3 < \ldots$$

with

$$s_{j_k} \geq L - \varepsilon \quad \text{for} \quad k = 1, 2, 3, \ldots$$

b) Since $L_n \to L$ we have

$$L_n \leq L + \varepsilon \quad \text{for} \quad n \geq J(\varepsilon) \ .$$

Since

$$L_n = \sup_{j \geq n} s_j$$

we have

$$s_j \leq L_n \quad \text{for} \quad j \geq n \ .$$

Therefore,

$$s_j \leq L + \varepsilon \quad \text{for} \quad j \geq J(\varepsilon) \ .$$

$\diamond$

Lemma 6.3 *Let $a_j > 0$ for all j and set*

$$\limsup_{j \to \infty} \frac{a_{j+1}}{a_j} = Q_1 \ ,$$

$$\liminf_{j \to \infty} \frac{a_{j+1}}{a_j} = Q_2 \ ,$$

$$\limsup_{j \to \infty} a_j^{1/j} = L_1 \ ,$$

$$\liminf_{j \to \infty} a_j^{1/j} = L_2 \ .$$

Then we have

$$Q_2 \leq L_2 \leq L_1 \leq Q_1 \ .$$

Proof: We will show that $L_1 \leq Q_1 =: Q$. (The proof of the inequality $Q_2 \leq L_2$ is similar.) Set $q_n := a_{n+1}/a_n$. Let $\varepsilon > 0$. Since $\limsup_{n \to \infty} q_n = Q$ there exists $N = N_\varepsilon$ so that $q_n \leq Q + \varepsilon$ for all $n \geq N$. Thus,

$$a_{n+1} \leq (Q + \varepsilon) a_n, \quad n \geq N \ .$$

It follows that

$$a_{N+j} \leq (Q + \varepsilon)^j a_N = (Q + \varepsilon)^{N+j} \, \frac{a_N}{(Q + \varepsilon)^N}, \quad j \geq 0 \ .$$

Therefore,

$$a_{N+j}^{1/(N+j)} \leq (Q + \varepsilon) M^{1/(N+j)}, \quad j \geq 0 \ ,$$

with

$$M = \frac{a_N}{(Q+\varepsilon)^N} \; .$$

Since $M^{1/(N+j)} \to 1$ as $j \to \infty$ it follows that

$$a_k^{1/k} \le Q + 2\varepsilon \quad \text{for} \quad k \ge K(\varepsilon) \; .$$

This implies that $L_1 \le Q + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, one obtains that $L_1 \le Q$. $\diamond$

A simple implication of the previous lemma is:

Lemma 6.4 *Let $a_j > 0$ for all j. If*

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = Q$$

then the sequence

$$a_j^{1/j}$$

also converges to Q.

Proof: We have $Q_1 = Q_2 = Q$ in the previous lemma. $\diamond$

Example: Let $a_j = j$ for $j = 1, 2, \ldots$ Since

$$\frac{j+1}{j} \to 1 \quad \text{as} \quad j \to \infty$$

it follows that $j^{1/j} \to 1$ as $j \to \infty$.

Another proof of $j^{1/j} \to 1$ as $j \to \infty$ goes as follows: We know that

$$\ln(j^{1/j}) = \frac{1}{j} \ln j \to 0 \quad \text{as} \quad j \to \infty \; .$$

Therefore,

$$j^{1/j} = e^{\ln(j^{1/j})} \to e^0 = 1 \quad \text{as} \quad j \to \infty \; .$$

6.4 The Radius of Convergence of a Power Series: Hadamard's Formula

Hadamard gave a formula for the radius of convergence r of a power series $\sum a_j z^j$. The formula has more theoretical than practical value. In other words, one often uses it in proofs, but it is less useful for computing r.

Theorem 6.4 *(Hadamard) Let $\sum a_j z^j$ have radius of convergence equal to r where $0 \le r \le \infty$. Then we have:*

$$\frac{1}{r} = \limsup_{j\to\infty} |a_j|^{1/j}$$

with the conventions

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0 \; .$$

Proof of Hadamard's Formula: Let r denote the radius of convergence of the power series $\sum a_j z^j$. Set

$$L = \limsup_{j\to\infty} |a_j|^{1/j} \ .$$

Assume $0 < L < \infty$. (The cases $L = 0$ and $L = \infty$ can be treated similarly.)

a) Let $|z| > 1/L$. We have $L|z| > 1$. There exists $\varepsilon > 0$ with

$$(L - \varepsilon)|z| > 1 \ .$$

By Lemma 6.2 a) there exist infinitely many j with $|a_j|^{1/j} \geq L - \varepsilon$, thus

$$|a_j|^{1/j}|z| > 1, \quad |a_j||z|^j > 1$$

for infinitely many j. It follows that the series $\sum_{j=1}^\infty a_j z^j$ diverges for $|z| > 1/L$. This implies that $r \leq 1/L$.

(Reason: If $r > 1/L$ then there exists z with $r > |z| > 1/L$, and one obtains a contradiction.)

b) Let

$$|z| < \frac{1}{L}, \quad L|z| < 1 \ .$$

There exists $\varepsilon > 0$ with

$$(L + \varepsilon)|z| =: q < 1 \ .$$

By Lemma 6.2 b) we have

$$|a_j|^{1/j} \leq L + \varepsilon \quad \text{for} \quad j \geq J(\varepsilon) \ .$$

It follows that

$$|a_j|^{1/j}|z| \leq (L + \varepsilon)|z| = q < 1 \quad \text{for} \quad j \geq J(\varepsilon) \ ,$$

thus

$$|a_j||z|^j \leq q^j \quad \text{for} \quad j \geq J(\varepsilon) \quad \text{where} \quad 0 < q < 1 \ .$$

The series $\sum_{j=1}^\infty a_j z^j$ converges. Therefore, $|z| \leq r$. This implies that $r \geq 1/L$.

(Reason: If $r < 1/L$ then there exists z with $r < |z| < 1/L$, and one obtains a contradiction.)

In a) we have shown that $r \leq 1/L$ and in b) we have shown that $r \geq 1/L$. The equation $r = 1/L$ follows. $\diamond$

6.5 Matrix–Valued Analytic Functions and Hadamard's Formula for the Spectral Radius of a Matrix

This section can be skipped.

An expression like $\limsup_{j\to\infty} |a_j|^{1/j}$ also comes up in matrix theory.

In this section we assume that $\|\cdot\|$ denotes a vector norm on $\mathbb{C}^m$. The corresponding matrix norm for matrices $A \in \mathbb{C}^{m\times m}$ is defined by

$$\begin{aligned} \|A\| &= \max\{\|Au\| \ : \ u \in \mathbb{C}^m, \|u\| = 1\} \\ &= \min\{C \geq 0 \ : \ \|Au\| \leq C\|u\| \quad \text{for all} \quad u \in \mathbb{C}^m\} \end{aligned}$$

Let $A_j \in \mathbb{C}^{m\times m}$ denote a sequence a square matrices. We consider the series

$$\sum_{j=0}^{\infty} z^j A_j \tag{6.7}$$

with variable $z \in \mathbb{C}$. The partial sums are the matrices

$$S_n(z) = \sum_{j=0}^{n} z^j A_j \ . \tag{6.8}$$

As $n \to \infty$, we may consider convergence of $S_n(z)$ in the space of matrices $\mathbb{C}^{m\times m}$ or, alternatively, we may consider convergence of the m^2 scalar series

$$\sum_{j=0}^{\infty} z^j (A_j)_{\mu\nu}, \quad 1 \leq \mu, \nu \leq m \ , \tag{6.9}$$

where $(A_j)_{\mu\nu}$ denotes the matrix entries of A_j.

With arguments as in the proof of Theorem 6.4, the following result can be shown:

Theorem 6.5 *Set*

$$q = \limsup_{j\to\infty} \|A_j\|^{1/j} \ .$$

a) If $|z| < \frac{1}{q}$ then the series (6.7) converges in $\mathbb{C}^{m\times m}$. If $|z| > \frac{1}{q}$ then the series (6.7) diverges in $\mathbb{C}^{m\times m}$.

b) If $|z| < \frac{1}{q}$ then the m^2 scalar series (6.9) converge in $\mathbb{C}$. If $|z| > \frac{1}{q}$ then at least one of the m^2 scalar series (6.9) diverges.

Of particular interest is the case where $A \in \mathbb{C}^{m\times m}$ is a fixed matrix and $A_j = A^j$, i.e., $A_0 = I, A_1 = A, A_2 = A^2$, etc.

We denote the set of eigenvalues of A by

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$$

and denote the spectral radius of A by

$$\rho(A) = \max_j |\lambda_j| \ .$$

Theorem 6.6 *(Hadamard) For any matrix $A \in \mathbb{C}^{m\times m}$ we have*

$$\rho(A) = \lim_{j\to\infty} \|A^j\|^{1/j} = \inf_j \|A^j\|^{1/j} \ . \tag{6.10}$$

Proof: First note that $\rho(A) \le \|A\|$ and

$$(\rho(A))^j = \rho(A^j) \le \|A^j\| \ ,$$

thus

$$\rho(A) \le \|A^j\|^{1/j}, \quad j = 1, 2, \dots \ .$$

Let $\varepsilon > 0$ be arbitrary and set

$$B = B_\varepsilon = \frac{1}{\rho(A) + \varepsilon} A \ .$$

Then $\rho(B) < 1$ and, by a theorem of linear algebra, $B^j \to 0$ as $j \to \infty$. In particular, there exists $J = J_\varepsilon \in \mathbb{N}$ with

$$\|B^j\| \le 1 \quad \text{for} \quad j \ge J \ .$$

This yields that

$$\frac{1}{(\rho(A) + \varepsilon)^j} \|A^j\| \le 1 \quad \text{for} \quad j \ge J \ .$$

Therefore,

$$\rho(A) \le \|A^j\|^{1/j} \le \rho(A) + \varepsilon \quad \text{for} \quad j \ge J_\varepsilon \ .$$

Since $\varepsilon > 0$ was arbitrary, the formula (6.10) is shown. $\diamond$

Linear Algebra Argument: Let $B \in \mathbb{C}^{m \times m}$, $\rho(B) < 1$. We claim that $B^j \to 0$ as $j \to \infty$. There exists $T \in \mathbb{C}^{m \times m}$ so that

$$T^{-1} B T = \Lambda + R$$

where Λ is diagonal and R is strictly upper triangular. Then let

$$D_\varepsilon = diag(1, \varepsilon, \dots, \varepsilon^{m-1}) \ .$$

Obtain

$$D_\varepsilon^{-1} (\Lambda + R) D_\varepsilon = \Lambda + \mathcal{O}(\varepsilon) \ .$$

One obtains that

$$\|D_\varepsilon^{-1} (\Lambda + R) D_\varepsilon\| < 1$$

if $\varepsilon > 0$ is small enough. The claim $B^j \to 0$ follows.

Power Series: Consider the powers series

$$\sum_{j=0}^{\infty} A^j z^j \in \mathbb{C}^{m \times m}$$

where $A \in \mathbb{C}^{m \times m}$. We claim that the series converges if $|z|\rho(A) < 1$ and diverges if $|z|\rho(A) \ge 1$.

Proof: Set $B = zA$, thus $\rho(B) = |z|\rho(A)$. If $|z|\rho(A) < 1$ then $\rho(B) < 1$ and

$$\sum_{j=0}^{\infty} B^j = (I-B)^{-1} \ .$$

If $|z|\rho(A) \geq 1$ then $\rho(B) \geq 1$ and

$$\sum_{j=0}^{\infty} B^j$$

diverges.

6.6 Differentiation of Power Series

Let $\sum_{j=0}^{\infty} a_j z^j$ have radius of convergence equal to $r > 0$. Then the function

$$f(z) = \sum_{j=0}^{\infty} a_j z^j = a_0 + a_1 z + a_2 z^2 + \ldots$$

is defined for $z \in D = D(0,r)$. Also, the convergence is uniform on any compact subset of D. Therefore, by Theorem 6.3 the limit function $f(z)$ is continuous in D. More is true as we will show below: The formally differentiated power series has the same radius of convergence as the power series for $f(z)$, and the formally differentiated series converges to the complex derivative of $f(z)$.

Let

$$\begin{aligned} g(z) &:= \sum_{j=1}^{\infty} j a_j z^{j-1} \\ &= a_1 + 2a_2 z + 3a_3 z^2 + \ldots \\ &= \frac{1}{z} \sum_{j=1}^{\infty} j a_j z^j, \quad z \neq 0 \ , \end{aligned}$$

be obtained by differentiating the series for $f(z)$ term by term. We claim that the radius of convergence for $g(z)$ equals r and that $f(z)$ has the complex derivative $g(z)$:

Lemma 6.5 *The series $\sum_{j=0}^{\infty} a_j z^j$ and the series $\sum_{j=1}^{\infty} j a_j z^{j-1}$ have the same radius of convergence.*

Proof: This follows from Hadamard's formula and the following lemma. $\diamond$

Lemma 6.6

$$\lim_{j\to\infty} j^{1/j} = 1$$

Proof: We have shown this above, but give another simple proof here. For $t \geq 0$ we have

$$e^t \geq 1 + \frac{t^2}{2}, \quad e^{-t} t^2 \leq 2 \ ,$$

thus

$$\lim_{t\to\infty} e^{-t} t = 0 \ .$$

With

$$t = \ln j, \quad e^{-t} = \frac{1}{j}$$

obtain that

$$\ln(j^{1/j}) = \frac{1}{j} \ln j = e^{-t} t \ ,$$

thus

$$\ln(j^{1/j}) \to 0 \quad \text{as} \quad j \to \infty \ .$$

This implies that

$$j^{1/j} = e^{\ln(j^{1/j})} \to 1 \quad \text{as} \quad j \to \infty \ .$$

$\diamond$

Remark: The result $j^{1/j} \to 1$ also follows from Lemma 6.4.

The lemma together with Hadamard's formula imply that the series for $f(z)$ and $g(z)$ have the same radius of convergence.

Theorem 6.7 *Let $f(z) = \sum_{j=0}^\infty a_j z^j$ be holomorphic in $D(0,r)$ and let $g(z) := \sum_{j=1}^\infty j a_j z^{j-1}$. Then*

$$f'(z) = g(z) \quad \textit{for} \quad |z| < r \ .$$

Proof: Let z with $|z| < r$ be fixed. Fix r_1 with $|z| < r_1 < r$. In the following, we let $h \in \mathbb{C}, h \neq 0$, be so small that

$$|z+h| \leq |z| + |h| \leq r_1 < r \ .$$

Set

$$s_n(z) = \sum_{j=0}^n a_j z^j \quad \text{and} \quad \eta_n(z) = \sum_{j=n+1}^\infty a_j z^j$$

and let $\varepsilon > 0$ be given. Then, using that $f(w) = s_n(w) + \eta_n(w)$, we have for all $n = 0, 1, \ldots$

$$\begin{aligned} \Big| \frac{1}{h}(f(z+h) - f(z)) - g(z) \Big| \quad & \leq \quad \Big| \frac{1}{h}(s_n(z+h) - s_n(z)) - s_n'(z) \Big| + |s_n'(z) - g(z)| \\ & \quad + \Big| \frac{1}{h}(\eta_n(z+h) - \eta_n(z)) \Big| \\ & =: \quad A + B + C \end{aligned}$$

To estimate the term C we use the following lemma:

Lemma 6.7 *Let $a, b \in \mathbb{C}$ and let $M = \max\{|a|, |b|\}$. Then we have*

$$|a^j - b^j| \leq |a - b| j M^{j-1} \quad \textit{for} \quad j = 1, 2 \ldots$$

Proof of lemma: This follows from

$$a^j - b^j = (a-b)(a^{j-1} + a^{j-2}b + \ldots + b^{j-1}) \ .$$

$\diamond$

Applying the lemma, we obtain

$$|(z+h)^j - z^j| \le |h| j (|z| + |h|)^{j-1} \le |h| j \, r_1^{j-1} \ .$$

Therefore,

$$C \le \sum_{j=n+1}^{\infty} j|a_j| r_1^{j-1} \le \varepsilon$$

for $n \ge N_1 = N_1(\varepsilon)$. Here we use that $r_1 < r$ and absolute convergence of the series $\sum_{j=1}^{\infty} j a_j z^{j-1}$ for $z \in D(0,r)$.

Also,

$$B = |s_n'(z) - g(z)| \le \varepsilon$$

for $n \ge N_2 = N_2(\varepsilon)$. This follows from the fact that the power series defining $g(z)$ converges in $D(0,r)$ and $|z| < r$.

Fix $n = \max\{N_1, N_2\}$. Then, since $s_n(z)$ is a polynomial, there exists $\delta > 0$ with

$$A = \Big| \frac{1}{h}(s_n(z+h) - s_n(z)) - s_n'(z) \Big| \le \varepsilon$$

for $0 < |h| \le \delta$. To summarize, given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$\Big| \frac{1}{h}(f(z+h) - f(z)) - g(z) \Big| \le 3\varepsilon \quad \text{for} \quad 0 < |h| \le \delta \ .$$

This proves the theorem. $\diamond$

One can apply the previous theorem repeatedly and obtain the following result: If $\sum a_j z^j$ has radius of convergence $r > 0$ then the function

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad |z| < r \ ,$$

is infinitely often complex differentiable and all derivatives can be obtained by differentiating the series term by term:

$$\begin{aligned} f'(z) &= \sum_{j=1}^{\infty} j a_j z^{j-1} \\ f''(z) &= \sum_{j=2}^{\infty} j(j-1) a_j z^{j-2} \end{aligned}$$

etc. The power series for each derivative also has radius of convergence equal to r. In particular, we have that

$$\begin{aligned} f(0) &= a_0 \\ f'(0) &= a_1 \\ f''(0) &= 2a_2 \\ f'''(0) &= 2\cdot 3a_3 \\ f^{(k)}(0) &= k!\,a_k \end{aligned}$$

This implies that the coefficients of a power series are uniquely determined by the function represented by the series. Precisely:

Lemma 6.8 *Assume that*

$$\begin{aligned} f(z) &= \sum_{j=0}^\infty a_j z^j, \quad |z| < r_f \\ g(z) &= \sum_{j=0}^\infty b_j z^j, \quad |z| < r_g \end{aligned}$$

where $r_f > 0$ and $r_g > 0$ denote the radii of convergence. If, for some $r > 0$,

$$f(z) = g(z) \quad \textit{for all} \quad z \quad \textit{with} \quad |z| < r$$

then $a_j = b_j$ for all j. Therefore, $r_f = r_g$ and $f(z) = g(z)$ for all z with $|z| < r_f$.

Summary: Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Let $D(z_0, \rho) \subset U$ and let $0 < r < \rho$. Let Γ denote the boundary curve of $D(z_0, r)$. Set

$$a_j = \frac{1}{j!}\, f^{(j)}(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta \ .$$

We then have

$$f(z) = \sum_{j=0}^\infty a_j (z - z_0)^j \quad \text{for} \quad z \in D(z_0, \rho) \ . \tag{6.11}$$

The convergence of the series is absolute for $z \in D(z_0, \rho)$ and uniform for $|z - z_0| \le r < \rho$.

In particular, if R denotes the radius of convergence of the series (6.11), then $R \ge \rho$ as long as $D(z_0, \rho) \subset U$. One obtains that $R = \infty$ if $U = \mathbb{C}$. If $U \ne \mathbb{C}$ then the complement

$$U^c = \mathbb{C} \setminus U$$

is a non–empty, closed set. One obtains that

$$R \ge dist(z_0, U^c) \ .$$

Example 6.7: Consider the function

$$f(z) = \frac{e^z}{z^2 + 9}, \quad z \in U \ ,$$

with

$$U = \mathbb{C} \setminus \{3i, -3i\} \ .$$

Expansion of the function about $z_0 = 4$ yields a series

$$f(z) = \sum_{j=0}^{\infty} a_j (z-4)^j, \quad |z-4| < R \ .$$

By the previous considerations, the radius of convergence is at least $R = 5$. (Here $R = 5$ is the distance between the expansion point $z_0 = 4$ and the pole–set $\{3i, -3i\}$.) If the power series would converge in $D(4, R')$ with $R' > 5$ then the function $f(z)$ would be bounded near $\pm 3i$, which is not true. It follows that the radius of convergence of the series is exactly $R = 5$.

Example 6.8: The geometric sum

$$\sum_{j=0}^{\infty} z^j$$

has radius of convergence equal to $r = 1$. The value of the series is

$$f(z) = \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad |z| < 1 \ .$$

The function $f(z) = \frac{1}{1-z}$ is holomorphic in $U = \mathbb{C} \setminus \{1\}$. If we expand the function $f(z)$ about $z_0 = i/2$, we obtain a series of the form

$$f(z) = \sum_{j=0}^{\infty} a_j \Big(z - \frac{i}{2} \Big)^j \ . \tag{6.12}$$

Since we know that $z_1 = 1$ is the only singularity of $f(z)$, the radius r of convergence of the series (6.12) is the distance between $z_0 = i/2$ and $z_1 = 1$. Thus,

$$r = \frac{1}{2} \sqrt{5} \ .$$

We can determine the precise form of the expansion (6.12) as follows:

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \frac{1}{1 - \frac{i}{2} - (z - \frac{i}{2})} \\ &= \frac{1}{1 - \frac{i}{2}} \Big(1 - \frac{z - \frac{i}{2}}{1 - \frac{i}{2}} \Big)^{-1} \\ &= \sum_{j=0}^{\infty} a_j \Big(z - \frac{i}{2} \Big)^j \end{aligned}$$

with

$$a_j = \Big(1 - \frac{i}{2} \Big)^{-j-1} \ .$$

Using the quotient criterion, it is easy to confirm that the radius of convergence is $r = \frac{1}{2}\sqrt{5}$. To see this, note that

$$|a_{j+1}/a_j|^2 = \left|1 - \frac{i}{2}\right|^{-2} = \left(1 + \frac{1}{4}\right)^{-1} = \frac{4}{5} \ .$$

Example 6.9: Consider the series

$$g(z) = \sum_{j=0}^{\infty} b_j z^j$$

where

$$b_j = \frac{2 + \sin(j)}{3 + \cos(j^2)} \ .$$

Since

$$\frac{1}{4} \leq b_j \leq \frac{3}{2}$$

it follows from Hadamard's formula that the radius of convergence is $r = 1$. In this case, we do not know a simple analytic expression for $g(z)$. If we expand $g(z)$ about $z_0 = i/2$ we can say that the radius of convergence is at least $\frac{1}{2}$ and not larger than $\frac{3}{2}$. But it will be difficult to determine the radius precisely.

7 The Cauchy Estimates and Implications

Summary: For a complex differentiable function $f(z)$ one can bound derivatives $f'(z), f''(z)$, etc. in terms of values of the function. Here the constants in the bounds do not depend on the function f, but on some distance.

The Cauchy estimates express bounds of derivatives of f in terms of values of f. The estimates have many implications. We will use them to prove Liouville's theorem: Every bounded holomorphic function is constant. Liouville's theorem will then be used to prove the fundamental theorem of algebra.

7.1 The Cauchy Estimates

Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $\bar{D}(z_0, r) \subset U$. We have

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < r$$

with

$$a_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta, \quad \Gamma = \partial D(z_0, r) \ ,$$

and

$$f^{(j)}(z_0) = j! \, a_j \ .$$

See Theorem 4.11.

Clearly, the curve Γ has length $2\pi r$. Therefore, noting that

$$|\zeta - z_0| = r \quad \text{for} \quad \zeta \in \Gamma \ ,$$

we obtain the following bound:

$$|f^{(j)}(z_0)| \le \frac{j!}{r^j} \max_{|\zeta - z_0| = r} |f(\zeta)|, \quad j = 0, 1, \ldots \tag{7.1}$$

The above estimates are called Cauchy estimates:

Theorem 7.1 *(Cauchy Estimates) Let $f \in H(U)$ where U is an open subset of $\mathbb{C}$. If $\bar{D}(z_0, r) \subset U$ then the estimates (7.1) hold.*

7.2 Liouville's Theorem

Theorem 7.2 *(Liouville) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded. Then f is constant.*

Proof: We have, for all $z \in \mathbb{C}$,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{with} \quad a_j = \frac{1}{j!} f^{(j)}(0) \ .$$

By the Cauchy estimates:

$$|f^{(j)}(0)| \leq \frac{j!}{r^j} M(r)$$

with

$$M(r) = \max_{|\zeta|=r} |f(\zeta)| \ .$$

By assumption, $M(r)$ is bounded as $r \to \infty$. Therefore, if $j \geq 1$, then the term $M(r)/r^j$ goes to zero as $r \to \infty$ and consequently $a_j = 0$ for $j \geq 1$. It follows that $f(z) \equiv a_0$. $\diamond$

The following generalization says that if an entire function $f(z)$ grows at most like $|z|^k$ for $z \to \infty$, then $f(z)$ is a polynomial of degree less than or equal to k. For short: Entire functions with polynomial growth are polynomials.

Theorem 7.3 *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Assume that there are positive constants C, R and q with*

$$|f(z)| \leq C|z|^q \quad \textit{for} \quad |z| \geq R \ .$$

If k denotes the integer with $k \leq q < k+1$ then f is a polynomial of degree less than or equal to k.

Proof: By an estimate as in the previous proof obtain that $a_j = 0$ for $j > k$. $\diamond$

7.3 The Fundamental Theorem of Algebra

Theorem 7.4 *Let $p(z) = a_0 + a_1 z + \ldots + a_k z^k$ with $a_k \neq 0$, i.e., $p(z)$ is a polynomial of degree k. If $k \geq 1$ then there exists $z_1 \in \mathbb{C}$ with $p(z_1) = 0$.*

Proof: It is easy to check (see below) that $|p(z)| \to \infty$ as $|z| \to \infty$ because $p(z)$ has a positive degree. If a zero z_1 of $p(z)$ would not exist, then

$$f(z) = \frac{1}{p(z)}$$

would be a bounded entire function. By Liouville's theorem, $f(z) = const$, thus $p(z) = const$, a contradiction. $\diamond$

For completeness, we show here that $|p(z)| \to \infty$ as $|z| \to \infty$: Write

$$p(z) = a_k z^k + q(z) \quad \text{where} \quad a_k \neq 0$$

and

$$q(z) = a_0 + a_1 z + \ldots + a_{k-1} z^{k-1} \ .$$

Let

$$M := |a_0| + |a_1| + \ldots + |a_{k-1}| \ .$$

Then, for all z with $|z| \geq 1$,

$$|q(z)| \leq M|z|^{k-1} \ .$$

Therefore, for $|z| \geq 1$,

$$
\begin{aligned}
|p(z)| &\geq |a_k||z|^k - |q(z)| \\
&\geq |a_k||z|^k - M|z|^{k-1} \\
&= |z|^{k-1}(|a_k||z| - M) \\
&= \frac{1}{2}\,|a_k||z|^k + \Big(\frac{1}{2}\,|a_k||z|^k - M\Big)
\end{aligned}
$$

It follows that

$$|p(z)| \geq \frac{1}{2}\,|a_k||z|^k$$

if $|z| \geq 1$ and $|z| \geq 2M/|a_k|$.

Extension: We want to show that every polynomial $p(z) = \sum_{j=0}^k a_j z^j$ of degree k can be factorized:

$$p(z) = a_k(z - z_1)\cdots(z - z_k)\ .$$

This follows from Theorem 7.4 and the following lemma.

Lemma 7.1 *Let $p(z) = \sum_{j=0}^k a_j z^j$ denote a polynomial of degree k where $k \geq 2$. Further, let $z_1 \in \mathbb{C}$ be a zero of the polynomial $p(z)$, i.e, $p(z_1) = 0$. Then there is a polynomial $q(z)$ of degree $k - 1$ with*

$$p(z) = (z - z_1)q(z)\ .$$

Proof: Using the binomial formula for $(a + b)^j$, we write

$$
\begin{aligned}
p(z) &= \sum_{j=0}^k a_j z^j \\
&= \sum_{j=0}^k a_j \Big((z - z_1) + z_1\Big)^j \\
&= \sum_{j=0}^k b_j (z - z_1)^j
\end{aligned}
$$

where $b_k = a_k$. Since $0 = p(z_1) = b_0$ we obtain

$$p(z) = (z - z_1)\Big(b_1 + b_2(z - z_1) + \ldots + b_k(z - z_1)^{k-1}\Big)\ .$$

This proves the lemma. $\diamond$

Clearly, if $k - 1 \geq 1$, we can apply Theorem 7.4 to the polynomial $q(z)$ which occurs in the factorization $p(z) = (z - z_1)q(z)$. This process can then be repeated. This proves:

Theorem 7.5 *(Fundamental Theorem of Algebra) Let $p(z) = a_0 + a_1 z + \ldots + a_k z^k$ with $a_k \neq 0$, i.e., $p(z)$ is a polynomial of degree k. Let $k \geq 1$. Then there are k (not necessarily distinct) numbers $z_1, z_2, \ldots, z_k \in \mathbb{C}$ with*

$$p(z) = a_k(z - z_1)\cdots(z - z_k)\ .$$

7.4 The Zeros of $p(z)$ and $p'(z)$

This section can be skipped.

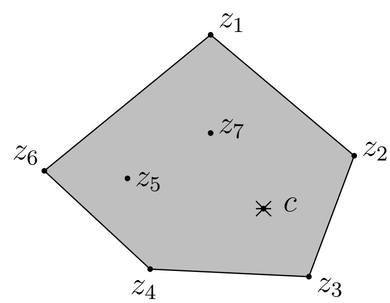

Figure 7.1: Convex hull of zeros of $p(z)$

Let

$$p(z) = a(z - z_1) \cdots (z - z_n) = a \prod_{j=1}^{n} (z - z_j)$$

denote a polynomial of degree $n \geq 2$. We claim: If c is a zero of the derivative $p'(z)$, then c lies in the convex hull of $z_1, \ldots, z_n$, i.e., c can be written in the form

$$c = \sum_j \alpha_j z_j \quad \text{with} \quad \alpha_j \geq 0 \quad \text{and} \quad \sum_j \alpha_j = 1 \; .$$

To show this, we may assume that $c \neq z_j$ for all j, i.e., $p(c) \neq 0$. (If $c = z_j$ then the claim is trivial.) We have

$$\frac{p'(z)}{p(z)} = \sum_j \frac{1}{z - z_j} \; , \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_n\} \; , \tag{7.2}$$

and the assumption $p'(c) = 0$ yields (using that $|w|^2 = w\bar{w}$):

$$0 = \sum \frac{1}{c - z_j} = \sum \frac{\bar{c} - \bar{z}_j}{|c - z_j|^2} \; .$$

Therefore,

$$0 = \sum \frac{c - z_j}{|c - z_j|^2} \; .$$

Set

$$\gamma_j = |c - z_j|^{-2} > 0$$

and obtain

$$c \sum_k \gamma_k = \sum_j \gamma_j z_j \; .$$

Therefore,

$$c = \frac{\sum_j \gamma_j z_j}{\sum_k \gamma_k} = \sum_j \alpha_j z_j$$

with

$$\alpha_j = \frac{\gamma_j}{\sum_k \gamma_k} \; .$$

We have shown:

Theorem 7.6 *Let $p(z)$ be a polynomial of degree n with zeros $z_1, \ldots, z_n$. (The z_j are not necessarily distinct.) Any zero c of $p'(z)$ lies in the convex hull of $z_1, \ldots, z_n$.*

Another simple implication of (7.2) is the following: Assume that z is a complex number with $p(z) \neq 0$ and $p'(z) \neq 0$. Then (7.2) yields

$$\Big| \frac{p'(z)}{p(z)} \Big| \leq n \max_j \frac{1}{|z - z_j|} \; ,$$

thus

$$n \Big| \frac{p(z)}{p'(z)} \Big| \geq \min_j |z - z_j| \; .$$

This says that for every z with $p'(z) \neq 0$ the closed disk

$$\bar{D}(z, R) \quad \text{with} \quad R = n \Big| \frac{p(z)}{p'(z)} \Big|$$

contains at least one zero z_j of $p(z)$. Here n is the degree of p.

The reason is simple: If

$$R \geq \min_j |z - z_j|$$

then there exists z_j with $R \geq |z - z_j|$.

Theorem 7.7 *Let $p(z)$ be a polynomial of degree n. Let $z \in \mathbb{C}$ and $p'(z) \neq 0$. Set*

$$R_z = n \Big| \frac{p(z)}{p'(z)} \Big| \; .$$

The closed disk

$$\bar{D}(z, R_z)$$

contains at least one zero of p.

8 Morera's Theorem and Locally Uniform Limits of Holomorphic Functions; Stirling's Formula for the Γ–Function

Summary: Let $U \subset \mathbb{C}$ be open and simply connected. If $f : U \to \mathbb{C}$ is holomorphic then, by Cauchy's Theorem,

$$\int_\Gamma f(z)\,dz = 0 \tag{8.1}$$

for every closed curve Γ in U. Morera's Theorem is a converse: If $f : U \to \mathbb{C}$ is continuous and (8.1) holds for every closed curve Γ in U then f is holomorphic on U.

Morera's Theorem is very useful if one wants to prove the holomorphy of a limit function $f(z)$ of a sequence of holomorphic functions $f_n(z)$. One needs the concept of *local uniform convergence* of a sequence of functions. We will use it to show that the Gamma–function $\Gamma(z)$ is holomorphic for $\mathrm{Re}\, z > 0$. We will also show Stirling's formula

$$\Gamma(x+1) = \Big(\frac{x}{e}\Big)^x \sqrt{2\pi x}\Big(1 + \mathcal{O}(x^{-1})\Big) \quad \text{as} \quad x \to \infty \ ,$$

which is used in statistics.

8.1 On Connected Sets

If (X, d) is a metric space, one calls X disconnected if one can write $X = X_1 \cup X_2$ where X_1 and X_2 are nonempty, disjoint, open subsets of X:

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = \emptyset, \quad X_1 \neq \emptyset \neq X_2, \quad X_j \text{ open} \ .$$

Otherwise, X is called connected. The study of a function f defined on a metric space X can typically be reduced to the study of f on the connected components of X. Therefore, without much loss of generality, one may often assume that X is connected.

For a complicated subset X of $\mathbb{R}^2$ it may not be easy to determine if it connected or disconnected. For example, consider the set $X = X_1 \cup X_2$ where

$$X_1 = \{(0, y) \ : \ -1 \le y \le 1\}$$

and

$$X_2 = \{(x, \sin(1/x)) \ : \ x > 0\} \ .$$

One could believe that X is disconnected, but it is not. Note that the subset X_2 of X is not closed in X.

Since we will only deal with open subsets U of $\mathbb{C}$, the issue of connectedness is simple. One can show that an open subset U of $\mathbb{C}$ is connected if and only if for any two points P, Q in U there is a smooth curve Γ in U from P to Q.

Suppose $U \subset \mathbb{C}$ is disconnected and $U = U_1 \cup U_2$ where the U_j are nonempty, disjoint, and open. Then, if $g \in H(U_1), h \in H(U_2)$, the function $f : U \to \mathbb{C}$ defined by

$$f(z) = g(z) \quad \text{for} \quad z \in U_1, \quad f(z) = h(z) \quad \text{for} \quad z \in U_2 \ ,$$

is holomorphic on U. This says that the behavior of any $f \in H(U)$ on the set U_1 may be completely unrelated to the behavior of f on U_2. In other words, it suffices to study holomorphic maps on open, *connected* sets.

8.2 Morera's Theorem

Morera's theorem is a converse of Cauchy's integral theorem. It is very useful when studying convergence of sequences and series of holomorphic functions.

Theorem 8.1 *(Morera) Let $U \subset \mathbb{C}$ be open and connected. Let $f : U \to \mathbb{C}$ be continuous. Assume that*

$$\int_\Gamma f(z)\,dz = 0$$

for all closed, piecewise smooth curves Γ in U. Then there is a holomorphic function $F : U \to \mathbb{C}$ with $F' = f$. In particular, f is holomorphic.

Proof: Fix $P_0 \in U$ and, for any $P \in U$, let ψ_P denote a curve in U from P_0 to P. Define

$$F(P) = \int_{\psi_P} f(z)\,dz\ .$$

Note: Because of the assumption $\int_\Gamma f(z)\,dz = 0$ for any closed curve Γ in U, the value $F(P)$ is well-defined: The value $F(P)$ does not depend on the choice of ψ_P as long as ψ_P lies in U and goes from P_0 to P.

Fix any $P \in U$. We will prove that $F'(P) = f(P)$. There is $r > 0$ with $D(P, r) \subset U$. Let $0 < |h| < r$. Define the curve γ_h by

$$\gamma_h(t) = P + th, \quad 0 \le t \le 1\ .$$

Then the curve

$$\psi_{P+h} - (\psi_P + \gamma_h)$$

is closed. Therefore,

$$\begin{aligned} F(P+h) - F(P) &= \int_{\gamma_h} f(z)\,dz \\ &= \int_0^1 f(P+th)h\,dt\ , \end{aligned}$$

thus

$$\frac{1}{h}\Big(F(P+h) - F(P)\Big) = \int_0^1 f(P+th)\,dt\ .$$

The function $t \to f(P+th)$ converges to $f(P)$ as $h \to 0$, uniformly for $0 \le t \le 1$. (This follows from the continuity of f in P.) Therefore,

$$\frac{1}{h}\Big(F(P+h) - F(P)\Big) \to f(P) \quad \text{as} \quad h \to 0\ .$$

$\diamond$

8.3 Modes of Convergence of a Sequence of Functions

Let X denote any non–empty set and let $f_0, f_1, f_2, \ldots$ and f denote functions from X to $\mathbb{C}$. What does it mean that the sequence f_n converges to f as $n \to \infty$? Different definitions are used, leading to different notions of convergence. The most commonly used notions are pointwise convergence and uniform convergence. Recall:

Definition 1: The sequence f_n converges to f pointwise on X if for every $z \in X$ and every $\varepsilon > 0$ there exists $N = N(\varepsilon, z) \in \mathbb{N}$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$.

Definition 2: The sequence f_n converges to f uniformly on X if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq N$ and all $z \in X$.

It turns out that both theses concepts are not completely perfect in complex analysis. The concept of pointwise convergence is too weak: One cannot integrate the limit relation $f_n(z) \to f(z)$ if the convergence is only pointwise. On the other hand, the concept of uniform convergence is too restrictive, because it typically does not hold on the whole domain where the functions f_n and f are defined. For example, let

$$f_n(z) = \sum_{j=0}^{n} z^j \quad \text{and} \quad f(z) = \frac{1}{1-z} \quad \text{for} \quad z \in D(0,1) \; .$$

Then f_n converges pointwise to f on $D(0,1)$, but not uniformly. The convergence is uniform, however, on any subdomain $D(0,r)$ with $0 < r < 1$.

The following two notions of convergence, which turn out to be equivalent, are very useful in complex analysis. Let $U \subset \mathbb{C}$ be an open set and let $f_n, f : U \to \mathbb{C}$ be functions.

Definition 3: The sequence f_n converges to f *uniformly on compact sets in* U if the following holds: For every compact set $K \subset U$ and for every $\varepsilon > 0$ there exists $N = N(\varepsilon, K)$ so that

$$|f_n(z) - f(z)| < \varepsilon$$

for all $n \geq N$ and all $z \in K$.

Definition 4: The sequence f_n converges to f *locally uniformly in* U if the following holds: For every $z_0 \in U$ there is a neighborhood $D(z_0, r) \subset U$ so that for every $\varepsilon > 0$ there is $N = N(\varepsilon, z_0)$ with

$$|f_n(z) - f(z)| < \varepsilon$$

for all $n \geq N$ and all $z \in D(z_0, r)$.

Remark: If one replaces U by a general metric space, the two notions of *uniform convergence on compact sets* and *local uniform convergence*, may differ from one another.

Theorem 8.2 *Let U denote an open subset of $\mathbb{C}$ and let $f_n, f : U \to \mathbb{C}$ be functions. The sequence f_n converges to f uniformly on compact sets in U if and only if it converges to f locally uniformly in U.*

Proof: 1) Assume that f_n converges to f uniformly on compact sets in U. Let $z_0 \in U$. There exists $r > 0$ with $\bar{D}(z_0, r) \subset U$; etc.

2) Assume that f_n converges to f locally uniformly in U. Let $K \subset U$ be compact. For every $z \in K$ there exists $r_z > 0$ so that f_n converges to f uniformly on $D(z, r_z)$. The sets $D(z, r_z)$ for $z \in K$ form an open cover of K. Since K is compact there are finitely many sets

$$D_j = D(z_j, r_{z_j}), \quad j = 1, \ldots, J \ ,$$

whose union covers K. For every $\varepsilon > 0$ and for every j there exists $N_j = N(\varepsilon, j)$ with

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for} \quad n \geq N_j$$

if $z \in D_j$. Set $N(\varepsilon) := max_j N_j$ then

$$|f_n(z) - f(z)| < \varepsilon \quad \text{for} \quad n \geq N(\varepsilon)$$

and $z \in K$. $\diamond$

We have proved that local uniform convergence in U of a sequence $f_n : U \to \mathbb{C}$ is equivalent to uniform convergence on compact sets in U. Next, let us consider series of functions $f_j(z)$.

Normal convergence of a series of functions. Let $U \subset \mathbb{C}$ denote an open set and let $f_j : U \to \mathbb{C}$ denote a sequence of functions. The series

$$\sum_{j=1}^{\infty} f_j(z), \quad z \in U \ , \tag{8.2}$$

has the partial sums

$$s_n(z) = \sum_{j=1}^{n} f_j(z), \quad n = 1, 2, \ldots$$

One says that the series (8.2) converges uniformly on the compact set $K \subset U$ if the sequence of partial sums $s_n(z)$ converges uniformly on K.

Example: We claim that the series

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{z+j}, \quad z \in U := \mathbb{C} \setminus \{-1, -2, \ldots\} \ ,$$

converges uniformly on every compact set $K \subset U$, but the convergence is not absolute. Proof: If $K \subset U$ is compact then $K \subset \bar{D}(0, R)$ for some $R > 0$. If $j \geq 2R$ and $z \in K$ then $|z| \leq R$ and

$$\begin{aligned} |z+j| &\geq j - |z| \geq j/2 \\ |z+j+1| &\geq j + 1 - |z| \geq j/2 \end{aligned}$$

thus

$$\Big| \frac{1}{z+j} - \frac{1}{z+j+1} \Big| \leq \frac{4}{j^2} \quad \text{for} \quad j \geq 2R \ .$$

This can be used to show that the sequence of partial sums $s_n(z)$ is a Cauchy sequence with respect to the maximum norm on K, defined by

$$|f|_K = \sup \Big\{ |f(z)| \ : \ z \in K \Big\} \ .$$

It is clear that the series $\sum \frac{(-1)^j}{z+j}$ does not converge absolutely. A very useful convergence concept for a series of functions is normal convergence.

Definition 5: The series (8.2) converges normally on U if for every compact set $K \subset U$ the series

$$\sum_{j=1}^{\infty} |f_j|_K$$

converges.

Example: Assume that the power series

$$\sum_{j=0}^{\infty} a_j z^j$$

has radius of convergence $r > 0$. We claim that the series converges normally in $D(0,r)$. If $K \subset D(0,r)$ is compact then

$$K \subset \bar{D}(0, r_1)$$

for some $0 < r_1 < r$. For all $z \in K$ we have

$$|a_j z^j| \le |a_j|\, r_1^j, \quad j = 0, 1, 2, \ldots$$

and the series

$$\sum_{j=0}^{\infty} |a_j| r_1^j$$

converges by Abel's Lemma, Lemma 6.1. Therefore, if $f_j(z) = a_j\, z^j$, then

$$|f_j|_K \le |a_j| r_1^j \ ,$$

and

$$\sum_{j=0}^{\infty} |f_j|_K \ .$$

converges.

A simple and important convergence theorem for holomorphic functions is stated next. Its proof is based on Morera's theorem and the Cauchy estimates.

Theorem 8.3 *Let $U \subset \mathbb{C}$ be open; let $f_n, f : U \to \mathbb{C}$ be functions. Assume that all f_n are holomorphic. If f_n converges to f locally uniformly in U, then f is also holomorphic. Furthermore, f_n' converges to f' locally uniformly in U.*

Proof: 1) First note that the continuity of all f_n and the local uniform convergence of f_n to f implies that f is continuous.

2) Let $D = D(z_0, r)$ be any disk in U. Let Γ be any closed curve in D. Then, by Cauchy's theorem,

$$\int_\Gamma f_n(z)\, dz = 0$$

for all n. Since Γ is compact, the f_n converge to f uniformly on Γ. It follows that

$$\int_\Gamma f(z)\,dz = 0\ .$$

By Morera's theorem, f is holomorphic in D. Since D was an arbitrary open disk in U, the function f is holomorphic in U.

3) Let $z_0 \in U$. There exists $r > 0$ so that $\bar{D}(z_0, 2r) \subset U$. Then, by Cauchy's estimate (7.1) we have for $z \in D(z_0, r)$,

$$\begin{aligned} |f_n'(z) - f'(z)| &\leq \frac{1}{r} \max_{|\zeta - z| = r} |f_n(\zeta) - f(\zeta)| \\ &\leq \frac{1}{r} \max_{\zeta \in \bar{D}(z_0, 2r)} |f_n(\zeta) - f(\zeta)| =: M_n\ . \end{aligned}$$

As $n \to \infty$, the maximum M_n converges to zero since f_n converges to f uniformly on $\bar{D}(z_0, 2r)$. Also, M_n is uniform for all $z \in D(z_0, r)$. This proves the theorem. $\diamond$

8.4 Integration with Respect to a Parameter

Theorem 8.4 *Let U be an open subset of $\mathbb{C}$ and let $F : U \times [a, b] \to \mathbb{C}$ denote a function. Here $[a, b]$ is a compact interval in $\mathbb{R}$. Assume that F is continuous on $U \times [a, b]$ and that $z \to F(z, t)$ is holomorphic on U for every fixed t. Then*

$$f(z) = \int_a^b F(z, t)\,dt, \quad z \in U\ ,$$

is holomorphic on U.

Proof: 1) Let $\bar{D} = \bar{D}(z_0, r) \subset U$. Since $F(x, t)$ is uniformly continuous on $\bar{D} \times [a, b]$ it follows that the function $f(z)$ is continuous.

2) Let $D = D(P, r)$ be any open disk in U and let Γ be a smooth closed curve in D. By Cauchy's theorem,

$$\int_\Gamma F(z, t)\,dz = 0 \quad \text{for all} \quad a \leq t \leq b\ .$$

We have

$$\begin{aligned} \int_\Gamma f(z)\,dz &= \int_\Gamma \int_a^b F(z, t)\,dt\,dz \\ &= \int_a^b \int_\Gamma F(z, t)\,dz\,dt \\ &= 0\ . \end{aligned}$$

Therefore, by Morera's theorem, the function $f(z)$ is holomorphic in D. Since D is an arbitrary disk in U, the function $f(z)$ is holomorphic in U.

Note that, in the second equation, we have exchanged the order of integration. Let us justify this. If Γ has the parameterization $\gamma(s), c \leq s \leq d$, then

$$
\begin{aligned}
\int_\Gamma \int_a^b F(z,t)\,dt\,dz &= \int_c^d \int_a^b F(\gamma(s),t)\gamma'(s)\,dt\,ds \\
&= \int_a^b \int_c^d F(\gamma(s),t)\gamma'(s)\,ds\,dt \\
&= \int_a^b \int_\Gamma F(z,t)\,dz\,dt
\end{aligned}
$$

Here, in the second step, the continuous function $(s,t) \to F(\gamma(s),t)\gamma'(s)$ is integrable over $[c,d] \times [a,b]$, and Fubini's theorem justifies to exchange the order of integration. $\diamond$

8.5 Application to the Γ–Function: Analyticity in the Right Half–Plane

Let $H_r = \{z = x + iy \ : \ x > 0\}$ denote the open right half–plane. For $z \in H_r$ define Euler's Γ–function by

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt \ . \tag{8.3}
$$

We will prove that $\Gamma(z)$ is holomorphic on H_r.

For $\varepsilon > 0$ and $z \in \mathbb{C}$ define

$$
\Gamma_\varepsilon(z) = \int_\varepsilon^{1/\varepsilon} t^{z-1} e^{-t}\,dt \ .
$$

Note: If $t > 0$ then

$$
t = e^{\ln t}
$$

and

$$
t^z = e^{z \ln t} \ .
$$

For every fixed $t > 0$, the function

$$
z \to t^{z-1} e^{-t} = e^{(z-1)\ln t} e^{-t}
$$

is entire. Also,

$$
(z,t) \to t^{z-1} e^{-t} = e^{(z-1)\ln t} e^{-t}
$$

is continuous on $\mathbb{C} \times [\varepsilon, \frac{1}{\varepsilon}]$. Therefore, by Theorem 8.4, each function $\Gamma_\varepsilon(z)$ is entire.

Fix $0 < a < b < \infty$ and consider the vertical strip

$$
S_{a,b} = \{z = x + iy \ : \ a \le x \le b, \ y \in \mathbb{R}\} \ .
$$

For $z = x + iy \in S_{a,b}$ and $0 < \varepsilon \le 1$ we have

$$
\begin{aligned}
|\Gamma(z) - \Gamma_\varepsilon(z)| &\le \int_0^\varepsilon t^{x-1} e^{-t}\,dt + \int_{1/\varepsilon}^\infty t^{x-1} e^{-t}\,dt \\
&\le \int_0^\varepsilon t^{a-1}\,dt + \int_{1/\varepsilon}^\infty t^{b-1} e^{-t}\,dt \\
&=: R(\varepsilon)
\end{aligned}
$$

(Note: If $0 < t \le 1$ then $\ln t \le 0$. Therefore, $0 < a \le x$ yields that $a \ln t \ge x \ln t$, thus $t^x \le t^a$.)

It is clear that $R(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore,

$$\sup_{z \in S_{a,b}} |\Gamma(z) - \Gamma_\varepsilon(z)| \to 0 \quad \text{as} \quad \varepsilon \to 0 \ .$$

If $K \subset H_r$ is an arbitrary compact set, then there exist $0 < a < b < \infty$ with $K \subset S_{a,b}$. It follows that $\Gamma_\varepsilon(z)$ converges to $\Gamma(z)$ as $\varepsilon \to 0$, uniformly on compact subsets K of H_r. This implies that $\Gamma(z)$ is holomorphic on H_r.

Remarks: 1) We will show later that $\Gamma(z)$ can be continued as a holomorphic function defined for $z \in U$ where

$$U = \mathbb{C} \setminus \{0, -1, -2, \ldots\} \ .$$

The extended function, also denoted by $\Gamma(z)$, has a simple pole at each $n \in \{0, -1, -2, \ldots\}$.

The integral representation (8.3) for $\Gamma(z)$ only holds for $\mathrm{Re}\, z > 0$, however, since the integral does not exist if $\mathrm{Re}\, z \le 0$. The singularity of the function $t \to t^{z-1}$ at $t = 0$ is not integrable if $\mathrm{Re}\, z \le 0$.

2) Consider

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad \text{for} \quad 0 < x < \infty \ .$$

For $0 < x << 1$ we have

$$\Gamma(x) \sim \int_0^1 t^{x-1}\, dt = \frac{1}{x}\, t^x \Big|_{t=0}^{t=1} = \frac{1}{x} \ .$$

This suggests that, for $z \sim 0$,

$$\Gamma(z) = \frac{1}{z} + \sum_{j=0}^\infty a_j z^j$$

where the series converges for $z \sim 0$. This can in fact be shown. The above representation holds for $|z| < 1$. The function $\Gamma(z)$ has a simple pole at $z = 0$ with

$$Res(\Gamma(z), z = 0) = 1 \ .$$

3) Consider

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt \quad \text{for} \quad x >> 1 \ .$$

The term t^x is very large for large t. In fact, one can show that $\Gamma(x+1)$ grows faster than $e^{\alpha x}$ as $x \to \infty$, for any $\alpha > 0$.

Stirling's formula says that

$$\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} \to 1 \quad \text{as} \quad x \to \infty \ . \tag{8.4}$$

For any $\alpha > 0$,

$$\ln\left(\frac{x^x}{e^{\alpha x}}\right) = x(\ln x - \alpha) \to \infty \quad \text{as} \quad x \to \infty \ .$$

Therefore, using (8.4),

$$\frac{\Gamma(x+1)}{e^{\alpha x}} \to \infty \quad \text{as} \quad x \to \infty \ .$$

Thus, $\Gamma(x)$ grows faster than any exponential $e^{\alpha x}$. On the other hand, if $\varepsilon > 0$, then

$$\ln\Big(\frac{x^x}{e^{(x^{1+\varepsilon})}}\Big) = x(\ln x - x^{\varepsilon}) \to -\infty \quad \text{as} \quad x \to \infty \ .$$

Therefore, using (8.4),

$$\frac{\Gamma(x+1)}{e^{(x^{1+\varepsilon})}} \to 0 \quad \text{as} \quad x \to \infty \ .$$

Thus, for any $\varepsilon > 0$ the function $e^{(x^{1+\varepsilon})}$ grows faster than $\Gamma(x)$ as $x \to \infty$.

4) Let $z = x + iy, x > 0, y \in \mathbb{R}$. We have

$$\Gamma(z) = \int_0^\infty t^{x-1} t^{iy} e^{-t}\, dt$$

where

$$t^{iy} = e^{iy \ln t} = \cos(y \ln t) + i \sin(y \ln t) \ .$$

Let us try to understand the formula

$$\mathrm{Re}\,\Gamma(z) = \int_0^\infty t^{x-1} \cos(y \ln t) e^{-t}\, dt \quad \text{for} \quad z = x + iy, \quad x > 0 \ .$$

For $y \neq 0$ the function $\cos(y \ln t))$ varies rapidly in the interval $0 < t < \infty$ as $t \to 0$ and as $t \to \infty$.

Fix $y \neq 0$ and let $x = 0$. The integral

$$\int_0^\infty t^{-1} \cos(y \ln t) e^{-t}\, dt$$

does not exist since the singularity at $t = 0$ is not integrable. However, the integrand varies rapidly as $t \to 0$, leading to cancellations. This is an intuitive reason why the limit

$$\lim_{x \to 0+} \Gamma(x + iy) =: \Gamma(iy)$$

exists for $y \neq 0$ and $\Gamma(z)$ can be continued analytically into parts of the left half–plane.

8.6 Stirling's Formula

Consider the Gamma–function for real positive x,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt \ .$$

Stirling's formula,

$$\Gamma(x+1) \sim \Big(\frac{x}{e}\Big)^x \sqrt{2\pi x} \ ,$$

gives an approximation for $\Gamma(x+1)$ which is valid for large x. Precisely:

Theorem 8.5 *As $x \to \infty$ we have*

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\Big(1 + \mathcal{O}(x^{-1})\Big) \ . \tag{8.5}$$

Proof: By definition, the term $\mathcal{O}(x^{-1})$ is a function for which the following holds: There are constants $C > 0$ and $x_0 > 0$ so that

$$|\mathcal{O}(x^{-1}| \le C/x \quad \text{for} \quad x > x_0 \ .$$

To prove (8.5) we first make simple linear substitutions:

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^x e^{-t}\, dt \quad \text{(substitute } t = xs, dt = xds) \\ &= x^{x+1} \int_0^\infty s^x e^{-sx}\, ds \quad \text{(use that } s = e^{\ln s}) \\ &= x^{x+1} \int_0^\infty e^{x(\ln s - s)}\, ds \quad \text{(substitute } s = 1+u, ds = du) \\ &= x^{x+1} \int_{-1}^\infty e^{x(\ln(1+u)-1-u)}\, du \\ &= \left(\frac{x}{e}\right)^x x \int_{-1}^\infty e^{x(\ln(1+u)-u)}\, du \ . \end{aligned}$$

We have to analyze the integral

$$J(x) = \int_{-1}^\infty e^{x\phi(u)}\, du \quad \text{for} \quad x >> 1$$

where

$$\phi(u) = \ln(1+u) - u \quad \text{for} \quad u > -1 \ .$$

We must show that

$$J(x) = \sqrt{\frac{2\pi}{x}} + \mathcal{O}(x^{-3/2}) = \sqrt{\frac{2\pi}{x}}\Big(1 + \mathcal{O}(x^{-1}\Big) \ . \tag{8.6}$$

Note: If x is large, then the main contribution to the integral defining $J(x)$ comes from the u–interval where $\phi(u)$ is maximal.

Clearly, $\phi(0) = 0$ and

$$\phi'(u) = \frac{1}{1+u} - 1, \quad \phi''(u) = -\frac{1}{(1+u)^2} < 0 \ .$$

Therefore, the function $\phi(u)$ attains its maximum at $u = 0$. Since

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots \quad \text{for} \quad |u| < 1$$

we have

$$\phi(u) = -\frac{u^2}{2} + \mathcal{O}(u^3) \quad \text{for} \quad u \sim 0 \ .$$

Here, and in the following, the term $\mathcal{O}(u^j)$ denotes a smooth function of u which satisfies an estimate $|\mathcal{O}(u^j)| \le C|u|^j$ for $|u| \le 1/2$.

If one neglects the $\mathcal{O}(u^3)$–term then one obtains

$$\begin{aligned} J(x) &\sim \int_{-1}^{\infty} e^{-xu^2/2}\,du \quad (\text{substitute } \ v = \sqrt{x/2}\,u) \\ &\sim \sqrt{\frac{2}{x}} \int_{-\infty}^{\infty} e^{-v^2}\,dv \\ &= \sqrt{\frac{2\pi}{x}} \end{aligned}$$

To prove (8.6) we have to be precise about the error terms.

Details 1: Fix a small constant $c > 0$. It is not difficult to show that

$$\begin{aligned} J(x) &= \int_{-1}^{\infty} e^{x\phi(u)}\,du \\ &= \int_{-c}^{c} e^{x\phi(u)}\,du + error(x) \end{aligned}$$

where

$$|error(x)| \le e^{-\kappa x} \quad \text{for} \quad x \ge x_0 \ .$$

Here $\kappa > 0$ and x_0 is sufficiently large. This holds since there are positive constants c_1, c_2 with

$$\phi(u) \le -c_1 - c_2(u - c) \quad \text{for} \quad u \ge c \ ,$$

thus

$$\int_{c}^{\infty} e^{x\phi(u)}\,du \le e^{-c_1 x} \int_{0}^{\infty} e^{-c_2 x u}\,du \ .$$

The integral

$$\int_{-1}^{-c} e^{x\phi(u)}\,du$$

can be estimated similarly. Therefore,

$$J(x) = \int_{-c}^{c} e^{x\phi(u)}\,du + \mathcal{O}(e^{-\kappa x}) \quad \text{as} \quad x \to \infty \ .$$

Details 2: We have

$$\begin{aligned} \phi(u) &= -\frac{u^2}{2}\Big(1 - \frac{2u}{3} + \mathcal{O}(u^2)\Big) \\ &= -\frac{u^2}{2}\Big(1 - \frac{u}{3} + \mathcal{O}(u^2)\Big)^2 \end{aligned}$$

for $|u| \le c$. In the integral

$$\begin{aligned} J_1(x) &= \int_{-c}^{c} e^{x\phi(u)}\,du \\ &= \int_{-c}^{c} e^{-(x/2)u^2(1-u/3+\mathcal{O}(u^2))^2}\,du \end{aligned}$$

use the substitution

$$u\Big(1-\frac{u}{3}+\mathcal{O}(u^2)\Big)=y\ .$$

From

$$\begin{aligned} u &= \frac{y}{1-u/3+\mathcal{O}(u^2)} \\ &= y(1+u/3+\mathcal{O}(u^2)) \\ &= y(1+y/3+\mathcal{O}(y^2)) \end{aligned}$$

obtain that

$$du=\Big(1+\frac{2}{3}\,y+\mathcal{O}(y^2)\Big)dy\ .$$

Therefore,

$$J_1(x)=\int_{y_1}^{y_2} e^{-xy^2/2}\Big(1+\frac{2}{3}\,y+\mathcal{O}(y^2)\Big)\,dy$$

where

$$y_1=-c+\mathcal{O}(c^2),\quad y_2=c+\mathcal{O}(c^2)\ .$$

Details 3: As in Details 1, the interval $y_1 \le y \le y_2$ can be changed to $-\infty < y < \infty$. The error is $\mathcal{O}(e^{-\kappa x})$. One obtains:

$$\begin{aligned} J_1(x) &= \int_{-\infty}^{\infty} e^{-xy^2/2}\Big(1+\frac{2}{3}y+\mathcal{O}(y^2)\Big)\,dy+\mathcal{O}(e^{-\kappa x}) \\ &= \sqrt{\frac{2\pi}{x}}+\int_{-\infty}^{\infty} e^{-xy^2/2}\mathcal{O}(y^2)\,dy+\mathcal{O}(e^{-\kappa x}) \end{aligned}$$

In the next integral use the substitution $\sqrt{x}y=q$ and obtain

$$\int_{-\infty}^{\infty} e^{-xy^2/2}y^2\,dy=x^{-3/2}\int_{-\infty}^{\infty} e^{-q^2/2}q^2\,dq=Cx^{-3/2}\ .$$

This proves the formula (8.6) and Stirling's formula. $\diamond$

Remark: According to [Whittaker, Watson, p. 253]:

$$\Gamma(x+1)=\Big(\frac{x}{e}\Big)^x\sqrt{2\pi x}\,\Big(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}+\mathcal{O}(x^{-5})\Big)\quad\text{as}\quad x\to\infty\ .$$

9 Zeros of Holomorphic Functions and the Identity Theorem; Analytic Continuation

Summary: Let $U \subset \mathbb{C}$ be open and connected; let $f, g : U \to \mathbb{C}$ denote two holomorphic functions. The Identity Theorem gives a rather simple condition which implies that f and g are identical, i.e., $f(z) = g(z)$ for all $z \in U$. The condition is that the set $Z = \{z \in U \ : \ f(z) = g(z)\}$ has an accumulation point $P \in Z$, i.e., there exists a sequence $z_n \in Z \setminus \{P\}$ with $z_n \to P$.

We first prove connectedness of the interval $[0,1]$. The result will be used to prove the Identity Theorem.

Lemma 9.1 *Let $A \subset [0,1]$. Assume:*
a) $0 \in A$.
b) A is open in $[0,1]$, i.e., for every $t \in A$ there is $\varepsilon > 0$ such that

$$\{s \in [0,1] \ : \ |s - t| < \varepsilon\} \subset A \ .$$

c) A is closed in $[0,1]$, i.e., if $t_n \in A$ converges to $t \in [0,1]$, then $t \in A$.
Under these assumptions we have $A = [0,1]$.

Proof: Suppose $B = A^c = [0,1] \setminus A$ is not empty. Then let

$$\beta = \inf B \ .$$

Since $0 \in A$ and A is open in $[0,1]$ we have that $\beta > 0$. Also, $[0, \beta) \subset A$. Since A is closed, it follows that $\beta \in A$. Therefore, $[0, \beta] \subset A$. If $\beta = 1$, then A^c is empty, which contradicts our assumption. Thus, $\beta < 1$. But then, since $\beta \in A$ and A is open in $[0,1]$, there is $\varepsilon > 0$ such that $[0, \beta + \varepsilon) \subset A$. This contradicts the definition, $\beta = \inf A^c$. $\diamond$

Definition: Let $S \subset \mathbb{C}$ be non–empty. Let $P \in \mathbb{C}$. The point P is called an accumulation point of S if there exists a sequence of points $z_n \in S \setminus \{P\}$ with $z_n \to P$. Here P may or may not be a point of S. If $P \in S$ and P is not an accumulation point of S, then P is called an isolated point of S.

The following theorem is called the **Identity Theorem**. It implies that two holomorphic functions, $f, g \in H(U)$, are identical on U if U is connected and if the set of all $z \in U$ with $f(z) = g(z)$ has an accumulation point in U. In particular, if $f(z) = g(z)$ for all z in an open disk in U or if $f(z) = g(z)$ for all z on a line segment of positive length, then f and g are identical on U.

Theorem 9.1 *(Identity Theorem) Let U be an open and connected subset of $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be holomorphic. Let*

$$Z = \{z \in U \ : \ f(z) = 0\}$$

be the set of points in U where f is zero. If Z has an accumulation point belonging to U, then $f \equiv 0$.

Proof: a) Let $P \in U$ be an accumulation point of Z and let $z_n \in Z$ with $z_n \to P$, $z_n \neq P$. Let

$$f(z) = \sum_{j=0}^{\infty} a_j (z - P)^j, \quad |z - P| < r \ .$$

We claim that $a_j = 0$ for all j. Otherwise, let

$$a_0 = a_1 = \ldots = a_J = 0, \quad a_{J+1} \neq 0 \ .$$

Then we have

$$\begin{aligned} f(z) &= (z-P)^{J+1}(a_{J+1} + a_{J+2}(z-P) + \ldots) \\ &= (z-P)^{J+1} g(z) \end{aligned}$$

with $g(z)$ holomorphic in $D(P,r)$ and $g(P) \neq 0$. There exists $\varepsilon > 0$ so that $g(z) \neq 0$ for $|z-P| < \varepsilon$. Therefore,

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z-P| < \varepsilon \ .$$

This contradicts

$$z_n \to P, \quad z_n \neq P, \quad f(z_n) = 0 \ .$$

b) Let

$$V = \{z \in U \ : \ f^{(j)}(z) = 0 \quad \text{for all} \quad j\} \ .$$

We have shown that $P \in V$ and claim that $V = U$. To show this, let $Q \in U$ be arbitrary. Let $\gamma : [0,1] \to U$ be a continuous function with

$$\gamma(0) = P, \quad \gamma(1) = Q \ .$$

Such a function γ exists since U is a connected set. Let

$$A = \{t \in [0,1] \ : \ \gamma(t) \in V\} \ .$$

We have that $0 \in A$ since $\gamma(0) = P \in V$. If $t \in A$ then $\gamma(t) \in V$, and therefore $f \equiv 0$ in a neighborhood of the point $\gamma(t)$. This implies that A is open in $[0,1]$. If $t_n \in A$ and $t_n \to t$, then

$$f^{(j)}(\gamma(t_n)) = 0$$

for all n and all j. This yields that

$$f^{(j)}(\gamma(t)) = 0$$

for all j. Therefore, $t \in A$, thus A is closed. By the previous lemma, we have $A = [0,1]$. Therefore, $Q \in V$. $\diamond$

Remark: We can use a different argument for part b) of the proof if we use the definition of connectedness of U from topology. The set V is closed in U since all $f^{(j)}$ are continuous. Also, if $z \in V$ then f is zero in a neighborhood of z. Therefore, V is open in U. Since $P \in V$ we have that $V \neq \emptyset$. The connectedness of U then implies that $V = U$ showing that f is zero on U.

Analytic Continuation: Let $U \subset V \subset \mathbb{C}$ where U and V are open sets and where V is connected. Let $f \in H(U), g \in H(V)$. The function g is called an analytic continuation of U in V if $g(z) = f(z)$ for all $z \in U$. The above arguments imply that f has at most one analytic continuation in V. If an analytic continuation (in an open connected set) exists, then it is unique.

10 Isolated Singularities and Laurent Expansion

Summary: If $f(z)$ is a holomorphic function in the punctured disk $D(P,r) \setminus \{P\}$ then f has an isolated singularity at P. There are three types: An isolated singularity can be removable, it can be a pole, or it can be essential. The Casorati–Weierstrass Theorem says that for $0 < \varepsilon < r$ the image set $f(D(P,\varepsilon) \setminus \{P\})$ is dense in $\mathbb{C}$ if the singularity at P is essential.

If f is holomorphic on $D(P,r) \setminus \{P\}$ then f can be written as a Laurent series,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j \quad \text{for} \quad 0 < |z-P| < r \ .$$

The singularity at P is essential if and only if there are infinitely many negative $j \in \mathbb{Z}$ with $a_j \neq 0$.

More generally, a Laurent series

$$\sum_{j=-\infty}^{\infty} a_j (z-P)^j$$

converges in an annulus

$$A(P, r_1, r_2) = \Big\{ z \in \mathbb{C} \ : \ r_1 < |z-P| < r_2 \Big\}$$

where $0 \leq r_1 < r_2 \leq \infty$.

10.1 Classification of Isolated Singularities

Let $P \in \mathbb{C}$ and let $r > 0$. Then the set

$$D(P,r) \setminus \{P\}$$

is a so–called punctured disk, a disk where the center is removed. If $f = f(z)$ is holomorphic in the set $D(P,r) \setminus \{P\}$ for some $r > 0$, then one says that f has an isolated singularity at P.

For simplicity of notation, let $P = 0$. There are three cases:

Case 1: There exists $\varepsilon > 0$ and $M > 0$ with

$$|f(z)| \leq M \quad \text{for} \quad 0 < |z| \leq \varepsilon \ .$$

Case 2: $|f(z)| \to \infty$ as $z \to 0$, i.e., for all $R > 0$ there exists $\varepsilon > 0$ with

$$|f(z)| \geq R \quad \text{for} \quad 0 < |z| \leq \varepsilon \ .$$

Case 3: Neither Case 1 nor Case 2 holds.

Terminology: Assume that f has an isolated singularity at P, i.e., f is a holomorphic function in $D(P,r) \setminus \{P\}$ for some $r > 0$. In Case 1, one says that f has a removable singularity at P. This terminology is justified by Riemann's theorem on removable singularities, which we will prove below. In Case 2, one says that f has a pole at P. In Case 3 one says that f has an essential singularity at P.

Example 1: The function

$$f(z) = \frac{z^2 - 9}{z - 3} \quad \text{for} \quad z \neq 3$$

has an isolated singularity at $z = 3$. For $z \neq 3$ we have

$$f(z) = z + 3 \; .$$

Case 1 holds. By setting $f(3) = 6$ we can remove the singularity of f at $z = 3$. The point $z = 3$ is a removable singularity of the function $f(z) = (z^2 - 9)/(z - 3)$.

Example 2: The function

$$f(z) = \frac{1}{z^2} \quad \text{for} \quad z \neq 0$$

has an isolated singularity at $z = 0$. Case 2 holds. The point $z = 0$ is a pole of order 2 of the function $f(z) = 1/z^2$.

Example 3: The function

$$f(z) = e^{1/z} \quad \text{for} \quad z \neq 0$$

has an isolated singularity at $z = 0$. We claim that Case 3 holds. To show this, let

$$a_n = \frac{1}{in}, \quad b_n = \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Then we have

$$f(a_n) = e^{in}, \quad |f(a_n)| = 1 \; ,$$

and

$$f(b_n) = e^n \; .$$

Since $a_n \to 0$ and $|f(a_n)| = 1$ for all n, Case 2 does not hold. Since $b_n \to 0$ and $f(b_n) \to \infty$. Case 1 does not hold. The point $z = 0$ is an essential singularity of the function $f(z) = e^{1/z}$.

10.2 Removable Singularities

Theorem 10.1 *(Riemann's Theorem on Removable Singularities) Let* $f \in H\Big(D(0, r) \setminus \{0\}\Big)$ *for some* $r > 0$*. Assume that Case 1 holds, i.e.,* f *is bounded near the isolated singularity at* $P = 0$*:* $|f(z)| \leq M$ *for* $0 < |z| \leq \varepsilon$*. Then*

$$\lim_{z \to 0} f(z) =: f_0$$

exists and the extended function, $f_e(z)$*, defined by*

$$f_e(z) = f(z) \quad \text{for} \quad 0 < |z| < r, \quad f_e(0) = f_0 \; ,$$

is holomorphic in $D(0, r)$*.*

Proof: Set

$$g(z) = z^2 f(z) \quad \text{for} \quad 0 < |z| < r, \quad g(0) = 0 \ .$$

Clearly, g is holomorphic at every z with $0 < |z| < r$ and g is continuous at $z = 0$. We will show that g is also complex differentiable at $z = 0$.

For $0 < |h| < \varepsilon$ we have

$$\begin{aligned} \Big|\frac{1}{h}(g(h) - g(0))\Big| &= \Big|\frac{1}{h}g(h)\Big| \\ &= |h||f(h)| \\ &\leq M|h| \ . \end{aligned}$$

Therefore, $g'(0)$ exists and is zero. Since g is holomorphic in $D(0,r)$ we can write

$$g(z) = a_0 + a_1 z + a_2 z^2 + \ldots \quad \text{for} \quad |z| < r \ .$$

Also, since $g(0) = g'(0) = 0$, we have $a_0 = a_1 = 0$. Therefore,

$$g(z) = z^2(a_2 + a_3 z + \ldots) \quad \text{for} \quad |z| < r \ .$$

Here the power series converges for $|z| < r$. Since $g(z) = z^2 f(z)$ for $0 < |z| < r$ it follows that

$$f(z) = a_2 + a_3 z + \ldots \quad \text{for} \quad 0 < |z| < r \ .$$

This implies that $\lim_{z \to 0} f(z)$ exists, is equal to $f_0 := a_2$, and that the extended function $f_e(z)$ is holomorphic in $D(0,r)$. This proves the theorem. $\diamond$

10.3 Theorem of Casorati–Weierstrass on Essential Singularities

The following result is known as the Casorati–Weierstrass Theorem:

Theorem 10.2 *Let f be a holomorphic function defined on $D(P,r) \setminus \{P\}$ and assume that f has an essential singularity at P. Then, for any $0 < \delta < r$, the set*

$$f\Big(D(P,\delta) \setminus \{P\}\Big)$$

is dense in $\mathbb{C}$.

Proof: Suppose this does not hold. Then fix $0 < \delta < r$ so that the set

$$f\Big(D(P,\delta) \setminus \{P\}\Big)$$

is not dense in $\mathbb{C}$. This means that there exists $Q \in \mathbb{C}$ and $\varepsilon > 0$ with

$$|f(z) - Q| \geq \varepsilon \quad \text{for} \quad 0 < |z - P| < \delta \ .$$

Set

$$g(z) = \frac{1}{f(z) - Q} \quad \text{for} \quad 0 < |z - P| < \delta \ .$$

We have $|g(z)| \leq \frac{1}{\varepsilon}$. By Riemann's removability theorem,

$$\lim_{z \to P} g(z) =: g_0$$

exists.

Case 1: $g_0 \neq 0$. In this case,

$$\lim_{z \to P} (f(z) - Q) = \frac{1}{g_0} \ .$$

This implies that $f(z)$ is bounded near P, which contradicts the assumption that f has an essential singularity at P.

Case 2: $g_0 = 0$. In this case,

$$\lim_{z \to P} |f(z) - Q| = \infty \ .$$

It follows that f has a pole at P, which contradicts the assumption that f has an essential singularity at P. $\diamond$

Remark: A deeper result is **Picard's Big Theorem:**

Theorem 10.3 *Under the same assumptions as in the Casorati–Weierstrass theorem, we have*

$$f\Big(D(P,\delta) \setminus \{P\}\Big) = \mathbb{C} \text{ for every } \delta \text{ with } 0 < \delta < r$$

or, for some $Q \in \mathbb{C}$,

$$f\Big(D(P,\delta) \setminus \{P\}\Big) = \mathbb{C} \setminus \{Q\} \text{ for every } \delta \text{ with } 0 < \delta < r \ .$$

In other words, only the following two possibilities exist:

Possibility 1:
For any $w \in \mathbb{C}$ *and any* $0 < \delta < r$ *the equation* $f(z) = w$ *has infinitely many solutions* $z = z_n$ *with* $0 < |z_n - P| < \delta$.

Possibility 2:
There is a point $Q \in \mathbb{C}$ *so that for any* $w \in \mathbb{C} \setminus \{Q\}$ *and any* $0 < \delta < r$ *the equation* $f(z) = w$ *has infinitely many solutions* $z = z_n$ *with* $0 < |z_n - P| < \delta$.

Example 1: Let $f(z) = e^{1/z}, z \neq 0$. Clearly, f has an essential singularity at $P = 0$. Here we can directly verify that possibility 2 holds with $Q = 0$. If $w \in \mathbb{C}, w \neq 0$, is given, then we can write

$$w = re^{i\theta} = e^{\ln r + i\theta + 2\pi i n}$$

for any $n \in \mathbb{Z}$. If

$$z_n := \frac{1}{\ln r + i\theta + 2\pi i n}$$

then $f(z_n) = w$ and $|z_n| < \delta$ if $|n|$ is large. This shows that the function

$$f(z) = e^{1/z}, \quad z \in \mathbb{C} \setminus \{0\} \ ,$$

has the following property: Given any $w \in \mathbb{C} \setminus \{0\}$ and given any $\delta > 0$, there are infinitely many points z_n with $0 < |z_n| < \delta$ and $f(z_n) = w$. In other words: In any neighborhood of its essential singularity at $P = 0$, the function $f(z) = e^{1/z}$ attains every value $w \in \mathbb{C}$, except for $w = 0$, infinitely many times.

Example 2: $f(z) = \sin(1/z), z \neq 0$. Again, f has an essential singularity at $P = 0$. In this case, for any $w \in \mathbb{C}$ and any $\delta > 0$ the equation $f(z) = w$ has infinitely many solutions $z = z_n$ with $0 < |z_n| \leq \delta$. Proof: We solve

$$\sin \alpha = \frac{1}{2i}\Big(e^{i\alpha} - e^{-i\alpha}\Big) = w$$

by setting

$$q = e^{i\alpha} \ .$$

The equation becomes

$$q - \frac{1}{q} = 2iw \quad \text{or} \quad q^2 - 2iwq - 1 = 0 \ .$$

Clearly, given any $w \in \mathbb{C}$ there exists a solution $q \in \mathbb{C}, q \neq 0$. The equation

$$e^{i\alpha} = q$$

has the solutions

$$\alpha_n = \alpha_{par} + 2\pi n, \quad n \in \mathbb{Z} \ ,$$

where α_{par} is a particular solution. For $n \in \mathbb{Z}$ with $\alpha_n \neq 0$ set

$$z_n = \frac{1}{\alpha_n} = \frac{1}{\alpha_{par} + 2\pi n} \ .$$

We have

$$e^{i/z_n} = e^{i\alpha_n} = q$$

and obtain that

$$\begin{aligned} \sin(1/z_n) &= \sin \alpha_n \\ &= \frac{1}{2i}\left(e^{i\alpha_n} - e^{-i\alpha_n}\right) \\ &= \frac{1}{2i}\left(q - \frac{1}{q}\right) \\ &= w \end{aligned}$$

Also, $|z_n| < \delta$ for large $|n|$.

10.4 Laurent Series

10.4.1 Terminology

An expression

$$\sum_{j=-\infty}^{\infty} a_j (z-P)^j \tag{10.1}$$

is called a Laurent series centered at P. The series (10.1) is called convergent at z if the limits

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_j(z-P)^j =: L_1$$

and

$$\lim_{n\to\infty} \sum_{j=-n}^{-1} a_j(z-P)^j =: L_2$$

exist. If these limits exist then the value of (10.1) is $L_1 + L_2$.

Assume that $0 < r_2 \le \infty$ is the radius of convergence of the power series

$$\sum_{j=0}^{\infty} a_j(z-P)^j$$

and $0 < 1/r_1 \le \infty$ is the radius of convergence of the power series

$$\sum_{j=1}^{\infty} a_{-j} w^j \; .$$

Then the series

$$\sum_{j=-\infty}^{-1} a_j(z-P)^j$$

converges for $r_1 < |z-P| < \infty$ and defines a holomorphic function for $r_1 < |z-P| < \infty$. If

$$0 \le r_1 < r_2 \le \infty$$

then the Laurent series (10.1) converges for $r_1 < |z-P| < r_2$ and defines a holomorphic function in the annulus

$$A = A(P, r_1, r_2) = \{z \; : \; r_1 < |z-P| < r_2\} \; .$$

The set $A(P, r_1, r_2)$ is called the annulus centered at P with inner radius r_1 and outer radius r_2.

We will prove below that, conversely, every function $f(z)$, which is holomorphic in an annulus, can be written as a Laurent series:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z-P)^j \quad \text{for} \quad z \in A(P, r_1, r_2) \; .$$

The coefficients a_j are uniquely determined.

10.4.2 Characterization of Isolated Singularities in Terms of Laurent Expansions

If the holomorphic function f has an isolated singularity at P then P is removable or a pole or an essential singularity. We will prove that these three possibilities have a simple characterization in terms of the Laurent expansion of f in $D(P, r) \setminus \{P\}$.

Let $A = A(P, 0, r) = D(P, r) \setminus \{P\}$. We will show: If $f : A \to \mathbb{C}$ is holomorphic, then f has a Laurent expansion in A,

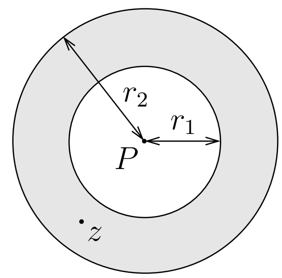

Figure 10.1: The annulus $A(P, r_1, r_2)$

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j, \quad z \in A \ ,$$

where the a_j are uniquely determined. Clearly, there are three cases:

Case A: $a_j = 0$ for all $j < 0$.

Case B: There exists $J < 0$ with $a_J \neq 0$ and $a_j = 0$ for all $j < J$. We will show below that this case holds if and only if f has a pole at P; one says that f has a pole of order $|J|$ at P.

Case C: There are infinitely many $j < 0$ with $a_j \neq 0$.

We will prove:

Theorem 10.4 *Let $f = f(z)$ be holomorphic in $D(P, r) \setminus \{P\}$,*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j \quad \text{for} \quad 0 < |z-P| < r \ .$$

Then f has a removable singularity at P if and only if Case A holds; f has a pole at P if and only if Case B holds; f has an essential singularity at P if and only if Case C holds.

10.4.3 Convergence of Laurent Series

Theorem 10.5 *Assume the Laurent series $\sum_j a_j (z-P)^j$ converges for $z = z_1$ and $z = z_2$ with*

$$r_1 = |z_1 - P| < r_2 = |z_2 - P| \ .$$

Then the series converges for all z with

$$r_1 < |z - P| < r_2 \ .$$

Furthermore, the series

$$\sum_{j=0}^{\infty} a_j (z-P)^j =: g(z)$$

converges absolutely for $|z-P| < r_2$ and the series

$$\sum_{j=-\infty}^{-1} a_j(z-P)^j =: h(z)$$

converges absolutely for $|z-P| > r_1$. *Also,*

$$\sum_{j=0}^{n} a_j(z-P)^j \to g(z) \quad as \quad n\to\infty$$

normally in $D(P, r_2)$ *and*

$$\sum_{j=-n}^{-1} a_j(z-P)^j \to h(z) \quad as \quad n\to\infty$$

normally for $|z-P| > r_1$, *i.e., in* $A(P, r_1, \infty)$.

Proof: This follows, essentially, from Abel's Lemma for power series. $\diamond$

10.4.4 Examples

1) The Laurent series

$$f(z) = \sum_{j=-10}^{\infty} \frac{z^j}{j^2+1}$$

converges for $0 < |z| < 1$. The annulus of convergence is $A(0,0,1)$. The function $f(z)$ has a pole of order 10 at $z=0$.

2) The Laurent series

$$f(z) = \sum_{j=-\infty}^{50} 2^j z^j$$

converges if

$$2^{-1}|z^{-1}| < 1$$

and diverges if

$$2^{-1}|z^{-1}| > 1 \ .$$

Thus, convergence holds for $|z| > \frac{1}{2}$. The annulus of convergence is $A(0, \frac{1}{2}, \infty)$. The function $f(z)$ does not have an isolated singularity at $z=0$.

3) In the following example we show that the Laurent expansion of a function $f(z)$ in an annulus $A(P, r_1, r_2)$ not only depends on P, but also on r_1 and r_2. Consider the function

$$f(z) = \frac{1}{(1-z)(2-z)} = \frac{1}{1-z} - \frac{1}{2-z}, \quad z \in \mathbb{C} \setminus \{1,2\} \ .$$

It can be written as a Laurent series, centered at $z=0$, in

$$\begin{aligned} A_1 &= A(0,0,1) \\ A_2 &= A(0,1,2) \\ A_3 &= A(0,2,\infty) \end{aligned}$$

a) The expansion in A_1 is the Taylor expansion about 0: We have

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \quad |z| < 1 \ ,$$

and

$$\begin{aligned} \frac{1}{2-z} &= \frac{1}{2(1-z/2)} \\ &= \frac{1}{2}\sum_{j=0}^{\infty} 2^{-j} z^j, \quad |z| < 2 \ . \end{aligned}$$

Therefore,

$$f(z) = \sum_{j=0}^{\infty} (1 - 2^{-j-1}) z^j, \quad |z| < 1 \ .$$

b) To obtain the Laurent expansion in A_2 we write

$$\begin{aligned} \frac{1}{1-z} &= -\frac{1}{z}\frac{1}{1-1/z} \\ &= -\frac{1}{z}\sum_{j=0}^{\infty} z^{-j} \end{aligned}$$

for $|z| > 1$. Together with the expansion of $1/(2-z)$ of the previous case:

$$f(z) = -\frac{1}{z}\sum_{j=0}^{\infty} z^{-j} - \sum_{j=0}^{\infty} 2^{-j-1} z^j \quad \text{for} \quad 1 < |z| < 2 \ .$$

c) To obtain the Laurent expansion in A_3 we write for $|z| > 2$:

$$\begin{aligned} -\frac{1}{2-z} &= \frac{1}{z}\frac{1}{1-2/z} \\ &= \frac{1}{z}\sum_{j=0}^{\infty} 2^j z^{-j} \end{aligned}$$

Therefore,

$$f(z) = \frac{1}{z}\sum_{j=0}^{\infty} (2^j - 1) z^{-j} \ .$$

10.4.5 Laurent Expansion: Uniqueness

Let $P = 0$, for simplicity. Let $0 \le r_1 < r_2 \le \infty$ and let

$$A = A(0, r_1, r_2)$$

denote an annulus. Assume that

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \quad z \in A \; . \tag{10.2}$$

Since the convergence is normal in A, the function $f(z)$ is holomorphic in A. Let $r_1 < r < r_2$ and let

$$\gamma(t) = re^{it}, \quad 0 \le t \le 2\pi \; .$$

We claim that

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz, \quad n \in \mathbb{Z} \; .$$

The proof is easy: Since the convergence of the series (10.2) is uniform on γ, we can exchange summation and integration. Therefore,

$$\int_\gamma \frac{f(z)}{z^{n+1}}\, dz = \sum_j a_j \int_\gamma \frac{z^j}{z^{n+1}}\, dz = 2\pi i\, a_n \; .$$

This result shows that the coefficients a_j of the expansion (10.2) are uniquely determined by the function $f(z)$.

10.4.6 Laurent Expansion: Existence

Let $A = A(0, r_1, r_2)$ denote the annulus as above and let $f : A \to \mathbb{C}$ be holomorphic. Let $z \in A$ be arbitrary. Choose real numbers s_1 and s_2 with

$$r_1 < s_1 < |z| < s_2 < r_2 \; .$$

Let

$$\gamma_1(t) = s_1 e^{it}, \quad \gamma_2(t) = s_2 e^{it}, \quad 0 \le t \le 2\pi \; .$$

We claim that

$$2\pi i f(z) = \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta =: r.h.s. \; .$$

In order to show this, we deform the curves $-\gamma_1$ and γ_2 so that the right–hand side becomes

$$r.h.s. = \int_{\gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

with

$$\gamma_\varepsilon = z + \varepsilon e^{it}, \quad 0 \le t \le 2\pi \; .$$

Writing

$$f(\zeta) = f(z) + (f(\zeta) - f(z))$$

and taking the limit $\varepsilon \to 0$ one obtains that

$$r.h.s. = 2\pi i\, f(z)\ .$$

The Laurent expansion of $f(z)$ can now be obtained by using the geometric sum formula. The details are as follows. We have

$$2\pi i f(z) = Int_2 - Int_1$$

with

$$Int_k = \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad k = 1, 2\ .$$

Consider Int_2 first. We have $|\zeta| > |z|$, thus

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta(1 - z/\zeta)} \\ &= \frac{1}{\zeta} \sum_{j=0}^{\infty} \Big(\frac{z}{\zeta}\Big)^j \end{aligned}$$

Therefore,

$$Int_2 = \sum_{j=0}^{\infty} a_j z^j$$

with

$$a_j = \int_{\gamma_2} \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta\ .$$

When considering Int_1, we note that $|\zeta| < |z|$. Therefore,

$$\begin{aligned} \frac{1}{\zeta - z} &= -\frac{1}{z} \frac{1}{1 - \zeta/z} \\ &= -\frac{1}{z} \sum_{j=0}^{\infty} \Big(\frac{\zeta}{z}\Big)^j \end{aligned}$$

This yields

$$Int_1 = \sum_{j=0}^{\infty} b_j z^{-j-1}$$

with

$$b_j = -\int_{\gamma_1} f(\zeta)\zeta^j\, d\zeta\ .$$

We summarize:

Theorem 10.6 *Let $A = A(P, r_1, r_2)$ denote an open annulus and let $f \in H(A)$. There are uniquely determined coefficients $a_j, j \in \mathbb{Z}$, so that*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j \quad \textit{for} \quad z \in A \ . \tag{10.3}$$

This series representation of f is called the Laurent expansion of f in A.

10.4.7 Local Behavior and Laurent Expansion

Assume that f has an isolated singularity at P. There are three cases: P is a removable singularity; P is a pole; or P is an essential singularity. These notions have been defined in Section 10.1 in terms of the local behavior of f near P. We now characterize the three cases in terms of the Laurent expansion of f near P.

Theorem 10.7 *Let f be a holomorphic function defined in $D(P, r) \setminus \{P\}$,*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j, \quad 0 < |z-P| < r \ .$$

a) The point P is a removable singularity of f if and only if $a_j = 0$ for all $j < 0$.
b) The point P is a pole of f if and only if there exists $J < 0$ with

$$a_J \neq 0 \quad \textit{and} \quad a_j = 0 \quad \textit{for all} \quad j < J \ .$$

c) The point P is an essential singularity of f if and only if there are infinitely many $j < 0$ with $a_j \neq 0$.

Proof: For simplicity, assume $P = 0$.
a) If P is removable, then $a_j = 0$ for all $j < 0$ by Riemann's removability theorem. The converse is trivial.
b) First assume that J exists, i.e., with $J = -k$,

$$f(z) = z^{-k}(a_{-k} + a_{-k+1}z + \ldots) = z^{-k} g(z) \ .$$

The function $g(z)$ has a removable singularity at $z = P$ and $|g(z)| \geq \frac{1}{2}|a_{-k}|$ for $|z-P| < \varepsilon$. It follows that $f(z)$ has a pole at $z = P$. Conversely, let $|f(z)| \to \infty$ as $z \to P$. Set $g(z) = 1/f(z)$ for $0 < |z-P| < \varepsilon$ and apply Riemann's theorem to $g(z)$. Obtain that, for some $m \geq 0$,

$$g(z) = z^m(b_m + b_{m+1}z + \ldots), \quad b_m \neq 0 \ .$$

This yields that

$$f(z) = z^{-m} Q(z)$$

where $Q(z)$ has a holomorphic extension at $z = P$. The statement c) now follows trivially. $\diamond$

Terminology: If

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j, \quad 0 < |z-P| < r \ ,$$

is a holomorphic function in $D(P, r) \setminus \{P\}$ then

$$\sum_{j=-\infty}^{-1} a_j (z-P)^j$$

is called the singular part of f (about P). Note that the singular part defines a holomorphic function in $\mathbb{C} \setminus \{P\}$. The coefficient

$$a_{-1} = Res(f, P)$$

is called the residue of f at P.

11 The Calculus of Residues; Evaluation of Integrals; Partial Fraction Decompositions

Summary: Let $\gamma_r(\theta) = re^{i\theta}, 0 \le \theta \le 2\pi$, parameterize the circle of radius r centered at the origin. We know that

$$\int_{\gamma_r} z^j\,dz = \begin{cases} 0 & \text{for} \quad j \in \mathbb{Z},\ j \neq -1 \\ 2\pi i & \text{for} \quad j = -1 \end{cases}$$

Therefore, if $f \in H\Big(D(0,R) \setminus \{0\}\Big)$ has the Laurent expansion

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j \quad \text{for} \quad 0 < |z| < R$$

then

$$\int_{\gamma_r} f(z)\,dz = 2\pi i\, a_{-1} \quad \text{for} \quad 0 < r < R\ .$$

For this reason, the residue

$$a_{-1} = Res(f, 0)$$

is very important.

In Section 11.1 we discuss methods to calculate residues. In Section 11.2 we use residues to evaluate integrals.

For $a \in \mathbb{C} \setminus \mathbb{Z}$ the function

$$q(z) = \frac{\cot(\pi z)}{(z-a)^2} = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}, \quad z \in \mathbb{C} \setminus (\mathbb{Z} \cup \{a\})\ ,$$

has a simple pole at each integer j and a double pole at $z = a$ unless $\cos(\pi a) = 0$. If $\cos(\pi a) = 0$ then the pole of $q(z)$ at $z = a$ is simple.

In Section 11.3 we will apply residue calculus to $q(z)$ to obtain the partial fraction decomposition

$$\frac{\pi^2}{\sin^2(\pi a)} = \sum_{j=-\infty}^{\infty} \frac{1}{(j-a)^2}, \quad a \in \mathbb{C} \setminus \mathbb{Z}\ .$$

This partial fraction decomposition can be used to obtain that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z}\ .$$

This is the partial fraction decomposition of the function $\pi \cot(\pi z)$. It will be used in the next Chapter to evaluate the Zeta–function at even integers.

11.1 Computation of Residues

Let f be holomorphic in

$$D(P,r) \setminus \{P\} = \{z \ : \ 0 < |z-P| < r\} \ ,$$

i.e., f has an isolated singularity at P. We have shown that f has a Laurent expansion in $D(P,r) \setminus \{P\}$:

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-P)^j, \quad 0 < |z-P| < r \ ,$$

where the coefficients a_j are uniquely determined. The coefficient

$$a_{-1} = Res(f,P)$$

is called the residue of f at P.

11.1.1 The Case of a Simple Pole

If $a_{-1} \neq 0$, but $a_j = 0$ for all $j \leq -2$, then the point P is a simple pole of the function $f(z)$. If f has a simple pole at P then one can write

$$f(z) = \frac{g(z)}{z-P}, \quad 0 < |z-P| < r \ ,$$

where, after extension, g is holomorphic in $D(P,r)$. In this case,

$$a_{-1} = Res(f,P) = g(P) \ .$$

Example A: Let

$$f(z) = \frac{e^z}{(z-1)(z-2)} \ .$$

To determine $Res(f,1)$ we write

$$f(z) = \frac{g(z)}{z-1} \quad \text{with} \quad g(z) = \frac{e^z}{z-2} \ .$$

Therefore,

$$Res(f,1) = g(1) = -e \ .$$

To determine $Res(f,2)$ we write

$$f(z) = \frac{g(z)}{z-2} \quad \text{with} \quad g(z) = \frac{e^z}{z-1} \ .$$

Therefore,

$$Res(f,2) = g(2) = e^2 \ .$$

A general result about the residue at a simple pole is the following:

Lemma 11.1 *Let* $f, g \in H(D(P,r))$ *with*

$$g(P) = 0, \quad g'(P) \neq 0, \quad f(P) \neq 0 \ .$$

Then the quotient function

$$q(z) = \frac{f(z)}{g(z)}, \quad 0 < |z - P| < \varepsilon \ ,$$

has a simple pole at $z = P$ *and*

$$Res(q, P) = \frac{f(P)}{g'(P)} \ .$$

Proof: We have

$$\begin{aligned} g(z) &= g'(P)(z-P) + \mathcal{O}((z-P)^2) \\ &= (z-P)g'(P)\Big(1 + \mathcal{O}(z-P)\Big) \ . \end{aligned}$$

Therefore,

$$q(z) = \frac{f(z)}{g(z)} = \frac{1}{z-P}\Big(\frac{f(P)}{g'(P)} + \mathcal{O}(z-P)\Big) \ .$$

$\diamond$

Example B: Let $a \in \mathbb{C} \setminus \mathbb{Z}$. We apply the lemma to

$$q(z) = \frac{\cot(\pi z)}{(z-a)^2} = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}, \quad z \in \mathbb{C} \setminus (\mathbb{Z} \cup \{a\})$$

and determine $Res(q, j)$ for $j \in \mathbb{Z}$. The denominator

$$g(z) = (z-a)^2 \sin(\pi z)$$

has a simple zero at each integer $z = j \in \mathbb{Z}$ and we have

$$g'(j) = \pi (j-a)^2 \cos(\pi j) \neq 0 \ ,$$

thus

$$Res(q, j) = \frac{\cos(\pi j)}{g'(j)} = \frac{1}{\pi (j-a)^2}, \quad j \in \mathbb{Z} \ .$$

11.1.2 Poles of Order 2

If $f(z)$ has a pole of order 2 has $z = P$ then

$$f(z) = \frac{\alpha}{(z-P)^2} + \frac{\beta}{z-P} + h(z) \quad \text{for} \quad 0 < |z-P| < r$$

where $\alpha \neq 0$ and $h \in H(D(P,r))$. Obtain

$$(z-P)^2 f(z) = \alpha + \beta\,(z-P) + (z-P)^2 h(z)$$

and

$$\frac{d}{dz}\Big((z-P)^2 f(z)\Big)\Big|_{z=P} = \beta = Res(f,P)\ .$$

This formula for the residue can be generalized to poles of any order k.

11.1.3 Poles of Order $\le k$

Let $k \ge 1$ and assume that f has a pole of order less than or equal to k at P,

$$f(z) = \sum_{j=-k}^{\infty} a_j (z-P)^j \quad \text{for} \quad 0 < |z-P| < r\ .$$

Then we have

$$(z-P)^k f(z) = a_{-k} + a_{-k+1}(z-P) + a_{-k+2}(z-P)^2 + \ldots =: g(z)$$

and $g(z)$ has a removable singularity at P.

Since

$$g(z) = a_{-k} + a_{-k+1}(z-P) + \ldots + a_{-1}(z-P)^{k-1} + a_0(z-P)^k + \ldots$$

we have

$$(d/dz)^{k-1} g(z)\Big|_{z=P} = (k-1)!\ a_{-1}\ .$$

Here $a_{-1} = Res(f,P)$.

One obtains:

Lemma 11.2 *If $f(z)$ has a pole of order less than or equal to k at P then*

$$Res(f,P) = \frac{1}{(k-1)!}\Big(\frac{d}{dz}\Big)^{k-1}\Big((z-P)^k f(z)\Big)\Big|_{z=P}\ .$$

Example C: Consider the same function as in the previous example,

$$q(z) = \frac{\cot(\pi z)}{(z-a)^2} = \frac{\cos(\pi z)}{(z-a)^2 \sin(\pi z)}, \quad z \in \mathbb{C} \setminus (\mathbb{Z} \cup \{a\})$$

where $a \in \mathbb{C} \setminus \mathbb{Z}$. We want to determine

$$Res(q,a)\ .$$

First assume that $\cos(\pi a) \ne 0$. Then $q(z)$ has a double pole at $z = a$ and we apply Lemma 11.2 with $k = 2$.

We have

$$\begin{aligned} Res(q,a) &= \frac{d}{dz}\Big((z-a)^2 q(z)\Big)\Big|_{z=a} \\ &= \Big(\frac{d}{dz}\cot(\pi z)\Big)\Big|_{z=a} \\ &= -\frac{\pi}{\sin^2(\pi a)}\ . \end{aligned}$$

Second, assume that $\cos(\pi a) = 0$. In this case the function $q(z)$ has a simple pole at $z = a$. We use the Taylor expansion of $\cot(\pi z)$ about $z = a$:

$$\cot(\pi z) = \cot(\pi a) + \Big(\frac{d}{dz}\cot(\pi z)\Big)\Big|_{z=a}(z-a) + \mathcal{O}\Big((z-a)^2\Big) \ .$$

Since

$$\Big(\frac{d}{dz}\cot(\pi z)\Big)\Big|_{z=a} = -\frac{\pi}{\sin^2(\pi a)}$$

one obtains that (assuming $\cos(\pi a) = 0$):

$$\begin{aligned}\cot(\pi z) &= \cot(\pi a) - \frac{\pi}{\sin^2(\pi a)}\,(z-a) + \mathcal{O}\Big((z-a)^2\Big) \\ &= -\frac{\pi}{\sin^2(\pi a)}\,(z-a) + \mathcal{O}\Big((z-a)^2\Big)\end{aligned}$$

Therefore,

$$\begin{aligned}q(z) &= \frac{\cot(\pi z)}{(z-a)^2} \\ &= -\frac{\pi}{\sin^2(\pi a)}\cdot\frac{1}{z-a} + \mathcal{O}(1)\end{aligned}$$

It follows that

$$Res(q,a) = -\frac{\pi}{\sin^2(\pi a)}$$

if $\cot(\pi a) = 0$.

For later reference we summarize the results of Examples B and C:

Lemma 11.3 *Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and let*

$$q(z) = \frac{\cot(\pi z)}{(z-a)^2} = \frac{\cos(\pi z)}{(z-a)^2\sin(\pi z)}, \quad z \in \mathbb{C} \setminus (\mathbb{Z} \cup \{a\}) \ .$$

We have

$$Res(q,j) = \frac{1}{\pi(j-a)^2} \quad \textit{for} \quad j \in \mathbb{Z}$$

and

$$Res(q,a) = -\frac{\pi}{\sin^2(\pi a)} \ .$$

Example D: Let

$$f(z) = \frac{e^z}{(z-1)^3} \ .$$

The function has a pole of order 3 at $P = 1$. By Lemma 11.2 we have

$$
\begin{aligned}
Res(f,1) &= \frac{1}{2}\Big(\frac{d^2}{dz^2}\, e^z\Big)\Big|_{z=1} \\
&= \frac{e}{2}
\end{aligned}
$$

It may be difficult to remember Lemma 11.2. One often can proceed more directly using Taylor expansion. Application to Example D: Let

$$g(z) = (z-1)^3 f(z) = e^z \ .$$

We make a Taylor expansion of $g(z) = e^z$ about $z = 1$. We have

$$g(z) = g(1) + g'(1)(z-1) + \frac{1}{2} g''(1)(z-1)^2 + \ldots$$

If $g(z) = e^z$ then $g^{(j)}(1) = e$ for all j. Therefore,

$$
\begin{aligned}
f(z) &= (z-1)^{-3}\Big(e + e(z-1) + \frac{e}{2}(z-1)^2 + \ldots\Big) \\
&= e(z-1)^{-3} + e(z-1)^{-2} + \frac{e}{2}(z-1)^{-1} + \ldots
\end{aligned}
$$

It follows that

$$Res(f,1) = \frac{e}{2} \ .$$

11.2 Calculus of Residues

Let $U \subset \mathbb{C}$ be an open set and let $P \in U$. Let $f \in H(U \setminus \{P\})$, thus f has an isolated singularity at P. Let

$$\gamma_\varepsilon(t) = P + \varepsilon e^{it}, \quad 0 \le t \le 2\pi \ .$$

Assume that $\varepsilon > 0$ is so small that $\gamma_\varepsilon \subset U$ and the curve γ_ε encircles only the singularity P of f, but no other singularities of f. In this case,

$$\int_{\gamma_\varepsilon} f(z)\, dz = 2\pi i\, a_{-1}$$

with

$$a_{-1} = Res(f,P) \ .$$

Together with Cauchy's theorem, which allows the deformation of curves in regions where f is holomorphic, this yields a very powerful tool for the evaluation of integrals. We formalize this in the residue theorem.

Theorem 11.1 *(Residue Theorem) Let $U \subset \mathbb{C}$ be an open set and let $\Gamma \subset U$ be a simply closed curve which is positively oriented. Let V denote the region encircled by Γ and assume that $V \subset U$. Let $P_1, \ldots, P_k \in V$ and let $f \in H(U \setminus \{P_1, \ldots, P_k\})$. Then we have*

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{j=1}^k Res(f, P_j) \ .$$

11.2.1 Direct Applications of the Residue Theorem

In Examples 1 to 3 we evaluate integrals directly using residue calculus.

Example 1: Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, denote the parameterized unit circle. We want to evaluate

$$I = \int_\gamma z^2 \sin(1/z)\, dz \ .$$

We have

$$\sin w = w - \frac{1}{6} w^3 + \dots \ ,$$

thus

$$\sin(1/z) = z^{-1} - \frac{1}{6} z^{-3} + \dots \ ,$$

thus

$$z^2 \sin(1/z) = z - \frac{1}{6} z^{-1} + \dots \ .$$

Therefore,

$$Res\Big(z^2 \sin(1/z), z = 0\Big) = -\frac{1}{6}$$

and

$$I = -\frac{2\pi i}{6} = -\frac{\pi i}{3} \ .$$

Example 2: Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. We want to evaluate

$$I = \int_\gamma \frac{5z-2}{z(z-1)}\, dz \ .$$

We have

$$f(z) = \frac{1}{z} \cdot \frac{5z-2}{z-1} = \frac{1}{z-1} \cdot \frac{5z-2}{z} \ ,$$

thus

$$Res(f, 0) = 2$$

and

$$Res(f, 1) = 3 \ .$$

It follows that

$$I = (2+3) 2\pi i = 10\pi i \ .$$

Example 3: Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. We want to evaluate

$$I = \int_\gamma \frac{\sinh z}{z^4}\, dz \ .$$

We have

$$\begin{aligned} e^z &= 1 + z + \frac{z}{2} + \frac{z^3}{6} + \ldots \\ e^{-z} &= 1 - z + \frac{z}{2} - \frac{z^3}{6} + \ldots \end{aligned}$$

thus

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) = z + \frac{z^3}{6} + \frac{z^5}{5!} + \ldots$$

Therefore,

$$\frac{\sinh z}{z^4} = z^{-3} + \frac{z^{-1}}{6} + \frac{z}{5!} + \ldots$$

This yields that

$$Res\Big(\frac{\sinh z}{z^4}, z = 0\Big) = \frac{1}{6}$$

and

$$I = \frac{\pi i}{3} \ .$$

11.2.2 Use of the Substitution $z = e^{it}$

Integrals involving trigonometric functions can sometimes be rewritten as complex line integrals and then be evaluated using the calculus of residues.

In the following example we use the substitution

$$z(t) = e^{it}, \quad 0 \le t \le 2\pi \ ,$$

to turn an integral involving a trigonometric function into an integral along the unit circle, $\mathcal{C}_1$.

Example 4: For $a > 1$ evaluate

$$I = \int_0^\pi \frac{dt}{a + \cos t} \ .$$

Using the symmetry $\cos t = \cos(2\pi - t)$ and $\cos t = (e^{it} + e^{-it})/2$ we obtain

$$2I = \int_0^{2\pi} \frac{dt}{a + (e^{it} + e^{-it})/2} \ .$$

The function $z(t) = e^{it}$, $0 \le t \le 2\pi$, parameterizes the unit circle $\mathcal{C}_1$ and $dz = ie^{it}\, dt = iz\, dt$. If $f(z)$ is a continuous function on $\mathcal{C}_1$, then

$$\int_{\mathcal{C}_1} f(z)\,dz = \int_0^{2\pi} f(e^{it}) ie^{it}\,dt \ .$$

Therefore, to evaluate $2I$, we define $f(z)$ by

$$f(z)iz = \frac{1}{a + (z+1/z)/2}$$

and obtain

$$f(z) = \frac{2}{i} \cdot \frac{1}{z^2 + 2az + 1} \ .$$

Therefore,

$$2I = \int_0^{2\pi} \frac{dt}{a + (e^{it} + e^{-it})/2} = \int_{\mathcal{C}_1} f(z)\,dz \ .$$

This yields that

$$I = \frac{1}{i} \int_{\mathcal{C}_1} \frac{dz}{z^2 + 2az + 1} \ .$$

Thus we have written the integral I as a complex line integral. We now evaluate I using residue calculus. The solutions of

$$z^2 + 2az + 1 = 0$$

are

$$z_1 = -a + \sqrt{a^2 - 1}, \quad z_2 = -a - \sqrt{a^2 - 1}$$

with $z_1 z_2 = 1$, thus

$$z_2 < -1 < z_1 < 0 \ .$$

Set

$$g(z) = \frac{1}{z^2 + 2az + 1} = \frac{1}{(z - z_1)(z - z_2)} \ ,$$

thus

$$Res(g, z_1) = \frac{1}{z_1 - z_2} = \frac{1}{2\sqrt{a^2 - 1}} \ .$$

Therefore,

$$I = 2\pi i \, Res\Big(\frac{g}{i}, z_1\Big) = \frac{\pi}{\sqrt{a^2 - 1}} \ .$$

Note: We have $I = I(a) \to \infty$ as $a \to 1+$. This is expected since $a + \cos t = 0$ for $t = \pi$ if $a = 1$.

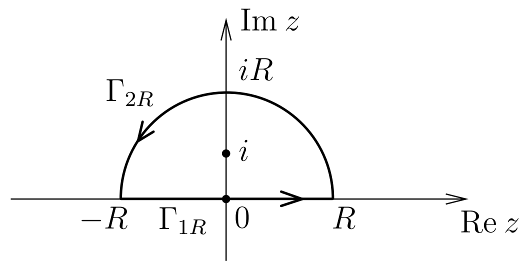

Figure 11.1: Integrals: Examples 5 and 6

11.2.3 Integrals over $-\infty < x < \infty$

Example 5: We know from calculus that

$$I := \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \ . \tag{11.1}$$

In calculus, one uses that $(d/dx)\arctan x = (1+x^2)^{-1}$. Let us obtain (11.1) using the calculus of residues. Let

$$\Gamma_{1R}(x) = x, \quad -R \le x \le R$$

and

$$\Gamma_{2R}(t) = Re^{it}, \quad 0 \le t \le \pi \ .$$

Then $\Gamma_R = \Gamma_{1R} + \Gamma_{2R}$ is a closed curve, consisting of the part $-R \le x \le R$ of the x–axis and a semi–circle in the upper half–plane.

We assume $R > 1$. Then, by residue calculus,

$$\begin{aligned} \int_{\Gamma_R} \frac{dz}{z^2+1} &= \int_{\Gamma_R} \frac{dz}{(z-i)(z+i)} \\ &= \frac{2\pi i}{2i} \\ &= \pi \end{aligned}$$

We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_{\Gamma_{1R}} \frac{dz}{z^2+1}$$

and the corresponding integral along Γ_{2R} tends to zero as $R \to \infty$. Therefore, $I = \pi$.

Example 6: We claim that for $a > 0$:

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{a^2+x^2}\, dx = \frac{\pi}{a}\, e^{-a} \ .$$

A crude simple bound for the integral follows from

$$|I| \le \int_{-\infty}^{\infty} \frac{1}{a^2+x^2} = \frac{\pi}{a} \ .$$

Let Γ_{1R}, Γ_{2R}, and Γ_R be defined as in Example 5. One should note that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

becomes exponentially large in the upper half–plane: If $z = x + iy$, then $|e^{iz}| = e^{-y} \leq 1$ for $y \geq 0$, but

$$|e^{-iz}| = e^y, \quad y \geq 0 \; .$$

Thus we cannot directly proceed as in the previous example, because the integral of $\cos z/(a^2+z^2)$ along Γ_{2R} does not converge to zero as $R \to \infty$. Instead, we recall that $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$, thus

$$\cos z = \mathrm{Re}\, e^{iz} \quad \text{for} \quad z = x \in \mathbb{R} \; .$$

Therefore,

$$I = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{iz}}{a^2 + z^2}\, dz \; .$$

We have

$$g(z) := \frac{e^{iz}}{a^2 + z^2} = \frac{e^{iz}}{(z - ia)(z + ia)}$$

with

$$Res(g, ia) = \frac{e^{-a}}{2ia} \; .$$

Therefore, for $R > a$,

$$\begin{aligned} \int_{\Gamma_R} g(z)\, dz &= 2\pi i\, Res(g, ia) \\ &= \frac{\pi}{a}\, e^{-a} \; . \end{aligned}$$

It remains to show that

$$\int_{\Gamma_{2R}} g(z)\, dz \to 0 \quad \text{as} \quad R \to \infty \; . \tag{11.2}$$

Note that $|e^{iz}| \leq 1$ in the upper half–plane. Also, if $|z| = R \geq 2a$ then

$$|a^2 + z^2| \geq |z|^2 - a^2 \geq \frac{3}{4}\, R^2 \; ,$$

thus

$$|g(z)| \leq \frac{4}{3}\, R^{-2}.$$

This implies (11.2).

11.2.4 Extensions Using Jordan's Lemma

Example 7: We claim that, for $a > 0$,

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{a^2 + x^2}\, dx = \pi\, e^{-a} \ .$$

Here, by definition,

$$I = \lim_{R \to \infty} \int_{-R}^{R} \frac{x \sin x}{a^2 + x^2}\, dx \ .$$

Let Γ_{1R}, Γ_{2R}, and Γ_R be defined as in Example 5. Since $e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$, we have

$$\sin z = \operatorname{Im} e^{iz} \quad \text{for} \quad z = x \in \mathbb{R} \ .$$

Setting

$$g(z) = \frac{z e^{iz}}{a^2 + z^2} = \frac{z e^{iz}}{(z - ia)(z + ia)} \tag{11.3}$$

we have

$$I = \operatorname{Im} \lim_{R \to \infty} \int_{\Gamma_{1R}} g(z)\, dz \ .$$

Since

$$g(z) = \frac{1}{z - ia} \cdot \frac{z e^{iz}}{z + ia}$$

we have

$$Res(g, ia) = \frac{1}{2}\, e^{-a} \ ,$$

Therefore, for $R > a$:

$$\int_{\Gamma_R} g(z)\, dz = 2\pi i\, Res(g, ia) = \pi i e^{-a} \ .$$

It remains to prove (11.2) for the function $g(z)$ defined in (11.3). Note that the estimate of the previous example, $|g(z)| \le C R^{-2}$ for $z \in \Gamma_{2R}$, does not hold here. We must estimate the integral along Γ_{2R} more carefully.

Theorem 11.2 *(Jordan's Lemma) Recall that Γ_{2R} denotes the semi–circle with parameterization*

$$z(t) = R e^{it}, \quad 0 \le t \le \pi \ .$$

Let $\bar{\mathbb{H}}$ denote the closed upper half–plane and let $f : \bar{\mathbb{H}} \to \mathbb{C}$ be a continuous function. Let

$$M_R = \max\{|f(z)| \ : \ z \in \Gamma_{2R}\}$$

and assume that $M_R \to 0$ as $R \to \infty$. Then we have

$$I_R := \int_{\Gamma_{2R}} f(z) e^{iz}\, dz \to 0 \quad \text{as} \quad R \to \infty \ .$$

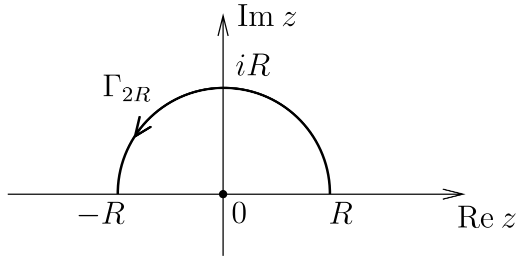

Figure 11.2: Contour for Jordan's Lemma

Proof: Noting that

$$z(t) = R(\cos t + i\sin t) \quad \text{and} \quad |z'(t)| = R$$

we have

$$\begin{aligned} |I_R| &\leq M_R \int_0^\pi |e^{iz(t)}|\, R\, dt \\ &= RM_R \int_0^\pi e^{-R\sin t}\, dt \\ &= 2RM_R \int_0^{\pi/2} e^{-R\sin t}\, dt\ . \end{aligned}$$

Since

$$\sin t \geq \frac{2t}{\pi} \quad \text{for} \quad 0 \leq t \leq \frac{\pi}{2}$$

we have, with $c = 2R/\pi$:

$$\begin{aligned} \int_0^{\pi/2} e^{-R\sin t}\, dt &\leq \int_0^{\pi/2} e^{-ct}\, dt \\ &\leq \frac{1}{c} \\ &= \frac{\pi}{2R} \end{aligned}$$

Therefore,

$$|I_R| \leq \pi M_R \to 0 \quad \text{as} \quad R \to \infty\ .$$

$\diamond$

Applying Jordan's Lemma with

$$f(z) = \frac{z}{a^2 + z^2} \quad \text{where} \quad a > 0$$

one obtains that (11.2) holds for

$$g(z) = \frac{ze^{iz}}{a^2+z^2} \ .$$

This completes the proof of the formula

$$\int_{-\infty}^{\infty} \frac{x \sin x}{a^2+x^2}\, dx = \pi\, e^{-a} \ .$$

11.2.5 A Pole on the Real Axis

Example 8: We want to show

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \ .$$

The integral exists as an improper Riemann integral. Note that the function $f(z) = \frac{\sin z}{z}$ does not have a pole, but we will integrate the function e^{iz}/z, which does have a pole at $z = 0$.

We first discuss the existence of the integral. The integral $\int_\pi^\infty \frac{\sin x}{x}\, dx$ does not exist as a proper Riemann or Lebesgue integral since the integrand decays too slowly. To see this, note that, for $j = 1, 2, \ldots$

$$|\sin x| \geq \frac{1}{\sqrt{2}} \quad \text{for} \quad \pi(j+\frac{1}{4}) \leq x \leq \pi(j+\frac{3}{4})$$

Therefore,

$$\frac{|\sin x|}{x} \geq \frac{1}{\sqrt{2}} \frac{1}{\pi(j+1)} =: \frac{c}{j+1} \quad \text{for} \quad \pi(j+\frac{1}{4}) \leq x \leq \pi(j+\frac{3}{4})$$

where $c = 1/(\sqrt{2}\pi)$. It follows that

$$\int_{\pi j}^{\pi(j+1)} \frac{|\sin x|}{x}\, dx \geq \frac{c}{j+1} \cdot \frac{\pi}{2} \ .$$

Since $\sum_{j=1}^\infty \frac{1}{j+1} = \infty$ one obtains that

$$\int_\pi^\infty \frac{|\sin x|}{x}\, dx = \infty \ .$$

A theorem of integration theory implies that the integral

$$\int_\pi^\infty \frac{\sin x}{x}\, dx$$

does not exist as a Lebesgue or Riemann integral.

However, for $1 < R < \infty$:

$$\begin{aligned} \int_1^R \frac{\sin x}{x}\, dx &= -\frac{1}{x} \cos x \Big|_1^R - \int_1^R \frac{1}{x^2} \cos x\, dx \\ &= -\frac{1}{R} \cos R + \cos 1 - \int_1^R \frac{1}{x^2} \cos x\, dx \end{aligned}$$

Therefore, the limit

$$\lim_{R\to\infty} \int_1^R \frac{\sin x}{x}\, dx$$

exists since the integral

$$\int_1^\infty \frac{1}{x^2}\, \cos x\, dx$$

is finite.

By definition,

$$I := P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x}\, dx \tag{11.4}$$

where $P.V.$ stands for principle value. It is common to drop the $P.V.$ notation and to say that the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

exists as an improper integral, defined by (11.4).

Computation of I: We have

$$I = \lim_{R\to\infty,\, \varepsilon\to 0} I(R,\varepsilon)$$

with

$$I(R,\varepsilon) = \int_{-R}^{-\varepsilon} \frac{\sin x}{x}\, dx + \int_{\varepsilon}^{R} \frac{\sin x}{x}\, dx \ .$$

Also, for $x = z \in \mathbb{R}$:

$$\frac{\sin x}{x} = \mathrm{Im}\left(\frac{e^{iz}}{z}\right) \ ,$$

thus

$$I(R,\varepsilon) = \mathrm{Im}\Big(\int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\, dz + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\, dz\Big) \ .$$

The term in brackets is

$$K(R,\varepsilon) := \int_{\Gamma_{-R,-\varepsilon}+\Gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\, dz \ .$$

Let Γ denote the closed curve shown in Figure 11.3:

$$\Gamma = \Gamma_{-R,-\varepsilon} + \Gamma_{-\varepsilon,\varepsilon} + \Gamma_{\varepsilon,R} + \Gamma_{2R} \ .$$

By Cauchy's theorem,

$$\int_\Gamma \frac{e^{iz}}{z}\, dz = 0 \ .$$

Therefore,

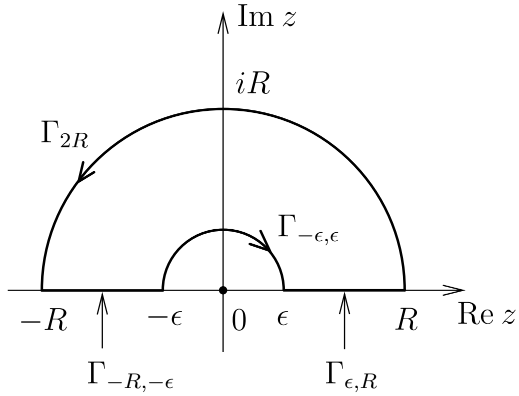

Figure 11.3: Contour for $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$, Example 8

$$K(R,\varepsilon) := -\int_{\Gamma_{-\varepsilon,\varepsilon}+\Gamma_{2R}} \frac{e^{iz}}{z}\, dz \ .$$

By Jordan's lemma, [3] the integral along Γ_{2R} tends to zero as $R \to \infty$. Also,

$$\frac{e^{iz}}{z} = \frac{1}{z} + g(z)$$

where $g(z)$ is holomorphic near $z = 0$. Therefore,

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{-\varepsilon,\varepsilon}} \frac{e^{iz}}{z}\, dz = -\pi i \ .$$

One obtains that

$$\lim_{R\to\infty,\, \varepsilon\to 0} K(R,\varepsilon) = \pi i \ ,$$

thus $I = \pi$.

Remarks on Fourier transforms: Let $\chi_J(x)$ denote the characteristic function of the interval $J = [-1, 1]$. Its Fourier transform is

[3]Jordan's Lemma is applied with $f(z) = \frac{1}{z}$. The function $1/z$ is not continuous in the closed upper half–plane, but the proof of Jordan's Lemma shows that the singularity at $z = 0$ is not important.

$$
\begin{aligned}
\hat{\chi}_J(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_J(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik}\, e^{-ikx}\Big|_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik}\, (e^{-ik} - e^{ik}) \\
&= \sqrt{\frac{2}{\pi}}\, \frac{1}{k}\, \frac{1}{2i}\, (e^{ik} - e^{-ik}) \\
&= \sqrt{\frac{2}{\pi}}\, \frac{\sin k}{k}
\end{aligned}
$$

for $k \in \mathbb{R}, k \neq 0$. The function $\hat{\chi}_J(k)$ is not integrable over $\mathbb{R}$ since $1/k$ decays too slowly. The inverse Fourier transform of $\hat{\chi}_J(k)$ exists only in the principle value sense. We have for the inverse Fourier transform of $\hat{\chi}_J(k)$:

$$
g(x) := \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin k}{k}\, e^{ikx}\, dk \ .
$$

In Example 8 we have shown that $\int_{-\infty}^{\infty} (\sin k)/k\, dk = \pi$ and obtain

$$
g(0) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}}\, \pi = 1 \ .
$$

This is to be expected since $\chi_J(0) = 1$. One can also say that the formula $\int_{-\infty}^{\infty} (\sin k)/k\, dk = \pi$ is a special case of the Fourier inversion theorem applied to the function $\chi_J(x)$.

11.2.6 Use of a Second Path

Example 9: For $0 < a < 1$:

$$
\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin(\pi a)} \ . \tag{11.5}
$$

This integral will be used in Chapter 13 to show the reflection property of the Γ–function.

Let $f(z) = \frac{e^{az}}{1+e^z}$. Consider the rectangle $\mathcal{R}$ with corners at

$$
-R, \quad R, \quad R + 2\pi i, \quad -R + 2\pi \quad \text{where} \quad R > 0 \ .
$$

Denote the positively oriented boundary curve of $\mathcal{R}$ by

$$
\Gamma_R = \Gamma_{1R} + \Gamma_{2R} + \Gamma_{3R} + \Gamma_{4R} \ .
$$

The pieces have parameterizations

$$
\begin{aligned}
\Gamma_{1R} &: \ z(x) = x, \quad -R \le x \le R \\
-\Gamma_{3R} &: \ z(x) = x + 2\pi i, \quad -R \le x \le R \\
\Gamma_{2R} &: \ z(y) = R + iy, \quad 0 \le y \le 2\pi \\
-\Gamma_{4R} &: \ z(y) = -R + iy, \quad 0 \le y \le 2\pi
\end{aligned}
$$

The function $f(z) = e^{az}/(1+e^z)$ has one singularity in the rectangle $\mathcal{R}$. The singularity is a simple pole at $P = \pi i$ and, using Lemma 11.1,

$$Res(f, \pi i) = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i} \ .$$

By the residue theorem:

$$\int_{\Gamma_R} f(z)\, dz = -2\pi i\, e^{a\pi i} \ . \tag{11.6}$$

It is not difficult to show that

$$Q_R := \int_{\Gamma_{2R}+\Gamma_{4R}} f(z)\, dz \to 0 \quad \text{as} \quad R \to \infty \ .$$

(See Details below.) Set

$$I_R := \int_{\Gamma_{1R}} f(z)\, dz = \int_{-R}^{R} f(x)\, dx \ .$$

The main trick of the whole approach is that the integral I_R occurs again when one integrates along Γ_{3R}:

$$\int_{-\Gamma_{3R}} f(z)\, dz = e^{2\pi ai} \int_{-R}^{R} f(x)\, dx = e^{2\pi ai} I_R \ .$$

Therefore, using (11.6):

$$2\pi i\, Res(f, \pi i) = -2\pi i\, e^{a\pi i} = I_R(1 - e^{2\pi ai}) + Q_R \ .$$

This implies that

$$I_R = \frac{2\pi i}{e^{\pi ai} - e^{-\pi ai}} + \tilde{Q}_R = \frac{\pi}{\sin(\pi a)} + \tilde{Q}_R$$

where $\tilde{Q}_R \to 0$ as $R \to \infty$. As $R \to \infty$ one obtains (11.5).

Details: For $z \in \Gamma_{2R}$ we have $z = R + iy,\ 0 \le y \le 2\pi$, and

$$f(z) = \frac{e^{az}}{1+e^z} = \frac{e^{aR}\, e^{iay}}{1 + e^R\, e^{iy}} \ .$$

Therefore, since $0 < a < 1$:

$$|f(z)| \le \frac{e^{aR}}{e^R - 1} \to 0 \quad \text{as} \quad R \to \infty \ .$$

It follows that

$$\int_{\Gamma_{2R}} f(z)\, dz \to 0 \quad \text{as} \quad R \to \infty \ .$$

For $z \in \Gamma_{4R}$ we have $z = -R + iy, 0 \le y \le 2\pi$, and

$$f(z) = \frac{e^{az}}{1+e^z} = \frac{e^{-aR}\, e^{iay}}{1 + e^{-R}\, e^{iy}} \ .$$

The convergence

$$\int_{\Gamma_{4R}} f(z)\,dz \to 0 \quad \text{as} \quad R \to \infty$$

follows since $a > 0$.

11.3 Derivation of a Partial Fraction Decomposition via Integration

Example 10: Let $a \in \mathbb{C} \setminus \mathbb{Z}$ and consider the function

$$q(z) = \frac{\cot(\pi z)}{(z-a)^2}, \quad z \in \mathbb{C} \setminus (\mathbb{Z} \cup \{a\}) \ .$$

The function $q(z)$ has a simple pole at each integer j and a double pole at $z = a$ unless $\cos(\pi a) = 0$. Also, by Lemma 11.3:

$$Res(q, j) = \frac{1}{\pi (j-a)^2}, \quad j \in \mathbb{Z} \ ,$$

and

$$Res(q, a) = -\frac{\pi}{\sin^2(\pi a)} \ .$$

For positive integers n, let γ_n denote the boundary curve of the rectangle in Figure 11.4. Assume that $n > |a|$. By the residue theorem,

$$\frac{1}{2\pi i} \int_{\gamma_n} q(z)\,dz = \sum_{j=-n}^{n} \frac{1}{\pi (j-a)^2} \ - \ \frac{\pi}{\sin^2(\pi a)} \ . \tag{11.7}$$

By estimating the integrand $q(z)$ on γ_n we will prove that

$$\int_{\gamma_n} q(z)\,dz \to 0 \quad \text{as} \quad n \to \infty \ .$$

Therefore,

$$\sum_{j=-\infty}^{\infty} \frac{1}{\pi (j-a)^2} = \frac{\pi}{\sin^2(\pi a)} \quad \text{for} \quad a \in \mathbb{C} \setminus \mathbb{Z} \ .$$

The following lemma will be used to bound $\cot(\pi z)$ on γ_n. If one sets

$$Q := e^{2\pi i z}$$

then

$$\cot(\pi z) = \frac{\frac{1}{2}(e^{i\pi z} + e^{-i\pi z})}{\frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})} = i\, \frac{Q+1}{Q-1} \ .$$

Therefore, in order to bound $|\cot(\pi z)|$ for $z \in \gamma_n$, we have to bound $|Q-1|$ away from zero for $z \in \gamma_n$. We show:

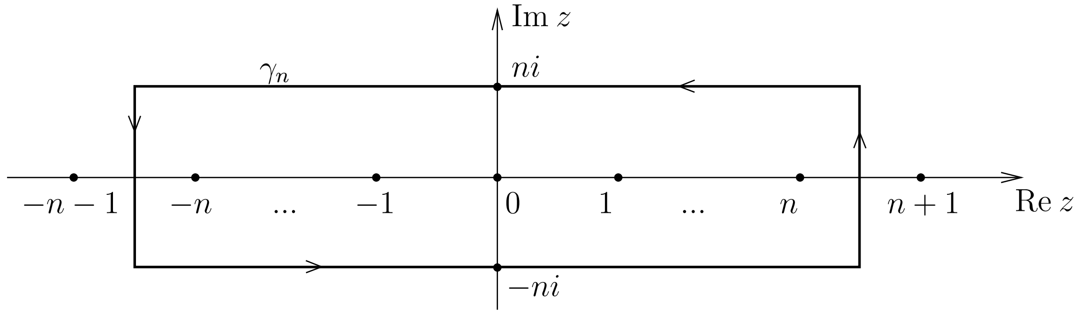

Figure 11.4: Contour γ_n for $\frac{1}{2\pi i}\int_{\gamma_n}\frac{\cot(\pi z)}{(z-a)^2}dz$

Lemma 11.4 *For $n = 1, 2, \dots$ let $z \in \gamma_n$ and set $Q = e^{2\pi iz}$. Then we have*

$$|Q-1| \geq \frac{1}{2} \ .$$

Proof: a) Let $z = (n+\frac{1}{2}) + iy, y \in \mathbb{R}$. We have

$$Q = e^{2\pi i(n+\frac{1}{2})}e^{-2\pi y} = -e^{-2\pi y} < 0 \ ,$$

thus $|Q-1| > 1$.

The same argument works for $z = -(n+\frac{1}{2}) + iy, y \in \mathbb{R}$.

b) Let $z = x + ni, x \in \mathbb{R}$. We have

$$Q = e^{2\pi ix}e^{-2\pi n}, \quad |Q| \leq e^{-2\pi} < \frac{1}{2} \ .$$

c) Let $z = x - ni, x \in \mathbb{R}$. We have

$$Q = e^{2\pi ix}e^{2\pi n}, \quad |Q| \geq e^{2\pi} > 2 \ .$$

This proves the lemma. $\diamond$

Lemma 11.5 *For $n = 1, 2, \dots$ we have*

$$|\cot(\pi z)| \leq 6 \quad \textit{for all} \quad z \in \gamma_n \ .$$

Proof: With $Q = e^{2\pi iz}$ we have

$$\cot(\pi z) = \frac{\frac{1}{2}(e^{i\pi z} + e^{-i\pi z})}{\frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})} = i\ \frac{Q+1}{Q-1} \ .$$

By the previous lemma, $|Q-1| \geq \frac{1}{2}$.

Case 1: $|Q| \geq 2$, thus $1 \leq \frac{1}{2}|Q|$. We have

$$\begin{aligned} |Q+1| &\leq |Q|+1 \leq \frac{3}{2}|Q| \\ |Q-1| &\geq |Q|-1 \geq \frac{1}{2}|Q| \end{aligned}$$

thus

$$\left|\frac{Q+1}{Q-1}\right| \leq 3 \ .$$

Case 2: $|Q| \leq 2$. Recall that $|Q-1| \geq \frac{1}{2}$. We have

$$\left|\frac{Q+1}{Q-1}\right| \leq \frac{3}{\frac{1}{2}} = 6 \ .$$

This proves the lemma. $\diamond$

Let Ω be a compact subset of the open set $U = \mathbb{C} \setminus \mathbb{Z}$. Let $a \in \Omega$. There exists a constant C, depending on Ω but not on a, so that

$$|q(z)| \leq \frac{C}{n^2} \quad \text{for} \quad z \in \gamma_n, \quad a \in \Omega \ ,$$

for $n \geq N = N(\Omega)$. The detailed argument is as follows: If $z \in \gamma_n$, then $|z| \geq n$. Since Ω is bounded, there exists $N(\Omega) \in \mathbb{N}$ with

$$2|a| \leq N(\Omega) \quad \text{for all} \quad a \in \Omega \ .$$

If $n \geq N(\Omega)$ then $n \geq 2|a|$, thus

$$|z-a| \geq |z| - |a| \geq n - \frac{n}{2} = \frac{n}{2} \ .$$

This implies that

$$\frac{1}{|(z-a)^2|} \leq \frac{4}{n^2} \quad \text{for} \quad n \geq N(\Omega) \quad \text{and} \quad a \in \Omega \ .$$

Using the previous lemma one obtains that

$$|\int_{\gamma_n} q(z)\,dz| \leq \frac{C_1}{n} \quad \text{for} \quad n \geq N(\Omega) \ .$$

This proves that

$$\Big| \sum_{j=-n}^{n} \frac{1}{(j-a)^2} \ - \ \frac{\pi^2}{\sin^2(\pi a)} \Big| \leq \frac{\pi C_1}{n} \quad \text{for} \quad n \geq N(\Omega) \ .$$

We now write $a = -z$ and obtain:

Theorem 11.3 *We have*

$$\lim_{n\to\infty} \sum_{j=-n}^{n} \frac{1}{(z-j)^2} = \frac{\pi^2}{\sin^2(\pi z)} \quad \textit{for} \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

The convergence is uniform on every compact subset of $\mathbb{C} \setminus \mathbb{Z}$.

The above formula is also written as

$$\sum_{j=-\infty}^{\infty} \frac{1}{(z-j)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \; . \tag{11.8}$$

The left–hand side is called the partial fraction decomposition of the meromorphic function

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \; .$$

The Special Value $z = \frac{1}{2}$. By substituting special values for z into (11.8) one can obtain interesting (and uninteresting) results. For $z = \frac{1}{2}$ obtain:

$$\begin{aligned} \pi^2 &= 4 \sum_0^\infty \frac{1}{(2j+1)^2} + 4 \sum_{-\infty}^{-1} \frac{1}{(2j+1)^2} \\ &= 8(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots) \; , \end{aligned}$$

thus

$$\sum_{j=0}^\infty \frac{1}{(2j+1)^2} = \frac{1}{1^1} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} \; .$$

With a trick we can also evaluate the following series:

$$\begin{aligned} S &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \\ &= \frac{\pi^2}{8} + \frac{1}{4}\Big(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\Big) \\ &= \frac{\pi^2}{8} + \frac{1}{4} S \end{aligned}$$

Therefore, $\frac{3}{4} S = \frac{\pi^2}{8}$ and $S = \frac{\pi^2}{6}$. We have shown that

$$\zeta(2) = \sum_{j=1}^\infty \frac{1}{j^2} = \frac{\pi^2}{6} \; .$$

Here the Riemann zeta–function is defined by

$$\zeta(z) = \sum_{j=1}^\infty \frac{1}{j^z} \quad \text{for} \quad \mathrm{Re}\, z > 1 \; .$$

Another proof of the partial fraction decomposition (11.8).
Let

$$f(z) = \frac{\pi^2}{\sin^2(\pi z)} \quad \text{and} \quad g(z) = \sum_{j=-\infty}^\infty \frac{1}{(z-j)^2} \quad \text{for} \quad z \in U := \mathbb{C} \setminus \mathbb{Z} \; .$$

Both functions f and g are holomorphic and 1–periodic on U. Also, both functions have a pole of order 2 at each $j \in \mathbb{Z}$ with singular part

$$\frac{1}{(z-j)^2} \ .$$

Therefore, after removing the singularities, the function

$$h(z) = f(z) - g(z)$$

is entire. We will use Liouville's Theorem to show that $h(z)$ is constant. Growth estimates of $f(z)$ and $g(z)$ imply that the constant is zero.

Bounds for $|f(z)|$ and $|g(z)|$. Since the functions are 1–periodic and satisfy

$$f(\bar{z}) = \bar{f}(z), \quad g(\bar{z}) = \bar{g}(z)$$

it suffices to derive bounds in the strip

$$S = \left\{ z = x + iy \ : \ |x| \le \frac{1}{2}, \ y \ge 1 \right\} \ .$$

A bound for $|f(z)|$ in S.

We have

$$2i \sin(\pi z) = e^{i\pi z} - e^{-i\pi z} = e^{i\pi x} e^{-\pi y} - e^{-i\pi x} e^{\pi y} \ ,$$

thus

$$2|\sin(\pi z)| \ge e^{\pi y} - e^{-\pi y} \ge \frac{1}{2}\, e^{\pi y} \quad \text{for} \quad y \ge 1 \ .$$

Therefore,

$$|f(z)| \le 16\pi^2 e^{-2\pi y} \quad \text{for} \quad z \in S \ .$$

A bound for $|g(z)|$ in S.

We have $|z-j|^2 = (j-x)^2 + y^2$ and

$$|j - x| \ge \Big| |j| - |x| \Big| \ge \frac{1}{2}\, |j| \quad \text{for} \quad |x| \le \frac{1}{2} \ .$$

Therefore,

$$|z-j|^2 \ge \frac{1}{4}\, (j^2 + 4y^2)$$

and

$$\begin{aligned} |g(z)| \ &\le \ 4 \sum_{j=-\infty}^{\infty} \frac{1}{j^2 + 4y^2} \\ &= \ \frac{1}{y^2} + 8 \sum_{j=1}^{\infty} \frac{1}{j^2 + 4y^2} \end{aligned}$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$$

it follows that $|g(z)|$ is bounded in S. Also, given $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ so that

$$8 \sum_{j=N+1}^{\infty} \frac{1}{j^2} \le \varepsilon \ .$$

Therefore,

$$\begin{aligned} |g(z)| &\le \frac{1}{y^2} + 8 \sum_{j=1}^{N} \frac{1}{j^2 + 4y^2} + \varepsilon \\ &\le (2N+1)\frac{1}{y^2} + \varepsilon \\ &\le 2\varepsilon \end{aligned}$$

for $y \ge y_\varepsilon$. This shows that $|g(z)|$ is bounded in S and $|g(z)| \to 0$ as $y \to \infty$.

Boundedness of the entire function $h(z) = f(z) - g(z)$ follows. By Liouville's Theorem we have $h(z) \equiv const$. Since $f(z) \to 0$ and $g(z) \to 0$ as $y \to \infty$ the constant is zero. This proves that $f(z) \equiv g(z)$.

11.4 The Partial Fraction Decomposition of $\pi \cot(\pi z)$

We want to show that the partial fraction decomposition of the meromorphic function

$$\pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} \ ,$$

can be obtained by integrating (11.8). First note that for $z \in \mathbb{C} \setminus \mathbb{Z}$:

$$\begin{aligned} \frac{d}{dz} \pi \cot(\pi z) &= -\frac{\pi^2}{\sin^2(\pi z)} \\ \frac{d}{dz}(z-j)^{-1} &= -(z-j)^{-2} \end{aligned}$$

We define

$$t_n(z) := \sum_{j=-n}^{n} (z-j)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

By Theorem 11.3 we have

$$\lim_{n \to \infty} \frac{d}{dz} t_n(z) = \frac{d}{dz} \pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} \ , \tag{11.9}$$

where the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

We have

$$\begin{aligned} t_n(z) &= \sum_{j=-n}^{n} \frac{1}{z-j} \\ &= \frac{1}{z} + \sum_{j=1}^{n} \Big(\frac{1}{z-j} + \frac{1}{z+j} \Big) \\ &= \frac{1}{z} + \sum_{j=1}^{n} \frac{2z}{z^2-j^2} \end{aligned}$$

and

$$t_n(z) \to t(z) := \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2-j^2}$$

normally on $U := \mathbb{C} \setminus \mathbb{Z}$. Therefore,

$$t_n'(z) \to t'(z)$$

normally on U. (This follows, essentially, from Cauchy's inequalities.)

Using (11.9) it follows that

$$\frac{d}{dz}\, \pi \cot(\pi z) = t'(z)$$

on U. Therefore, the function

$$h(z) := \pi \cot(\pi z) - t(z)$$

is constant on U. Take $z = \frac{1}{2}$. We have $\cot(\pi/2) = 0$. Also,

$$\begin{aligned} t_n(1/2) &= \sum_{k=0}^{n} \frac{1}{\frac{1}{2}-k} + \sum_{j=1}^{n} \frac{1}{\frac{1}{2}+j} \\ &= \sum_{k=0}^{n-1} \frac{1}{\frac{1}{2}-k} + \frac{1}{\frac{1}{2}-n} + \sum_{j=1}^{n} \frac{1}{\frac{1}{2}+j} \\ &= \sum_{j=1}^{n} \frac{1}{\frac{1}{2}-j-1} + \frac{1}{\frac{1}{2}-n} + \sum_{j=1}^{n} \frac{1}{\frac{1}{2}+j} \\ &= \frac{1}{\frac{1}{2}-n} \end{aligned}$$

The last equation holds since

$$\frac{1}{\frac{1}{2}-j-1} + \frac{1}{\frac{1}{2}+j} = 0 \; .$$

This shows that $t_n(1/2) \to 0$ as $n \to \infty$, thus $t(1/2) = 0$.

We have shown that

$$\lim_{n\to\infty} \sum_{j=-n}^{n} \frac{1}{z-j} = \pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

In this case, it is not good to write the result as

$$\sum_{j=-\infty}^{\infty} \frac{1}{z-j} = \pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} \ ,$$

since the series

$$\sum_{j=1}^{\infty} \frac{1}{z-j}$$

does not converge for any z. However,

$$\begin{aligned} t_n(z) &= \sum_{j=-n}^{n} \frac{1}{z-j} \\ &= \frac{1}{z} + \sum_{j=1}^{n} \frac{2z}{z^2-j^2} \ . \end{aligned}$$

One obtains:

$$\frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2-j^2} = \pi \cot(\pi z), \quad z \in \mathbb{C} \setminus \mathbb{Z} \ . \tag{11.10}$$

This is the partial fraction decomposition of $\pi \cot(\pi z)$.

11.5 Summary of Examples

Example 1: Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, denote the parameterized unit circle. Then we have

$$\int_\gamma z^2 \sin(1/z)\, dz = -\frac{\pi i}{3} \ .$$

Example 2: Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. We have

$$\int_\gamma \frac{5z-2}{z(z-1)}\, dz = 10\pi i \ .$$

Example 3: Let $\gamma(t) = 2e^{it}$. We have

$$\int_\gamma \frac{\sinh z}{z^4}\, dz = \frac{\pi i}{3} \ .$$

Example 4: For $a > 1$:

$$\int_0^\pi \frac{dt}{a+\cos t} = \frac{\pi}{\sqrt{a^2-1}} \ .$$

Example 5: We know from calculus that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi \ ,$$

which can also be obtained using residues.

Example 6: For $a > 0$:

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2+x^2}\, dx = \frac{\pi}{a}\, e^{-a} \ .$$

Example 7: For $a > 0$:

$$\int_{-\infty}^{\infty} \frac{x \sin x}{a^2+x^2}\, dx = \pi\, e^{-a} \ .$$

(This requires Jordan's lemma.)

Example 8: We have:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \ .$$

The integral exists as an improper Riemann integral.

Example 9: For $0 < a < 1$:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin(\pi a)} \ .$$

Example 10: Let $z \in \mathbb{C} \setminus \mathbb{Z}$. Then we have the partial fraction decomposition:

$$\sum_{j=-\infty}^{\infty} \frac{1}{(z-j)^2} = \frac{\pi^2}{\sin^2(\pi z)} \ .$$

This follows by integrating the function

$$q(\zeta) = \frac{\cot(\pi\zeta)}{(\zeta+z)^2}$$

along a closed rectangle γ_n for $n \to \infty$.

Example 11: For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have:

$$\frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2-j^2} = \pi \cot(\pi z) \ .$$

This partial fraction decomposition can be obtained by integrating the partial fraction decomposition of the previous example.

11.6 Practice Problems

Problem 1: Prove or disprove:

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}\ .$$

Problem 2: Prove or disprove:

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx = 0\ .$$

Problem 3: Prove or disprove:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}\ .$$

Problem 4:

Let $-1 < \alpha < 1$. Prove or disprove:

$$\int_{0}^{\infty} \frac{x^\alpha}{x^2+1}\,dx = \frac{\pi}{2\cos(\pi\alpha/2)}\ .$$

12 The Bernoulli Numbers, the Values $\zeta(2m)$, and the Sums of Powers

Summary: Bernoulli number $B_\nu, \nu = 0,1,2,\ldots$ occur in some interesting formulas. The numbers B_ν can be defined by the Taylor expansion

$$\frac{z}{e^z - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!}\, z^\nu, \quad |z| < 2\pi \ .$$

Using the function $z/(e^z-1)$ and its relation to $\cot z$ we will derive the Taylor expansion

$$\pi z \cot(\pi z) = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{(2\pi)^{2j}}{(2j)!} B_{2j}\, z^{2j}, \quad |z| < 1 \ , \tag{12.1}$$

where the Bernoulli numbers come up.

Using the partial fraction decomposition of $\pi \cot(\pi z)$ derived in Section 11.1 we also have

$$\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2} \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{Z} \ . \tag{12.2}$$

Using equality of the right–hands sides of (12.1) and (12.2) one can derive explicit formulas for the values of the ζ–function at even integers, i.e., for

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}}, \quad m = 1,2,3,\ldots$$

The Bernoulli numbers show up in the value of $\zeta(2m)$.

The Bernoulli numbers also occur in formulas for sums of powers,

$$S_k(n-1) = \sum_{j=1}^{n-1} j^k = 1 + 2^k + 3^k + \ldots + (n-1)^k, \quad k = 1,2,3,\ldots$$

12.1 The Bernoulli Numbers

The function $g(z)$ defined by

$$g(z) = z/(e^z - 1) \quad \text{for} \quad 0 < |z| < 2\pi, \quad g(0) = 1 \ ,$$

is holomorphic in $D(0, 2\pi)$. We write its Taylor series as

$$g(z) = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!}\, z^\nu, \quad |z| < 2\pi \ , \tag{12.3}$$

where the numbers B_ν are, by definition, the Bernoulli numbers. Since

$$\begin{aligned} g(z) &= \frac{1}{1 + \frac{1}{2}z + \frac{1}{6}z^2 + \ldots} \\ &= 1 - \frac{1}{2}z + \ldots \end{aligned}$$

it follows that

$$B_0 = 1, \quad B_1 = -\frac{1}{2} \ .$$

Lemma 12.1 *The function*

$$h(z) = g(z) + \frac{z}{2}$$

is even. Consequently,

$$B_\nu = 0 \quad for \quad \nu \geq 3, \quad \nu \quad odd \ .$$

Proof: We must show that

$$g(-z) - \frac{z}{2} = g(z) + \frac{z}{2} \ ,$$

i.e.,

$$g(-z) - g(z) = z \ .$$

We have

$$\begin{aligned} g(-z) - g(z) &= \frac{-z}{e^{-z}-1} - \frac{z}{e^z-1} \\ &= z\Big(\frac{1}{1-e^{-z}} - \frac{1}{e^z-1}\Big) \\ &= z\Big(\frac{e^z}{e^z-1} - \frac{1}{e^z-1}\Big) \\ &= z \ . \end{aligned}$$

◇

One can compute the Bernoulli numbers using a recursion. First recall the binomial coefficients

$$\binom{n}{\nu} = \frac{n!}{\nu!(n-\nu)!}$$

We claim:

Lemma 12.2 *For $n \geq 1$ we have*

$$\sum_{\nu=0}^{n} \binom{n+1}{\nu} B_\nu = 0 \ .$$

Proof: We have, for $0 < |z| < 2\pi$:

$$\begin{aligned} 1 &= \frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} \\ &= \Big(\sum_{\mu=0}^{\infty} \frac{z^\mu}{(\mu+1)!} \Big) \cdot \Big(\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} \, z^\nu \Big) \\ &= \sum_{\mu,\nu=0}^{\infty} \frac{B_\nu}{\nu!(\mu+1)!} \, z^{\mu+\nu} \quad \text{(with } \mu = n - \nu\text{)} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{\nu=0}^{n} \frac{B_\nu}{\nu!(n+1-\nu)!} \Big) \, z^n \end{aligned}$$

Therefore,

$$\sum_{\nu=0}^{n} \frac{B_\nu}{\nu!(n+1-\nu)!} = 0 \quad \text{for} \quad n \geq 1 \; .$$

Since

$$\left(\begin{array}{c} n+1 \\ \nu \end{array} \right) = \frac{(n+1)!}{\nu!(n+1-\nu)!}$$

the lemma is proved. ⋄

Using Pascal's triangle, we can compute the binomial coefficients. Then, using the previous lemma and $B_0 = 1$ we obtain:

For $n = 1$:

$$B_0 + 2B_1 = 0, \quad \text{thus} \quad B_1 = -\frac{1}{2} \; .$$

For $n = 2$:

$$B_0 + 3B_1 + 3B_2 = 0, \quad \text{thus} \quad B_2 = \frac{1}{6} \; .$$

For $n = 3$:

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \quad \text{thus} \quad B_3 = 0 \; .$$

For $n = 4$:

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0, \quad \text{thus} \quad B_4 = -\frac{1}{30} \; .$$

Continuing this process, one obtains the following non–zero Bernoulli numbers:

$$\begin{aligned} B_6 &= \frac{1}{42} \\ B_8 &= -\frac{1}{30} \\ B_{10} &= \frac{5}{66} \\ B_{12} &= -\frac{691}{2730} \\ B_{14} &= \frac{7}{6} \end{aligned}$$

etc.

Remark: The sequence $|B_{2\nu}|$ is unbounded since otherwise the series (12.3) would have an infinite radius of convergence. More precisely, by Hadamard's formula,

$$\limsup_{\nu\to\infty} \Big(|B_{2\nu}|/(2\nu)! \Big)^{1/(2\nu)} = \frac{1}{2\pi} \ .$$

Also, we will see below that $(-1)^{\nu+1} B_{2\nu} > 0$. Thus, the sign pattern observed for B_2 to B_{14} continuous.

12.2 The Taylor Series of $z \cot z$ in Terms of Bernoulli Numbers

Recall that

$$g(w) = \frac{w}{e^w - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} \, w^\nu \ .$$

We now express the Taylor series for $z \cot z$ about $z = 0$ in terms of Bernoulli numbers. Note that

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\
\cot z &= i \, \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \\
&= i \, \frac{e^{2iz} + 1}{e^{2iz} - 1} \\
&= i \, \frac{e^{2iz} - 1 + 2}{e^{2iz} - 1} \\
&= i \Big(1 + \frac{2}{e^{2iz} - 1} \Big) \quad \text{for} \quad 0 < |z| < \pi \ .
\end{aligned}$$

Therefore,

$$\cot z = i + \frac{1}{z} \cdot \frac{2iz}{e^{2iz} - 1} \quad \text{for} \quad 0 < |z| < \pi \ ,$$

thus, for $|z| < \pi$:

$$\begin{aligned}
z \cot z &= iz + g(2iz) \\
&= iz + 1 - \frac{1}{2}(2iz) + \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!} \, (2iz)^\nu \quad (\text{set } \nu = 2j) \\
&= 1 + \sum_{j=1}^{\infty} (-1)^j \, \frac{2^{2j}}{(2j)!} \, B_{2j} \, z^{2j} \ .
\end{aligned}$$

We substitute πz for z and summarize:

Lemma 12.3 *If B_ν denotes the sequence of the Bernoulli numbers, then we have the Taylor series expansion*

$$\pi z \cot(\pi z) = 1 + \sum_{j=1}^{\infty} (-1)^j \frac{(2\pi)^{2j}}{(2j)!} B_{2j} z^{2j} \quad \textit{for} \quad |z| < 1 \ . \tag{12.4}$$

12.3 The Values $\zeta(2m), m = 1, 2, \ldots$

For $\mathrm{Re}\, s > 1$ the Riemann Zeta–function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ .$$

In Section 11.4 we have shown the following partial fraction decomposition (also called Mittag–Leffler expansion):

$$\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

(See equation (11.10).)

Therefore,

$$\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

Here, for $|z| < 1$:

$$\begin{aligned} \frac{z^2}{n^2 - z^2} &= \frac{(z/n)^2}{1 - (z/n)^2} \\ &= \sum_{m=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} \end{aligned}$$

Therefore,

$$\begin{aligned} \pi z \cot(\pi z) &= 1 - 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} && (12.5) \\ &= 1 - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} && (12.6) \\ &= 1 - 2 \sum_{m=1}^{\infty} \Big(\sum_{n=1}^{\infty} \frac{1}{n^{2m}} \Big) z^{2m} && (12.7) \\ &= 1 - 2 \sum_{m=1}^{\infty} \zeta(2m)\, z^{2m} && (12.8) \end{aligned}$$

Comparing the expressions (12.8) and (12.4), we obtain the following result about the values of the Riemann ζ–function at positive even integers. (This result was already known to Euler in 1734. Nothing similar has ever been derived for the zeta–values at odd integers.)

Theorem 12.1 *For $m = 1, 2, \ldots$ the value of $\zeta(2m)$ is*

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{2}\,(-1)^{m+1}\,\frac{(2\pi)^{2m}}{(2m)!}\,B_{2m} \ . \tag{12.9}$$

Remark: Since, clearly, $\zeta(2m) > 0$ we obtain that $(-1)^{m+1}B_{2m} > 0$.

Examples:

For $m = 1$ we have $B_2 = \frac{1}{6}$, thus

$$\zeta(2) = \frac{(2\pi)^2}{2\cdot 2}\cdot\frac{1}{6} = \frac{\pi^2}{6} \ .$$

For $m = 2$ we have $B_4 = -\frac{1}{30}$, thus

$$\zeta(4) = \frac{(2\pi)^4}{2\cdot 4!}\cdot\frac{1}{30} = \frac{\pi^4}{90} \ .$$

For $m = 3$ we have $B_6 = \frac{1}{42}$, thus

$$\zeta(6) = \frac{(2\pi)^6}{2\cdot 6!}\cdot\frac{1}{42} = \frac{\pi^6}{945} \ .$$

For $m = 4$ one obtains

$$\zeta(8) = \frac{\pi^8}{9450} \ .$$

12.4 Sums of Powers and Bernoulli Numbers

It is not difficult to show the following formulae by induction in n:

$$\begin{aligned}
S_1(n-1) \equiv \sum_{j=1}^{n-1} j &= \frac{1}{2}\,n^2 - \frac{1}{2}\,n \\
S_2(n-1) \equiv \sum_{j=1}^{n-1} j^2 &= \frac{1}{3}\,n^3 - \frac{1}{2}\,n^2 + \frac{1}{6}\,n \\
S_3(n-1) \equiv \sum_{j=1}^{n-1} j^3 &= \frac{1}{4}\,n^4 - \frac{1}{2}\,n^3 + \frac{1}{4}\,n^2 + 0\,n
\end{aligned}$$

Recalling that

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0 \ ,$$

we notice that the three formulae have the pattern:

$$\sum_{j=1}^{n-1} j^k = \frac{1}{k+1}\,n^{k+1} - \frac{1}{2}\,n^k + \ldots + B_k\,n \ ,$$

but it is not obvious how the general formula should read.

Define the sum

$$S_k(n-1) = \sum_{j=1}^{n-1} j^k$$

where $k = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$. We claim that, for every fixed integer $k \geq 1$, the sum $S_k(n-1)$ is a polynomial

$$\Phi_k(n)$$

of degree $k+1$ in the variable n and that the coefficients of $\Phi_k(n)$ can be obtained in terms of Bernoulli numbers. Precisely:

Theorem 12.2 *For every integer $k \geq 1$, let $\Phi_k(n)$ denote the polynomial of degree $k+1$ given by*

$$\Phi_k(n) = \frac{1}{k+1} \sum_{\mu=0}^{k} \binom{k+1}{\mu} B_\mu n^{k+1-\mu} \ .$$

Then we have

$$S_k(n-1) = \Phi_k(n) \quad \textit{for all} \quad n = 1, 2, \cdots \ .$$

Remark: Writing out a few terms of $\Phi_k(n)$, the theorem says that

$$S_k(n-1) = \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k + \frac{1}{k+1} \binom{k+1}{2} B_2 n^{k-1} + \cdots + B_k n \ .$$

Proof of Theorem: The trick is to write the finite geometric sum

$$E_n(w) = 1 + e^w + e^{2w} + \cdots + e^{(n-1)w}$$

in two ways and then to compare coefficients. We have

$$\begin{aligned} E_n(w) &= \sum_{j=0}^{n-1} e^{jw} \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{j^k}{k!} w^k \\ &= \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{n-1} j^k \Big) \frac{1}{k!} w^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} S_k(n-1) \, w^k \end{aligned}$$

Here we have used the convention $0^0 = 1$.

On the other hand, we have

$$
\begin{aligned}
E_n(w) &= \frac{e^{nw}-1}{e^w-1} \\
&= \frac{w}{e^w-1}\cdot\frac{e^{nw}-1}{w} \\
&= \Big(\sum_{\mu=0}^{\infty}\frac{B_\mu}{\mu!}\,w^\mu\Big)\cdot\Big(\sum_{\lambda=0}^{\infty}\frac{n^{\lambda+1}}{(\lambda+1)!}\,w^\lambda\Big) \\
&= \sum_{k=0}^{\infty}\Big(\sum_{\mu+\lambda=k}\frac{B_\mu}{\mu!(\lambda+1)!}\,n^{\lambda+1}\Big)\,w^k
\end{aligned}
$$

Comparison yields that

$$
\begin{aligned}
S_k(n-1) &= \sum_{\mu+\lambda=k}\frac{k!}{\mu!(\lambda+1)!}\,B_\mu\,n^{\lambda+1} \quad (\text{with } \lambda = k-\mu) \\
&= \frac{1}{k+1}\sum_{\mu=0}^{k}\frac{(k+1)!}{\mu!(k+1-\mu)!}\,B_\mu\,n^{k+1-\mu}
\end{aligned}
$$

This proves the claim since

$$
\binom{k+1}{\mu} = \frac{(k+1)!}{\mu!(k+1-\mu)!}\ .
$$

◇

13 Properties of the Γ–Function

13.1 Extension of the Domain of Definition of $\Gamma(z)$ Using the Functional Equation

The Γ–function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad \text{for} \quad \operatorname{Re} z > 0 \; .$$

It is a holomorphic function in the half–plan $\operatorname{Re} z > 0$ satisfying the functional equation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for} \quad \operatorname{Re} z > 0 \; .$$

The functional equation follows through integration by parts,

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty t^z e^{-t}\, dt \\ &= -t^z e^{-t}\Big|_0^\infty + z \int_0^\infty t^{z-1} e^{-t}\, dt \\ &= z\Gamma(z) \end{aligned}$$

Therefore,

$$\Gamma(z) = \frac{1}{z}\,\Gamma(z+1) \quad \text{for} \quad \operatorname{Re} z > 0 \; .$$

The right–hand side is defined for

$$\operatorname{Re} z > -1, \quad z \neq 0 \; .$$

If one sets

$$r_0(z) := \frac{1}{z}\,\Gamma(z+1) \quad \text{for} \quad \operatorname{Re} z > -1, \quad z \neq 0 \; ,$$

then one obtains a holomorphic function $r_0(z)$ defined in the region

$$\{z \;:\; \operatorname{Re} z > -1,\ z \neq 0\} \; .$$

The function $r_0(z)$ agrees with the Γ–function for $\operatorname{Re} z > 0$. The identity theorem yields that $r_0(z)$ satisfies the functional equation

$$r_0(z+1) = z r_0(z) \quad \text{for} \quad \operatorname{Re} z > -1, \quad z \neq 0 \; .$$

One extends the domain of definition of Γ by setting

$$\Gamma(z) = r_0(z) \quad \text{for} \quad \operatorname{Re} z > -1, \quad z \neq 0 \; .$$

The process can be repeated: Set

$$\Gamma(z) = \frac{1}{z}\,\Gamma(z+1) \quad \text{for} \quad -2 < \operatorname{Re} z \le -1, \quad z \notin \{0, -1\} \; .$$

etc.

Repeating the process, one obtains a holomorphic function $\Gamma(z)$ in the region

$$\mathbb{C} \setminus \{0, -1, -2, \ldots\} \; .$$

The extended Γ–function satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z) \quad \text{for} \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \; .$$

Since $\Gamma(x)$ is real for $x > 0$, the extended Γ–function is real for

$$x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \; .$$

Also, since $\Gamma(x) > 0$ for every $x > 0$, it follows that

$$\begin{aligned} \Gamma(x) < 0 \quad &\text{for} \quad -1 < x < 0 \; , \\ \Gamma(x) > 0 \quad &\text{for} \quad -2 < x < -1 \; , \end{aligned}$$

etc. The function $\Gamma(x), x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ is sketched in Figure 13.1.

13.2 Extension of the Domain of Definition of $\Gamma(z)$ Using Series Expansion

One can extend the definition of Γ also as follows. First, assume again that $\mathrm{Re}\, z > 0$ and write

$$\begin{aligned} \Gamma(z) &= \int_0^1 t^{z-1} e^{-t}\, dt + \int_1^\infty t^{z-1} e^{-t}\, dt \\ &=: g(z) + h(z) \end{aligned}$$

It is easy to show that the formula

$$h(z) = \int_1^\infty t^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C} \; ,$$

defines an entire function. (Use that $t^{z-1} = e^{(\ln t)(z-1)}$ and apply Cauchy's theorem and Morera's theorem. See Section 8.4.)

In the formula defining $g(z)$ we write out the exponential series and interchange summation and integration. Thus, for $\mathrm{Re}\, z > 0$:

$$\begin{aligned} g(z) &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} \int_0^1 t^{z-1} t^j\, dt \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} \frac{1}{z+j} \end{aligned}$$

The infinite series converges for every

$$z \in U := \mathbb{C} \setminus \{0, -1, -2, \ldots\} \; .$$

The convergence of

$$g_n(z) = \sum_{j=0}^n \frac{(-1)^j}{j!} \frac{1}{z+j}$$

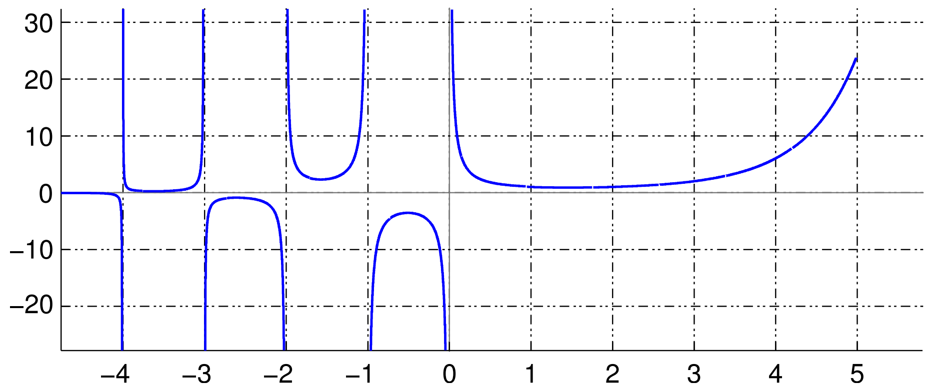

Figure 13.1: Gamma function on the real axis

to $g(z)$ is normal in U. Thus, $g \in H(U)$.

To summarize, the formula

$$\Gamma(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} + \int_1^{\infty} t^{z-1} e^{-t}\, dt$$

defines $\Gamma(z)$ as a holomorphic function in $U = \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Poles of Γ: For every $k = 0, 1, 2, \ldots$ we have

$$g(z) = \frac{(-1)^k}{k!} \frac{1}{z+k} + \sum_{j=0,\, j \neq k}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z+j} \ .$$

Here the infinite sum is holomorphic in a neighborhood of $z = k$. It follows that $\Gamma(z)$ has a simple pole at every number $z_k = -k$ for $k = 0, 1, 2, \ldots$ Also,

$$Res(\Gamma, -k) = \frac{(-1)^k}{k!} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

13.3 The Reflection Formula

Let $0 < a < 1$. We have shown in Chapter 11, Example 9 that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx = \frac{\pi}{\sin(\pi a)} \ .$$

We will use this to prove the so–called reflection formula for the Γ–function:

Theorem 13.1 *We have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \textit{for all} \quad z \in \mathbb{C} \setminus \mathbb{Z} \ .$$

Proof: Using the identity theorem for holomorphic functions, it sufficies to prove that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{for} \quad 0 < s < 1 \ .$$

We have

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$$

and

$$\begin{aligned}
\Gamma(1-s) &= \int_0^\infty t^{-s} e^{-t}\, dt \quad \text{(rename } t = u\text{)} \\
&= \int_0^\infty u^{-s} e^{-u}\, du \quad \text{(substitute } u = tv \text{ for fixed } t > 0\text{)} \\
&= t \int_0^\infty (tv)^{-s} e^{-tv}\, dv \quad \text{for} \quad t > 0 \ .
\end{aligned}$$

Obtain that

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty t^{s-1} e^{-t} \Gamma(1-s)\, dt \\
&= \int_0^\infty t^{s-1} e^{-t} \Big(t \int_0^\infty (tv)^{-s} e^{-tv}\, dv \Big)\, dt \\
&= \int_0^\infty \int_0^\infty v^{-s} e^{-(1+v)t}\, dv\, dt \\
&= \int_0^\infty v^{-s} \int_0^\infty e^{-(1+v)t}\, dt\, dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\, dv \\
&= \int_0^\infty \frac{v^{1-s}}{1+v} \frac{dv}{v} \quad \text{(substitute } v = e^x,\ \frac{dv}{v} = dx\text{)} \\
&= \int_{-\infty}^\infty \frac{e^{(1-s)x}}{1+e^x}\, dx \\
&= \frac{\pi}{\sin(\pi(1-s))} \\
&= \frac{\pi}{\sin(\pi s)}
\end{aligned}$$

This proves the reflection formula for $0 < s < 1$. $\diamond$

13.4 Special Values of $\Gamma(z)$

We have

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 \ .$$

Using the functional equation:

$$\begin{aligned}
\Gamma(1+1) &= 1 \cdot \Gamma(1) = 1 \\
\Gamma(2+1) &= 2 \cdot \Gamma(2) = 2 \\
\Gamma(3+1) &= 3 \cdot \Gamma(3) = 2 \cdot 3
\end{aligned}$$

etc. In general,

$$\Gamma(n+1) = n!, \quad n \in \mathbb{Z}_+ \ .$$

From the reflection formula one obtains that

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \ .$$

Then one can use the functional equation to compute $\Gamma(n+\frac{1}{2})$ for every $n \in \mathbb{N}$:

$$\begin{aligned}
\Gamma(\frac{1}{2}+1) &= \frac{1}{2}\cdot\Gamma(\frac{1}{2}) = \frac{1}{2}\cdot\sqrt{\pi} \\
\Gamma(\frac{3}{2}+1) &= \frac{3}{2}\cdot\Gamma(\frac{3}{2}) = \frac{1\cdot 3}{2\cdot 2}\cdot\sqrt{\pi} \\
\Gamma(\frac{5}{2}+1) &= \frac{5}{2}\cdot\Gamma(\frac{5}{2}) = \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\cdot\sqrt{\pi}
\end{aligned}$$

In general, for all $n \in \mathbb{Z}_+$:

$$\begin{aligned}
\Gamma(n+\frac{1}{2}) &= \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n}\cdot\sqrt{\pi} \\
&= \frac{1\cdot(2\cdot 1)\cdot 3\cdot(2\cdot 2)\cdot 5\cdot\ldots\cdot(2n-1)\cdot(2\cdot n)}{2^{2n}n!}\cdot\sqrt{\pi} \\
&= \frac{(2n)!}{4^n\, n!}\sqrt{\pi}
\end{aligned}$$

13.5 Applications

The Γ–function is used in many formulas.

Example 1: Using the substitution $x^2 = t, dt = 2xdx$, one obtains:

$$\begin{aligned}
\int_0^\infty x^{2n}e^{-x^2}\,dx &= \frac{1}{2}\int_0^\infty x^{2n-1}\,e^{-x^2}\,2xdx \\
&= \frac{1}{2}\int_0^\infty t^{n+\frac{1}{2}-1}\,e^{-t}\,dt \\
&= \frac{1}{2}\,\Gamma(n+\frac{1}{2})
\end{aligned}$$

Recall that the integral

$$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{2}\,\Gamma(1+\frac{1}{2}) = \frac{\sqrt{\pi}}{4}$$

appears in the error term of Stirling's formula.

Example 2: Using the substitution

$$t = -\ln x \quad \text{for} \quad 0 < x \le 1, \quad e^{-t} = x, \quad dx = -e^{-t}dt \ ,$$

one obtains for $\operatorname{Re} z > -1$:

$$
\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\, dt \\
&= -\int_0^\infty t^z(-e^{-t})\, dt \\
&= \int_0^1 (-\ln x)^z\, dx
\end{aligned}
$$

In particular, for $n = 0, 1, 2, \ldots$

$$\int_0^1 (-\ln x)^n\, dx = \Gamma(n+1) = n!\ .$$

13.6 The Function $\Delta(z) = 1/\Gamma(z)$

We know that $\Gamma(n) > 0$ for all $n \in \mathbb{N}$. Also, the reflection formula implies that $\Gamma(z) \neq 0$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Therefore,

$$\Gamma(z) \neq 0 \quad \text{for all} \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}\ .$$

Since Γ has a (simple) pole at every $k = 0, -1, -2, \ldots$ one can use Riemann's theorem on removable singularities (Theorem 10.1) to show that the function $\Delta(z)$ defined by

$$
\begin{aligned}
\Delta(z) &= 1/\Gamma(z) \quad \text{for} \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \\
\Delta(z) &= 0 \quad \text{for} \quad z \in \{0, -1, -2, \ldots\}
\end{aligned}
$$

is entire. Weierstrass based his theory of the Γ–function on the investigation of $\Delta(z)$.

13.7 Log–Convexity of $\Gamma(x)$

We know that

$$\Gamma : (0, \infty) \to (0, \infty)$$

is a C^∞–function.

Theorem 13.2 *For all $x > 0$:*

$$\frac{d^2}{dx^2} \ln \Gamma(x) > 0\ .$$

Proof: If $\phi(x) := \ln \Gamma(x)$ then

$$\phi' = \frac{\Gamma'}{\Gamma}, \quad \phi'' = \frac{\Gamma''\Gamma - \Gamma'^2}{\Gamma^2}\ .$$

We must show that

$$\Gamma''(x)\Gamma(x) > \Gamma'^2(x) \quad \text{for} \quad x > 0\ . \tag{13.1}$$

We have

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

with

$$t^x = e^{x \ln t} \quad \text{for} \quad t > 0 \ .$$

Since

$$\begin{aligned} \frac{d}{dx}\, t^x &= (\ln t) e^{x \ln t} \\ \frac{d^2}{dx^2}\, t^x &= (\ln t)^2 e^{x \ln t} \end{aligned}$$

we obtain:

$$\begin{aligned} \Gamma'(x) &= \int_0^\infty (\ln t) t^{x-1} e^{-t}\, dt \\ \Gamma''(x) &= \int_0^\infty (\ln t)^2 t^{x-1} e^{-t}\, dt \end{aligned}$$

For fixed $0 < x < \infty$ define the quadratic

$$g(u) = u^2 \Gamma(x) + 2u\Gamma'(x) + \Gamma''(x), \quad u \in \mathbb{R} \ .$$

The above expressions for $\Gamma(x)$ and its derivatives yield:

$$g(u) = \int_0^\infty \Big\{ u^2 + 2u \ln t + (\ln t)^2 \Big\} t^{x-1} e^{-t}\, dt$$

Here

$$u^2 + 2u \ln t + (\ln t)^2 = (u + \ln t)^2 > 0 \quad \text{for} \quad u \neq \ln t \ .$$

This implies that

$$g(u) > 0 \quad \text{for all} \quad u \in \mathbb{R} \ .$$

Since

$$g'(u) = 2u\Gamma(x) + 2\Gamma'(x)$$

the function $g(u)$ attains its minimum at

$$u_0 = -\Gamma'(x)/\Gamma(x) \ .$$

Evaluating $g(u)$ at $u = u_0$ one obtains:

$$\begin{aligned} \min_u g(u) &= g(u_0) \\ &= -\frac{\Gamma'^2(x)}{\Gamma(x)} + \Gamma''(x) \end{aligned}$$

Since

$$\min g(u) > 0$$

we have shown (13.1), and the theorem is proved. $\diamond$

13.8 Summary

The formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

defines $\Gamma(z)$ for $\mathrm{Re}\, z > 0$ as an analytic function. We have $z\Gamma(z) = \Gamma(z+1)$ and $\Gamma(n+1) = n!$ for $n = 0, 1, 2 \ldots$ Using the formula

$$\Gamma(z) = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \frac{1}{z+j} + \int_1^\infty t^{z-1} e^{-t} \, dt$$

one obtains the analytic continuation of $\Gamma(z)$ in

$$U := \mathbb{C} \setminus \{0, -1, -2, \ldots\} \; .$$

The function $\Gamma \in H(U)$ has a simple pole at $-k$ for $k = 0, 1, 2, \ldots$ and $Res(\Gamma, -k) = \frac{(-1)^k}{k!}$.

The reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \; ,$$

holds. It implies that $\Gamma(1/2) = \sqrt{\pi}$ and that $\Gamma(z) \neq 0$ for all $z \in U$. The function $\Delta(z) = 1/\Gamma(z)$ is entire.

For real x, $x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, the value $\Gamma(x)$ is real. We have

$$(d/dx)^2 \ln \Gamma(x) > 0 \quad \text{for} \quad x > 0$$

and

$$\Gamma(x) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left(1 + \mathcal{O}(x^{-1})\right) \quad \text{as} \quad x \to \infty \; ,$$

which is Stirling's formula proved in Section 8.6.

14 Log Functions

Summary: The main branch of the log–function is defined on $U := \mathbb{C} \setminus (-\infty, 0]$ by

$$\log z = \ln|z| + i\theta \quad \text{where} \quad z = |z|e^{i\theta}, \quad -\pi < \theta < \pi \ .$$

For many other simply connected regions one can also use polar coordinates. Another possibility is to invert the exponential function or to integrate the function $1/w$.

14.1 The Main Branch of $\log(z)$: Use of Polar Coordinates

Let $U = \mathbb{C} \setminus (-\infty, 0]$. The main branch of the complex logarithm can be introduced as a function defined on U as follows: Take any $z \in U$ and write

$$z = re^{i\theta} = e^{\ln r + i\theta} \quad \text{where} \quad r = |z| > 0 \quad \text{and} \quad -\pi < \theta < \pi \ .$$

The real numbers $r > 0$ and $\theta \in (-\pi, \pi)$ are uniquely determined. Then we have

$$\log z = \ln r + i\theta \ .$$

If $z = x + iy$ then $r = (x^2 + y^2)^{1/2}$ and

$$\theta = \arctan(y/x) \ .$$

Here one must choose the correct branch of the arctan–function and must be careful when $x = 0$. One obtains

$$\log(x + iy) = \ln\Big((x^2 + y^2)^{1/2}\Big) + i \arctan(y/x) \ .$$

With some effort (in particular for $x = 0$) one can use the Cauchy–Riemann equations to prove that the function $\log(x + iy)$ is holomorphic on U.

From the point of view of complex variables, there is a better way to introduce $\log z, z \in U$, namely as the inverse of e^w. We will do this below. To construct $\log z$ we will use the formula

$$\log z = \int_{\Gamma_z} \frac{dw}{w}, \quad z \in U \ ,$$

where Γ_z is a curve in U from $z_0 = 1$ to z.

14.2 Auxiliary Results

Recall the following (see Theorems 4.8 and 4.9):

Theorem 14.1 *Let $U \subset \mathbb{C}$ be open and simply connected and let $g \in H(U)$. Fix $z_0 \in U$ and, for every $z \in U$ choose a curve Γ_z in U from z_0 to z. Then:*

1. The function

$$f(z) = \int_{\Gamma_z} g(w)\, dw, \quad z \in U \ ,$$

is well–defined, i.e., it does not depend on the particular choice of Γ_z.

2. We have $f \in H(U)$ and $f'(z) = g(z), z \in U$.

3. $f(z_0) = 0$.

Lemma 14.1 *Let $U \subset \mathbb{C}$ be open and connected and let $g \in H(U)$. Assume that $g'(z) = 0$ for all $z \in U$. Then $g(z)$ is constant in U.*

Proof: Fix $z_0 \in U$ and let $z \in U$ be arbitrary. Choose a curve Γ_z in U from z_0 to z. We have

$$g(z) - g(z_0) = \int_{\Gamma_z} g'(w)\,dw = 0 \ ,$$

thus $g(z) = g(z_0)$. $\diamond$

14.3 The Main Branch of the Complex Logarithm: Inversion of $w \to e^w$

Theorem 14.2 *Let $U = \mathbb{C} \setminus (-\infty, 0]$. There is a unique function $L \in H(U)$ with the following two properties:*

1. *$L(1) = 0$;*
2. *$e^{L(z)} = z$ for all $z \in U$.*

This function $L(z)$ is denoted by

$$L(z) = \log z, \quad z \in U \ ,$$

and is called the main branch of the complex logarithm. The function $L(z) = \log z$ satisfies $L'(z) = 1/z, z \in U$, and we have

$$L(x) = \ln x := \int_1^x \frac{ds}{s} \quad \textit{for} \quad 0 < x < \infty \ .$$

Proof: Let Γ_z denote a curve in U from $z_0 = 1$ to $z \in U$.

Uniqueness of L: Suppose $L \in H(U)$ has the properties 1. and 2. We have

$$e^{L(z)} L'(z) = 1 \quad \text{for} \quad z \in U \ ,$$

thus

$$\begin{aligned} L'(z) &= e^{-L(z)} \\ &= \frac{1}{z} \end{aligned}$$

in U. Therefore,

$$\begin{aligned} L(z) &= L(z) - L(1) \\ &= \int_{\Gamma_z} \frac{dw}{w} \end{aligned}$$

The value of the integral does not depend on the curve Γ_z in U from $z_0 = 1$ to z since the region U is simply connected.

Existence of L: Define

$$L(z) = \int_{\Gamma_z} \frac{dw}{w}, \quad z \in U \ .$$

We then have $L(1) = 0$ and $L'(z) = \frac{1}{z}, z \in U$. Therefore,

$$\frac{d}{dz}\Big(ze^{-L(z)}\Big) = e^{-L(z)} - ze^{-L(z)}L'(z) = 0 \ .$$

This shows that

$$ze^{-L(z)} = const$$

At $z_0 = 1$ we obtain

$$const = 1\, e^0 = 1 \ ,$$

thus $e^{L(z)} = z$.

For $0 < x < \infty$ we have

$$L(x) = \int_1^x \frac{dw}{w} = \ln x \ .$$

$\diamond$

Lemma 14.2 *We have*

$$\log(1+z) = \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j}\, z^j \quad for \quad |z| < 1 \ .$$

Proof: The derivative of the left–hand side is

$$l'(z) = \frac{1}{1+z}, \quad |z| < 1 \ .$$

The derivative of the right–hand side is

$$\begin{aligned} r'(z) &= \sum_{j=1}^\infty (-1)^{j-1}\, z^{j-1} \\ &= \sum_{k=0}^\infty (-z)^k \\ &= \frac{1}{1+z}, \quad |z| < 1 \end{aligned}$$

It follows that $l(z) - r(z)$ is constant. Also,

$$l(0) - r(0) = \log(1) - 0 = 0 \ ,$$

thus $r(z) \equiv l(z)$ for $|z| < 1$. $\diamond$

Discontinuity along the negative real axis: Fix $-\infty < x < 0$. We have

$$x = |x| e^{i\pi} = |x| e^{-i\pi} \ .$$

Consider the circle γ of radius $r = |x|$ with parameterization

$$z(\theta) = re^{i\theta}, \quad -\pi \le \theta \le \pi \ .$$

On the circle γ consider the points

$$z_\varepsilon = re^{i(\pi-\varepsilon)} \quad \text{and} \quad w_\varepsilon = re^{i(-\pi+\varepsilon)} \quad \text{for} \quad 0 < \varepsilon << 1 \ .$$

As $\varepsilon \to 0+$ we have

$$z_\varepsilon \to re^{i\pi} = -r = x$$

and

$$w_\varepsilon \to re^{-i\pi} = -r = x \ .$$

As $\varepsilon \to 0+$ the points z_ε and w_ε both converge to x. However,

$$\begin{aligned} \log z_\varepsilon &= \ln r + i(\pi - \varepsilon) \to \ln r + i\pi \\ \log w_\varepsilon &= \ln r + i(-\pi - \varepsilon) \to \ln r - i\pi \end{aligned}$$

Since $z_\varepsilon \to x$ and $w_\varepsilon \to x$, but $\log z_\varepsilon$ and $\log w_\varepsilon$ have different limits, one cannot extend the function $\log z$ defined on $U = \mathbb{C} \setminus (-\infty, 0]$ continuously to the axis $-\infty < x < 0$. Also, since $\ln r \to -\infty$ as $r \to 0+$ the function $\log z$ is singular at $z = 0$.

14.4 Complex Logarithms in Other Simply Connected Regions

Theorem 14.3 *Let $V \subset \mathbb{C}$ be open and simply connected. Assume that $0 \notin V$. Fix $z_0 \in V$ and write*

$$z_0 = r_0 e^{i\theta_0} \quad \textit{where} \quad r_0 > 0 \quad \textit{and} \quad \theta \in \mathbb{R} \ .$$

Then there is a unique function $L \in H(V)$ with

1. *$L(z_0) = \ln(r_0) + i\theta_0$;*
2. *$e^{L(z)} = z$ for all $z \in V$.*

This function $L(z)$ satisfies $L'(z) = 1/z, z \in V$.

Proof: Let Γ_z denote a curve in V from z_0 to $z \in V$.

Uniqueness of L. Suppose $L \in H(U)$ satisfies the conditions 1. and 2. We have

$$L'(z)e^{L(z)} = 1,$$

thus

$$\begin{aligned} L'(z) &= e^{-L(z)} \\ &= \frac{1}{z} \end{aligned}$$

in V. Therefore,

$$L(z) - L(z_0) = \int_{\Gamma_z} \frac{dw}{w} \ .$$

This shows that $L(z)$ is uniquely determined.

Existence of L. Define

$$L(z) = \ln(r_0) + i\theta_0 + \int_{\Gamma_z} \frac{dw}{w}, \quad z \in V \ .$$

We then have $L(z_0) = \ln(r_0) + i\theta_0$ and $L'(z) = \frac{1}{z}, z \in V$. Therefore,

$$\frac{d}{dz}\Big(ze^{-L(z)}\Big) = e^{-L(z)} - zL'(z)e^{-L(z)} = 0 \ .$$

This shows that

$$ze^{-L(z)} = const$$

At $z = z_0$ we have

$$e^{L(z_0)} = r_0 e^{i\theta} = z_0 \ ,$$

thus

$$const = z_0 e^{-L(z_0)} = 1 \ .$$

This proves that

$$e^{L(z)} = z, \quad z \in V \ .$$

$\diamond$

We call the function $L(z)$ the logarithm in V with normalization $L(z_0) = \ln(r_0) + i\theta_0$. If we drop the dependency on the normalization in our notation, we write

$$L(z) = \log_V(z), \quad z \in V \ .$$

In particular, we have shown the existence statement of the following theorem:

Theorem 14.4 *Let $V \subset \mathbb{C}$ be open and simply connected. Assume that $0 \notin V$. Then there exists a function $L \in H(V)$ with $e^{L(z)} = z$ for all $z \in V$. Any such function satisfies $L'(z) = 1/z$ in V.*

If $L_1, L_2 \in H(V)$ satisfy $e^{L_1(z)} = e^{L_2(z)}$ for all $z \in V$, then there exists $n \in \mathbb{Z}$ with

$$L_1(z) = L_2(z) + 2\pi i n, \quad z \in V \ . \tag{14.1}$$

Proof: We only have to show (14.1). We know that $e^w = 1$ holds if and only if $w = 2\pi i n$ for some $n \in \mathbb{Z}$. Therefore,

$$e^{L_1(z) - L_2(z)} = 1$$

implies that $L_1(z) - L_2(z) = 2\pi i n(z), n(z) \in \mathbb{Z}$. However, since $n(z) \in H(V)$, the function $n(z)$ is constant. $\diamond$

Definition: If $V \subset \mathbb{C}$ is an open set and if $L \in H(V)$ then we call L a logarithm on V if $e^{L(z)} = z$ for all $z \in V$.

Using the above terminology, Theorem 14.4 says that a logarithm exists on V if V is simply connected and $0 \notin V$. Furthermore, any two logarithms on V differ by an integer multiple of $2\pi i$.

14.5 Argument Functions

Let $V \subset \mathbb{C}$ be open and simply connected and assume that $0 \notin V$. Let $L \in H(V)$ denote a logarithm on V and write

$$L(z) = L_R(z) + iL_I(z)$$

with real functions L_R and L_I. We have

$$z = e^{L_R(z)} e^{iL_I(z)},$$

thus

$$|z| = e^{L_R(z)}, \quad L_R(z) = \ln|z| \ .$$

Definition: Let $V \subset \mathbb{C}$ denote an open set. A C^∞–function

$$\arg : V \to \mathbb{R}$$

is called an argument function on V if

$$z = e^{\ln|z| + i \arg(z)} \quad \text{for all} \quad z \in V \ .$$

Our results say that an argument function exists on V if V is simply connected and $0 \notin V$. In fact, $\arg(z) = \operatorname{Im} L(z)$ is an argument function on V if L is a logarithm on V. Furthermore, any two argument functions on V differ by an integer multiple of $2\pi i$.

15 Extensions of Cauchy's Theorem in a Disk

Summary: Let $\subset \mathbb{C}$ be a region and let $f \in H(U)$. If γ_0 is a curve in U which can be smoothly deformed into the curve γ_1, without leaving U and without changing endpoints, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz \ .$$

We give a formal proof.

15.1 Homotopic Curves

In the following, let $U \subset \mathbb{C}$ be a region, i.e., U is open and connected.

Let $\gamma_0(t), \gamma_1(t), a \le t \le b$, denote parameterizations of two curves in U with

$$\gamma_0(a) = \gamma_1(a) = P, \quad \gamma_0(b) = \gamma_1(b) = Q \ .$$

Thus, γ_0 and γ_1 have the same starting point, P, and the same endpoint, Q.

Definition: The curve γ_0 is homotopic to the curve γ_1 in U (with fixed end points), if there exists a continuous function

$$\gamma : [0,1] \times [a,b] \to U$$

with the following properties:

1) For $a \le t \le b$:

$$\gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t) \ .$$

2) For $0 \le s \le 1$:

$$\gamma(s,a) = P, \quad \gamma(s,b) = Q \ .$$

3) For every parameter $s \in [0,1]$ the function

$$t \to \gamma(s,t), \quad a \le t \le b \ ,$$

is continuous and piecewise C^1.

Terminology: The function $\gamma(s,t)$ is called a homotopy (with fixed end points). The parameter s is called the homotopy parameter and t is called the curve parameter. Intuitively, γ describes a continuous deformation of the curve γ_0 into γ_1.

We will only consider homotopies with *fixed end points.* Therefore we will drop the term.

15.2 Cauchy's Theorem

Theorem 15.1 *Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. If γ_0 and γ_1 are two curves in U which are homotopic in U then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz \ .$$

Proof: a) The set

$$K = \gamma\Big([0,1] \times [a,b]\Big)$$

is a compact subset of U. Assume that $U^c = \mathbb{C} \setminus U$ is not empty. (Otherwise, the following will be trivial.) Let

$$\varepsilon := dist(K, U^c) = inf\{|k - z| \ : \ k \in K, z \in U^c\}$$

denote the distance between K and U^c. Since K is compact and U^c is closed and $K \cap U^c = \emptyset$, it follows that $\varepsilon > 0$. (Proof of this statement: If $\varepsilon = 0$ then, for every $n \in \mathbb{N}$ there is $k_n \in K$ and $z_n \in U$ with

$$|k_n - z_n| < \frac{1}{n} \ .$$

For a subsequence, $k_n \to k$ and $z_n \to z$. Since $k \in K$ and $z \in U^c$ and $|k - z| = 0$, one obtains a contradiction to $K \cap U^c = \emptyset$.)

It follows that

$$D(\gamma(s,t), \varepsilon) \subset U$$

for all $(s,t) \in [0,1] \times [a,b]$.

b) Since γ is uniformly continuous, there exists $\delta > 0$ with

$$|s - s'| + |t - t'| < \delta \quad \Rightarrow \quad |\gamma(s,t) - \gamma(s',t')| < \varepsilon \ .$$

c) Choose $N \in \mathbb{N}$ so large that

$$\frac{1}{N} + \frac{b-a}{N} < \delta \ .$$

Define a grid in

$$\mathcal{Q} = [0,1] \times [a,b]$$

by

$$s_j = \frac{j}{N}, \quad t_k = a + (b-a)\frac{k}{N}, \quad 0 \le j, k \le N \ .$$

The rectangle $\mathcal{Q}$ is partitioned into the sub-rectangles

$$\mathcal{Q}_{jk} = [s_j, s_{j+1}] \times [t_k, t_{k+1}] \ .$$

If (s,t) and (s',t') are two points in $\mathcal{Q}_{jk}$, then

$$|s - s'| + |t - t'| < \delta \ .$$

Therefore,

$$\gamma(\mathcal{Q}_{jk}) \subset D(\gamma(s_j, t_k), \varepsilon) \subset U \ .$$

d) Set

$$\gamma_{s_j}(t) = \gamma(s_j, t), \quad a \le t \le b \ .$$

We claim that

$$\int_{\gamma_{s_j}} f\,dz = \int_{\gamma_{s_{j+1}}} f\,dz\ .$$

To show this, we apply Cauchy's integral theorem in the disks

$$D(\gamma(s_j,t_k),\varepsilon)$$

successively for $k = 0, 1, \ldots, N-1$ to deform the curve γ_{s_j} into $\gamma_{s_{j+1}}$. Since the deformation takes place in disks that lie in U, the integral does not change. $\diamond$

Definition: Let U be a region in $\mathbb{C}$ and let $\gamma_0(t), a \le t \le b$, be a closed curve in U. Then γ_0 is called **null–homotopic in** U if γ_0 is homotopic in U (with fixed endpoints) to the constant curve $\gamma_1(t)$ defined by

$$\gamma_1(t) \equiv \gamma_0(a) = \gamma_0(b), \quad a \le t \le b\ .$$

The following three theorems are different versions of Cauchy's Theorem.

Theorem 15.2 *Let U be a region in $\mathbb{C}$ and let γ be null–homotopic in U. If $f \in H(U)$ then*

$$\int_\gamma f(z)\,dz = 0\ .$$

Definition: A region U in $\mathbb{C}$ is called simply connected if every closed curve in U is null–homotopic in U.

Theorem 15.3 *Let U be a simply connected region in $\mathbb{C}$. If γ is a closed curve in U and $f \in H(U)$ then*

$$\int_\gamma f(z)\,dz = 0\ .$$

Theorem 15.4 *Let U be a region in $\mathbb{C}$. (It is not assumed that U is simply connected.) Let $f \in H(U)$. There exists a function $F \in H(U)$ with $F' = f$ in U if and only if*

$$\int_\gamma f(z)\,dz = 0$$

for every closed curve γ in U.

16 The General Residue Theorem and the Argument Principle

Summary: If an equation gets perturbed, what happens to the solutions of the equation? The implicit function theorem gives an important result. It's proof is based on contraction.

Complex variables has a different tool, the winding number or index, which is an integer. If an integer gets slightly perturbed, it remains unchanged. This can be used to obtain information about solutions of an equation under perturbations. One has to be precise about the multiplicities of zeros of holomorphic functions and relate their multiplicities to an integral. The result is called the argument principle.

16.1 Remarks on Solutions of Equations under Perturbations

A general questions of mathematics, vaguely formulated, is the following: Suppose u_0 is the solution of an equation and the equation gets perturbed by ε. Will the perturbed equation have a solution $u(\varepsilon)$ near u_0? A precise result of this nature is formalized in the implicit function theorem, which is itself based on completeness of the underlying solution space (all Cauchy sequences converge) and contraction. To formalize ideas, assume that

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

is a smooth map and consider the equation

$$F(u, \lambda) = 0 \ . \tag{16.1}$$

Here we consider λ as a vector of parameters in the parameter space $\mathbb{R}^m$. The solutions u lie in the state space $\mathbb{R}^n$. The space $\mathbb{R}^n$ is also the space of right-hand sides so that, for fixed $\lambda \in \mathbb{R}^m$, the system $F(u, \lambda) = 0$ has n scalar unknowns and n scalar equations. Suppose that

$$F(u_0, \lambda_0) = 0 \ ,$$

where $u_0 \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}^m$, i.e., for $\lambda = \lambda_0$ the equation (16.1) has the solution u_0. Let $\lambda = \lambda_0 + \varepsilon$ where $\varepsilon \in \mathbb{R}^m$ is small in norm. We ask if the equation

$$F(u, \lambda_0 + \varepsilon) = 0 \tag{16.2}$$

has a solution $u = u(\varepsilon) \sim u_0$. To ensure that this is true, we assume that the Jacobian

$$A := F_u(u_0, \lambda_0) \in \mathbb{R}^{n \times n}$$

is nonsingular. Then, proceeding formally, we try to find a solution u of (16.2) of the form

$$u = u_0 + \delta, \quad \delta \in \mathbb{R}^n \ ,$$

where the vector δ is small in norm. We have, formally,

$$\begin{aligned} 0 &= F(u_0 + \delta, \lambda_0 + \varepsilon) \\ &= F(u_0, \lambda_0) + A\delta + F_\lambda(u_0, \lambda_0)\varepsilon + Q(\delta, \varepsilon) \end{aligned}$$

where

$$|Q(\delta, \varepsilon)| \le C(|\delta|^2 + |\varepsilon|^2) \ .$$

Since $F(u_0, \lambda_0) = 0$ we obtain

$$\delta = -A^{-1} F_\lambda(u_0, \lambda_0)\varepsilon - A^{-1} Q(\delta, \varepsilon) \ .$$

This is a fixed point equation for δ, suggesting the iteration

$$\delta^{j+1} = -A^{-1} F_\lambda(u_0, \lambda_0)\varepsilon - A^{-1} Q(\delta^j, \varepsilon), \quad \delta^0 = -A^{-1} F_\lambda(u_0, \lambda_0)\varepsilon \ .$$

If ε is small enough, one can use a contraction argument to show that the equation

$$F(u_0 + \delta, \lambda_0 + \varepsilon) = 0$$

has a unique small solution $\delta \in \mathbb{R}^n$. This is made precise by the implicit function theorem.

Complex variables offers another tool, different from contraction, to study the solutions of an equation under perturbation. The tool is, ultimately, Cauchy's integral theorem, which allows us to count the number of zeros of a function in terms of an integral. The idea is as follows: If the function is perturbed slightly, the integral only changes slightly. Since the integral is an integer, it does not change at all and, consequently, the number of zeros of the perturbed function equals the number of zeros of the unperturbed function. See Rouché's Theorem in the next chapter.

In a more general form, this tool is developed further in degree theory, an advanced topic of analysis and topology. [4]

16.2 The Winding Number or Index

Let $\gamma(t), a \le t \le b$, be a parameterization of a closed curve in $\mathbb{C}$, thus $\gamma(a) = \gamma(b)$. We denote the curve parameterized by γ also by γ. Let $P \in \mathbb{C} \setminus \gamma$, i.e., P is a point in the complex plane that does not lie on the curve γ.

The number

$$\begin{aligned} Ind_\gamma(P) &:= \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - P} \\ &= \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s) - P}\, ds \end{aligned}$$

is called the index of γ w.r.t. P or the winding number of γ w.r.t. P. Intuitively, $Ind_\gamma(P)$ counts how many times γ winds around P in the positive sense. If $Ind_\gamma(P)$ is negative, then γ winds around P clockwise.

It is not completely trivial to prove that the index defined above is always an integer.

Lemma 16.1 *Under the above assumptions, the number $Ind_\gamma(P)$ is an integer.*

Proof: Set

$$g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - P}\, ds, \quad a \le t \le b \ .$$

We have $g(a) = 0$ and

[4] An interesting result of index theory is Brower's fixed point theorem: If $K \subset \mathbb{R}^n$ is compact and convex and if $f : K \to K$ is a continuous function, then there exists $P \in K$ with $f(P) = P$, i.e., f has a fixed point.

$$\frac{1}{2\pi i}\, g(b) = Ind_\gamma(P) \ .$$

Define

$$\phi(t) := e^{-g(t)}(\gamma(t) - P), \quad a \le t \le b \ .$$

We will prove that $\phi(t)$ is constant. We have

$$\begin{aligned} \phi(a) &= e^{-g(a)}(\gamma(a) - P) \\ &= \gamma(a) - P \end{aligned}$$

and

$$\begin{aligned} \phi(b) &= e^{-g(b)}(\gamma(b) - P) \\ &= e^{-g(b)}(\gamma(a) - P) \ . \end{aligned}$$

In the last equation we have used that $\gamma(a) = \gamma(b)$, which holds since the curve γ is assumed to be closed.

Note that the definition of $g(t)$ yields that

$$g'(t) = \gamma'(t)(\gamma(t) - P)^{-1} \ .$$

We use this to prove that $\phi(t)$ is constant:

$$\begin{aligned} \phi'(t) &= e^{-g(t)}\Big(- g'(t)\Big)(\gamma(t) - P) + e^{-g(t)}\gamma'(t) \\ &= e^{-g(t)}\Big(- \gamma'(t)\Big) + e^{-g(t)}\gamma'(t) \\ &= 0 \ . \end{aligned}$$

We obtain that

$$\phi(a) = \phi(b) \ .$$

Therefore,

$$\begin{aligned} \gamma(a) - P &= \phi(a) \\ &= \phi(b) \\ &= e^{-g(b)}(\gamma(a) - P) \end{aligned}$$

and $\phi(a) - P \neq 0$ yields that

$$e^{-g(b)} = 1 \ .$$

Therefore,

$$g(b) = 2\pi i n \quad \text{for some} \quad n \in \mathbb{Z} \ .$$

Finally,

$$Ind_\gamma(P) = \frac{g(b)}{2\pi i} = n \in \mathbb{Z} \ .$$

$\diamond$

Remark: It is not easy to formalize our intuition about the winding number. It is difficult to prove Jordan's Lemma, which seems to be quite obvious. Jordan's Lemma: Let $\Gamma \subset \mathbb{R}^2$ denote a Jordan curve, i.e., there exists a bijective continuous map $\phi : S^1 \to \Gamma$. Here $S^1 = \{(x,y) \ : \ x^2+y^2=1\}$ is the unit circle. Then there exists two open connected subsets A, B of $\mathbb{R}^2$ with

$$\mathbb{R}^2 \setminus \Gamma = A \cup B, \quad A \cap B = \emptyset \ ,$$

where A is bounded and B is unbounded.

16.3 The General Residue Theorem

Recall that an open connected set $U \subset \mathbb{C}$ is called a region. Also, recall that a closed curve γ in U is called null–homotopic in U if one can deform γ continuously to a point in U where the deformations of γ all lie in U.

Theorem 16.1 *Let U be a region in $\mathbb{C}$. Let $P_1, \ldots P_J \in U$ be J distinct points in U and let*

$$f \in H\Big(U \setminus \{P_1, \ldots, P_J\}\Big) \ .$$

Let γ be a closed curve in U which is null–homotopic in U and avoids the points P_j, i.e.,

$$P_j \notin \gamma, \quad j = 1, \ldots, J \ .$$

Under these assumptions:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^J Res(f, P_j)\, Ind_\gamma(P_j) \ .$$

Proof: For $0 < |z - P_j| < \varepsilon$:

$$f(z) = \sum_{k=-\infty}^{-1} a_k^{(j)} (z - P_j)^k + g_j(z)$$

where $g_j \in H(D(P_j, \varepsilon))$ and

$$a_{-1}^{(j)} = Res(f, P_j) \ .$$

The singular part of the Laurent expansion of f near P_j is:

$$s_j(z) = \sum_{k=-\infty}^{-1} a_k^{(j)} (z - P_j)^k \ ;$$

this function is holomorphic in $\mathbb{C} \setminus \{P_j\}$. (See the results on Laurent expansions.) Therefore,

$$g(z) := f(z) - \sum_{j=1}^J s_j(z)$$

can be extended to a holomorphic function in U, i.e., the singularity of g at every point P_j is removable.

One obtains:

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma g(z)\,dz + \sum_{j=1}^J \int_\gamma s_j(z)\,dz \\ &= \sum_{j=1}^J a_{-1}^{(j)} \int_\gamma \frac{dz}{z-P_j} \\ &= \sum_{j=1}^J Res(f,P_j)\, 2\pi i\, Ind_\gamma(P_j)\end{aligned}$$

$\diamond$

16.4 Zero–Counting of Holomorphic Maps

We show here that the zeros of a holomorphic function $f(z)$ can be counted (according to their multiplicity) by an integral. This is very useful if one perturbs the function $f(z)$ or if one counts the solutions z_j of the perturbed equation

$$f(z) - w = 0$$

for small $w \in \mathbb{C}$ instead of the zeros of f.

16.4.1 The Multiplicity of a Zero

Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. We assume that f is not identically zero. If $z_0 \in U$ and $f(z_0) = 0$ then z_0 is called a zero of f. For $|z - z_0| < \varepsilon$ we can write:

$$\begin{aligned}f(z) &= \sum_{j=M}^\infty a_j (z-z_0)^j \\ &= (z-z_0)^M h(z)\end{aligned}$$

where $M \geq 1$ and $a_M \neq 0$. The function $h(z)$ is holomorphic in

$$D(z_0, \varepsilon)$$

and we have, for sufficiently small ε:

$$h(z) \neq 0 \quad \text{for} \quad |z - z_0| \leq \varepsilon\ .$$

The number M is called the multiplicity of the zero z_0 of f. We write

$$M = mult_f(z_0)$$

and note that

$$f^{(j)}(z_0) = 0 \quad \text{for} \quad j = 0, \ldots, M-1, \quad f^{(M)}(z_0) = a_M\, M! \neq 0\ .$$

Note: If the holomorphic function $f(z)$ has a zero of multiplicity M at z_0 then there exists $\varepsilon > 0$ so that

$$f^{(j)}(z) \neq 0 \quad \text{for} \quad 0 < |z - z_0| < \varepsilon, \quad 0 \le j \le M-1 \ ,$$

and

$$f^{(M)}(z) \neq 0 \quad \text{for} \quad 0 \le |z - z_0| < \varepsilon \ .$$

16.4.2 The Zeros of a Holomorphic Function in a Disk

Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. We assume that f is not identically zero.

Let $\bar{D} = \bar{D}(P, r) \subset U$ be a closed disk in U. We assume that $f(z) \neq 0$ for all $z \in \partial \bar{D}$, i.e., f has no zero on the boundary of the disk $\bar{D}$. Let

$$\gamma(t) = P + re^{it}, \quad 0 \le t \le 2\pi \ ,$$

denote the positively oriented boundary curve of $\bar{D}$.

Let $z_1, \ldots, z_J$ denote the distinct zeros of f in the open disk $D = D(P, r)$ with multiplicities

$$M_j = mult_f(z_j) \ .$$

The following result is called the **argument principle for holomorphic functions**.

Theorem 16.2 *Under the above assumptions:*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^J M_j \ , \tag{16.3}$$

i.e., the integral can be used to count the zeros of f encircled by γ according to their multiplicities.

Proof: If $\varepsilon > 0$ is sufficiently small, then the curve

$$\gamma_{j\varepsilon}(t) = z_j + \varepsilon e^{it}, \quad 0 \le t \le 2\pi \ ,$$

encircles the zero z_j, but no other zero of f. We have

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^J \int_{\gamma_{j\varepsilon}} \frac{f'(z)}{f(z)}\, dz \ .$$

Fix j and set $M = M_j$. From

$$f(z) = (z - z_j)^M\, h(z) \quad \text{for} \quad z \in D_j := D(z_j, \varepsilon)$$

with

$$h \in H(D_j), \quad h(z) \neq 0 \quad \text{for} \quad z \in \bar{D}_j \ ,$$

we obtain:

$$f'(z) = M(z - z_j)^{M-1} h(z) + (z - z_j)^M h'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{M}{z - z_j} + g(z), \quad g \in H(D_j) \ .$$

Therefore,

$$\int_{\gamma_{j\varepsilon}} \frac{f'(z)}{f(z)}\, dz = 2\pi i\, M = 2\pi i\, M_j \ .$$

This proves the claim. $\diamond$

Remark: In Section 17.6 we will consider the argument principle for meromorphic functions.

Interpretation of Equation (16.3) in Terms of the Image Curve $f(\gamma)$**:** Let

$$\mu(t) = f(\gamma(t)) = f(P + re^{it}), \quad 0 \le t \le 2\pi \ ,$$

denote the image of the curve γ under the map f. Then $\mu(t) \neq 0$ for $0 \le t \le 2\pi$ since, by assumption, f has no zero on ∂D. The winding number of the curve $\mu(t), 0 \le t \le 2\pi$, w.r.t. the point 0 is

$$\begin{aligned}
Ind_\mu(0) &= \frac{1}{2\pi i}\int_\mu \frac{dw}{w} \quad (w = \mu(t),\ dw = \mu'(t)\, dt) \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\mu'(t)}{\mu(t)}\, dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\, dt \quad (z = \gamma(t), dz = \gamma'(t)\, dt) \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz
\end{aligned}$$

In other words, the left–hand side of (16.3) is the number of times by which the point $f(\gamma(t))$ moves counterclockwise around 0 when t changes from 0 to 2π. (If f has no zero in the disk $D(P, r)$ then $Ind_\mu(0) = 0$ and the curve $\mu(t) = f(\gamma(t)), 0 \le t \le 2\pi$, does not go around the point 0.)

We obtain the following reformulation of Theorem 16.2:

Theorem 16.3 *Let $\bar{D}(P, r) \subset U$ and let $f \in H(U)$. Assume that f has no zero on $\partial D(P, r)$. Then the number of zeros of f in $D(P, r)$ (counting multiplicities) equals the number of times by which the curve*

$$\mu(t) = f(P + re^{it}), \quad 0 \le t \le 2\pi \ ,$$

winds counterclockwise around $w = 0$.

Example: Let $f(z) = z^3$ and let $D(0, 1)$ denote the unit circle. The boundary curve of $D(0, 1)$ is

$$\gamma(t) = e^{it}, \quad 0 \le t \le 2\pi \ .$$

The f–image of this curve is

$$\mu(t) = e^{3it}, \quad 0 \le t \le 2\pi \ .$$

Then $\mu(t)$ winds three times counterclockwise around $w = 0$. The function $f(z) = z^3$ has three zeros (counting multiplicities) in $D(0, 1)$.

16.4.3 The Argument Principle and Log–Functions

The result of Theorem 16.2 is often called the *argument principle*. To explain this, we first make some remarks on log–functions.

If $r > 0$ we denote by $\ln r = \int_0^r \frac{dx}{x}$ the usual real natural logarithm of r. We know that $\frac{d}{dr} \ln r = \frac{1}{r}$ for $r > 0$.

The function

$$r \to \ln r, \quad r > 0 \ ,$$

cannot be extended as a holomorphic function $\log w$ defined for all $w \in \mathbb{C} \setminus \{0\}$. Otherwise, by the identity theorem,

$$\frac{d}{dw} \log w = \frac{1}{w}, \quad w \neq 0 \ .$$

However, we know that

$$\int_\gamma \frac{dw}{w} = 2\pi i \neq 0 \ ,$$

where $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$.

Log–functions in a simply connected region. If $W \subset \mathbb{C}$ is a *simply connected* region with $0 \notin W$, then we can make a continuous choice for $arg(w), w \in W$, and write

$$w = re^{i\,arg(w)}, \quad r = |w| > 0, \quad w \in W \ .$$

We define

$$\log_W(w) = \ln r + i\,arg(w), \quad w \in W \ ,$$

and obtain

$$e^{\log_W(w)} = w, \quad w \in W \ .$$

Now let us make the same assumptions as in the previous subsection: Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. We assume that f is not identically zero. Let $\bar{D} = \bar{D}(P, r) \subset U$ be a closed disk in U. We assume that $f(z) \neq 0$ for all $z \in \partial\bar{D}$. Let

$$\gamma(t) = P + re^{it}, \quad 0 \leq t \leq 2\pi \ ,$$

denote the positively oriented boundary curve of $\bar{D}$.

Let us assume that f has at least one zero in D. Then the curve

$$w(t) = f(\gamma(t)), \quad 0 \leq t \leq 2\pi \ ,$$

winds around *zero*, and we cannot define $\log w(t)$ consistently for $0 \leq t \leq 2\pi$.

Make a subdivision of the interval $0 \leq t \leq 2\pi$ by choosing points

$$t_0 = 0 < t_1 < \ldots < t_K = 2\pi$$

and let

$$\gamma_k(t) = \gamma(t), \quad t_{k-1} \leq t \leq t_k \quad \text{for} \quad k = 1, \ldots, K \ .$$

We have

$$\gamma = \gamma_1 + \ldots + \gamma_K \ .$$

We make the subdivision fine enough so that each curve $\gamma_k(t)$ lies in a simply connected region W_k with $0 \notin W_k$. On W_k we have a log–function, which we call $\log_k(w)$. We have

$$\frac{d}{dw} \log_k(w) = \frac{1}{w}, \quad w \in W_k \ .$$

If z is chosen so that $f(z) \in W_k$ then

$$\frac{d}{dz} \log_k(f(z)) = \frac{f'(z)}{f(z)} \ .$$

Therefore,

$$\int_{\gamma_k} \frac{f'(z)}{f(z)}\, dz = \log_k f(\gamma(t_k)) - \log_k f(\gamma(t_{k-1})) = \log_k(w_k/w_{k-1})$$

with

$$w_k := f(\gamma(t_k)), \quad 0 \le k \le K \ .$$

(Note that $w_0 = w_K$ since $\gamma(t_0) = \gamma(0) = \gamma(2\pi) = \gamma(t_k)$.)

Write

$$w_k = r_k e^{i\, arg_k(w_k)} \ .$$

We then have

$$\int_{\gamma_k} \frac{f'(z)}{f(z)}\, dz = \log_k w_k/w_{k-1}) = \ln(r_k/r_{k-1}) + i\Big(arg_k(w_k) - arg_k(w_{k-1})\Big) \ .$$

Summation over k from 1 to K yields that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^K \ln(r_k/r_{k-1}) + i \sum_{k=1}^K \Big(arg_k(w_k) - arg_k(w_{k-1})\Big) \ .$$

The first part involves logs of real numbers:

$$\begin{aligned} \sum_{k=1}^K \ln(r_k/r_{k-1}) &= \ln\Big(\frac{r_1}{r_0} \cdot \frac{r_2}{r_1} \cdot \ldots \cdot \frac{r_K}{r_{K-1}}\Big) \\ &= \ln 1 \\ &= 0 \end{aligned}$$

Here we have used that $w_0 = w_K$, thus $r_0 = r_K$.

The real number

$$\sum_{k=1}^K \Big(arg_k(w_k) - arg_k(w_{k-1})\Big)$$

is the total change of argument of the function $w(t) = f(\gamma(t))$ as t goes from 0 to 2π. In other words,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\,i\sum_{k=1}^{K}\Big(arg_k(w_k) - arg_k(w_{k-1})\Big)$$

is the number of times by which $w(t)$ moves around zero when t goes from 0 to 2π. This confirms our earlier interpretation of the left–hand side of the above equation.

Example: Let

$$f(z) = z^3(z-1)^2, \quad z \in \mathbb{C}\ .$$

Let $D = D(0,2)$ and let $\gamma(t) = 2e^{it}, 0 \le t \le 2\pi$, denote the boundary curve of D. By Theorem 16.2 we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = 5$$

since f has five zeros in D. This is easily confirmed by the residue theorem: Since

$$f'(z) = 3z^2(z-1)^2 + 2z^3(z-1)$$

we have

$$\frac{f'(z)}{f(z)} = \frac{3}{z} + \frac{2}{z-1}\ .$$

The curve with parameterization

$$\mu(t) = 8e^{3it}(2e^{it}-1)^2, \quad 0 \le t \le 2\pi\ ,$$

is the image of $\gamma(t)$ under f. By Theorem 16.3 the curve $\mu(t)$ winds five times counterclockwise around zero. See the figure below. Note that $\mu(0) = 8$ and $\mu(\pi) = -72$.

16.5 The Change of Argument and Zeros of Polynomials

Let $U \subset \mathbb{C}$ denote an open set and let $f \in H(U)$. Let Γ denote a curve in U and assume that $f(z) \neq 0$ for all $z \in \Gamma$. The real number

$$\mathrm{Im}\int_\Gamma \frac{f'(z)}{f(z)}\,dz =: \Delta_\Gamma \arg f$$

is called the change of argument of f along Γ.

Interpretation: Let $z(t), a \le t \le b$, denote a parameterization of Γ. The curve Γ goes from $A = z(a)$ to $B = z(b)$. First assume that $f(\Gamma) \subset W$, where $W \subset \mathbb{C}$ is an open set, and that $\log w$ is a logarithm on W. We have, for z near Γ,

$$\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}\ ,$$

thus

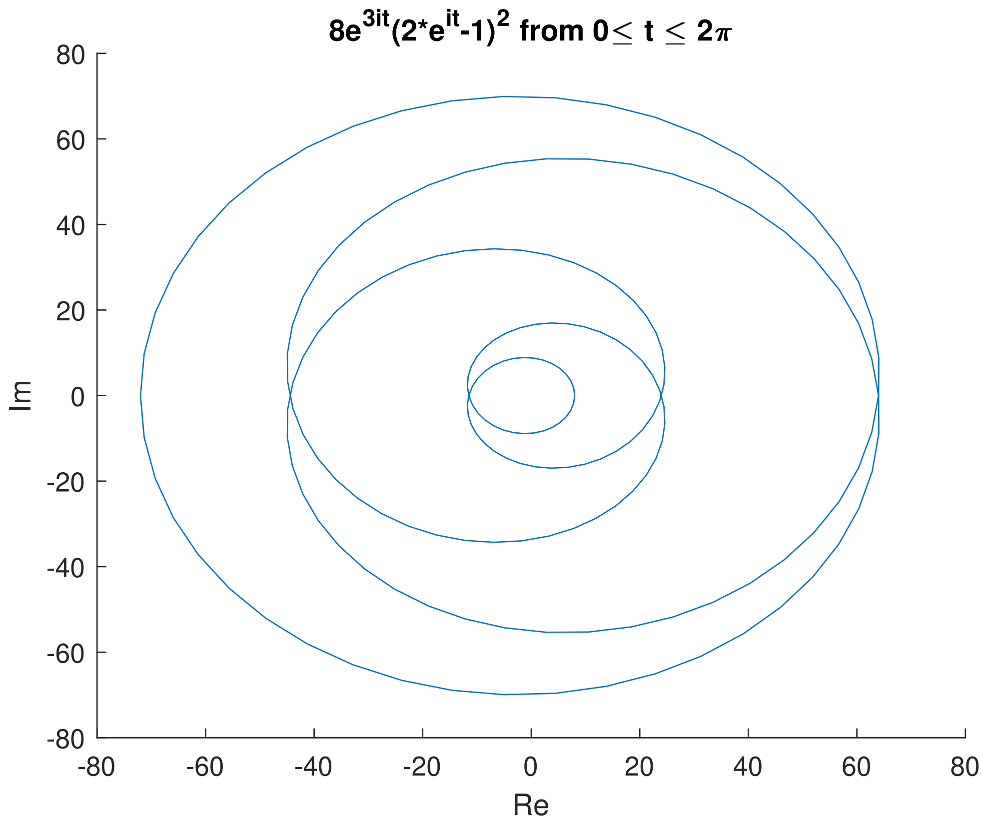

Figure 16.1: Graph of Image Curve

$$\begin{aligned}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \log f(B) - \log f(A)\\ &= \ln\Big|\frac{f(B)}{f(A)}\Big| + i\Big(\arg f(B) - \arg f(A)\Big)\ ,\end{aligned}$$

thus

$$\begin{aligned}\Delta_\Gamma \arg f &= \mathrm{Im} \int_\Gamma \frac{f'(z)}{f(z)}\,dz\\ &= \arg f(B) - \arg f(A)\ .\end{aligned}$$

Note that the difference in argument does not depend on the specific argument function on W. A main assumption is that $f(\Gamma) \subset W$ and that a log–function exists on W.

In the general case, let Γ denote a curve in U with parameterization $z(t), a \le t \le b$. Let $f \in H(U)$ and assume $f(z) \neq 0$ for $z \in \Gamma$. Choose a subdivision

$$t_0 = a < t_1 < \ldots < t_K = b$$

and obtain

$$\begin{aligned}
\Delta_\Gamma \arg f &= \operatorname{Im} \int_\Gamma \frac{f'(z)}{f(z)}\, dz \\
&= \operatorname{Im} \sum_k \int_{\Gamma_k} \frac{f'(z)}{f(z)}\, dz \\
&= \sum_k \Big(\arg{}_k f(z_k) - \arg{}_k f(z_{k-1}) \Big) \ .
\end{aligned}$$

Here $z_k = z(t_k), z_{k-1} = z(t_{k-1})$ and Γ_k is the curve with parameterization $z(t), t_{k-1} \le t \le t_k$. Also, $f(\Gamma_k) \subset W_k$ and $\arg{}_k(w)$ is an argument function on W_k.

Example 1: Let Γ_R denote the curve, along the imaginary axis, parameterized by $z(t) = it, -R \le t \le R$, and let Γ denote the whole imaginary axis with parameterization $z(t) = it, -\infty < t < \infty$.

Consider the polynomial $f(z) = z + 1$ with the simple zero $z_1 = -1$ to the left of Γ. We note that $f(\Gamma)$ lies in the right half–plane and we can work with the main branch, $\log w$.

We have

$$\begin{aligned}
\int_{\Gamma_R} \frac{dz}{z+1} &= \log(iR+1) - \log(-iR+1) \\
&= \ln(R^2+1)^{1/2} + i\theta_R - (\ln(R^2+1)^{1/2} - i\theta_R)
\end{aligned}$$

with

$$\theta_R = \frac{\pi}{2} - \alpha_R, \quad \alpha_R = \arctan(1/R) = \mathcal{O}(1/R) \ .$$

Therefore,

$$\Delta_{\Gamma_R} \arg (z+1) = \pi + \mathcal{O}(1/R)$$

and, as $R \to \infty$,

$$\Delta_\Gamma \arg (z+1) = \pi \ .$$

Similarly, if z_1 is any point to the left of Γ, one finds that

$$\Delta_\Gamma \arg (z - z_1) = \pi \ .$$

Example 2: Let Γ_R and Γ denote the same curves as in Example 1 and consider the polynomial $f(z) = z - 1$ with zero $z_2 = 1$ to the right of Γ. In this case, $f(\Gamma)$ lies to the left of the imaginary axis, and the main branch, $\log w$, is not defined along $f(\Gamma)$. One obtains

$$\begin{aligned}
\Delta_{\Gamma_R} \arg (z-1) &= \Big(\frac{\pi}{2} + \alpha_R\Big) - \Big(\frac{3\pi}{2} - \alpha_R\Big) \\
&= -\pi + \mathcal{O}(1/R)
\end{aligned}$$

and, as $R \to \infty$,

$$\Delta_\Gamma \arg\,(z-1) = -\pi \ .$$

Similarly, if z_2 is any point to the right of Γ, one finds that

$$\Delta_\Gamma \arg\,(z-z_2) = -\pi \ .$$

Example 3: As in Examples 1 and 2, let Γ denote the imaginary axis, $z(t) = it, -\infty < t < \infty$. Let

$$f(z) = \Pi_{j=1}^n (z - z_j)$$

denote a polynomial and assume that none of the zeros z_j of f lies on Γ. Since

$$\frac{f'(z)}{f(z)} = \sum \frac{1}{z - z_j}$$

one obtains that

$$\Delta_\Gamma \arg f = \pi(p - q)$$

if p of the zeros of f lie to the left and $q = n - p$ of the zeros of f lie to the right of Γ.

With a change of variables, one obtains the following result.

Theorem 16.4 *Let Γ denote the straight line with parameterization*

$$z(t) = A + Bt, \quad -\infty < t < \infty \ ,$$

where A, B are complex numbers and $B \neq 0$. If $f(z)$ is any polynomial without a zero on Γ then we have

$$\Delta_\Gamma \arg f = \pi(p - q)$$

if f has p zeros to the left and q zeros to the right of Γ.

17 Applications and Extensions of the Argument Principle

Summary: The argument principle can be used to study how zeros of a holomorphic function are perturbed if the function is perturbed. We will use this to prove the **Open Mapping Theorem**.

In Section 17.4 we give a generalization of the argument principle and will use it to show that the local inverse of a holomorphic function is holomorphic if the inverse exists.

If $f(z)$ and $p(z)$ are holomorphic functions, where $p(z)$ is a perturbation term, how are the solutions of the perturbed equation $f(z) + p(z) = 0$ related to the solutions of the unperturbed equation $f(z) = 0$? Rouché's Theorem gives an important result.

17.1 Perturbation of an Equation: An Example

We first consider a simple example. Let

$$f(z) = z^3, \quad z \in \mathbb{C} \ .$$

Let $D = D(0,1)$ denote the unit disk with boundary curve γ. In this case, the function $f(z) = z^3$ has the zero

$$z_0 = 0$$

of multiplicity $M = 3$. We have

$$\frac{f'(z)}{f(z)} = \frac{3}{z}$$

and Theorem 16.2 yields that

$$\frac{1}{2\pi i} \int_\gamma \frac{3z^2}{z^3} \, dz = 3 \ .$$

Consider the perturbed equation for the unknown z:

$$z^3 = w$$

where $w \in \mathbb{C}$ is small in absolute value, $w = re^{i\theta}, r > 0, -\pi < \theta \leq \theta$.

The solutions are

$$\begin{aligned} z_1(w) &= r^{1/3} \, e^{i\theta/3} \\ z_2(w) &= r^{1/3} \, e^{i\theta/3} \, e^{2\pi i/3} \\ z_3(w) &= r^{1/3} \, e^{i\theta/3} \, e^{4\pi i/3} \end{aligned}$$

These are simple zeros of the function

$$g(z) = z^3 - w \ .$$

Theorem 16.2 applied to $g(z)$ yields that

$$\frac{1}{2\pi i} \int_\gamma \frac{3z^2}{z^3 - w} \, dz = 3 \quad \text{if} \quad |w| < 1 \ .$$

In this example, the triple zero $z_0 = 0$ of $f(z) = z^3$ splits into three simple zeros for the perturbed function $g(z) = z^3 - w$ if $w \neq 0$. We want to generalize this result.

17.2 Perturbation of a Multiple Zero

We make the same assumptions as in 16.4.2: U is a region in $\mathbb{C}$; $f \in H(U)$ is not identically zero. Let $z_0 \in U$ be a zero of f of multiplicity M. We choose $r > 0$ with

a) $\bar{D} = \bar{D}(z_0, r) \subset U$;

b) $f(z) \neq 0$ for $0 < |z - z_0| \leq r$;

c) $f'(z) \neq 0$ for $0 < |z - z_0| \leq r$.

Let $\gamma(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$. Set $\eta := \min\{|f(z)| \ : \ |z - z_0| = r\}$, thus $\eta > 0$. We consider the equation

$$f(z) = w, \quad z \in D(z_0, r) \ ,$$

where $w \in \mathbb{C}$ with $|w| < \eta$ is given.

Let us make a plausibility consideration first: We have $w \sim 0$. Also, for z close to z_0:

$$f(z) \sim a_M (z - z_0)^M, \quad a_M \neq 0 \ .$$

We must solve

$$a_M (z - z_0)^M \sim w \ ,$$

i.e.,

$$(z - z_0)^M \sim \frac{w}{a_M} =: \rho e^{i\theta} \quad \text{where} \quad -\pi < \theta \leq \pi \ .$$

If $\rho > 0$ then the equation for $q \in \mathbb{C}$

$$q^M = \frac{w}{a_M} = \rho e^{i\theta}$$

has M distinct solutions q_j:

$$\begin{aligned} q_1 &= \rho^{1/M} e^{i\theta/M} \\ q_2 &= q_1 e^{2\pi i/M} \\ q_3 &= q_1 e^{4\pi i/M} \\ \dots &= \dots \\ q_M &= q_1 e^{(M-1)2\pi i/M} \end{aligned}$$

We expect that the equation

$$f(z) = w$$

has M distinct solutions

$$z_j(w) \sim z_0 + q_j, \quad j = 1, \dots, M$$

if $|w|$ is small, but $w \neq 0$.

Theorem 17.1 *Let U denote a region in $\mathbb{C}$ and let $f \in H(U)$. Assume that f is not identically zero. Let $z_0 \in U$ denote a zero of f of multiplicity M. Choose $r > 0$ so that the disk $D = D(z_0, r)$ satisfies the conditions $\bar{D}(z_0, r) \subset U$; $f(z) \neq 0$ for $0 < |z - z_0| \leq r$; $f'(z) \neq 0$ for $0 < |z - z_0| \leq r$. Define $\eta := \min\{|f(z)| \; : \; |z - z_0| = r\}$ and let $0 < |w| < \eta$. Then the equation $f(z) = w$ has M distinct solutions $z_1, \ldots, z_M$ in D. Every z_j is a simple zero of the function $g(z) = f(z) - w$, i.e., $g'(z_j) = f'(z_j) \neq 0$.*

Proof: Let $g(z) = f(z) - w$. If $|w| < \eta$ and $|z - z_0| = r$ then

$$|g(z)| \geq |f(z)| - |w| \geq \eta - |w| > 0 \; .$$

Therefore, the function

$$F(w) := \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w} \, dz, \quad |w| < \eta \; ,$$

is integer valued. Here γ denotes the positively oriented circle of radius r centered at z_0.

We know that $F(w)$ is the number of zeros of $g(z)$ in $D(z_0, r)$, where the zeros are counted according to their multiplicity.

We claim that $F(w)$ is holomorphic for $|w| < \eta$. In fact, for $|z - z_0| = r$ we have

$$\begin{aligned} \frac{1}{f(z) - w} &= \frac{1}{f(z)} \cdot \frac{1}{1 - w/f(z)} \\ &= \sum_{j=0}^\infty \frac{w^j}{(f(z))^{j+1}} \end{aligned}$$

For every fixed w with $|w| < \eta$ the convergence is uniform for $z \in \gamma$. This yields:

$$F(w) = \frac{1}{2\pi i} \sum_{j=0}^\infty b_j w^j$$

with

$$b_j = \int_\gamma \frac{f'(z)}{(f(z))^{j+1}} \, dz \; .$$

A holomorphic function that is integer–valued is constant. One obtains that

$$F(w) \equiv M \; .$$

It follows that the number of zeros of $g(z)$ in D is M if zeros are counted according to their multiplicity.

Now let $0 < |w| < \eta$ and let $z_1 \in D$ be a zero of g. Then $f(z_1) = w$, thus $z_1 \neq z_0$. It follows that $f'(z_1) \neq 0$; thus all zeros of $g(z)$ are simple. The equation $f(z) = w$ has M distinct zeros $z_1, \ldots, z_M \in D(z_0, r)$ if $0 < |w| < \eta$. $\diamond$

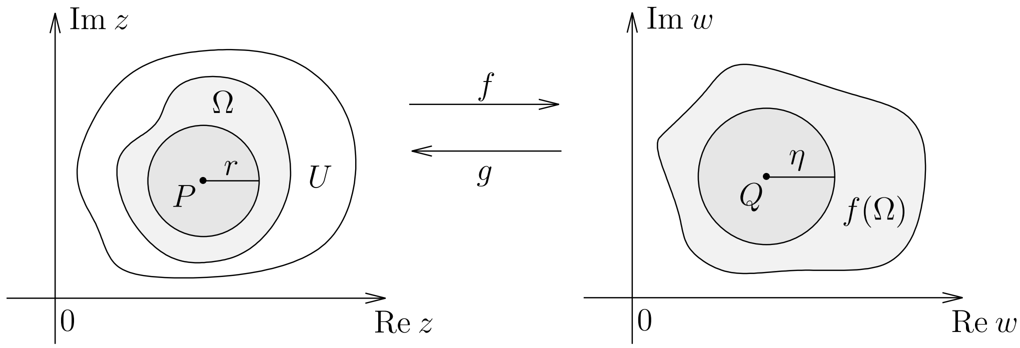

Figure 17.1: Open Mapping Theorem

17.3 The Open Mapping Theorem

If U and V are metric spaces (or, more generally, topological spaces) and $f : U \to V$ is a map, then f is called *open* if the set $f(\Omega)$ is an open subset of V whenever Ω is an open subset of U. (This notion is different from continuity. A map $f : U \to V$ can be shown to be continuous on U if and only if $f^{-1}(W)$ is an open subset of U whenever W is an open subset of V. Here $f^{-1}(W) = \{u \in U \ : \ f(u) \in W\}$.)

The following result is know as the **Open Mapping Theorem** of complex analysis. (There is another Open Mapping Theorem of functional analysis, which is different.) [5]

Theorem 17.2 *Let U be a region in $\mathbb{C}$ and let $f \in H(U)$ be a non–constant function. Then the mapping $f : U \to \mathbb{C}$ is open.*

Remark: Such a result is not true in $\mathbb{R}$. For example, if $f(x) = x^2$, then $f(\mathbb{R}) = [0, \infty)$. The set $[0, \infty)$ is not open in $\mathbb{R}$.

Proof: Let $\Omega \subset U$ be an open non–empty set. We must show that $f(\Omega)$ is open. To this end, let $Q \in f(\Omega)$ be an arbitrary point. We must show that there exists $\varepsilon > 0$ with $D(Q, \varepsilon) \subset f(\Omega)$.

Since $Q \in f(\Omega)$ there exists $P \in \Omega$ with $f(P) = Q$. We will apply Theorem 17.1 to the function

$$h(z) = f(z) - Q, \quad z \in \Omega \ .$$

(The function $h(z)$ is not identically zero since $f(z)$ is not constant.) Note that $h(P) = f(P) - Q = 0$.

Let M denote the multiplicity of the zero P of the function $h(z)$. There exists $r > 0$ with:

a) $\bar{D}(P, r) \subset \Omega$;

b) $f(z) \neq Q$ for $0 < |z - P| \leq r$;

c) $f'(z) \neq 0$ for $0 < |z - P| \leq r$.

Let $\eta := \min\{|f(z) - Q| \ : \ |z - P| = r\}$, thus $\eta > 0$. If $|v| < \eta$ then the equation

$$f(z) = Q + v$$

has M solutions $z_j \in D(P, r) \subset \Omega$. In particular, if $Q + v \in D(Q, \eta)$ then $Q + v$ lies in $f(\Omega)$. This says that $D(Q, \eta) \subset f(\Omega)$, proving the theorem. $\diamond$

[5] Open mapping theorem of functional analysis: Let X and Y be Banach spaces and let $T : X \to Y$ be linear, continuous and onto. Then T is an open mapping.

17.4 Extension of Theorem 16.2

The following is a useful generalization of Theorem 16.2. The assumptions are similar to those in Theorem 16.2, but a general function $\phi \in H(U)$ appears in Theorem 17.3. (In Theorem 16.2 the corresponding function is $\phi(z) \equiv 1$.)

Theorem 17.3 *Let U be a region and let $f, \phi \in H(U)$. Let $\bar{D} = \bar{D}(P,r) \subset U$. Assume that*

$$f(z) \neq 0 \quad \text{for} \quad |z-P| = r$$

and let $\gamma(t) = P + re^{it}$, $0 \le t \le 2\pi$. Let $z_1, \ldots, z_J$ denote the distinct zeros of f in D with multiplicities $M_j = mult_f(z_j)$. Then we have

$$\frac{1}{2\pi i} \int_\gamma \phi(z) \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^J M_j \phi(z_j) \ .$$

Proof: Let $\varepsilon > 0$ be small enough and let

$$\gamma_{j\varepsilon}(t) = z_j + \varepsilon e^{it}, \quad 0 \le t \le 2\pi \ .$$

Fix j and let $M = M_j$. In the following, the functions $h_k(z)$ are holomorphic for $|z - z_j| < \varepsilon$. We have

$$\begin{aligned} f(z) &= (z-z_j)^M h_1(z), \quad h_1(z_j) \neq 0 \ , \\ f'(z)/f(z) &= M(z-z_j)^{-1} + h_2(z) \\ \phi(z) &= \phi(z_j) + h_3(z), \quad h_3(z_j) = 0 \\ \phi(z) f'(z)/f(z) &= M\phi(z_j)(z-z_j)^{-1} + h_4(z) \end{aligned}$$

This implies that

$$\int_{\gamma_{j\varepsilon}} \phi(z) \frac{f'(z)}{f(z)} \, dz = 2\pi i \, M_j \, \phi(z_j) \ .$$

The theorem follows by summing over j. $\diamond$

Special Case: Assume $J = 1, M_1 = 1, \phi(z) \equiv z$. Then we have

$$\frac{1}{2\pi i} \int_\Gamma z \frac{f'(z)}{f(z)} \, dz = z_1$$

where z_1 is the unique simple zero of f in $D(P,r)$. This case is used in the proof of Theorem 17.4 below.

17.5 Local Inverses of Holomorphic Functions

Let U be a region in $\mathbb{C}$ and let $f \in H(U)$. We ask for conditions under which the mapping $f : U \to \mathbb{C}$ is $1-1$. First assume that $f'(z_0) = 0$ for some $z_0 \in U$. If $Q := f(z_0)$ then z_0 is a zero of multiplicity $M \ge 2$ of the function

$$h(z) = f(z) - Q, \quad z \in U \ .$$

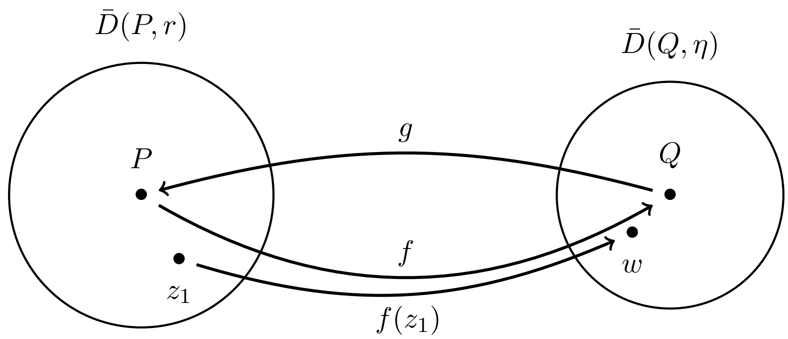

Figure 17.2: Local Inversion

Choose any $v \in \mathbb{C}, v \neq 0$, with $|v|$ small. By Theorem 17.1 there are M *distinct* points $z_1, \ldots, z_M \in D(z_0, r)$ which have the same image under f, thus

$$f(z_j) = Q + v, \quad j = 1, \ldots, M \ .$$

One obtains that f cannot be $1-1$ if $f'(z_0) = 0$ for some z_0.

Now assume that $f'(z) \neq 0$ for all $z \in U$. The example $f(z) = e^z$ shows that f may still fail to be *globally* $1-1$ since

$$e^0 = e^{2\pi i} = 1 \ .$$

However, as we will prove below, if $f'(z_0) \neq 0$, then f is *locally* $1-1$ near z_0. This means that there exists $\varepsilon > 0$ so that f is $1-1$ on $D(z_0, \varepsilon)$. In addition, if $Q = f(z_0)$, then the local inverse of f is defined and holomorphic in a disk $D(Q, \eta)$ for some $\eta > 0$.

Theorem 17.4 *(local inversion of holomorphic functions) Let $f \in H(U)$. Let $P \in U$ with $f'(P) \neq 0$. Set $Q = f(P)$. Then there exists an open neighborhood U_0 of P with $U_0 \subset U$ and there exists an open disk $D(Q, \eta)$ so that the following holds:*

a) $f : U_0 \to D(Q, \eta)$ is $1-1$ and onto;

b) there is a unique function $g : D(Q, \eta) \to U_0$ which is $1-1$ and onto satisfying

$$f(g(w)) = w \quad \textit{for all} \quad w \in D(Q, \eta)$$

and

$$g(f(z)) = z \quad \textit{for all} \quad z \in U_0 \ .$$

This uniquely determined function g is holomorphic on $D(Q, \eta)$.

Proof: 1) Choose $r > 0$ with

a) $\bar{D}(P, r) \subset U$;

b) $f(z) \neq Q$ for $0 < |z - P| \leq r$;

c) $f'(z) \neq 0$ for $0 \leq |z - P| \leq r$.

If $\gamma(t) = P + re^{it}, 0 \leq t \leq 2\pi$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - Q} \, dz = 1 \ .$$

This holds since the equation $f(z) - Q = 0$ has precisely one solution z in $D(P, r)$, namely $z = P$, and the solution $z = P$ is simple since $f'(P) \neq 0$.

Set

$$\eta := \min\{|f(z) - Q| \ : \ |z - P| = r\} > 0 \ .$$

If $|w - Q| < \eta$ then the equation

$$f(z) = w$$

has a unique solution $z_1 \in D(P, r)$. We call this solution $z_1 = g(w)$. In this way we have defined a function

$$g : D(Q, \eta) \to D(P, r)$$

with

$$f(g(w)) = w \quad \text{for all} \quad w \in D(Q, \eta) \ .$$

This equation implies that g is $1 - 1$. We note that $g(Q) = P$ since P is the unique solution of the equation $f(z) = Q, z \in D(P, r)$.

2) Apply Theorem 17.3 with $\phi(z) \equiv z$ to obtain

$$\frac{1}{2\pi i} \int_\gamma z \cdot \frac{f'(z)}{f(z) - w} \, dz = g(w) \quad \text{for} \quad w \in D(Q, \eta) \ . \tag{17.1}$$

(Note that $z_1 = g(w)$ is the unique zero of the function $z \to f(z) - w$ in $D(P, r)$ and $z_1 = g(w)$ is a simple zero. Also, if $\phi(z) \equiv z$, then $\phi(g(w)) = g(w)$.)

We use the representation (17.1) of the function g to prove that g is holomorphic on $D(Q, \eta)$. To this end, note that for $z \in \gamma$ and $w \in D(Q, \eta)$:

$$f(z) - w = (f(z) - Q) - (w - Q)$$

with

$$|f(z) - Q| \geq \eta > |w - Q| \ .$$

Therefore,

$$\begin{aligned} \frac{1}{f(z) - w} &= \frac{1}{(f(z) - Q) - (w - Q)} \\ &= \frac{1}{f(z) - Q} \cdot \frac{1}{1 - \frac{w - Q}{f(z) - Q}} \\ &= \sum_{j=0}^{\infty} \frac{(w - Q)^j}{(f(z) - Q)^{j+1}} \end{aligned}$$

The convergence is uniform for $z \in \gamma$. Using the above series in (17.1) and exchanging summation and integration, we obtain the expansion

$$g(w) = \frac{1}{2\pi i} \sum_{j=0}^{\infty} b_j (w - Q)^j$$

with

$$b_j = \int_\gamma \frac{zf'(z)}{(f(z)-Q)^{j+1}}\, dz \; .$$

This proves that $g \in H(D(Q,\eta))$.

c) Define $U_0 := g(D(Q,\eta))$. Then, by the Open Mapping Theorem, U_0 is an open neighborhood of P and $U_0 \subset D(P,r) \subset U$. The remaining claims of the theorem are now easily verified: The mapping

$$g : D(Q,\eta) \to U_0$$

is $1-1$ and onto. If $z \in U_0$ is given, then there exists a unique $w \in D(Q,\eta)$ with $g(w) = z$. We have $f(g(w)) = w$, thus

$$g(f(g(w))) = g(w) \; .$$

Recalling that $z = g(w)$ this becomes:

$$g(f(z)) = z \quad \text{for all} \quad z \in U_0 \; .$$

This equation implies that f is $1-1$ on U_0.

Let $w \in D(Q,\eta)$ be given. Then $z := g(w) \in U_0$ satisfies $f(z) = f(g(w)) = w$. Thus we have shown that $f : U_0 \to D(Q,\eta)$ is $1-1$ and onto, with inverse function g. The uniqueness of g is trivial. $\diamond$

Remark: One can also prove the previous theorem by power series expansion. Assume $P = Q = 0$, for simplicity, and let

$$f(z) = \sum_{j=1}^\infty a_j z^j, \quad |z| < r, \quad a_1 \neq 0 \; .$$

We try to determine a function

$$g(w) = \sum_{k=1}^\infty b_k w^k, \quad |w| < \eta \; ,$$

with

$$|g(w)| < r \quad \text{and} \quad f(g(w)) = w \quad \text{for all} \quad |w| < \eta \; .$$

First proceeding formally, we write

$$\begin{aligned} f(g(w)) &= a_1(b_1 w + b_2 w^2 + \ldots) + a_2(b_1 w + b_2 w^2 + \ldots)^2 + \ldots \\ &= a_1 b_1 w + w^2(a_1 b_2 + a_2 b_1^2) + w^3(a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) + \ldots \end{aligned}$$

The condition $f(g(w)) = w$ yields that

$$a_1 b_1 = 1, \quad \text{thus} \quad b_1 = 1/a_1 \; .$$

Further,

$$a_1b_2 + a_2b_1^2 = 0, \quad \text{thus} \quad b_2 = -a_2b_1^2/a_1 \ ,$$

and

$$a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 = 0, \quad \text{thus} \quad b_3 = -\frac{1}{a_1}(2a_2b_1b_2 + a_3b_1^3) \ .$$

This process can be continued. The b_k are determined recursively. One then has to prove that the series

$$g(w) = \sum_{k=1}^{\infty} b_k w^k$$

has a positive radius of convergence.

17.6 The Argument Principle for Meromorphic Functions

Roughly speaking, a function which is holomorphic except for poles is called meromorphic. Let us be more precise.

Definition: A set $S \subset \mathbb{C}$ is called discrete if for all $z \in S$ there exists $r > 0$ with $S \cap D(z,r) = \{z\}$.

Example: Let $S = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$. Then S is a discrete set. The set $S \cup \{0\}$ is not discrete.

Definition: Let $U \subset \mathbb{C}$ be open. Assume that $S \subset U$ is a discrete subset of $\mathbb{C}$ which is closed in U, i.e., if $z_n \in S$ and $z_n \to z \in U$, then $z \in S$. Let $f \in H(U \setminus S)$. The function f is called meromorphic in U with singular set S if every $z_j \in S$ is a pole of f. Often one simply says that f is meromorphic in U and writes $f \in M(U)$.

Example: Let $S = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$. This set is not closed as a subset of $\mathbb{C}$. However, if $U = \{z = x + iy \ : \ x > 0\}$ denotes the right half–plane, then S is closed in U.

Example: Let $p(z)$ and $q(z)$ be polynomials which have no common zero. The rational function

$$f(z) = \frac{p(z)}{q(z)}$$

is meromorphic in $\mathbb{C}$ with singular set

$$S = \{z_j \ : \ q(z_j) = 0\} \ .$$

Example: The function

$$f(z) = \frac{1}{\sin(\pi z)}$$

is meromorphic in $\mathbb{C}$ with singular set $S = \mathbb{Z}$.

Example: Let

$$S = \{z_n = \frac{1}{n\pi} \ : \ n \in \mathbb{Z}, \ n \neq 0\}$$

and let

$$S_0 = S \cup \{0\} \ .$$

Note that 0 is an accumulation point of S and of S_0. Consider the function

$$f(z) = \frac{1}{\sin(1/z)}, \quad z \in \mathbb{C} \setminus S_0 \; .$$

Clearly, f is holomorphic on $\mathbb{C} \setminus S_0$. The function f is not meromorphic on $\mathbb{C} \setminus S_0$ since the singularity at $z = 0$ is not isolated. The singularity at $z = 0$ is neither a pole nor an essential singularity. If $U = \mathbb{C} \setminus \{0\}$, then f is meromorphic on U with singular set S.

Theorem 17.5 *Let $U \subset \mathbb{C}$ be open and let $f \in M(U)$. Let $\bar{D} = \bar{D}(P, r) \subset U$ be a closed disk in U and assume that f has no zero and no pole on the boundary ∂D of D. Let $\gamma(t) = P + re^{it}, 0 \le t \le 2\pi$, denote the boundary curve of D. Let $z_1, \ldots, z_J$ denote the distinct zeros of f in D with multiplicities $M_j = mult_f(z_j)$ and let $p_1, \ldots, p_K$ denote the distinct poles of f in D with orders $N_k = ord_f(p_k)$. Then we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{J} M_j - \sum_{k=1}^{K} N_k \; .$$

In other words,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \#(zeros) - \#(poles) \; .$$

Here the zeros and poles of f in D are counted with their multiplicities.

Proof: The proof is similar to the proof of Theorem 16.2. We only note that if p_k is a pole of order $N = N_k$ of f, then we have for $0 < |z - p_k| < \varepsilon$:

$$f(z) = a_{-N}(z - p_k)^{-N}(1 + h_1(z)), \quad a_{-N} \neq 0 \; ,$$

and

$$f'(z) = (-N)a_{-N}(z - p_k)^{-N-1}(1 + h_2(z)) \; ,$$

thus

$$\frac{f'(z)}{f(z)} = \frac{-N}{z - p_k} + h_3(z) \; .$$

Here $h_{1,2,3}$ are holomorphic near p_k. The claim then follows as in the proof of Theorem 16.2. $\diamond$

17.7 Rouché's Theorem and Hurwitz's Theorem

Rouché's Theorem [6] is very useful if one studies the solutions of an equation $f(z) = 0$ under perturbations of f. In the following theorem the perturbed equation is $g(z) = 0$.

Theorem 17.6 *(Rouché) Let $U \subset \mathbb{C}$ be open and let $f, g \in H(U)$. Let $\bar{D}(P, r) \subset U$. Assume that f and g are close to each other in the sense that*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \quad for \quad |z - P| = r \; . \tag{17.2}$$

[6] Eugene Rouché (1832–1910) was a French mathematician.

Then f and g have the same number of zeros in $D(P,r)$ where zeros are counted with their multiplicities. In other words, if $\gamma(t) = P + re^{it}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz \ . \tag{17.3}$$

Proof: Let $|z-P| = r$. Then (17.2) implies that

$$f(z) \neq 0 \neq g(z) \ .$$

Therefore the integrals in (17.3) are defined. We claim that, for $|z-P| = r$, the complex number

$$\lambda := \frac{f(z)}{g(z)}$$

does not belong to $(-\infty, 0]$. Otherwise,

$$\begin{aligned} \Big|\frac{f(z)}{g(z)} - 1\Big| &= |\lambda - 1| \\ &= -\lambda + 1 \\ &= \Big|\frac{f(z)}{g(z)}\Big| + 1 \end{aligned}$$

Multiplying by $|g(z)|$ one obtains that

$$|f(z) - g(z)| = |f(z)| + |g(z)|$$

in contradiction to the assumption (17.2).

Consider the function

$$f_t(z) = tf(z) + (1-t)g(z), \quad z \in U \ ,$$

for $0 \le t \le 1$. If $|z-P| = r$ then $f_t(z) \neq 0$ since, otherwise, one obtains that $f(z)/g(z)$ is negative. It follows that

$$I(t) = \frac{1}{2\pi i}\int_\gamma \frac{f_t'(z)}{f_t(z)}\,dz, \quad 0 \le t \le 1 \ ,$$

is integer valued and continuous. Therefore, $I(0) = I(1)$, proving the theorem. $\diamond$

Example: Let $f(z) = z^7 + 5z^3 - z - 2$ and $g(z) = 5z^3$. For $|z| = 1$ we have

$$|f(z) - g(z)| = |z^7 - z - 2| \le 4$$

and

$$|g(z)| = 5 \ .$$

Therefore, the assumption (17.3) holds for $P = 0$ and $r = 1$. Clearly, $g(z) = 5z^3$ has a zero of multiplicity 3 in $D(0,1)$, and has no other zero. Therefore, by Rouché's theorem, the polynomial $f(z)$ has exactly three zeros z_j with $|z_j| < 1$, counting multiplicities. These three zeros are not necessarily distinct.

Rouché's Theorem is often formulated somewhat differently by assuming that the function $g(z)$ has the form

$$g(z) = f(z) + p(z)$$

where $p(z)$ perturbs $f(z)$.

Theorem 17.7 *(Rouché, 2nd version) Let $U \subset \mathbb{C}$ be open and let $f, p \in H(U)$. Let $\bar{D}(P, r) \subset U$. Assume that $p(z)$ is small so that*

$$|p(z)| < |f(z)| + |f(z) + p(z)| \quad for \quad |z - P| = r \ . \tag{17.4}$$

Then f and $f + p$ have the same number of zeros in $D(P, r)$ where zeros are counted with their multiplicities. In other words, if $\gamma(t) = P + re^{it}$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + p'(z)}{f(z) + p(z)}\, dz \ . \tag{17.5}$$

The following theorem of Hurwitz [7] is often used in limit arguments.

Theorem 17.8 *(Hurwitz) Let U be a region and let $f_n \in H(U)$ for $n = 1, 2, \ldots$ Assume that $f_n(z)$ converges locally uniformly to $f(z)$. (Thus, $f \in H(U)$.) If $f_n(z) \neq 0$ for all $z \in U$ and all $n = 1, 2, \ldots$, then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in U$.*

Proof: Suppose that f is not identically zero, but $f(P) = 0$ for some $P \in U$. Let M denote the multiplicity of the zero P of f,

$$M = mult_f(P) \geq 1 \ .$$

There exists $r > 0$ with $\bar{D}(P, r) \subset U$ and $f(z) \neq 0$ for $0 < |z - P| \leq r$. Let $\gamma(t) = P + re^{it}, 0 \leq t \leq 2\pi$. One obtains that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = M, \quad \text{but} \quad \frac{1}{2\pi i} \int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = 0$$

for all n. As $n \to \infty$, the quotient $f_n'(z)/f_n(z)$ converges uniformly on γ to $f'(z)/f(z)$, and we obtain a contradiction. $\diamond$

Details: Details regarding uniform convergence on the curve γ: For every $z \in \gamma$ there exists $\varepsilon(z) > 0$ so that $f_n'(\zeta)/f_n(\zeta)$ converges uniformly on $D(z, \varepsilon(z))$ to $f'(\zeta)/f(\zeta)$. Since the curve γ is a compact set, there exist finitely many points $z_1, \ldots, z_J$ so that

$$\gamma \subset \cup_{j=1}^J D(z_j, \varepsilon(z_j)) \ ,$$

and uniform convergence on γ follows.

[7] Adolf Hurwitz (1859–1919) was a German mathematician.

17.8 An Application of Rouché's Theorem

Lemma 17.1 *Let*

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

denote a normalized polynomial. Then there exists $z \in \mathbb{C}$ with

$$|p(z)| \geq 1 \quad \textit{and} \quad |z| = 1 \ .$$

Proof: Set $f(z) = z^n$ and

$$g(z) = -\Big(a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \Big) \ ,$$

thus

$$p(z) = f(z) - g(z) \ .$$

Note that $|f(z)| = |z^n| = 1$ for $|z| = 1$. We may assume that $g(z)$ is not identically zero. (Otherwise the claim is trivial.) Suppose that

$$|p(z)| = |f(z) - g(z)| < 1 \quad \text{for all} \quad z \quad \text{with} \quad |z| = 1 \ .$$

Then, by Rouché's Theorem, the functions $f(z) = z^n$ and $g(z)$ have the same number of zeros in $D(0,1)$. However, $f(z) = z^n$ has a zero of multiplicity n at $z = 0$, and $g(z)$ has only $n-1$ zeros. This contradiction proves that there exists $z \in \mathbb{C}$ with $|z| = 1$ and $|p(z)| \geq 1$. $\diamond$

17.9 Another Proof of the Fundamental Theorem of Algebra

Let

$$p(z) = z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

denote a normalized polynomial of degree $n \geq 1$. We claim that $p(z)$ has n zeros. The function $g(z) = z^n$ has n zeros and

$$|g(z)| = R^n \quad \text{for} \quad |z| = R \ .$$

If R is large then

$$|p(z) - g(z)| \leq C R^{n-1} < R^n = |g(z)| \quad \text{for} \quad |z| = R \geq C \ .$$

By Rouché's Theorem the functions $p(z)$ and $g(z) = z^n$ have the same number of zeros in $D(0,R)$ if R is large.

18 Matrix–Valued and Operator Valued Analytic Functions

Summary: Let $A \in \mathbb{C}^{n\times n}$ denote a square matrix and let $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ denote the set of eigenvalues of A. The matrix valued holomorphic function

$$(zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A) \ ,$$

is called the resolvent of A. It is an important generalization of the scalar function $\frac{1}{z-a}$ where $a \in \mathbb{C}$. We will show: If $\Gamma \subset \mathbb{C} \setminus \sigma(A)$ is a positively oriented simply closed curve then the matrix

$$P_A := \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} \, dz$$

is the projector onto U along V where U is the sum of the generalized eigenspaces to the eigenvalues λ_j inside Γ and V is the sum of the generalized eigenspaces to the eigenvalues λ_j outside Γ.

If $a \in \mathbb{C}$ lies inside Γ then, under suitable assumptions on the holomorphic function $\phi(z)$, we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{\phi(z)}{z - a} \, dz = \phi(a) \ .$$

If the eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie inside Γ then one can use the corresponding formula

$$\frac{1}{2\pi i} \int_\Gamma \phi(z)(zI - A)^{-1} \, dz = \phi(A)$$

to obtain the matrix $\phi(A)$. Under suitable assumptions, generalizations to unbounded linear operators A on Banach spaces are possible and one can study e^{At} using the resolvent of A.

18.1 Outline and Examples

Let $\gamma(t), a \leq t \leq b$, denote a parameterization of a simply closed positively oriented curve in $\mathbb{C}$. We denote the curve again by γ. If λ is a complex number, $\lambda \notin \gamma$, then we have by the residue theorem:

$$\frac{1}{2\pi i} \int_\gamma (z - \lambda)^{-1} \, dz = \begin{cases} 1, & \lambda \quad \text{inside} \quad \gamma \\ 0, & \lambda \quad \text{outside} \quad \gamma \end{cases} \tag{18.1}$$

It is interesting that one can generalize the formula to the case where λ is replaced by a matrix $A \in \mathbb{C}^{n\times n}$ or a more general operator defined on a dense subspace of a Banach space. We will consider here only the case of a matrix A, but generalizations are possible and important.

Let $A \in \mathbb{C}^{n\times n}$. With $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ we denote the set of distinct eigenvalues of A. (More generally, $\sigma(A)$ denotes the spectrum of the operator A.) The matrix valued function

$$(zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A) \ ,$$

is called the resolvent of A. By Cramer's rule, each matrix entry

$$((zI - A)^{-1})_{jk}$$

is a rational function of z defined for $z \in \mathbb{C} \setminus \sigma(A)$. See Section 18.6.

Assume that γ is a curve, as above, and $\lambda_j \notin \gamma$ for $j = 1, \ldots, s$. We set

$$P_A := \frac{1}{2\pi i}\int_\gamma (zI-A)^{-1}\,dz\ , \tag{18.2}$$

where the integral is defined elementwise, i.e.,

$$(P_A)_{jk} = \frac{1}{2\pi i}\int_\gamma \Big((zI-A)^{-1}\Big)_{jk}\,dz, \quad 1 \le j,k \le n\ .$$

Example 1: Let A denote the 4×4 diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}\ .$$

We have

$$(zI-A)^{-1} = \begin{pmatrix} (z-\lambda_1)^{-1} & & & \\ & (z-\lambda_2)^{-1} & & \\ & & (z-\lambda_3)^{-1} & \\ & & & (z-\lambda_4)^{-1} \end{pmatrix}\ .$$

Assume that $\lambda_{1,2}$ lie inside and $\lambda_{3,4}$ lie outside γ. Using (18.1) it is then clear that

$$P_A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\ .$$

We now give an interpretation of P_A: Let $e^1,\ldots,e^4$ denote the standard bases of $\mathbb{C}^4$. For the diagonal matrix A, the space $U = span\{e^1,e^2\}$ is the sum of the eigenspaces of $\lambda_{1,2}$ and $V = span\{e^3,e^4\}$ is the sum of the eigenspaces of $\lambda_{3,4}$. The matrix P_A is the projector onto U along V.

The result generalizes. Even if $A\in\mathbb{C}^{n\times n}$ is not diagonalizable, the matrix P_A defined in (18.2) is the projector onto a space U along a space V. Here U is the sum of the generalized eigenspaces of the eigenvalues inside γ and V is the sum of the generalized eigenspaces of the eigenvalues outside γ.

Definition 1: Let $A\in\mathbb{C}^{n\times n}$. If λ_j is an eigenvalue of A, then

$$E(\lambda_j) = \{u\in\mathbb{C}^n\ :\ (A-\lambda_j I)u = 0\}$$

is the geometric eigenspace of A to the eigenvalue λ_j and

$$G(\lambda_j) = \Big\{u\in\mathbb{C}^n\ :\ (A-\lambda_j I)^m u = 0 \quad \text{for} \quad \text{some} \quad m\in\{1,2,\ldots,n\}\Big\}$$

is the generalized eigenspace to λ_j.

Example 2: Let A denote the 2×2 matrix:

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} = \lambda_1 I + J$$

with

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

The only eigenvalue of A is λ_1, which is geometrically simple, but algebraically double. The geometric eigenspace is

$$E(\lambda_1) = span\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

and the generalized eigenspace is $G(\lambda_1) = \mathbb{C}^2$.

For $z \neq \lambda_1$ we have

$$zI - A = (z - \lambda_1)I - J = (z - \lambda_1)\Big(I - \frac{1}{z - \lambda_1}\, J\Big) .$$

Since $J^2 = 0$ one obtains:

$$(zI - A)^{-1} = \frac{1}{z - \lambda_1}\Big(I + \frac{1}{z - \lambda_1}\, J\Big) = \frac{1}{z - \lambda_1}\, I + \frac{1}{(z - \lambda_1)^2}\, J .$$

For

$$P_A = \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1}\, dz$$

one obtains $P_A = I$ if λ_1 lies inside and $P_A = 0$ if λ_1 lies outside γ.

This result indicates that the generalized eigenspace is important, not the geometric eigenspace. We will prove:

Theorem 18.1 *Let $A \in \mathbb{C}^{n\times n}$ denote a matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_s$. Let γ be a simply closed positively oriented curve in $\mathbb{C}$. Set*

$$U = G(\lambda_1) \oplus \ldots \oplus G(\lambda_k)$$

and

$$V = G(\lambda_{k+1}) \oplus \ldots \oplus G(\lambda_s)$$

where $\lambda_1, \ldots, \lambda_k$ lie inside and $\lambda_{k+1}, \ldots, \lambda_s$ lie outside γ. Then

$$P_A = \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1}\, dz \tag{18.3}$$

is the projector onto U along V.

We will prove the theorem in Section 18.4

The formula for P_A is useful if one studies perturbations of A. Assume, for example, that $A = A(w)$ depends analytically in a parameter $w \in \mathbb{C}$. The eigenvalues $\lambda_j(w)$ are continuous functions of w (if this is properly defined), but they are generally not smooth functions of w unless they are algebraically simple.[8]

[8] An eigenvalue λ of a matrix A is called algebraically simple, if λ is a simple zero of the characteristic polynomial $p(z) = det(zI - A)$.

The formula (18.3) shows, however, that $P_{A(w)}$ depends analytically on w as long as the eigenvalues $\lambda_j(w)$ do not cross γ. Thus, the projector P_A behaves better under perturbations of A than the eigenvalues of A.

Example 3: Let $A(w)$ denote a 2×2 matrix

$$A(w) = \begin{pmatrix} 0 & 1 \\ w & 0 \end{pmatrix} , \quad w \in \mathbb{C} .$$

The eigenvalues are

$$\lambda_1 = \sqrt{w}, \quad \lambda_2 = -\sqrt{w} .$$

These functions are not differentiable at $w = 0$ and are not analytic in $\mathbb{C} \setminus \{0\}$.

18.2 Analyticity of the Resolvent

Lemma 18.1 *Let $A \in \mathbb{C}^{n\times n}$ and let $\sigma(A)$ denote the set of eigenvalues of A. Then each matrix entry of the resolvent $(zI - A)^{-1}$,*

$$((zI - A)^{-1})_{jk} \tag{18.4}$$

is a rational function on $\mathbb{C} \setminus \sigma(A)$.

This result follows from Cramer's rule for the inverse of a matrix. See Section 18.6.

Another way to prove analyticity of the functions (18.4) uses the Neumann series. See Section 18.7. This proof generalizes to operators in Banach spaces.

18.3 Complementary Subspaces and Projectors

We want to make the concept of a projector onto a space U along a space V precise.

Definition 2: Let W be a vector space. Two subspaces U and V of W are called *complementary subspaces* of W if for every $w \in W$ there exists a unique $u \in U$ and a unique $v \in V$ with

$$w = u + v, \quad u \in U, \quad v \in V .$$

If U, V are complementary subspaces of W one writes

$$W = U \oplus V$$

and calls W the direct sum of U and V.

Definition 3: Let W be a vector space. A linear map $P : W \to W$ is called a projector if $P^2 = P$.

There is a close relation between pairs of complementary subspaces of W and projectors P from W into itself. The following is not difficult to prove:

Theorem 18.2 *1. Let U, V be complementary subspaces of W. The map $P : W \to W$ defined by*

$$Pw = u \quad \text{where} \quad w = u + v, \quad u \in U, \quad v \in V ,$$

is a projector. We have

$$
\begin{aligned}
U &= R(P) = \textit{ range of } P \\
V &= N(P) = \textit{ nullspace of } O
\end{aligned}
$$

The linear map P is called the projector onto U along V.

2. Let $P : W \to W$ be a projector. Then the subspaces

$$
U := R(P), \quad V := N(P) \ ,
$$

are complementary and the projector onto U along V is P.

3. If $P : W \to W$ is a projector, then $Q = I - P$ is also a projector. We have

$$
R(P) = N(Q), \quad N(P) = R(Q) \ .
$$

18.3.1 The Matrix Representation of a Projector

In the following, let U and V denote subspaces of $\mathbb{C}^n$ and assume that

$$
\mathbb{C}^n = U \oplus V \ .
$$

Let

$$
t^1, \ldots, t^r
$$

be a basis of U and let

$$
t^{r+1}, \ldots, t^n
$$

be a basis of V. Then

$$
T = (t^1, \ldots, t^n) \in \mathbb{C}^{n \times n}
$$

is a nonsingular matrix.

Lemma 18.2 *Under the above assumptions, the projector P onto U along V has the matrix representation*

$$
P = T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \ . \tag{18.5}
$$

Proof: Let P denote the projector onto U along V. If $w \in \mathbb{C}^n$ is any given vector, we write

$$
w = x_1 t^1 + \ldots + x_n t^n = Tx, \quad x \in \mathbb{C}^n \ ,
$$

and obtain

$$
u := Pw = x_1 t^1 + \ldots + x_r t^r \ .
$$

This holds since $Pt^j = t^j$ for $1 \le j \le r$ and $Pt^j = 0$ for $r+1 \le j \le n$. If we write x in the form

$$
x = \begin{pmatrix} x^I \\ x^{II} \end{pmatrix}, \quad x^I \in \mathbb{C}^r, \quad x^{II} \in \mathbb{C}^{n-r} \ ,
$$

then we have

$$
\begin{aligned}
u &= Pw \\
&= T \begin{pmatrix} x^I \\ 0 \end{pmatrix} \\
&= T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} x \\
&= T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} w \ .
\end{aligned}
$$

The equation $x = T^{-1}w$ holds since $w = Tx$ by the definition of x. This proves the formula (18.5) for the projector P onto U along V. $\diamond$

18.4 Proof of Theorem 18.1

First assume, for simplicity, that A is diagonalizable. This holds if and only if all generalized eigenspaces agree with the geometric eigenspaces, $G(\lambda_j) = E(\lambda_j)$ for $j = 1, \ldots, s$. In this case,

$$U = E(\lambda_1) \oplus \ldots \oplus E(\lambda_k)$$

and

$$V = E(\lambda_{k+1}) \oplus \ldots \oplus E(\lambda_s) \ .$$

Let $t^1, \ldots, t^r$ denote a basis of U, consisting of eigenvectors of A, and let $t^{r+1}, \ldots, t^n$ denote a basis of V, consisting of eigenvectors of A. Set

$$T = (t^1, \ldots, t^n) \in \mathbb{C}^{n \times n} \ .$$

We have

$$AT = T\Lambda$$

where

$$\Lambda = diag\Big(\lambda_1, \ldots \lambda_k, \lambda_{k+1}, \ldots, \lambda_s\Big) \ .$$

For $z \in \mathbb{C} \setminus \sigma(A)$:

$$
\begin{aligned}
A &= T\Lambda T^{-1} \\
zI - A &= T(zI - \Lambda)T^{-1} \\
(zI - A)^{-1} &= T(zI - \Lambda)^{-1} T^{-1}
\end{aligned}
$$

It follows that

$$\begin{aligned} P_A &= \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1}\, dz \\ &= \frac{1}{2\pi i}\, T \int_\gamma (zI - \Lambda)^{-1}\, dz\, T^{-1} \\ &= T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \end{aligned}$$

By Lemma 18.2 the matrix P_A is the projector onto U along V.

Next consider the general case where A is not necessarily diagonalizable. By Schur's Theorem and Blocking (or by transformation to Jordan normal form) there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$ so that $T^{-1}AT$ has block diagonal form:

$$T^{-1}AT = diag\Big(B_1, \ldots, B_k, B_{k+1}, \ldots, B_s\Big) =: B\ .$$

Here

$$B_j = \lambda_j I_{\alpha_j} + R_j, \quad j = 1, \ldots, s\ ,$$

where α_j is the dimension of the generalized eigenspace $G(\lambda_j)$ and

$$R_j \in \mathbb{C}^{\alpha_j \times \alpha_j}$$

is strictly upper triangular. Therefore,

$$R_j^m = 0 \quad \text{for} \quad m \geq \alpha_j\ .$$

From $T^{-1}AT = B$ obtain that

$$(zI - A)^{-1} = T(zI - B)^{-1}T^{-1}\ .$$

Consider a term

$$Q_j := \frac{1}{2\pi i} \int_\gamma (zI_{\alpha_j} - B_j)^{-1}\, dz\ .$$

We have for $z \neq \lambda_j$:

$$\begin{aligned} zI_{\alpha_j} - B_j &= (z - \lambda_j)I_{\alpha_j} - R_j \\ &= (z - \lambda_j)\Big(I_{\alpha_j} - \frac{1}{z - \lambda_j} R_j\Big) \end{aligned}$$

Therefore, for $z \in \mathbb{C} \setminus \sigma(A)$:

$$\Big(zI_{\alpha_j} - B_j\Big)^{-1} = \frac{1}{z - \lambda_j} \Big(I_{\alpha_j} + \sum_{m=1}^{\alpha_j - 1} (z - \lambda)^{-m} R_j^m\Big)\ .$$

It follows that

$$Q_j = I_{\alpha_j} \quad \text{for} \quad 1 \leq j \leq k$$

and

$$Q_j = 0 \quad \text{for} \quad k+1 \le j \le s \ .$$

Therefore,

$$\begin{aligned} P_A &= \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1}\, dz \\ &= \frac{1}{2\pi i}\, T \int_\gamma (zI - B)^{-1}\, dz\, T^{-1} \\ &= T \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \end{aligned}$$

We have shown that the matrix P_A is the projector onto U along V. $\diamond$

18.5 The Dunford–Taylor Integral

We first recall some familiar facts.

Let Γ denote a positively oriented, simply closed curve in $\mathbb{C}$. Then $\mathbb{C} \setminus \Gamma$ has two connected components, the interior of Γ and the exterior of Γ. These are denoted by

$$\text{int}\, \Gamma \quad \text{and} \quad \text{ext}\, \Gamma \ ,$$

respectively. Let $a \in \mathbb{C} \setminus \Gamma$. We have

$$\frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-a} = 1 \quad \text{if} \quad a \in \text{int}\, \Gamma$$

and

$$\frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-a} = 0 \quad \text{if} \quad a \in \text{ext}\, \Gamma \ .$$

Lemma 18.3 *Let $U \subset \mathbb{C}$ be an open set containing Γ and* int Γ. *Let $\phi \in H(U)$. Assuming $a \in$ int Γ, we have*

$$\frac{1}{2\pi i} \int_\Gamma \frac{\phi(z)}{z-a}\, dz = \phi(a) \ . \tag{18.6}$$

In particular, for $\phi(z) = z^j$, one obtains that

$$\frac{1}{2\pi i} \int_\Gamma \frac{z^j}{z-a}\, dz = a^j \quad \textit{for} \quad j = 0, 1, 2, \ldots$$

We want to generalize the formula (18.6) to the case where the number a is replaced by a matrix $A \in \mathbb{C}^{n \times n}$. Then $1/(z-a)$ will be replaced by

$$(zI - A)^{-1} \ .$$

From our previous results, we have:

Lemma 18.4 *Let Γ be a curve as above and let $A \in \mathbb{C}^{n\times n}$. Assuming that*

$$\sigma(A) \subset \text{int } \Gamma$$

we have

$$\frac{1}{2\pi i}\int_\Gamma (zI-A)^{-1}\,dz = I\ .$$

We now introduce a scalar function $\phi(z)$ multiplying $(zI-A)^{-1}$ in the integral, i.e., we consider the so-called Dunford–Taylor integral

$$\frac{1}{2\pi i}\int_\Gamma \phi(z)(zI-A)^{-1}\,dz\ . \tag{18.7}$$

Here it is assumed that $\phi: U \to \mathbb{C}$ is holomorphic in U where $U \subset \mathbb{C}$ is an open set containing Γ and $\text{int}\,\Gamma$.

Under suitable assumptions, the formula (18.7) can be used to *define* the matrix $\phi(A)$ in a reasonable way.

18.5.1 The Case of a Polynomial

We first prove:

Lemma 18.5 *Let Γ be a curve as above and let $A \in \mathbb{C}^{n\times n}$ with $\sigma(A) \subset int\,\Gamma$. For $j = 0, 1, \ldots$ we have*

$$\frac{1}{2\pi i}\int_\Gamma z^j(zI-A)^{-1}\,dz = A^j\ .$$

Proof: Write

$$\begin{aligned}
(zI)^j &= (zI-A+A)^j \\
&= (zI-A)^j + \ldots + A^j \\
&= \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} (zI-A)^k A^{j-k} \\
&= A^j + \sum_{k=1}^{j} \begin{pmatrix} j \\ k \end{pmatrix} (zI-A)^k A^{j-k}
\end{aligned}$$

Consider a term

$$(zI-A)^k A^{j-k}(zI-A)^{-1} = A^{j-k}(zI-A)^{k-1}\ .$$

If $k \geq 1$ then the above function is holomorphic as a function of z and the corresponding integral is zero. Thus, a nontrivial contribution is obtained for $k = 0$ only. One obtains

$$
\begin{aligned}
\int_\Gamma z^j (zI - A)^{-1}\, dz &= \int_\Gamma A^j (zI - A)^{-1}\, dz \\
&= A^j \int_\Gamma (zI - A)^{-1}\, dz \\
&= 2\pi i\, A^j
\end{aligned}
$$

This proves the lemma. $\diamond$

If

$$p(z) = \sum_{j=0}^N a_j z^j$$

is a polynomial, then one defines

$$p(A) = \sum_{j=0}^N a_j A^j \ .$$

Using the previous lemma, it is clear that

$$\frac{1}{2\pi i} \int_\Gamma p(z)(zI - A)^{-1}\, dz = p(A) \ . \tag{18.8}$$

18.5.2 The Case of a Power Series

Next let

$$\phi(z) = \sum_{j=0}^\infty a_j z^j, \quad |z| < \rho \ , \tag{18.9}$$

denote a convergent power series with radius of convergence ρ, $0 < \rho \le \infty$. We let

$$\phi_N(z) = \sum_{j=0}^N a_j z^j, \quad |z| < \rho$$

denote the partial sums of $\phi(z)$.

Lemma 18.6 *Let $\phi(z)$ denote the power series (18.9). If $\sigma(A) \subset D(0,\rho)$ then the sequence of matrices*

$$S_N := \phi_N(A) = \sum_{j=0}^N a_j A^j, \quad N = 1, 2, \ldots$$

converges in $\mathbb{C}^{n \times n}$. The limit is denoted by

$$\lim_{N\to\infty} S_N = \phi(A) = \sum_{j=0}^\infty a_j A^j \ .$$

Proof: Since the spectral radius of A is strictly less than ρ, there exists a vector norm $\| \cdot \|$ on $\mathbb{C}^n$ so that the corresponding matrix norm of A satisfies

$$r := \|A\| < \rho \ .$$

For $N > M \geq N(\varepsilon)$ we have (note that $\|A^j\| \leq \|A\|^j = r^j$):

$$\begin{aligned} \|S_N - S_M\| &= \| \sum_{j=M+1}^{N} a_j A^j \| \\ &\leq \sum_{j=M+1}^{N} |a_j| r^j \\ &\leq \varepsilon \ . \end{aligned}$$

Thus, S_N is a Cauchy sequence in $\mathbb{C}^{n\times n}$. $\diamond$

Let us connect this result with the Dunford–Taylor integral.

Theorem 18.3 *We make the same assumptions on A and $\phi(z)$ as in the previous lemma. Let Γ be a positively oriented, simply closed curve in $D(0,\rho)$ with $\sigma(A) \subset int\,\Gamma$. Then we have*

$$\frac{1}{2\pi i} \int_\Gamma \phi(z)(zI - A)^{-1}\, dz = \phi(A) \ . \qquad (18.10)$$

Here $\phi(A)$ is defined as the limit of the matrix sequence $\phi_N(A)$ considered in Lemma 18.5.

Proof: By (18.8) we have

$$\frac{1}{2\pi i} \int_\Gamma \phi_N(z)(zI - A)^{-1}\, dz = \phi_N(A)$$

for every finite $N = 1, 2, \ldots$. Taking the limit as $N \to \infty$, we obtain (18.10). $\diamond$

Example: Let $t \in \mathbb{R}$ be fixed and let $\phi(z) = e^{tz}$. If $A \in \mathbb{C}^{n\times n}$ is any matrix and if Γ is a positively oriented, simply closed curve surrounding $\sigma(A)$, then

$$e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{zt}(zI - A)^{-1}\, dz \ .$$

Here, by definition, $e^{tA} = \sum_{j=0}^{\infty} \frac{1}{j!} (tA)^j$.

18.5.3 A General Holomorphic Function

The formula

$$\frac{1}{2\pi i} \int_\Gamma \phi(z)(zI - A)^{-1}\, dz =: \phi(A) \qquad (18.11)$$

can be used to *define* $\phi(A)$ under more general assumptions than those of Theorem 18.3, where $\phi(z)$ was assumed to be a power series. All one needs is $\phi \in H(U)$ and a positively oriented, simply closed curve Γ in U with

$$\sigma(A) \subset int\,\Gamma \subset U \ .$$

Example: Let A be any nonsingular matrix. Choose a simply connected region U with

$$0 \notin U, \quad \sigma(A) \subset U \ .$$

We know that there exists a logarithm function $\log_U \in H(U)$ with

$$\exp(\log_U(z)) = z \quad \text{for all} \quad z \in U \ .$$

If Γ is a positively oriented, simply closed curve in U with $\sigma(A) \subset \text{ int } \Gamma$, then

$$\log_U(A) := \frac{1}{2\pi i} \int_\Gamma \log_U(z)(zI - A)^{-1}\, dz$$

is a well–defined matrix. We will prove that

$$\exp(\log_U(A)) = A \ .$$

Theorem 18.4 *Let $U \subset \mathbb{C}$ be open and simply connected and assume that $0 \notin U$. Let $\log(z)$ denote a holomorphic function on U with*

$$\exp(\log(z)) = z \quad \textit{for all} \quad z \in U \ .$$

Let $A \in \mathbb{C}^{n\times n}$ denote a nonsingular matrix with $\sigma(A) \subset U$ and let Γ denote a positively oriented simply closed curve in U surrounding $\sigma(A)$. Then the matrix

$$B := \frac{1}{2\pi i} \int_\Gamma \log(z)(zI - A)^{-1}\, dz$$

satisfies $e^B = A$.

To prove the theorem we will use the following result.

Theorem 18.5 *Let $U \subset \mathbb{C}$ be open and simply connected; let $f, g \in H(U)$. Let $A \in \mathbb{C}^{n\times n}$ denote a matrix with $\sigma(A) \subset U$ and let Γ denote a positively oriented simply closed curve in U surrounding $\sigma(A)$. If*

$$\begin{aligned} B_1 &:= \frac{1}{2\pi i} \int_\Gamma f(z)(zI - A)^{-1}\, dz \\ B_2 &:= \frac{1}{2\pi i} \int_\Gamma g(z)(zI - A)^{-1}\, dz \end{aligned}$$

then

$$B_1 B_2 = \frac{1}{2\pi i} \int_\Gamma f(z)g(z)(zI - A)^{-1}\, dz \ .$$

Proof: Using transformation to Jordan normal form it suffices to prove the theorem for matrices $A = \lambda I + J$ where

$$J = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{R}^{n\times n} \ .$$

In the following we assume that $A = \lambda I + J$ and let

$$f(z) = \sum_{j=0}^{\infty} a_j (z-\lambda)^j, \quad g(z) = \sum_{k=0}^{\infty} b_k (z-\lambda)^k \quad \text{for} \quad |z-\lambda| < r \ .$$

Then let

$$\begin{aligned} h(z) &:= f(z)g(z) \\ &= \Big(\sum_{j=0}^{\infty} a_j (z-\lambda)^j \Big) \cdot \Big(\sum_{k=0}^{\infty} b_k (z-\lambda)^k \Big) \\ &= \sum_{l=0}^{\infty} c_l (z-\lambda)^l \end{aligned}$$

with

$$c_l = \sum_{j=0}^{l} a_j b_{l-j} \ .$$

We have

$$B_1 = \frac{1}{2\pi i} \sum_{j=0}^{\infty} a_j \int_\Gamma (z-\lambda)^j (zI-A)^{-1} \, dz \ .$$

Here, for $z \neq \lambda$,

$$zI - A = (z-\lambda)I - J = (z-\lambda)\Big(I - \frac{1}{z-\lambda} J \Big) \ ,$$

thus

$$\begin{aligned} (zI-A)^{-1} &= \frac{1}{z-\lambda} \sum_{k=0}^{n-1} \frac{1}{(z-\lambda)^k} J^k \\ &= \sum_{k=0}^{n-1} (z-\lambda)^{-k-1} J^k \end{aligned}$$

Therefore,

$$\frac{1}{2\pi i} \int_\Gamma (z-\lambda)^j (zI-A)^{-1} \, dz = J^j$$

and

$$B_1 = \sum_{j=0}^{n-1} a_j J^j \ .$$

In the same way it follows that

$$B_2 = \sum_{k=0}^{n-1} b_k J^k$$

and

$$\frac{1}{2\pi i} \int_\Gamma f(z)g(z)(zI - A)^{-1}\, dz = \sum_{l=0}^{n-1} c_l J^l$$

with

$$c_l = \sum_{j=0}^{l} a_j b_{l-j} \ .$$

Also,

$$\begin{aligned} B_1 B_2 &= \Big(\sum_{j=0}^{n-1} a_j J^j\Big) \cdot \Big(\sum_{k=0}^{n-1} b_k J^k\Big) \\ &= \sum_{l=0}^{n-1}\sum_{j=0}^{n-1} a_j b_{l-j}\, J^j \\ &= \sum_{l=0}^{n-1} c_l J^l \end{aligned}$$

This proves that

$$\frac{1}{2\pi i} \int_\Gamma f(z)g(z)(zI - A)^{-1}\, dz = B_1 B_2$$

for $A = \lambda I + J$. The case of a general matrix A then follows by transformation to Jordan normal form. $\diamond$

Proof of Theorem 18.4: Recall that

$$B := \frac{1}{2\pi i} \int_\Gamma \log(z)(zI - A)^{-1}\, dz \ .$$

We claim that

$$\frac{1}{2\pi i} \int_\Gamma (\log z)^j (zI - A)^{-1}\, dz = B^j \quad \text{for} \quad j = 0, 1, 2 \ldots \tag{18.12}$$

We have $B^0 = I$ and (18.12) follows from Lemma 18.5 for $j = 0$. For $j \geq 2$ the formula (18.12) follows from Theorem 18.5 by induction in j. Obtain that

$$\frac{1}{2\pi i} \int_\Gamma \Big(\sum_{j=0}^{N} \frac{1}{j!}(\log z)^j\Big)(zI - A)^{-1}\, dz = \sum_{j=0}^{N} \frac{1}{j!}\, B^j \ .$$

We let $N \to \infty$ and note that

$$\sum_{j=0}^{\infty} \frac{1}{j!} (\log z)^j = \exp(\log z) = z \ .$$

Therefore,

$$\frac{1}{2\pi i} \int_{\Gamma} z(zI - A)^{-1} \, dz = \sum_{j=0}^{\infty} \frac{1}{j!} B^j = e^B \ .$$

Using Lemma 18.5 with $j = 1$ yields that $A = e^B$. $\diamond$

18.5.4 Remarks on Unbounded Operators

An important point of the formula

$$\frac{1}{2\pi i} \int_{\Gamma} \phi(z)(zI - A)^{-1} \, dz =: \phi(A) \tag{18.13}$$

is that one can use it for linear operators A more general than matrices, even for unbounded operators A that are densely defined in some Banach space. Such operators A appear when one formulates initial value problems for PDEs abstractly as

$$u_t = Au, \quad u(0) = u^{(0)}, \quad t \geq 0 \ .$$

The formal solution is

$$u(t) = e^{tA} u^{(0)}, \quad t \geq 0 \ ,$$

but if A is unbounded, one cannot use the exponential series to define e^{tA}. Instead, one considers the resolvent

$$(zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A) \ ,$$

and (under suitable assumptions) defines

$$e^{tA} := \frac{1}{2\pi i} \int_{\Gamma} e^{tz} (zI - A)^{-1} \, dz \ . \tag{18.14}$$

Typically, the spectrum $\sigma(A)$ is unbounded and Γ cannot surround $\sigma(A)$. Instead, Γ is chosen as an infinite line,

$$\Gamma \ : \ z(\xi) = b + i\xi, \quad -\infty < \xi < \infty \ ,$$

which must lie to the right of $\sigma(A)$. Since $dz = i \, d\xi$ one obtains

$$e^{tA} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{t(b+i\xi)} \Big((b + i\xi)I - A \Big)^{-1} d\xi \ . \tag{18.15}$$

(The Laplace transform of the scalar function e^{ta} is

$$\mathcal{L}(e^{ta})(s) = \int_0^{\infty} e^{-(s-a)t} \, dt = \frac{1}{s - a} \ .$$

The formulas (18.14) and (18.15) are versions of the inverse Laplace transform of an exponential.)

Details of these ideas lead to so–called semi–group theory, a part of functional analysis. The name *semi–group* arises since the family of operators

$$e^{tA}, \quad t \geq 0 \; ,$$

satisfies

$$e^{sA}e^{tA} = e^{(s+t)A} \quad \text{for all} \quad s,t \geq 0, \quad e^{0A} = I \; .$$

In other words, one can multiply the operators e^{tA}, obtaining the rules of associativity and commutativity. However, in general, the operator e^{tA} does not have an inverse since e^{-tA} does not exist for $t > 0$. Thus, the family of operators $e^{tA}, t \geq 0$, does not have the structure of a group.

18.6 Auxiliary Results: Determinants and Cramer's Rule

Determinants: For $A \in \mathbb{C}^{n \times n}$ the determinant is defined by

$$\det A = \sum_{\sigma \in S_n} \text{sgn}\,(\sigma) a_{1\sigma_1} \ldots a_{n\sigma_n} \; .$$

Here S_n denotes the group of all permutations of the set $\{1, 2, \ldots, n\}$.

A matrix $A \in \mathbb{C}^{n \times n}$ is non–singular if and only if $\det A \neq 0$.

The function

$$p_A(z) = \det\,(zI - A), \quad z \in \mathbb{C} \; ,$$

is a polynomial of degree n, the characteristic polynomial of A. By the fundamental theorem of algebra,

$$p_A(z) = \Pi_{j=1}^{s}(z - \lambda_j)^{\alpha_j} \; ,$$

where $\lambda_1, \ldots, \lambda_s$ are the distinct zeros of $p_A(z)$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}$ are the multiplicities of the zeros.

Eigenvalues: The numbers $\lambda_1, \ldots, \lambda_s$, introduced as the distinct zeros of the polynomial $p_A(z)$, are the distinct eigenvalues of the matrix A. The number $\alpha_j \in \mathbb{N}$ is the algebraic multiplicity of the eigenvalue λ_j for $j = 1, \ldots, s$. I.e., α_j is the dimension of the generalized eigenspace

$$G(\lambda_j) = \Big\{ u \in \mathbb{C}^n \; : \; (A - \lambda_j I)^m u = 0 \quad \text{for} \quad \text{some} \quad m \in \{1, 2, \ldots, n\} \Big\} \; .$$

Cramer's Rule: For $j, k \in \{1, 2, \ldots, n\}$ let $M_{jk} \in \mathbb{C}$ denote the determinant of the $(n-1) \times (n-1)$ matrix which is obtained by deleting the j–th row and the k–th column of A. Define the matrix $\hat{A} \in \mathbb{C}^{n \times n}$ by

$$(\hat{A})_{jk} = (-1)^{j+k} M_{jk} \quad \text{for} \quad j, k \in \{1, 2, \ldots\} \; .$$

If $\det A \neq 0$ then, by Cramer's Rule:

$$A^{-1} = \frac{1}{\det A}\,(\hat{A})^T \; .$$

Application to the Resolvent: Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ denote the set of eigenvalues of A. The function

$$\frac{1}{p_A(z)} = \frac{1}{\det(zI - A)}, \quad z \in \mathbb{C} \setminus \sigma(A) ,$$

is a rational function. It has a pole of order α_j at $z = \lambda_j$.

For every $j, k \in \{1, 2, \ldots, n\}$ and $z \in \mathbb{C} \setminus \sigma(A)$ we have

$$\Big((zI - A)^{-1}\Big)_{kj} = \frac{(-1)^{j+k}}{p_A(z)} M_{jk}(z)$$

where $M_{jk}(z)$ is the determinant of the $(n-1) \times (n-1)$ matrix which is obtained by deleting the j–th row and the k–th column of $zI - A$. It follows that $M_{jk}(z)$ is a polynomial of degree $\leq n - 1$.

Therefore, every entry of the resolvent

$$(zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A) ,$$

is a rational function of z. Poles only occur at the eigenvalues of A.

Since the degree of every polynomial $M_{jk}(z)$ is $\leq n - 1$ and the degree of $p_A(z)$ equals n, it follows that there exist constants $C > 0$ and $R > 0$ so that

$$|(zI - A)^{-1}| \leq \frac{C}{|z|} \quad \text{for} \quad z \in \mathbb{C} \quad \text{with} \quad |z| \geq R .$$

18.7 Auxiliary Results: Analyticity of the Resolvent

We will use the geometric sum formula for matrices: If $P \in \mathbb{C}^{n \times n}$ and $\|P\| < 1$ then

$$(I - P)^{-1} = \sum_{j=0}^{\infty} P^j .$$

Let $A \in \mathbb{C}^{n \times n}$ have the spectrum

$$\sigma(A) = \{\lambda_1, \ldots, \lambda_s\} .$$

We will give another proof of the analyticity of the resolvent

$$z \to (zI - A)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(A) .$$

(This proof generalizes to certain densely defined operators A in Banach spaces.)

Fix any $\lambda \in \mathbb{C} \setminus \sigma(A)$ and set

$$B = \lambda I - A .$$

The matrix B is nonsingular. Let $z \in \mathbb{C}$ and assume that

$$|z - \lambda| < \frac{1}{\|B^{-1}\|} .$$

We have

$$\begin{aligned} zI - A &= (z - \lambda)I + B \\ &= B\Big(I - (\lambda - z)B^{-1}\Big) \end{aligned}$$

Since

$$\|(\lambda - z)B^{-1}\| < 1$$

we can use the geometric sum formula and obtain

$$(zI - A)^{-1} = \sum_{j=0}^{\infty} (\lambda - z)^j (B^{-1})^{j+1} \quad \text{for} \quad |z - \lambda| < \frac{1}{\|B^{-1}\|} \ .$$

The formula shows that the resolvent $(zI - A)^{-1}$ is analytic in a neighborhood of any point $\lambda \in \mathbb{C} \setminus \sigma(A)$.

19 The Maximum Modulus Principle for Holomorphic Functions

Summary: Let $f \in H(U)$ where $U \subset \mathbb{C}$ is bounded region, i.e., U is open, connected, and bounded. In many cases the function $f(z)$ is also a continuous function on the closure of U, i.e., $f \in C(\bar{U})$, where $\bar{U} = U \cup \partial U$. Here ∂U is the boundary of U. If $f(z)$ is not a constant function then the maximum modulus principle says that

$$|f(z)| < M_0 \quad \text{for all} \quad z \in U$$

where

$$M_0 = \max\{|f(z)| \ : \ z \in \partial U\} \ .$$

This estimate is useful to obtain bounds for the absolute value of holomorphic functions.

If the region U is unbounded and $|f(z)|$ satisfies some growth restriction, then the maximum modulus principle may still be valid. We will give an example.

19.1 Local Maxima and the Maximum Principle in Bounded Regions

Definition: Let $U \subset \mathbb{C}$ be an open set and let $\phi : U \to \mathbb{R}$ be a real–valued function. Then $z_0 \in U$ is called a local maximum of ϕ if there exists $r > 0$ with $D(z_0, r) \subset U$ and

$$\phi(z) \le \phi(z_0) \quad \text{for all} \quad z \in D(z_0, r) \ .$$

We will apply this concept to functions $\phi(z) = |f(z)|$ where $f \in H(U)$.

Theorem 19.1 *(local maximum modulus principle for holomorphic functions) Let U be a region and let $f \in H(U)$. Assume that f is not constant. Then the function $|f(z)|$ does not attain any local maximum in U.*

Proof: 1) Suppose that $|f(z)|$ attains a local maximum at the point $z_0 \in U$, i.e.,

$$|f(z)| \le |f(z_0)| \quad \text{for} \quad |z - z_0| \le \varepsilon \ ,$$

for some $\varepsilon > 0$. The set $W := f(D(z_0, \varepsilon))$ is open by Theorem 17.2, the Open Mapping Theorem. Set $w_0 := f(z_0) \in W$. Since W is open there exists $\eta > 0$ so that $D(w_0, 2\eta) \subset W$. Let $w_0 = \rho e^{i\theta}$ and set

$$w_1 := (\rho + \eta)e^{i\theta} \ .$$

Then we have

$$|w_1| > |w| = |f(z_0)|$$

and $w_1 \in W$, a contradiction.

2) We give a second proof, not using the Open Mapping Theorem. Suppose that $|f(z)|$ attains a local maximum at z_0. We can write

$$f(z) = f(z_0) + \sum_{j=M}^{\infty} a_j (z - z_0)^j \quad \text{for} \quad |z - z_0| < 2\varepsilon \ ,$$

with $M \ge 1$ and $a_M \ne 0$.

Obtain that

$$f(z) = f(z_0) + (z - z_0)^M (a_M + h(z))$$

with $h(z_0) = 0$, thus

$$|h(z)| \leq \frac{1}{2}|a_M| \quad \text{for} \quad |z - z_0| \leq \varepsilon$$

if $\varepsilon > 0$ is sufficiently small.

Set $a_0 := f(z_0)$. We may assume $a_0 \neq 0$ since $|f(z)|$ attains a local maximum at z_0. Let

$$a_M/a_0 = \rho e^{i\theta} \quad \text{where} \quad \rho > 0 \quad \text{and} \quad \theta \in \mathbb{R} \ .$$

Set

$$z := z_0 + \varepsilon e^{i\phi}$$

where ϕ will be chosen below. We have

$$f(z) = a_0 \Big(1 + (z - z_0)^M (\frac{a_M}{a_0} + \frac{h(z)}{a_0}) \Big)$$

with

$$(z - z_0)^M \frac{a_M}{a_0} = \varepsilon^M \rho e^{iM\phi + i\theta} \ .$$

Choosing

$$\phi = -\frac{\theta}{M}$$

one obtains that

$$(z - z_0)^M \frac{a_M}{a_0} = \varepsilon^M \rho > 0 \ .$$

Also,

$$\Big| (z - z_0)^M \frac{h(z)}{a_0} \Big| \leq \varepsilon^M \, \frac{1}{2} \, \rho \ .$$

This yields that

$$\begin{aligned} |f(z)| &\geq |a_0|(1 + \varepsilon^M \rho) - |a_0| \, \frac{1}{2} \varepsilon^M \rho \\ &= |a_0|(1 + \frac{1}{2} \varepsilon^M \rho) \\ &> |a_0| \\ &= |f(z_0)| \end{aligned}$$

Thus $|f(z)|$ is not maximal at z_0. This contradiction proves the theorem. $\diamond$

Another form of the maximum modulus theorem is the following. As above, ∂U denotes the boundary of the set U.

Theorem 19.2 *Let U be a bounded region. Let $f \in H(U) \cap C(\bar{U})$ and set*

$$M_0 := |f|_{\partial U} = \max_{z \in \partial U} |f(z)| \; .$$

Then

$$|f(z)| < |f|_{\partial U} \quad \textit{for all} \quad z \in U$$

unless f is constant.

Proof: Let

$$M_1 := \max_{z \in \bar{U}} |f(z)| = |f(z_1)| \; .$$

First assume that $M_1 > M_0$. In this case, $|f(z)|$ attains a local maximum at a point $z_1 \in U$. By the previous theorem, f is constant, a contradiction.

Therefore, we may assume that $M_1 = M_0$. Again, if there exists $z_1 \in U$ with $|f(z_1)| = M_1$, then f is constant, a contradiction. It follows that $|f(z)| < M_0$ for all $z \in U$. $\diamond$

19.2 Some Results in Unbounded Regions

For some applications (in particular to the **Paley–Wiener Theorem** of Fourier analysis) it is important to extend the maximum modulus theorem to certain unbounded domains. A straightforward generalization is wrong, however.

Example: Let

$$U = \{z = re^{i\theta} \; : \; r > 0, \; |\theta| < \frac{\pi}{4}\}$$

and consider

$$f(z) = e^{(z^2)}, \quad z \in \mathbb{C} \; .$$

Clearly, $f \in H(U) \cap C(\bar{U})$. If $z \in \partial U$ then

$$z = x(1+i) \quad \text{or} \quad z = x(1-i), \quad x \geq 0 \; .$$

Therefore,

$$z^2 = \pm 2ix^2 \; ,$$

thus

$$|f(z)| = 1 \quad \text{for all} \quad z \in \partial U \; .$$

However, $f(x) = e^{x^2}$ is unbounded for $x > 0$. Thus, the values of $|f(z)|$ for $z \in U$ are not bounded by the boundary values of $|f(z)|$.

The following is an example of a **Phragmén–Lindelöf Theorem**. [9]

[9] Lars Phragmén (1863–1937) was a Swedish mathematician; Ernst Lindelöf (1870–1946) was a Finnish mathematician.

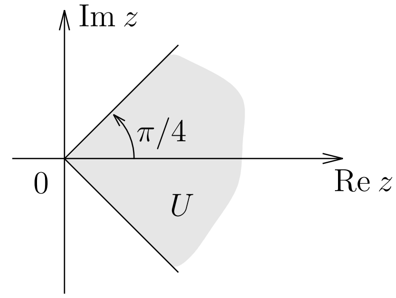

Figure 19.1: Region U in Theorem 19.3

Theorem 19.3 *Let U denote the unbounded region of the above example and let $f \in H(U) \cap C(\bar{U})$. Assume that $|f(z)| \le 1$ for all $z \in \partial U$ and assume that*

$$|f(z)| \le C e^{c|z|} \quad \textit{for all} \quad z \in \bar{U} \ , \tag{19.1}$$

where C and c are positive constants. Then the bound

$$|f(z)| \le 1$$

holds for all $z \in \bar{U}$.

Proof: For $z \in \bar{U}$ we can write

$$z = re^{i\theta} \quad \text{with} \quad r \ge 0 \quad \text{and} \quad |\theta| \le \frac{\pi}{4} \ ;$$

we define

$$z^{3/2} = r^{3/2}\, e^{i3\theta/2} = r^{3/2}\Big(\cos(3\theta/2) + i \sin(3\theta/2) \Big) \ .$$

(Note: For $z \in U$ we have $z = e^{\log z}, z^{3/2} = e^{(3/2) \log z}$, thus the function $z^{3/2}$ is holomorphic on U.)

Let $\varepsilon > 0$. With the above definition of $z^{3/2}$ we set

$$f_\varepsilon(z) := f(z) e^{-\varepsilon z^{3/2}}$$

and note that $f_\varepsilon \in H(U) \cap C(\bar{U})$. If $z = |z| e^{i\theta} \in \bar{U}$ then $|\theta| \le \frac{\pi}{4}$ and

$$3|\theta|/2 \le 3\pi/8 \ .$$

Therefore, for $z \in \bar{U}$:

$$\text{Re}\,(z^{3/2}) = r^{3/2} \cos(3\theta/2) \ge c_1 r^{3/2} \quad \text{where} \quad c_1 := \cos(3\pi/8) > 0 \ .$$

This implies that

$$|f_\varepsilon(z)| \le C e^{cr} e^{-\varepsilon c_1 r^{3/2}} \quad \text{for} \quad z \in \bar{U} \quad \text{where} \quad r = |z| \ .$$

The bound tends to zero as $r \to \infty$.

We may assume that f is not identically zero and set

$$M_\varepsilon := \sup_{z\in\bar{U}} |f_\varepsilon(z)| > 0 \ .$$

There exists $z_0 = z_0(\varepsilon) \in \bar{U}$ with

$$M_\varepsilon = |f_\varepsilon(z_0)| \ .$$

The existence of $z_0 = z_0(\varepsilon)$ follows since $|f_\varepsilon(z)|$ is smaller than M_ε if $|z|$ is large.

Note that for $z \in \partial U$ we have

$$z = re^{\pm i\pi/4} \ ,$$

thus

$$z^{3/2} = r^{3/2}\Big(\cos(3\pi/8) \pm i\sin(3\pi/8)\Big)$$

where

$$\cos(3\pi/8) = c_1 > 0 \ .$$

It follows that

$$|e^{-\varepsilon z^{3/2}}| \leq 1 \quad \text{for} \quad z \in \partial U \ ,$$

thus

$$|f_\varepsilon(z)| \leq 1 \quad \text{for} \quad z \in \partial U \ .$$

Suppose that $M_\varepsilon = |f_\varepsilon(z_0)| > 1$. Then the function $f_\varepsilon(z)$ attains its maximum at an interior point, at $z_0 \in U$, and we obtain a contradiction to the Open Mapping Theorem applied to the function $f_\varepsilon \in H(U) \cap C(\bar{U})$.

We conclude that $M_\varepsilon \leq 1$, which yields that

$$|f(z)| \leq |e^{\varepsilon z^{3/2}}| \quad \text{for} \quad z \in \bar{U} \ .$$

Here $\varepsilon > 0$ is arbitrary. It follows that $|f(z)| \leq 1$ for all $z \in \bar{U}$. $\diamond$

Remark: The assumption $|f(z)| \leq Ce^{c|z|}$ for $z \in \bar{U}$ can be replaced by the weaker assumption

$$|f(z)| \leq Ce^{c|z|^\alpha} \quad \text{for} \quad z \in \bar{U}$$

where $\alpha < \frac{3}{2}$.

20 Harmonic Functions

Summary: In two space dimensions, harmonic functions $u : U \to \mathbb{R}$ are solutions of the partial differential equation

$$\Delta u(x,y) \equiv u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,u) \in U \ .$$

Here $U \subset \mathbb{R}^2 \hat{=} \mathbb{C}$ is a region, an open and connected set. If $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic in U, then the functions $u(x,y)$ and $v(x,y)$ are harmonic conjugates on U.

If $u(x,y)$ is harmonic on U and U is a simply connected region, then a harmonic conjugate $v(x,y)$ of $u(x,y)$ exists on U, i.e., the function $u(x,y)$ is the real part of a function $f(x+iy)$ which is holomorphic on U. If the region U is not *simply* connected, then a function $u(x,y)$, harmonic on U, may not have a harmonic conjugate on U. The function

$$u(x,y) = \ln\Big(\sqrt{x^2+y^2}\Big), \quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \ ,$$

is an example.

Let $U \subset \mathbb{R}^2$ be a bounded region with boundary curve ∂U and let $u_0 : \partial U \to \mathbb{R}$ be a continuous function. The Dirichlet problem of PDEs is to determine a function $u \in C^2(U) \cap C(\bar{U})$ with

$$\Delta u = 0 \quad \text{in} \quad U \quad \text{and} \quad u = u_0 \quad \text{on} \quad \partial U \ .$$

If $U = D(0,1)$ is the open unit disk, then complex variables (essentially, Cauchy's integral formula) can be used to solve the Dirichlet problem. If $V \subset \mathbb{R}^2$ is a bounded region different from $D(0,1)$ and a biholomorphic map

$$f : D(0,1) \to V$$

is known which extends continuously to a map from $\bar{D}(0,1)$ to $\bar{V}$, then one can transform the Dirichlet problem on V to a Dirichlet problem on $D(0,1)$.

20.1 Basic Concepts: Harmonic Functions and Harmonic Conjugates

Let $U \subset \mathbb{R}^n$ be an open set and let $u : U \to \mathbb{R}$ be a C^2–function. The function $u \in C^2(U)$ is called harmonic in U if $\Delta u = 0$ in U. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

denotes the Laplace operator.

Applications: Stationary states of the heat equation $u_t = \Delta u$ are given by harmonic functions. If $\rho = \rho(x)$ is the charge density and $u = u(x)$ is the potential of the electric field generated by ρ, then (in suitable units) $-\Delta u = \rho$. This is Poisson's equation. In regions free of charge, the potential u is a harmonic function.

Theorem 20.1 *Let $U \subset \mathbb{C}$ be open and let $f \in H(U)$. Write $f(z) = u(x,y)+iv(x,y)$ for $z = x+iy$. Then $\Delta u = \Delta v = 0$.*

Proof: This follows directly from the Cauchy–Riemann equations

$$u_x = v_y, \ u_y = -v_x \quad \text{and} \quad v_{yx} = v_{xy} \ .$$

$\diamond$

In the following, let U be a region in $\mathbb{C}$. If $f = u + iv$ is holomorphic in U then one calls v a harmonic conjugate of u in U. Harmonic conjugates, if they exist, are unique up to a constant. To show this, assume that $f_1 = u+iv_1$ and $f_2 = u+iv_2$ are holomorphic in U. Then $f_1 - f_2 = i(v_1 - v_2)$ is also holomorphic. By the open mapping theorem, $f_1 - f_2$ is constant in the region U.

If U is simply connected and $\Delta u = 0$ in U, then u has a harmonic conjugate v in U; this is Theorem 20.2 below.

In Section 20.3 we consider the harmonic function $u(x,y) = \ln((x^2+y^2)^{1/2})$ in $U = \mathbb{R}^2 \setminus \{(0,0)\}$ to show that a harmonic conjugate does not always exist unless the domain U is *simply* connected.

An elementary observation is the following: Let v be a harmonic conjugate of u in U, i.e., $f = u + iv \in H(U)$. The Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

imply that

$$\begin{aligned}(u_x, u_y) \cdot (v_x, v_y) &= u_x v_x + u_y v_y \\ &= -u_x u_y + u_y u_x \\ &= 0\end{aligned}$$

In other words, at every point $(x, y) \in U$ the gradient vector $\nabla u(x,y) = (u_x, u_y)(x,y)$ is orthogonal to the gradient vector $\nabla v(x,y) = (v_x, v_y)(x,y)$. Therefore, the family of lines defined by

$$u(x,y) = c_1$$

is orthogonal to the family of lines

$$v(x,y) = c_2 \ .$$

In other words, every function $f = u + iv \in H(U)$ yields two families of mutually orthogonal coordinate lines in U.

Details: Let $(x(t), y(t)), a \le t \le b$, denote a parameterized line Γ where

$$u(x(t), y(t)) \equiv c_1 \quad \text{for} \quad a \le t \le b \ .$$

We have

$$\begin{aligned}0 &= u_x(x(t), y(t))x'(t) + u_y(x(t), y(t))y'(t) \\ &= \nabla u(x(t), y(t)) \cdot (x'(t), y'(t))\end{aligned}$$

Thus, $\nabla u(P)$ is orthogonal to the tangent vector $(x'(t), y'(t))$ of Γ at the point $P = (x(t), y(t))$. The orthogonality

$$\nabla u(P) \cdot \nabla v(P) = 0$$

implies that the two lines given by

$$u(x,y) \equiv c_1 \quad \text{and} \quad v(x,y) \equiv c_2$$

are orthogonal at their intersection point P.

Example: Let $f(z) = z^2$, thus

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy \ .$$

The equations

$$x^2 - y^2 = c_1$$

and

$$2xy = c_2$$

determine two families of hyperbolas. Each hyperbola of the family

$$y = \pm\sqrt{x^2 - c_1}$$

is orthogonal to each hyperbola

$$y = \frac{c_2}{2x}$$

of the other family at the intersection points.

Take $c_1 = -3$ and $c_2 = 4$, for example. The hyperbola Γ_1 given by

$$u(x,y) = x^2 - y^2 = -3 \quad \text{or} \quad y = \sqrt{x^2+3}$$

and the hyperbola Γ_2 given by

$$v(x,y) = 2xy = 4 \quad \text{or} \quad y = \frac{2}{x}$$

intersect at the point $P = (1,2)$. The tangent vector to Γ_1 at P is

$$\nabla u(P) = \Big(u_x(P), u_y(P)\Big) = (2,-4)$$

and the tangent vector to Γ_2 at P is

$$\nabla v(P) = \Big(v_x(P), v_y(P)\Big) = (4,2) \ .$$

Since

$$\nabla u(P) \cdot \nabla v(P) = 0$$

the hyperbolas Γ_1 and Γ_2 intersect orthogonally at P.

20.2 The Harmonic Conjugate in a Simply Connected Region

We begin with a simple lemma, showing uniqueness of harmonic conjugates up to a constant. The argument is elementary and does not use the open mapping theorem.

Lemma 20.1 *Let $U \subset \mathbb{C}$ be a region and let $u \in C^2(U, \mathbb{R})$ be harmonic. If v and w are harmonic conjugates of u in U, then $v(x,y) = w(x,y) + c$ in U for some constant c.*

Proof: Let $b(x,y) = v(x,y) - w(x,y)$. We have

$$v_x = -u_y \quad \text{and} \quad w_x = -u_y \ ,$$

thus $b_x \equiv 0$. Similarly, $b_y \equiv 0$ in U. Let P and Q denote two arbitrary points in U and let Γ be a curve in U from P to Q. Let $\gamma(t), 0 \le t \le 1$, parameterize Γ. Define the auxiliary function $h(t) = b(\gamma(t))$. We have

$$b(Q) - b(P) = h(1) - h(0) = \int_0^1 h'(t)\, dt \ .$$

Here, by the chain rule,

$$h'(t) = b_x(\gamma(t))\gamma_1'(t) + b_y(\gamma(t))\gamma_2'(t) \equiv 0 \ .$$

Therefore, $h(Q) = h(P)$. Fixing P and letting $Q \in U$ vary, we find that b is constant. $\diamond$

Existence of a harmonic conjugate is assured if the region U is *simply* connected.

Theorem 20.2 *Let $U \subset \mathbb{C}$ be a simply connected region and let $u \in C^2(U, \mathbb{R})$ be harmonic. Then there exists a function $v \in C^2(U, \mathbb{R})$ so that $f = u + iv$ is holomorphic in U.*

Proof: 1. (**real analysis proof** of the existence of v) We must show existence of a function $v \in C^2$ satisfying the Cauchy–Riemann equations:

$$v_x = -u_y, \quad v_y = u_x \ .$$

In terms of real analysis, we try to find a potential v of the vector field $\mathbf{F} = (-u_y, u_x)$, because the Cauchy–Riemann equations require that

$$\nabla v = (-u_y, u_x) \ .$$

The Jacobian of $\mathbf{F}$ is

$$J_{\mathbf{F}} = \begin{pmatrix} -u_{yx} & -u_{yy} \\ u_{xx} & u_{xy} \end{pmatrix} \ .$$

The assumption $u_{xx} + u_{yy} = 0$ yields that the Jacobian $J_{\mathbf{F}}$ is symmetric. Then, by a theorem of real analysis (see Theorem 20.3), the vector field $\mathbf{F} = (-u_y, u_x)$ has a potential in U. Any potential v of the vector field $\mathbf{F} = (-u_y, u_x)$ is a harmonic conjugate of u.

2. (**complex variables proof** of the existence of v) Suppose first that v is a harmonic conjugate of u and set $f = u + iv$. Then we have

$$f' = u_x + iv_x = u_x - iu_y \ .$$

In other words, f' can be determined in terms of u. This motivates to define

$$g := u_x - iu_y \ .$$

Let us prove that $g \in H(U)$: The Jacobian of g is

$$J_g = \begin{pmatrix} u_{xx} & u_{xy} \\ -u_{yx} & -u_{yy} \end{pmatrix} \ .$$

We note that the Cauchy–Riemann equations are fulfilled for the real and imaginary parts of g since

$$(\text{Re } g)_x = u_{xx}, \quad (\text{Im } g)_y = -u_{yy} = u_{xx} = (\text{Re } g)_x$$

and

$$(\text{Re } g)_y = u_{xy}, \quad (\text{Im } g)_x = -u_{yx} = -u_{xy} = -(\text{Re } g)_y \ .$$

Consequently, $g \in H(U)$. By Theorem 4.9, there exists $f \in H(U)$ with $f' = g$. Let $f = a(x,y) + ib(x,y)$. Then we have

$$f' = a_x + ib_x = a_x - ia_y$$

and the equation $f' = g = u_x - iu_y$ yields that

$$a_x = u_x, \quad a_y = u_y \ .$$

As shown in the proof of the previous lemma, this implies $a(x,y) = u(x,y) + c$ where c is a real constant. Since b is a harmonic conjugate of $a = u + c$, the function b is also a harmonic conjugate of u. Just note that $u + ib = a - c + ib = f - c$ is holomorphic. Thus we have shown that u has a harmonic conjugate in U. $\diamond$

In real analysis, one shows the following:

Theorem 20.3 *Let $U \subset \mathbb{R}^n$ be open and simply connected. Let $F : U \to \mathbb{R}^n$ be a C^1–vector field and assume that the Jacobian*

$$J_F(x) = \Big(\frac{\partial F_j(x)}{\partial x_i}\Big)_{1 \le i,j \le n}$$

is a symmetric matrix for all $x \in U$. Then F has a potential in U, i.e., there exists a scalar C^1–function $v : U \to \mathbb{R}$ with

$$\nabla v(x) = F(x), \quad x \in U \ .$$

20.3 A Harmonic Function in $\mathbb{C} \setminus \{0\}$ Without Harmonic Conjugate

In this section, let

$$U = \mathbb{C} \setminus \{0\} \quad \text{and} \quad U_1 = \mathbb{C} \setminus (-\infty, 0] \ .$$

Both sets are open and connected. The set U_1 is simply connected, but U is not simply connected.

We will show:

Lemma 20.2 *The function*

$$u(x,y) = \ln\Big((x^2 + y^2)^{1/2}\Big), \quad (x,y) \neq (0,0) \ ,$$

is harmonic in U, but does not have a harmonic conjugate in U.

Proof: 1. We first show that $\Delta u = 0$ an $U = \mathbb{C} \setminus \{0\}$. Recall the main branch of the complex logarithm defined in U_1: If $z \in U_1$ then

$$z = re^{i\theta}, \quad r > 0, \quad -\pi < \theta < \pi ,$$

and

$$f(z) := \log z = \ln r + i\theta .$$

If one writes

$$f(x+iy) = u(x,y) + iv(x,y) \quad \text{for} \quad x+iy \in U_1 ,$$

then

$$u(x,y) = \ln r = \ln\Big((x^2+y^2)^{1/2}\Big)$$

and

$$v(x,y) = \theta = \arctan(y/x) .$$

Here one must choose the correct branch of the arctan–function and the correct limiting values for $x = 0$.

Since $f \in H(U_1)$ we have

$$\Delta u = \Delta v = 0 \quad \text{in} \quad U_1 .$$

The function u is C^∞ in U, and one obtains that

$$\Delta u = 0 \quad \text{in} \quad U .$$

Of course, this can also be verified directly by calculus.

2. Next, we prove that u does not have a complex conjugate in $U = \mathbb{C} \setminus \{0\}$. Suppose that $w(x,y)$ is a complex conjugate of $u(x,y)$ in U. Thus

$$w_y = u_x, \; w_x = -u_y \quad \text{and} \quad \Delta w = 0 .$$

Then the function

$$g(x+iy) := u(x,y) + iw(x,y), \quad x+iy \in U ,$$

is holomorphic in U. Recall that $f(z) = \log z$ for $z \in U_1 = \mathbb{C} \setminus (-\infty, 0]$. We have, for $z \in U_1$,

$$f(z) - g(z) = i(v(x,y) - w(x,y)) .$$

By the open mapping theorem, one obtains that $f(z) - g(z) = const$ in U_1. Therefore,

$$f'(z) - g'(z) = 0 \quad \text{in} \quad U_1 .$$

Therefore,

$$g'(z) = \frac{1}{z}, \quad z \in U_1 .$$

By assumption, $g \in H(U)$, thus $g' \in H(U)$. Also, the function $\frac{1}{z}$ lies in $H(U)$. By the identity theorem, applied to functions in $H(U)$, we obtain that

$$g'(z) = \frac{1}{z}, \quad z \in U \ .$$

This would mean that the function $\frac{1}{z}$ has an antiderivative in U, namely $g(z)$. Then, if Γ is any closed curve in U, we would obtain that

$$\int_\Gamma \frac{dz}{z} = 0 \ .$$

Since this is not true, we conclude that $u(x,y)$ does not have a harmonic conjugate in U. $\diamond$

Real Analysis Arguments. We want to show the above lemma using arguments of real analysis. In the following, let

$$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$$

denote the main branch of the inverse tangent. Set

$$V = \mathbb{C} \setminus \{iy \ : \ y \in \mathbb{R}\} = \{z = x + iy \ : \ x, y \in \mathbb{R}, x \neq 0\} \ .$$

Lemma 20.3 *Define the functions*

$$u(x,y) = \ln\Big((x^2+y^2)^{1/2}\Big), \quad (x,y) \neq (0,0) \ ,$$

and

$$v(x,y) = \arctan(y/x), \quad x \neq 0 \ .$$

We have $\Delta u = 0$ in U, $\Delta v = 0$ in V and

$$u_x = v_y, \quad u_y = -v_x \quad in \quad V \ .$$

Thus, v is a harmonic conjugate of u in V.

Proof: Apply calculus to $u = \ln\Big((x^2+y^2)^{1/2}\Big)$:

$$\begin{aligned} u_x &= x(x^2+y^2)^{-1} \\ u_{xx} &= (x^2+y^2)^{-1} - 2x^2(x^2+y^2)^{-2} \\ u_y &= y(x^2+y^2)^{-1} \\ u_{yy} &= (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2} \end{aligned}$$

It follows that $\Delta u = 0$.

Also, if $v(x,y) = \arctan(y/x), x \neq 0$, then

$$\begin{aligned}
v_x &= \frac{1}{1+y^2/x^2}\cdot(-yx^{-2}) \\
&= -y(x^2+y^2)^{-1} \\
v_{xx} &= 2xy(x^2+y^2)^{-2} \\
v_y &= \frac{1}{1+y^2/x^2}\cdot x^{-1} \\
&= x(x^2+y^2)^{-1} \\
v_{yy} &= -2xy(x^2+y^2)^{-2}
\end{aligned}$$

It follows that $\Delta v = 0$ in V. We also obtain that

$$u_x = v_y, \quad u_y = -v_x$$

in V. $\diamond$

Let us prove that $u(x,y) = \ln((x^2+y^2)^{1/2})$ does not have a harmonic conjugate in $U = \mathbb{C}\setminus\{0\}$. Suppose that $w(x,y)$ is a harmonic conjugate of u in U. By Lemma 20.1 there are constants, c_1 and c_2, with

$$w(x,y) = \arctan(y/x) + c_1 \quad \text{for} \quad x > 0$$

and

$$w(x,y) = \arctan(y/x) + c_2 \quad \text{for} \quad x < 0 \; .$$

Fix $y = 1$, for example, and consider the limit as $x \to 0$. We obtain, for $x > 0$ and $x \to 0$:

$$w(0,1) = \frac{\pi}{2} + c_1 \; .$$

For $x < 0$ and $x \to 0$:

$$w(0,1) = -\frac{\pi}{2} + c_2 \; .$$

Now fix $y = -1$, for example, and again consider the limit as $x \to 0$. For $x > 0$ and $x \to 0$:

$$w(0,-1) = -\frac{\pi}{2} + c_1 \; .$$

For $x < 0$ and $x \to 0$:

$$w(0,-1) = \frac{\pi}{2} + c_2 \; .$$

Therefore,

$$\frac{\pi}{2} + c_1 = -\frac{\pi}{2} + c_2$$

and

$$-\frac{\pi}{2} + c_1 = \frac{\pi}{2} + c_2 \; .$$

The first equation requires that

$$c_1 - c_2 = -\pi$$

and the second equation requires that

$$c_1 - c_2 = \pi \ .$$

This contradiction implies that u does not have a harmonic conjugate in U though u has a harmonic conjugate in the open left half–plane (namely $v(x,y) = \arctan(y/x), x < 0$) and another harmonic conjugate in the open right half–plane (namely $v(x,y) = \arctan(y/x), x > 0$).

20.4 Dirichlet's Problem and the Poisson Kernel for the Unit Disk

Let $U \subset \mathbb{C}$ be a bounded region with boundary ∂U. Let $u_0 \in C(\partial U)$, i.e, u_0 is a continuous function on ∂U. We assume that u_0 is real valued. The Dirichlet problem for Laplace's equation is: Determine $u \in C^2(U) \cap C(\bar{U})$ with

$$\Delta u = 0 \quad \text{in} \quad U, \quad u = u_0 \quad \text{on} \quad \partial U \ . \tag{20.1}$$

If U is unbounded, one must specify additional conditions about the behavior of $u(x,y)$ for large (x,y). In this section we consider the Dirichlet problem (20.1) for

$$U = \mathbb{D} = D(0,1) \ ,$$

i.e., U is the unit disk $\mathbb{D}$. We let $\gamma(t) = e^{it}, 0 \le t \le 2\pi$, and denote the boundary curve of $\mathbb{D}$ by Γ.

Let $f \in H(D(0,1+\varepsilon))$ where $\varepsilon > 0$. By Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} \, dw \quad \text{for all} \quad z \in \mathbb{D} \ . \tag{20.2}$$

For $0 < |z| < 1$ let

$$z_1 = 1/\bar{z} \ .$$

The mapping

$$z = re^{i\theta} \to 1/\bar{z} = z_1 = \frac{1}{r} \, e^{i\theta}$$

is a reflection w.r.t. $\partial \mathbb{D}$, the boundary of the unit disk.

Since $|z_1| > 1$ we have

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z_1} \, dw \quad \text{for} \quad 0 < |z| < 1 \ . \tag{20.3}$$

Recall that $z_1 = 1/\bar{z}$. Therefore,

$$\frac{1}{w-z_1} = \frac{\bar{z}}{w\bar{z}-1} \quad \text{for} \quad w \in \Gamma \quad \text{and} \quad 0 < |z| < 1$$

and

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{\bar{z} f(w)}{\bar{z}w-1} \, dw, \quad z \in \mathbb{D} \ . \tag{20.4}$$

From (20.2) and (20.4) obtain that

$$f(z) = \int_\Gamma H(z,w) f(w)\, dw \quad \text{for} \quad z \in \mathbb{D}$$

where (for $|z| < 1, |w| = 1$):

$$\begin{aligned} H(z,w) &= \frac{1}{2\pi i}\Big(\frac{1}{w-z} + \frac{\bar z}{1-w\bar z}\Big) && (20.5)\\ &= \frac{1}{2\pi i}\,\frac{1-|z|^2}{w - z - w^2\bar z + w|z|^2} && (20.6)\\ &= \frac{1}{2\pi i w}\,\frac{1-|z|^2}{1 - \bar w z - w\bar z + |z|^2} && (20.7)\\ &= \frac{1}{2\pi i w}\,\frac{1-|z|^2}{|w-z|^2} && (20.8) \end{aligned}$$

With

$$w = e^{it}, \quad dw = iw\, dt$$

one obtains the formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, f(e^{it})\, dt$$

or, with $z = re^{i\theta}$:

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\, f(e^{it})\, dt\ .$$

One defines the Poisson kernel $P_r(\alpha)$ for the unit disk by

$$P_r(\alpha) = \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos\alpha + r^2}, \quad 0 \le r < 1, \quad \alpha \in \mathbb{R}\ .$$

Our derivation shows:

Lemma 20.4 *Let $f \in H(D(0,1+\varepsilon))$ for some $\varepsilon > 0$. Then we have*

$$f(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t) f(e^{it})\, dt$$

for $re^{i\theta} \in D(0,1)$.

Properties of the Poisson Kernel: 1. For $0 \le r < 1$ and all real α we have

$$\begin{aligned} 1 - 2r\cos\alpha + r^2 &= 1 - 2r + r^2 + 2r(1-\cos\alpha)\\ &\ge (1-r)^2\\ &> 0 \end{aligned}$$

thus

$$0 < P_r(\alpha) \le P_r(0) = \frac{1}{2\pi}\,\frac{1+r}{1-r} \ .$$

In particular, $P_r(0) \to \infty$ as $r \to 1-$.

2) Applying the previous lemma with $f \equiv 1$ yields that

$$\int_{-\pi}^{\pi} P_r(\alpha)\, d\alpha = 1 \quad \text{for} \quad 0 \le r < 1 \ . \tag{20.9}$$

3) Despite the fact that $P_r(0) \to \infty$ as $r \to 1-$, we will show that $P_r(\alpha) \to 0$ as $r \to 1-$ if α is bounded away from zero. A precise statement is:

Lemma 20.5 *For any $\delta_1 > 0, \varepsilon_1 > 0$ there exists $\eta > 0$ with*

$$P_r(\alpha) \le \varepsilon_1$$

if

$$0 < \delta_1 \le |\alpha| \le \pi \quad \textit{and} \quad 1 - \eta \le r < 1 \ .$$

Proof: For $\frac{1}{2} \le r < 1$ and $\delta_1 \le |\alpha| \le \pi$ we have:

$$\begin{aligned} 1 - 2r\cos\alpha + r^2 &= 1 - 2r + r^2 + 2r(1-\cos\alpha) \\ &\ge 1 - \cos\alpha \\ &\ge \delta_2 > 0 \end{aligned}$$

where $\delta_2 = 1 - \cos\delta_1$, i.e., δ_2 depends only on δ_1. Therefore,

$$P_r(\alpha) \le \frac{1-r^2}{2\pi\delta_2} \le \varepsilon_1 \quad \text{for} \quad 1 - \eta \le r < 1$$

if $\eta > 0$ is small enough. $\diamond$

We now use these properties of $P_r(\alpha)$ to prove the following result about the Poisson kernel.

Theorem 20.4 *Let $\mathbb{D} = D(0,1)$ denote the open unit disk and let $u_0 \in C(\partial\mathbb{D})$ be real valued. The function $u(z)$ defined for $z \in \bar{\mathbb{D}}$ by*

$$u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t) u_0(e^{it})\, dt \quad \textit{for} \quad 0 \le r < 1 \tag{20.10}$$

$$u(e^{i\theta}) = u_0(e^{i\theta}) \quad \textit{for} \quad r = 1 \tag{20.11}$$

solves the Dirichlet problem with boundary data u_0 on $\partial\mathbb{D}$. In particular:

a) $u \in C^\infty(\mathbb{D}) \cap C(\bar{\mathbb{D}})$;

b) $\Delta u = 0$ in $\mathbb{D}$.

To show that u is harmonic in $\mathbb{D}$, we use the following simple result:

Lemma 20.6 *Suppose that $g(z)$ is a holomorphic function in some open set V and let*

$$G(z) = g(\bar{z}) \quad \textit{for} \quad z \in V_1 = \{z \ : \ \bar{z} \in V\} \ .$$

Then the real and imaginary parts of G are harmonic in V_1.

Proof: If $g(x+iy) = u(x,y) + iv(x,y)$ then

$$G(x+iy) = u(x,-y) + iv(x,-y) \ .$$

$\diamond$

To prove that the function u defined by (20.10) is harmonic in $\mathbb{D}$, recall that our derivation shows:

$$u(z) = \int_\Gamma H(z,w) u_0(w)\, dw, \quad z \in \mathbb{D} \ , \tag{20.12}$$

where $H(z,w)$ is defined in (20.5). By the previous lemma, the real and imaginary parts of $z \to H(z,w)$ are harmonic in $\mathbb{D}$, for each fixed $w \in \gamma$. Since one can differentiate (20.12) under the integral sign, it follows that $\Delta u = 0$ in $\mathbb{D}$. (For another argument, using series, see the next section.)

We now show that the function $u(z)$ defined by (20.10) and (20.11) is continuous at every point $z_0 = e^{it_0}$.

Because of (20.9) we have

$$u(re^{i\theta}) - u(e^{it_0}) = \int_0^{2\pi} P_r(\theta - t)\Big(u_0(e^{it}) - u_0(e^{it_0})\Big)\, dt \ . \tag{20.13}$$

For given $\varepsilon > 0$ there exists $\delta > 0$ with

$$|u_0(e^{it}) - u_0(e^{it_0})| \le \varepsilon \quad \text{for} \quad |t - t_0| \le \delta \ . \tag{20.14}$$

We split the integral in (20.13):

$$\int_0^{2\pi} = \int_{|t-t_0|<\delta} + \int_{|t-t_0|>\delta} =: I_1 + I_2 \ .$$

Using (20.9) and (20.14) we have

$$I_1 \le \varepsilon \ .$$

To estimate I_2 we assume that $|\theta - t_0| < \delta/2$. Then the assumption $|t - t_0| > \delta$ yields that

$$|\theta - t| > \frac{\delta}{2} =: \delta_1 \ .$$

It follows that

$$I_2 \le 2|u_0|_\infty \cdot 2\pi \max_{\delta_1 \le |\alpha| \le \pi} P_r(\alpha) \ .$$

Using Lemma 20.5 we obtain that

$$I_2 \le \varepsilon \quad \text{for} \quad 1 - \eta \le r < 1$$

if $\eta > 0$ is sufficiently small. To summarize, if $\varepsilon > 0$ is given, then there exists $\delta > 0$ and $\eta > 0$ with

$$|u(re^{i\theta}) - u(e^{it_0})| \le 2\varepsilon$$

if $|\theta - t_0| < \delta/2$ and $1 - \eta \le r < 1$. Since u is continuous on ∂U this shows that u is continuous at $z_0 = e^{it_0}$. This completes the proof of Theorem 20.4. $\diamond$

Remark: We have derived the Poisson kernel for the unit disk,

$$P_r(\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}, \quad 0 \le r < 1, \quad \alpha \in \mathbb{R} \ .$$

For a disk of radius $R > 0$ the Poisson kernel is

$$P_r^{(R)}(\alpha) = \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\alpha + r^2}, \quad 0 \le r < R, \quad \alpha \in \mathbb{R} \ .$$

The solution of the Dirichlet problem,

$$\Delta u = 0 \quad \text{in} \quad D(0,R), \quad u(Re^{i\theta}) = u_0(Re^{i\theta}) \quad \text{for} \quad 0 \le \theta \le 2\pi \ ,$$

is

$$u(re^{i\theta}) = \int_0^{2\pi} P_r^{(R)}(\theta - t) u_0(Re^{i\theta})\, d\theta \quad \text{for} \quad 0 \le r < R \ ,$$

and

$$u(Re^{i\theta}) = u_0(Re^{i\theta}) \quad \text{for} \quad 0 \le \theta \le 2\pi \ .$$

20.5 The Poisson Kernel and Fourier Expansion

We have derived the Poisson kernel for the unit disk using Cauchy's integral formula. An alternative derivation proceeds via Fourier expansion.

Let $u_0 : \partial\mathbb{D} \to \mathbb{C}$ denote a continuous function. We want to determine a function

$$u \in C^2(\mathbb{D}) \cap C(\bar{\mathbb{D}})$$

with

$$\Delta u = 0 \quad \text{in} \quad \mathbb{D}, \quad u(z) = u_0(z) \quad \text{for} \quad |z| = 1 \ .$$

Set

$$g(t) = u_0(e^{it}), \quad t \in \mathbb{R} \ .$$

Then g is a continuous, 2π–periodic function and

$$g(t) = \sum_{k=-\infty}^{\infty} \hat{g}(k) e^{ikt}, \quad \hat{g}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} g(t)\, dt \ ,$$

is the Fourier expansion of $g(t)$. We ignore questions of convergence. We obtain, formally,

$$u_0(z) = \sum_{k=-\infty}^{\infty} \hat{g}(k) z^k, \quad |z| = 1 \ .$$

Note that, for $|z| = 1$ we have $z^{-1} = \bar{z}$, thus

$$z^k = \bar{z}^{|k|} \quad \text{for} \quad |z| = 1, \quad k < 0 \ .$$

Therefore, formally,

$$u_0(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k + \sum_{k=-\infty}^{-1} \hat{g}(k) \bar{z}^{|k|}, \quad |z| = 1 \ .$$

This second representation of the given function $u_0(z), z \in \partial\mathbb{D}$, has the advantage that every term

$$z^k \quad \text{for} \quad k \geq 0, \qquad \bar{z}^{|k|} \quad \text{for} \quad k \leq -1 \ ,$$

is a harmonic function in $\mathbb{D}$. In contrast, the function z^k has a pole at $z = 0$ if $k < 0$.

We claim that the solution of the Dirichlet problem is given by

$$u(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k + \sum_{k=-\infty}^{-1} \hat{g}(k) \bar{z}^{|k|} \quad \text{for} \quad |z| < 1 \tag{20.15}$$

and

$$u(z) = u_0(z) \quad \text{for} \quad |z| = 1 \ .$$

First note that the sequence of Fourier coefficients $\hat{g}(k)$ is bounded. This follows from the boundedness of the function $g(t) = u_0(e^{it}), t \in \mathbb{R}$.

Therefore,

$$u_1(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k \quad \text{for} \quad |z| < 1$$

and

$$u_2(z) = \sum_{k=-\infty}^{-1} \hat{g}(k) \bar{z}^{|k|} \quad \text{for} \quad |z| < 1$$

are harmonic functions in $\mathbb{D}$. (Note that $\bar{u}_2(x)$ is holomorphic in $\mathbb{D}$.) Thus, $u \in C^\infty(\mathbb{D})$ and $\Delta u = 0$ in $\mathbb{D}$.

It remains to prove that $u \in C(\bar{\mathbb{D}})$. To show this, we derive an integral representation of $u(z)$, the Poisson integral formula.

Setting $z = re^{i\theta}$ for $0 \leq r < 1$ we have

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi} \sum_{k=0}^{\infty} r^k \int_0^{2\pi} e^{ik(\theta-t)} g(t) dt \ + \ \frac{1}{2\pi} \sum_{k=-\infty}^{-1} r^{|k|} \int_0^{2\pi} e^{ik(\theta-t)} g(t) dt \\ &= \int_0^{2\pi} P_r(\theta - t) g(t) \, dt \end{aligned}$$

with

$$P_r(\alpha) = \frac{1}{2\pi} \sum_{k=0}^{\infty} r^k e^{ik\alpha} + \frac{1}{2\pi} \sum_{k=-\infty}^{-1} r^{|k|} e^{ik\alpha} \ .$$

We have used the integral formula for $\hat{g}(k)$ in (20.15) and have changed the order of summation and integration. This is allowed since the series converge uniformly in t for every fixed r with $0 \le r < 1$.

Set

$$w = re^{i\alpha} \ .$$

Then we have

$$\begin{aligned} 2\pi P_r(\alpha) &= \sum_{k=0}^\infty w^k + \sum_{k=1}^\infty \bar{w}^k \\ &= \frac{1}{1-w} + \frac{\bar{w}}{1-\bar{w}} \\ &= \frac{1-|w|^2}{1-w-\bar{w}+|w|^2} \\ &= \frac{1-r^2}{1-2r\cos\alpha+r^2} \end{aligned}$$

We have obtained the Poisson kernel for the unit disk $\mathbb{D}$ using Fourier expansion.

Remarks on Fourier Expansion: Let X denote the space of all 2π–periodic continuous functions

$$g : \mathbb{R} \to \mathbb{C} \ .$$

(More generally, one could take $X = L_2(0, 2\pi)$.) On X one defines the L_2–inner product and norm by

$$(u, v)_{L_2} = \int_0^{2\pi} \bar{u}(t)v(t)dt, \quad \|u\|_{L_2}^2 = (u, u)_{L_2} \ .$$

The functions in the sequence

$$e^{ikt}, \quad k \in \mathbb{Z} \ ,$$

are L_2–orthogonal to each other and

$$(e^{ijt}, e^{ikt})_{L_2} = 2\pi\delta_{jk} \ .$$

If $g \in X$ then its Fourier series is

$$\sum_{k=-\infty}^{\infty} \hat{g}(k)e^{ikt}$$

where

$$\hat{g}(k) = \frac{1}{2\pi}(e^{ikt}, u(t))_{L_2}, \quad k \in \mathbb{Z} \ ,$$

is the k–th Fourier coefficient of g. Let

$$S_n(t) = \sum_{k=-n}^{n} \hat{g}(k)e^{ikt}$$

denote the n–th partial sum of the Fourier series of g. Then it is known that

$$\|g - S_n\|_{L_2} \to 0 \quad \text{as} \quad n \to \infty \ ,$$

i.e., the Fourier series of g represents g in the L_2–sense. Pointwise convergence and convergence in maximum norm hold if $g \in C^1$, for example.

20.6 The Mean Value Property of Harmonic Functions

Let U be an open set and let $f \in H(U)$. If $\bar{D}(P,r) \subset U$ and $\gamma(t) = P + re^{it}$ then, by Cauchy's integral formula:

$$\begin{aligned} f(P) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-P}\, dz \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{it})\, dt \ . \end{aligned}$$

This says that $f(P)$ is the mean value of the values of f along the circle $\partial D(P,r)$.

Let $u : U \to \mathbb{R}$ be harmonic in U and let $\bar{D}(P,r) \subset U$, as above. Let $\varepsilon > 0$ and let $D(P, r+\varepsilon) \subset U$. In $D(P, r+\varepsilon)$ there exists a harmonic conjugate v of u. Applying the above equation to $f = u + iv$ and taking real parts, one obtains that

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{it})\, dt \ .$$

In other words, harmonic functions have the following mean value property: If $\bar{D}(P,r)$ lies in the region U where u is harmonic, then $u(P)$ equals the mean value of u on the circle $\partial D((P,r)$.

Example: Let $f(z) = e^z$ and take $P = 0, r = 1$. Cauchy's integral formula says that

$$1 = e^0 = \frac{1}{2\pi i} \int_\gamma \frac{e^z}{z}\, dz \ .$$

This also follows from the residue theorem, of course.

Using

$$z(t) = e^{it} = \cos t + i \sin t, \quad dz = iz\, dt, \quad \frac{dz}{z} = idt \ ,$$

one obtains the mean value formula

$$\begin{aligned} 2\pi &= \int_0^{2\pi} e^{\cos t + i \sin t}\, dt \\ &= \int_0^{2\pi} e^{\cos t} \Big(\cos(\sin t) + i \sin(\sin t) \Big)\, dt \end{aligned}$$

This yields that

$$I_1 := \int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt = 2\pi$$

and

$$I_2 := \int_0^{2\pi} e^{\cos t} \sin(\sin t) \, dt = 0 \; .$$

Let $h(t) = e^{\cos t} \sin(\sin t)$. Then $h(-t) = -h(t)$ and $h(t)$ has period 2π. Therefore, $I_2 = \int_{-\pi}^{\pi} h(t) \, dt = 0$ is obvious. The formula for $I_1 = 2\pi$ does not look obvious.

20.7 The Maximum Principle for Harmonic Functions

Let U be a bounded region and let $u \in C^2(U) \cap C(\bar{U})$ be a real valued function. Assume that $\Delta u = 0$ in U and that u is not constant. Let

$$M_1 = \max\{u(z) \; : \; z \in \bar{U}\} \; .$$

We claim that

$$u(P) < M_1$$

for all $P \in U$. Suppose the strict inequality $u(P) < M_1$ does not hold for some $P \in U$. Then we have $u(P) = M_1$, and P is a local maximum of U. Using the mean value property, one finds that for some $r > 0$:

$$u(z) = M_1 \quad \text{for} \quad |z - P| \le r \; .$$

Set

$$Z = \{z \in U \; : \; u(z) = M_1\} \; .$$

The above argument shows that Z is open. Also, by continuity, Z is closed in U. Since U is assumed to be connected, one obtains that $Z = U$. Thus, u is constant.

We can apply the same reasoning to $-u$ and obtain:

Theorem 20.5 *Let U be a bounded region and let $u \in C^2(U) \cap C(\bar{U})$ be harmonic in U. Assume that u is not constant. Then, for every $P \in U$:*

$$\min_{z \in \partial U} u(z) < u(P) < \max_{z \in \partial U} u(z) \; .$$

A simple implication is the following: If U is a bounded region, then the solution of the Dirichlet problem

$$\Delta u = f \quad \text{in} \quad U, \quad u = u_0 \quad \text{on} \quad \partial U \; ,$$

is unique (if the solution exists). (If u_1 and u_2 are two solutions, then $u = u_1 - u_2$ is harmonic in U and has zero boundary values. By Theorem 20.5 it follows that $u \equiv 0$.)

20.8 The Dirichlet Problem in More General Regions

Let us first summarize our results for the Dirichlet problem in the unit disk,

$$\mathbb{D} = D(0,1) \; .$$

Theorem 20.6 *Let $u_0 : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function. Then there is a unique function*

$$u \in C^2(\mathbb{D}) \cap C(\bar{\mathbb{D}})$$

with

$$\Delta u = 0 \quad in \quad \mathbb{D}, \quad u = u_0 \quad on \quad \partial\mathbb{D} .$$

For $z = re^{i\theta} \in \mathbb{D}$ the solution u is given by

$$u(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t) u_0(e^{it}) \, dt$$

where $P_r(\alpha)$ is the Poisson kernel for $\mathbb{D}$.

Let $V \subset \mathbb{C}$ be any bounded region and assume that there are holomorphic mappings

$$f : \mathbb{D} \to V, \quad g : V \to \mathbb{D}$$

which are $1 - 1$, onto, and inverse to each other. We also assume that f and g can be continuously extended as bijective mappings to the closures of $\mathbb{D}$ and V, respectively. We denote the extensions again by f and g. Thus we assume that

$$f : \bar{\mathbb{D}} \to \bar{V}, \quad g : \bar{V} \to \bar{\mathbb{D}} ,$$

are continuous, $1 - 1$, onto and

$$f(g(z)) = z \quad \text{for all} \quad z \in \bar{V} ,$$
$$g(f(w)) = w \quad \text{for all} \quad w \in \bar{\mathbb{D}} .$$

This implies that boundaries are mapped to boundaries:

$$f(\partial\mathbb{D}) = \partial V, \quad g(\partial V) = \partial\mathbb{D} .$$

We will discuss the existence and construction of such mappings f and g later in connection with the **Riemann Mapping Theorem**.

Now let $v_0 : \partial V \to \mathbb{R}$ be a given continuous function and consider the Dirichlet problem: Find

$$v \in C^2(V) \cap C(\bar{V})$$

with

$$\Delta v = 0 \quad \text{in} \quad V, \quad v = v_0 \quad \text{on} \quad \partial V .$$

We can transform this problem to the Dirichlet problem on $\mathbb{D}$ in the following way: Set

$$u_0(w) = v_0(f(w)), \quad w \in \partial\mathbb{D} .$$

This transforms the given boundary function v_0, defined on ∂V, to a boundary function u_0 defined on $\partial\mathbb{D}$.

Then let $u \in C^2(\mathbb{D}) \cap C(\bar{D})$ solve the Dirichlet problem in $\mathbb{D}$ with boundary data u_0. We claim that

$$v(z) = u(g(z)), \quad z \in \bar{V} \ ,$$

solves the Dirichlet problem in V. Clearly, if $z \in \partial V$, then $g(z) \in \partial \mathbb{D}$ and

$$\begin{aligned} v(z) &= u_0(g(z)) \\ &= v_0(f(g(z))) \\ &= v_0(z) \ , \end{aligned}$$

showing that v satisfies the boundary conditions. It remains to prove that v is harmonic in V. This follows from the following result.

Theorem 20.7 *Let U, V be regions and let $g : V \to U$ be holomorphic. Let $u_1 : U \to \mathbb{R}$ be harmonic in U. Then $v_1(z) = u_1(g(z))$ is harmonic in V.*

Proof: Fix $z_0 \in V$. We must show that $\Delta v_1(z_0) = 0$. We have $g(z_0) \in U$ and there is $r > 0$ with

$$D = D(g(z_0), r) \subset U \ .$$

Since u_1 is harmonic in D it has a harmonic conjugate u_2 in D. Then the function $u = u_1 + iu_2$ is holomorphic in D. It follows that the function $v(z) = u(g(z))$ is holomorphic in a neighborhood of z_0. Since v_1 is the real part of v, we conclude that v_1 is harmonic in a neighborhood of z_0. $\diamond$

21 Abel's Continuity Theorem

Outline: The function

$$f(z) = \log(1+z), \quad |z| < 1 \ ,$$

has the derivative

$$f'(z) = \frac{1}{1+z} = \sum_{j=0}^{\infty} (-1)^j z^j \quad \text{for} \quad |z| < 1$$

and the power series representation

$$f(z) = \log(1+z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \, z^{j+1}$$

holds for $|z| < 1$. What happens for $z = 1$? Does the equation

$$\log(1+1) = \ln 2 = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \tag{21.1}$$

hold? The answer is *yes*. Convergence of the above series holds since the series is alternating and the terms $p_j = \frac{1}{j+1}$ converge to zero monotonically. The equation (21.1) then follows from Abel's Continuity Theorem and the continuity of the function $f(z) = \log(1+z)$ at $z = 1$.

21.1 Alternating Series and Examples

Theorem 21.1 *(Convergence of Alternating Series) Let $p_j \in \mathbb{R}$ with $p_j \geq p_{j+1} > 0$ for all $j = 0, 1, 2, \ldots$ and $p_j \to 0$ as $j \to \infty$. Then the alternating series*

$$\sum_{j=0}^{\infty} (-1)^j p_j$$

converges.

Proof: Set

$$S_n := \sum_{j=0}^{n} (-1)^j p_j = p_0 - p_1 + p_2 - \ldots + (-1)^n p_n \ .$$

We have

$$\begin{aligned} S_{2n} &= p_0 - p_1 + \ldots + p_{2n} \\ S_{2n+1} &= S_{2n} - p_{2n+1} \end{aligned}$$

thus

$$S_{2n+1} < S_{2n} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Also,

$$\begin{aligned} S_{2n+2} &= S_{2n} - p_{2n+1} + p_{2n+2} \le S_{2n} \\ S_{2n+3} &= S_{2n+1} + p_{2n+2} - p_{2n+3} \ge S_{2n+1} \end{aligned}$$

Therefore,

$$S_{2n+1} \le S_{2n+3} < S_{2n+2} \le S_{2n} \quad \text{for all} \quad n \ .$$

Convergence

$$S_{2n} \to S^+, \quad S_{2n+1} \to S^-$$

follows. Since

$$-p_{2n+1} = S_{2n+1} - S_{2n} \to 0$$

one obtains that $S^+ = S^- =: S$ and

$$S_n = \sum_{j=0}^n (-1)^j p_j \to S \quad \text{as} \quad n \to \infty \ .$$

$\diamond$

Example 1: Let $f(x) = \ln(1+x)$ for $x > -1$. We have for $-1 < x < 1$:

$$\begin{aligned} f'(x) &= \frac{1}{1+x} = \sum_{j=0}^\infty (-1)^j x^j \\ f(x) &= \sum_{j=0}^\infty \frac{(-1)^j}{j+1} \, x^{j+1} \end{aligned}$$

Therefore,

$$\log(1+z) = \sum_{j=0}^\infty \frac{(-1)^j}{j+1} \, z^{j+1} \quad \text{for} \quad |z| < 1 \ .$$

By Theorem 21.1 the series

$$\sum_{j=0}^\infty \frac{(-1)^j}{j+1} \, x^{j+1}$$

converges for $x = 1$. By Abel's Continuity Theorem (Theorem 21.2) and continuity of the function $\ln(1+x)$ at $x = 1$ it follows that

$$\sum_{j=0}^\infty \frac{(-1)^j}{j+1} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2 \ .$$

Example 2: Let $f(x) = \arctan x$ for $x \in \mathbb{R}$. We have for $-1 < x < 1$:

$$\begin{aligned} f(x) &= \arctan x \\ f'(x) &= \frac{1}{1+x^2} = \sum_{j=0}^{\infty} (-1)^j x^{2j} \\ f(x) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\, x^{2j+1} \end{aligned}$$

Therefore,

$$\arctan z = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\, z^{2j+1} \quad \text{for} \quad |z| < 1 \ .$$

(Note that the radius of convergence of the above series cannot be larger than 1 since the function $f'(z) = \frac{1}{1+z^2}$ is singular at $z = i$.)

By Theorem 21.1 the series

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\, x^{2j+1}$$

converges for $x = 1$. By Abel's Continuity Theorem and continuity of the function $f(x) = \arctan x$ at $x = 1$ it follows that

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = \arctan 1 \ .$$

Since $\tan(\pi/4) = 1$ we have $\arctan 1 = \pi/4$, thus

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4} \ .$$

21.2 Abel's Theorem

Theorem 21.2 *(Abel's Continuity Theorem) Assume that the power series*

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

has the radius of convergence $R > 0$, defining the function $f \in H(D(0,R))$. Also assume that the series

$$\sum_{j=0}^{\infty} a_j z_0^j$$

converges for some z_0 with $|z_0| = R$. Set

$$f(z_0) = \sum_{j=0}^{\infty} a_j z_0^j \ .$$

Let $A, B \in D(0,R)$ denote two distinct points and let Δ denote the closed triangle with vertices A, B, z_0.

Abel's continuity theorem states that

$$f \in C(\Delta) \ .$$

Proof: Using simple transformations, it is not difficult to show that it suffices to prove the theorem under the following special assumptions:

$$R = 1, \quad z_0 = 1, \quad f(1) = 0 \ .$$

Also, for the points A, B defining Δ we may assume that

$$A = a + ib, \quad B = a - ib, \quad 0 < a, b < 1, \quad a^2 + b^2 < 1 \ .$$

For $\delta > 0$ set

$$\Delta_\delta = \Big\{ z \in \Delta \ : \ 0 < |1 - z| < \delta \Big\} \ .$$

We will prove that for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$|f(z)| < \varepsilon \quad \text{for all} \quad z \in \Delta_\delta \ .$$

It then follows that $f \in C(\Delta)$.

Set

$$s_\nu = a_0 + a_1 + \ldots + a_\nu \ ,$$

thus

$$a_0 = s_0, \quad a_j = s_j - s_{j-1} \quad \text{for} \quad j = 1, 2, \ldots$$

Recall that, by assumption, $s_\nu \to 0 = f(1)$ as $\nu \to \infty$. First let $z \in D(0,1)$ be arbitrary. We have

$$\begin{aligned} \sum_{j=0}^{n} a_j z^j &= s_0 + (s_1 - s_0) z + (s_2 - s_1) z^2 + \ldots + (s_n - s_{n-1}) z^n \\ &= s_0 (1 - z) + s_1 (z - z^2) + \ldots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \\ &= (1 - z)\Big(s_0 + s_1 z + \ldots + s_{n-1} z^{n-1} \Big) + s_n z^n \end{aligned}$$

Therefore,

$$f(z) = (1 - z) \sum_{j=0}^{\infty} s_j z^j, \quad |z| < 1 \ .$$

Let $\eta > 0$ be arbitrary. (Below we will make a proper choice for η.) Choose $N = N(\eta)$ so that

$$|s_j| < \frac{\eta}{2} \quad \text{for} \quad j > N \ .$$

The existence of $N = N(\eta)$ follows from the assumption that

$$
\begin{aligned}
0 &= f(1) \\
&= \sum_{n=0}^{\infty} a_n \\
&= \lim_{j\to\infty} \sum_{n=0}^{j} a_n \\
&= \lim_{j\to\infty} s_j
\end{aligned}
$$

Using that $|s_j| < \eta/2$ for $j > N$ one obtains the following estimates for $|z| < 1$:

$$
\begin{aligned}
|f(z)| &\le |1-z| \sum_{j=0}^{N} |s_j z^j| + \frac{\eta}{2} |1-z| \sum_{j=N+1}^{\infty} |z|^j \\
&\le |1-z| M(\eta) + \frac{\eta}{2} \frac{|1-z|}{1-|z|}
\end{aligned}
$$

where

$$
M(\eta) = \sum_{j=0}^{N(\eta)} |s_j| \ .
$$

So far we have only used the estimate $|z| < 1$ for z.

If $z \in \Delta_\delta$ then $|1-z| < \delta$ and we obtain the estimate

$$
|f(z)| \le \delta M(\eta) + \frac{\eta}{2} \frac{|1-z|}{1-|z|} \ . \tag{21.2}
$$

It remains to bound the quotient

$$
Q(z) = \frac{|1-z|}{1-|z|} \quad \text{for} \quad z \in \Delta_\delta \ .
$$

Auxiliary Estimate: Let $0 < \alpha_0 < \frac{\pi}{2}$ denote the angle at the point $z_0 = 1$ between the straight line from 1 to 0 and the straight line from 1 to $A = a + bi$.

Let $z \in \Delta, z \neq 1$, and denote the angle at 1 between the straight line from 1 to 0 and the straight line from 1 to z by α. We have

$$
|\alpha| \le \alpha_0, \quad \cos\alpha \ge \cos\alpha_0 =: c_0 > 0 \ .
$$

Consider the triangle with vertices $0, 1$ and z. Setting

$$
r = |1-z|, \quad d = d(r) = |z|
$$

we have by the cosine theorem of trigonometry

$$
d^2 = 1 + r^2 - 2r\cos\alpha \ .
$$

We will show the bound

$$\frac{|1-z|}{1-|z|} = \frac{r}{1-d(r)} \le C \quad \text{for} \quad z \in \Delta \tag{21.3}$$

for some constant $C > 0$. We have

$$d(r) = \sqrt{1+r^2-2rc} \quad \text{where} \quad c = \cos\alpha \ge c_0 > 0 \ .$$

The bound (21.3) is equivalent to (with $d = d(r)$):

$$\begin{aligned} r &\le C - Cd \\ Cd &\le C - r \\ C^2(1+r^2-2cr) &\le C^2 - 2Cr + r^2 \\ C^2r^2 + 2Cr &\le r^2 + 2crC^2 \end{aligned}$$

Clearly, since $r > 0$, the last estimate is equivalent to

$$C^2 r + 2C \le r + 2cC^2 \ . \tag{21.4}$$

We may restrict r to the interval

$$0 < r \le c_0 = \cos\alpha_0 \ .$$

Since $c = \cos\alpha \ge \cos\alpha_0 = c_0$ the estimate (21.4) holds if

$$C^2 c_0 + 2C \le 2c_0 C^2 \ .$$

Equivalently,

$$2C \le c_0 C^2 \ ,$$

i.e.,

$$C \ge \frac{2}{c_0} \ .$$

Thus we have proved the bound

$$Q(z) = \frac{|1-z|}{1-|z|} \le \frac{2}{c_0} \quad \text{with} \quad c_0 = \cos\alpha_0 > 0$$

for all

$$z \in \Delta \quad \text{with} \quad |1-z| = r \le c_0 \ .$$

Using the estimate (21.2) we have shown: If $\eta > 0$ and $0 < \delta \le c_0$ are chosen, then the following bound holds:

$$|f(z)| \le \delta M(\eta) + \frac{\eta}{2} \cdot \frac{2}{c_0} \quad \text{for} \quad z \in \Delta_\delta \ .$$

If $\varepsilon > 0$ is given, then choose $\eta > 0$ so that $\eta/c_0 < \varepsilon/2$. Then choose $0 < \delta \le c_0$ so that $\delta M(\eta) < \varepsilon/2$. Obtain that

$$|f(z)| < \varepsilon \quad \text{for} \quad z \in \Delta_\delta \ .$$

This complete the proof of Abel's theorem. $\diamond$

22 Quals

Jan. 2022, problem 4

Let $f \in H(\mathbb{C} \setminus \{0\}$. Assume that

$$f(n) = (-1)^n \quad \text{for} \quad n = 1, 2, \ldots$$

Prove that $\inf_{z \neq 0} |f(z)| = 0$.

Proof: Let

$$f(z) = \sum_{j=-\infty}^{\infty} a_j z^j \quad \text{for} \quad z \neq 0$$

denote the Laurent series of $f(z)$.

a) Assume that $f(z)$ has an essential singularity at $z = 0$. By Casorati–Weierstrass we have $\inf_{z \neq 0} |f(z)| = 0$.

b) Assume that

$$g(z) = f(1/z), \quad z \neq 0 \ ,$$

has an essential singularity at $z = 0$. Again, by Casorati–Weierstrass we have $\inf_{z \neq 0} |f(z)| = 0$.

c) If neither a) nor b) apply then

$$f(z) = \sum_{j=J}^{K} a_j z^j, \quad a_K \neq 0 \ ,$$

where J and K are finite.

If $K > 0$ then $|f(z)| \to \infty$ as $|z| \to \infty$. If $K = 0$ then $f(z) \to a_K$ as $|z| \to \infty$. If $K < 0$ then $f(z) \to 0$ as $|z| \to \infty$. In all three cases the assumption

$$f(n) = (-1)^n \quad \text{for} \quad n = 1, 2, \ldots$$

is violated.

23 Supplements

23.1 Euler's Solution of the Basel Problem (1734)

,

Consider the function

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \frac{z^4}{5!} \ldots \ .$$

The zeros of f are the numbers

$$\pm \pi j \quad \text{for} \quad j = 1, 2, \ldots$$

Therefore,

$$\begin{aligned} f(z) &= \Big(1 - \frac{z}{\pi}\Big)\Big(1 + \frac{z}{\pi}\Big)\Big(1 - \frac{z}{2\pi}\Big)\Big(1 + \frac{z}{2\pi}\Big) \ldots \\ &= \Big(1 - \frac{z^2}{\pi^2}\Big)\Big(1 - \frac{z^2}{2^2\pi^2}\Big) \ldots \\ &= 1 - \frac{z^2}{\pi^2}\Big(1 + \frac{1}{2^2} + \frac{1}{3^2} \ldots\Big) + \mathcal{O}(z^4) \end{aligned}$$

One obtains that

$$-\frac{1}{6} = -\frac{1}{\pi^2}\Big(1 + \frac{1}{2^2} + \frac{1}{3^2} \ldots\Big) \ ,$$

thus

$$1 + \frac{1}{2^2} + \frac{1}{3^2} \ldots = \frac{\pi^2}{6} \ .$$

Infinite products were considered about 100 years later by Weierstrass.

23.2 Application of $1/\zeta(2)$

Let $m, n \in \mathbb{N}$ be random numbers. We claim that

$$\textit{probability}\Big(g.c.d.(m,n) = 1\Big) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \ .$$

The product formula for the zeta–functions yields that

$$\frac{1}{\zeta(s)} = \Pi_p\Big(1 - \frac{1}{p^s}\Big) \quad \text{for} \quad \mathrm{Re}\, s > 1 \ ,$$

thus

$$\frac{1}{\zeta(2)} = \Pi_p\Big(1 - \frac{1}{p^2}\Big) \ .$$

If $n \in \mathbb{N}$ is a random numer and p is prime then the probability that p divides n equals $\frac{1}{p}$. If $m, n \in \mathbb{N}$ are random then the probability that p divides both, m and n, equals $\frac{1}{p^2}$.

The probability that p does not divide both numbers, m and n, equals

$$1 - \frac{1}{p^2} \ .$$

If p and q are distinct prime numbers then the probability that neither p nor q divides both number, m and n, equals

$$\left(1 - \frac{1}{p^2}\right) \cdot \left(1 - \frac{1}{q^2}\right) \ .$$

It follows that

$$probability\Big(g.c.d.(m, n) = 1\Big) = \Pi_p\Big(1 - \frac{1}{p^2}\Big) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \ .$$