Homework 2, Math. 562
Prof. Jens Lorenz, Instructor
Assigned: Feb. 11, 2022.   Due: Feb. 25, 2022

1) Let S denote the open half–strip

$$S = \left\{ z = x + iy \ : \ -\frac{\pi}{2} < x < \frac{\pi}{2},\ y > 0 \right\} .$$

a) Determine the image of S under the map $z \to \sin z$.
b) Determine the image of the boundary ∂S of S under the map $z \to \sin z$.
c) Is the function $f(z) = \sin z$ one–to–one on $\bar{S} = S \cup \partial S$? Justify your answer.

2) Let

$$w(q) = \int_0^q (1+\zeta)^{-1/2}(1-\zeta)^{-1/2}\, d\zeta$$

for $q \in \mathbb{R}$. How do you have to define the roots so that the function $q \to w(q)$ for $q \in \mathbb{R}$ inverts the function $z \to \sin z$ on ∂S?

3) Let

$$f(z) = \int_1^z (\zeta+1)^{-2/3}(\zeta-1)^{-2/3}\, d\zeta \ .$$

Choose the roots as in class. Determine the image of the extended real line $\mathbb{R} \cup \{\pm\infty\}$ under the map $z \to f(z)$.

4) Fix an angle α with $0 < \alpha < \pi$ and consider the function

$$f(z) = \int_0^z \zeta^{-(\pi-\alpha)/\pi}\, d\zeta \ .$$

Choose the root as in class.
a) Describe the image of the real line under the map $z \to f(z)$.
b) In this case it is not too difficult to describe the image of the open upper half–plane $\mathbb{H}$ under f. You can parametrize the straight line from 0 to z by

$$\zeta(t) = zt, \quad 0 \le t \le 1 \ ,$$

for example. Describe the image of $\mathbb{H}$ under the map f.