Homework 4, Math. 562
Prof. Jens Lorenz, Instructor
Assigned: March 20, 2023. Due: April 3, 2023

1) Let p_j denote the j–th prime, i.e., $p_1 = 2, p_2 = 3, p_3 = 5$, etc. Determine the following limits:

$$\lim_{n\to\infty} \Pi_{j=1}^{n}\Big(1 + \frac{1}{p_j}\Big)$$

and

$$\lim_{n\to\infty} \Pi_{j=1}^{n}\Big(1 - \frac{1}{p_j}\Big) \ .$$

Hint: Recall what we learned about $\sum_{j=1}^{\infty} \frac{1}{p_j}$.

2) Let Γ_c denote the straight line with parameterization

$$s(t) = c + it, \quad -\infty < t < \infty \ .$$

Prove the formula

$$\frac{1}{2\pi i}\int_{\Gamma_c} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 0 & \text{if} \quad 0 < a \le 1 \\ 1 - \frac{1}{a} & \text{if} \quad a \ge 1 \end{cases}$$

for $c > 0$.

Hint: This is a result in Chapter 7 of [Stein, Sharkarchi]. Carry out the details of the proof.

3) Define

$$Li(x) = \int_2^x \frac{dt}{\ln t} \quad \text{for} \quad x \ge 2 \ .$$

Prove that there exists a constant $C > 0$ so that

$$\Big| Li(x) - \frac{x}{\ln x}\Big| \le C\,\frac{x}{(\ln x)^2} \quad \text{for} \quad x \ge 2 \ .$$

Hint: Use the hints of [Stein, Sharkarchi], Chapter 7, Exercise 10.

4a) Use the Prime Number Theorem to prove that

$$\frac{\ln \pi(x)}{\ln x} \to 1 \quad \text{as} \quad x \to \infty \ .$$

b) Let p_j denote the j–th prime number. Prove that

$$\frac{p_j}{j \ln j} \to 1 \quad \text{as} \quad j \to \infty \ .$$

Hint: Use the PNT for $x = p_j$ and use a).