Problems and Remarks, Math. 562
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1) Let a_j denote a sequence of complex numbers. Then

$$f(s) = \sum_{j=1}^\infty \frac{a_j}{j^s} \tag{0.1}$$

is called a Dirichlet series.

Assume that $\operatorname{Re} s > 1$. Prove: If the sequence a_j is bounded then the series (0.1) converges absolutely and defines a function $f(s)$ which is holomorphic in the half–plane

$$H = \{ s \ : \ \operatorname{Re} s > 1 \} \ .$$

2) Let a_j and b_k denote two bounded sequences of complex numbers and let

$$f(s) = \sum_{j=1}^\infty \frac{a_j}{j^s}, \quad g(s) = \sum_{k=1}^\infty \frac{b_k}{k^s} \quad \text{for} \quad \operatorname{Re} s > 1 \ .$$

Prove that

$$f(s) g(s) = \sum_{n=1}^\infty \frac{c_n}{n^s} \quad \text{for} \quad \operatorname{Re} s > 1 \tag{0.2}$$

where

$$c_n = \sum_{jk=n} a_j b_k \ .$$

The sum defining c_n is the sum of all products $a_j b_k$ where j and k are positive integers with $jk = n$.

Prove that the series

$$\sum_{n=1}^\infty \frac{c_n}{n^s}$$

converges absolute for $\operatorname{Re} s > 1$ and that (0.2) holds.

Remark: One needs a theorem on the convergence of two absolutely convergent series and rearrangement of the product terms. Try to find and use such a theorem. You do not have to prove it.

3) **The Möbius function:**

$$\mu(n) = \begin{cases} 1 & \text{if} \ \ n = 1 \\ 1 & \text{if } n > 1 \text{ is the product of an even number of distinct primes} \\ -1 & \text{if } n > 1 \text{ is the product of an odd number of distinct primes} \\ 0 & \text{if } n \text{ contains a quadratic prime factor} \end{cases}$$

A property of the Möbius function:

Lemma 0.1 *The following holds:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{for} \quad n = 1 \\ 0 & \textit{for} \quad n > 1 \end{cases}$$

The product is taken over all $d \in \mathbb{N}$ which divide n.

Proof: The formula is obvious for $n = 1$. Let

$$n = p_1^{a_1} \dots p_k^{a_k}, \quad k \geq 1 \ ,$$

with distinct primes p_j and exponents $a_j \geq 1$.

Let $d|n$. If $d = 1$ then

$$\mu(d) = \mu(1) = 1 \ .$$

One obtains that

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \mu(1) + \mu(p_1) + \ldots + \mu(p_k) + \mu(p_1 p_2) + \ldots + \mu(p_{k-1} p_k) + \ldots + \mu(p_1 \cdots p_k) \\
&= 1 + \begin{pmatrix} k \\ 1 \end{pmatrix} (-1)^1 + \begin{pmatrix} k \\ 2 \end{pmatrix} (-1)^2 + \begin{pmatrix} k \\ 3 \end{pmatrix} (-1)^3 + \ldots + \begin{pmatrix} k \\ k \end{pmatrix} (-1)^k \\
&= (1-1)^k \\
&= (1-1)^k \\
&= 0
\end{aligned}$$

This proves the lemma. $\diamond$

4) Let

$$g(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \quad \text{for} \quad \operatorname{Re} s > 1 \ .$$

Use the results of 2) and 3) to show that

$$\zeta(s) g(s) = 1 \quad \text{for} \quad \operatorname{Re} s > 1 \ . \tag{0.3}$$

5) Recall Euler's product formula

$$\zeta(s) = \Pi_p (1 - p^{-s})^{-1} \quad \text{for} \quad \operatorname{Re} s > 1$$

where the product is taken over all primes p. Thus

$$\frac{1}{\zeta(s)} = \Pi_p (1 - p^{-s}) \quad \text{for} \quad \operatorname{Re} s > 1 \ .$$

Use this to give another proof of the formula

$$\frac{1}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \quad \text{for} \quad \operatorname{Re} s > 1$$

which agrees with (0.3).

Remarks: The function

$$M(n) = \sum_{k=1}^{n} \mu(k), \quad n \in \mathbb{N} \ ,$$

is called the Mertens function. Since the values $\mu(k)$ change rather randomly between plus one and minus one, one can expect that $|M(n)|$ should not be much larger than $\sqrt{n}$. (The zero values of $\mu(k)$ help, but not much.)

See graphs of the Mertens function on Wikipedia.

If you can prove that there is a constant $C > 0$ so that

$$|M(n)| \le C\sqrt{n} \quad \text{for all} \quad n \in \mathbb{N} \ , \tag{0.4}$$

then the function

$$g(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s}$$

is holomorphic for $\operatorname{Re} s > \frac{1}{2}$ and the equation

$$\zeta(s)g(s) = 1$$

holds for $\operatorname{Re} s > \frac{1}{2}$. Clearly, this implies that $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}$, and you have proved the Riemann hypothesis.

Actually, the Riemann hypothesis is equivalent to the slightly weaker estimate of the Mertens function:

For all $\varepsilon > 0$ there is a constant C_ε so that

$$|M(n)| \le \frac{C_\varepsilon}{n^{\frac{1}{2}+\varepsilon}} \quad \text{for all} \quad n \in \mathbb{N} \ .$$

If one looks at graphs of $M(n)$ it is hard to believe that such an estimate does not hold.