1) Let $p_j$ denote the $j$–th prime, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$, etc. Determine the following limits:

$$
\lim_{n \to \infty} \prod_{j=1}^{n} \left(1 + \frac{1}{p_j}\right)
$$

and

$$
\lim_{n \to \infty} \prod_{j=1}^{n} \left(1 - \frac{1}{p_j}\right).
$$

Hint: Recall what we learned about $\sum_{j=1}^{\infty} \frac{1}{p_j}$.

2) Let $\Gamma_c$ denote the straight line with parameterization

$$
s(t) = c + it, \quad \infty < t < \infty.
$$

Prove the formula

$$
\frac{1}{2\pi i} \int_{\Gamma_c} \frac{a^s}{s(s+1)} \, ds = \begin{cases} 
0 & \text{if } 0 < a \leq 1 \\
1 - \frac{1}{a} & \text{if } a \geq 1
\end{cases}
$$

for $c > 0$.

Hint: This is a result in Chapter 7 of [Stein, Sharkarchi]. Carry out the details of the proof.

3) Define

$$
Li(x) = \int_{2}^{x} \frac{dt}{\ln t} \quad \text{for} \quad x \geq 2.
$$

Prove that there exists a constant $C > 0$ so that

$$
\left| Li(x) - \frac{x}{\ln x} \right| \leq C \frac{x}{(\ln x)^{2}} \quad \text{for} \quad x \geq 2.
$$

Hint: Use the hints of [Stein, Sharkarchi], Chapter 7, Exercise 10.

4a) Use the Prime Number Theorem to prove that

$$
\frac{\ln \pi(x)}{\ln x} \to 1 \quad \text{as} \quad x \to \infty.
$$

b) Let $p_j$ denote the $j$–th prime number. Prove that

$$
\frac{p_j}{j \ln j} \to 1 \quad \text{as} \quad j \to \infty.
$$

Hint: Use the PNT for $x = p_j$ and use a).