Homework 1, Math. 562  
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1) Let $\Gamma$ denote the circle with parameterization $z(t) = 2e^{it}, 0 \leq t \leq 2\pi$, and let 
\[
A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}.
\]
a) Compute the projector 
\[
P = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} dz.
\]
Integrate the four matrix elements $((zI - A)^{-1})_{jk}$ along $\Gamma$ to compute $P$.
b) Determine the range of $P$ and the nullspace of $P$.
c) Write $P$ in the form 
\[
P = T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.
\]
Determine the matrix $T$.

2) Let $W$ be a vector space and let $P: W \to W$ be a projector, i.e., $P$ is linear and $P^2 = P$. Set 
\[
U = \text{range}(P), \quad V = \text{nullspace}(P).
\]
Prove that $W = U \oplus V$.

3) Let $W$ be a vector space with subspaces $U$ and $V$ satisfying $W = U \oplus V$. Define $P : W \to W$ as follows: 
If $w \in W$ and $w = u + v$ with $u \in U, v \in V$ then set $Pw = u$. Prove that $P$ is a projector and 
\[
U = \text{range}(P), \quad V = \text{nullspace}(P).
\]

4) Let 
\[
A = \begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}
\]
denote the matrix of Problem 1. Compute a matrix $B$ with $A = e^B$.

5) Let 
\[
A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.
\]
Let $\Gamma$ denote the circle with radius 1 centered at $z = 2$. Compute 
\[
B = \frac{1}{2\pi i} \int_{\Gamma} \log z (zI - A)^{-1} dz.
\]
Check if the equation $A = e^B$ holds or not.