Functions of a Complex Variable II
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Contents

1 Notations and History

1.1 Notations

$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$D(z_0, R)$	$\{z \ : \ \lvert z - z_0\rvert < R\}$: open disk of radius R centered at z_0
$\bar{D}(z_0, R)$	$\{z \ : \ \lvert z - z_0\rvert \le R\}$: closed disk of radius R centered at z_0
$\partial D(z_0, R)$	$\{z \ : \ \lvert z - z_0\rvert = R\}$: boundary of disk of radius R centered at z_0
$\mathbb{D}$	$\mathbb{D} = D(0,1) = \{z \ : \ \lvert z\rvert < 1\}$: open unit disk
$H(U)$	set of all holomorphic functions $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open
$C(U)$	set of all continuous functions $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is any set
$\ln(x)$	for real positive x: $\ln(x) = \int_1^x \frac{ds}{s}$
$\log(z)$	complex logarithm

1.2 History

Pafnuty Chebyshev, 1821–1899, Russian
Bernhard Riemann, 1826–1866, German
Elwin Christoffel, 1829–1900, German
Hermann Amandus Schwarz, 1843–1921, German
Gösta Mittag–Leffler, 1846–1927, Swedish
Jacques Hadamard, 1865–1963, French
Charles–Jean de la Vallée Poussin, 1866-1962, from Belgium
Laurent Schwartz, 1915–2002, French

2 The Schwarz Lemma and $Aut(\mathbb{D})$

Summary: An aim of this chapter is to study the group of automorphisms of the open unit disk $\mathbb{D} = D(0,1)$. An important tool is Schwarz Lemma.

We will show that the automorphism group $Aut(\mathbb{D})$ consists of all functions of the form

$$f(z) = \phi_c(z) \circ R_\alpha(z), \quad z \in \mathbb{D} \, ,$$

where $R_\alpha(z) = \alpha z$ with $|\alpha| = 1$ is a rotation and

$$\phi_c(z) = \frac{z-c}{1-\bar{c}z}, \quad z \in \mathbb{D} \, ,$$

with $|c| < 1$ is a special Möbius[1] transformation.

2.1 Schwarz Lemma

Let $\mathbb{D} = D(0,1)$ denote the open unit disk. In the Schwarz Lemma one considers functions $f \in H(\mathbb{D})$ with $f(0) = 0$ and $f(\mathbb{D}) \subset \bar{\mathbb{D}}$. The Schwarz Lemma then says that the estimate $|f(z)| \leq 1$ for all $z \in \mathbb{D}$ can be sharpened.

We recall the maximum modulus theorem, which will be used in the proof of the Schwarz Lemma.

Theorem 2.1 *(Maximum Modulus Theorem) Let $U \subset \mathbb{C}$ denote an open connected set and let $g \in H(U)$. If there exists a point $z_0 \in U$ with*

$$|g(z)| \leq |g(z_0)| \quad \text{for all} \quad z \in U$$

then $g(z)$ is constant in U. In other words, only constant holomorphic functions attain their maximal value in an open connected set.

This follows from the open mapping theorem: A holomorphic function $g(z)$ on an open, connected set U maps open subsets of U to open sets, unless $g(z)$ is constant.

Remark: The Schwarz Lemma is named after Hermann Amandus Schwarz, a German mathematician, 1843-1921. His name also occurs in the Cauchy–Schwarz inequality.

Theorem 2.2 *(Schwarz Lemma) Let $f \in H(\mathbb{D})$ satisfy*

a) $|f(z)| \leq 1$ for all $z \in \mathbb{D}$;

b) $f(0) = 0$.

Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. In addition, if

$$|f(z_0)| = |z_0|$$

for some $z_0 \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$, then f is a rotation, i.e.,

$$f(z) = \alpha z$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

[1] August Ferdinand Möbius, 1790-1868, was a German mathematician and astronomer.

Proof: Set

$$g(z) = \begin{cases} f(z)/z & \text{for} \quad 0 < |z| < 1 \ , \\ f'(0) & \text{for} \quad z = 0 \ . \end{cases}$$

Since $f(0) = 0$ we have

$$f(z) = \sum_{j=1}^{\infty} a_j z^j \quad \text{for} \quad |z| < 1$$

and obtain that $g \in H(\mathbb{D})$. Let $0 < \varepsilon < 1$ and consider g in the closed disk $\bar{D}(0, 1-\varepsilon) \subset \mathbb{D}$. Since $|f(z)| \leq 1$ in $\mathbb{D}$ one obtains that

$$|g(z)| \leq \frac{1}{1-\varepsilon} \quad \text{for} \quad |z| = 1 - \varepsilon \ .$$

By the maximum modulus theorem we conclude that

$$|g(z)| \leq \frac{1}{1-\varepsilon} \quad \text{for} \quad |z| \leq 1 - \varepsilon \ .$$

As $\varepsilon \to 0$ this yields that

$$|g(z)| \leq 1 \quad \text{for} \quad |z| < 1 \ .$$

Therefore,

$$|f(z)| \leq |z| \quad \text{for} \quad |z| < 1 \quad \text{and} \quad |f'(0)| \leq 1 \ .$$

Now assume that $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}, z_0 \neq 0$, or assume that $|f'(0)| = 1$. Then, using the definition of g, we have $|g(z_0)| = 1$ or $|g(0)| = 1$. In both cases, the absolute value of $g(z)$ attains a maximum at a point in $\mathbb{D}$, which implies that $g(z) \equiv \alpha$ is constant and $|\alpha| = 1$. Therefore, $f(z) = \alpha z$. $\diamond$

2.2 Biholomorphic Maps and Automorphisms

Terminology: A nonempty, open, connected subset $\Omega \subset \mathbb{C}$ is called a region.

Let U and V be regions in $\mathbb{C}$. A map $f : U \to V$ is called biholomorphic if f is $1-1$ and onto and f as well as f^{-1} are holomorphic. It is sufficient to assume that $f : U \to V$ is holomorphic, $1-1$ and onto. Then f^{-1} is automatically holomorphic, as we have proved in Math 561. We also recall that $f'(z) \neq 0$ for all $z \in U$ if f is biholomorphic.

Definition: Let U be a region in $\mathbb{C}$. If $f : U \to U$ is $1-1$, onto, and holomorphic (thus biholomorphic), then f is called an automorphism of U. The set of all automorphisms of U is denoted by $Aut(U)$. This set forms a group if the product of $f, g \in Aut(U)$ is defined as the composition, $f \circ g$.

Example 1: According to Homework 6, Problem 1, Math 561, the set $Aut(\mathbb{C})$ consists of all functions $f(z)$ of the form $f(z) = az + b$ where $a \neq 0$. To prove this the Casorati–Weierstrass theorem for essential singularities is useful.

2.3 The Automorphism Group of $\mathbb{D}$

Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. A rational function of the form

$$f(z) = \frac{az + b}{cz + d}$$

is called a Möbius transformation. (If $ad - bc = 0$ then $f(z)$ is constant.) A Möbius transformation is also called a linear fractional transformation.

We first consider some special Möbius transformations. Let $c \in \mathbb{C}$ and $|c| < 1$. Set

$$\phi_c(z) := \frac{z - c}{1 - \bar{c}z}, \quad z \in \mathbb{D} .$$

Clearly, $\phi_c \in H(\mathbb{D})$ for $|c| < 1$. Note that

$$\phi_c(z) \equiv z \quad \text{for} \quad c = 0 .$$

We prove:

Lemma 2.1 *Let $c \in \mathbb{C}, |c| < 1$. The following holds:*
a) $|\phi_c(z)| < 1$ for all $z \in \mathbb{D}$.
b) $\phi_c(\phi_{-c}(z)) = z$ for all $z \in \mathbb{D}$.
c) $\phi_c \in Aut(\mathbb{D})$

Proof: a) Clearly, $|\phi_c(z)| < 1$ is equivalent to

$$|1 - \bar{c}z|^2 > |z - c|^2 ,$$

which is equivalent to

$$(1 - \bar{c}z)(1 - c\bar{z}) > (z - c)(\bar{z} - \bar{c}) ,$$

which is equivalent to

$$1 + |c|^2|z|^2 > |z|^2 + |c|^2 . \tag{2.1}$$

Since

$$0 < (1 - |c|^2)(1 - |z|^2) = 1 + |c|^2|z|^2 - |z|^2 - |c|^2$$

the inequality (2.1) holds.

b) With

$$\phi_c(w) = \frac{w - c}{1 - \bar{c}w}, \quad \phi_{-c}(z) = \frac{z + c}{1 + \bar{c}z}$$

we have

$$\begin{aligned} \phi_c(\phi_{-c}(z)) &= \frac{\frac{z+c}{1+\bar{c}z} - c}{1 - \bar{c}\,\frac{z+c}{1+\bar{c}z}} \\ &= \frac{z + c - c - z|c|^2}{1 + \bar{c}z - \bar{c}z - |c|^2} \\ &= \frac{z(1 - |c|^2)}{1 - |c|^2} \\ &= z . \end{aligned}$$

c) Since $\phi_c(\phi_{-c}(z)) = z$ for all $z \in \mathbb{D}$, we obtain that ϕ_c is onto and ϕ_{-c} is $1-1$. Replacing c by $-c$ shows that ϕ_c is a bijection of $\mathbb{D}$. $\diamond$

Another set of automorphisms of $\mathbb{D}$ are the rotations R_α defined by

$$R_\alpha(z) = \alpha z, \quad |z| < 1 ,$$

for $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Clearly, $R_\alpha \in Aut(\mathbb{D})$ and

$$\phi_c \circ R_\alpha \in Aut(\mathbb{D}) \tag{2.2}$$

if $|c| < 1$ and $|\alpha| = 1$. The following theorem yields that *every* $f \in Aut(\mathbb{D})$ has the form (2.2).

Theorem 2.3 *Let $f \in Aut(\mathbb{D})$ and let $b = f(0)$. Then we have*

$$f = \phi_{-b} \circ R_\alpha = R_\alpha \circ \phi_{-b\bar{\alpha}} \tag{2.3}$$

for some α with $|\alpha| = 1$.

In particular, if $f \in Aut(\mathcal{D})$ satisfies $f(0) = 0$ then f is a rotation.

Proof: 1) First assume that $f(0) = 0$ and let $g = f^{-1}$. By Schwarz Lemma we conclude that $|f'(0)| \le 1$ and $|g'(0)| \le 1$. From $f(g(z)) = z$ for all $z \in \mathbb{D}$ we have

$$f'(g(z))g'(z) = 1 ,$$

and, in particular,

$$f'(0)g'(0) = 1 .$$

It follows that $|f'(0)| = 1$. Another application of Schwarz Lemma yields that $f(z) = \alpha z$ with $|\alpha| = 1$. Equation (2.3) holds with $b = 0$.

2) Let $b := f(0)$ and consider $h = \phi_b \circ f$. We have

$$h(0) = \phi_b(b) = 0 .$$

By Part 1) of the proof we conclude that $h = R_\alpha$ for some α with $|\alpha| = 1$. Therefore, $f = \phi_{-b} \circ R_\alpha$. In other words,

$$f(z) = \phi_{-b}(\alpha z) = \phi_{-b} \circ R_\alpha(z) .$$

3) The equations

$$\begin{aligned} f(z) &= \phi_{-b}(\alpha z) \\ &= \frac{\alpha z + b}{1 + \bar{b}\alpha z} \\ &= \alpha \frac{z + b\bar{\alpha}}{1 + \bar{b}\alpha z} \\ &= \alpha \phi_{-b\bar{\alpha}}(z) \end{aligned}$$

complete the proof of the theorem. $\diamond$

2.4 The Schwarz–Pick Lemma

This section can be skipped.

The automorphisms ϕ_c of $\mathbb{D}$, obtained for every $c \in \mathbb{C}$ with $|c| < 1$, allow to prove an extension of the Schwarz lemma. Note that

$$\phi_c(z) = \frac{z-c}{1-\bar{c}z}, \quad \phi_c'(z) = \frac{1-|c|^2}{(1-\bar{c}z)^2} \ .$$

Theorem 2.4 *(Schwarz–Pick) Let $f : \mathbb{D} \to \mathbb{D}, f \in H(\mathbb{D})$. (It is neither assumed that f is $1-1$ nor that f is onto nor that $f(0) = 0$.)*

1) For all $a \in \mathbb{D}$:

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2} \tag{2.4}$$

(If $a = 0$ and $f(0) = 0$ this reduces to the estimate $|f'(0)| \leq 1$ of the Schwarz lemma.)

2) If $a_1, a_2 \in \mathbb{D}$ then

$$\frac{|f(a_2)-f(a_1)|}{|1-\overline{f(a_1)}f(a_2)|} \leq \frac{|a_2-a_1|}{|1-\overline{a_1}a_2|} \ . \tag{2.5}$$

Proof: 1) Let $b = f(a)$ and set

$$F = \phi_b \circ f \circ \phi_{-a} \ .$$

Then we have $\phi_{-a}(0) = a$ and

$$\begin{aligned} F(0) &= \phi_b(f(a)) \\ &= \phi_b(b) \\ &= 0 \end{aligned}$$

The Schwarz lemma is applicable to F and implies that

$$|F'(0)| \leq 1 \ .$$

Further, by the chain rule,

$$F'(0) = \phi_b'(b) f'(a) \phi_{-a}'(0) \ .$$

Here

$$\phi_{-a}'(0) = 1 - |a|^2$$

and

$$\phi_b'(b) = \frac{1-|b|^2}{(1-|b|^2)^2} = \frac{1}{1-|b|^2} \ .$$

Therefore,

$$F'(0) = f'(a)\,\frac{1-|a|^2}{1-|b|^2} \ .$$

The estimate $|F'(0)| \le 1$ yields that

$$|f'(a)| \le \frac{1-|b|^2}{1-|a|^2} \; .$$

This proves (2.4).

2) Let $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Consider the function

$$F = \phi_{b_1} \circ f \circ \phi_{-a_1} \; .$$

As above, we have $F(0) = 0$ and the Schwarz lemma yields that $|F(z)| \le |z|$ for all $z \in \mathbb{D}$. Setting

$$w = \phi_{-a_1}(z), \quad z = \phi_{a_1}(w) \; ,$$

the estimate $|F(z)| \le |z|$ becomes

$$|\phi_{b_1}(f(w))| \le |\phi_{a_1}(w)| \; .$$

Using the definition of ϕ_c this reads:

$$\Big| \frac{f(w) - b_1}{1 - \overline{b_1} f(w)} \Big| \le \Big| \frac{w - a_1}{1 - \overline{a_1} w} \Big| \; .$$

If we use this estimate for $w = a_2$ we obtain (2.5). $\diamond$

2.5 Remarks

The formula

$$d(a,b) = \frac{|b-a|}{|1-\bar{a}b|}, \quad a, b \in \mathbb{D} \; ,$$

defines the so-called pseudo–hyperbolic metric on $\mathbb{D}$. (This is related to Poincaré geometry on $\mathbb{D}$.) The second part of the Schwarz–Pick Lemma then says that

$$d(f(a), f(b)) \le d(a,b) \quad \text{for all} \quad a, b \in \mathbb{D}$$

if $f \in H(\mathbb{D}), f : \mathbb{D} \to \mathbb{D}$. If $f \in Aut(\mathbb{D})$ then one can apply the estimate to f and f^{-1} to obtain that

$$d(f(a), f(b)) = d(a,b) \quad \text{for all} \quad a, b \in \mathbb{D} \; . \tag{2.6}$$

In fact, one can also prove a converse: If $f \in H(\mathbb{D}), f : \mathbb{D} \to \mathbb{D}$, and if (2.6) holds, then $f \in Aut(\mathbb{D})$.

The introduction of Non–Euclidean geometries was historically very important. Recall: Euclid of Alexandria ($\sim$ 365–300 B.C.) and his Parallel Postulate, the famous Fifth Postulate.

Immanuel Kant: *The concept of Euclidean space is by no means of empirical origin, but is an inevitable necessity of thought.* (Critique of Pure Reason, 1781)

Work of Carl Friedrich Gauss (1777–1855) and Janos Bolyai (1802–1860) on non–Euclidean geometry made it clear that Kant's position was wrong. One has to distinguish between geometry as a mathematical subject and geometry of physical space, which is a subject of observation. Newtonian physics assumes Euclidean space and a time variable independent of space. To create the theory of general relativity, it was necessary to overcome the doctrine of Euclidean space.

The space $\mathbb{D}$ with the metric $d(a,b)$ gives an example of Non–Euclidean geometry.

3 Linear Fractional Transformations and the Riemann Sphere

Summary: In complex variables, it is often useful to include the point ∞ in the domain of definition of a function and also as a possible value of a complex function. This leads to the introduction of the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2} \quad \text{with} \quad ad - bc \neq 0 \ ,$$

determines the Möbius transformation

$$\phi_A(z) = \frac{az+b}{cz+d}, \quad z \in \hat{\mathbb{C}} \ ,$$

which is biholomorphic on the Riemann sphere $\hat{\mathbb{C}}$. We will show that

$$\phi_{AB} = \phi_A \circ \phi_B$$

if $A, B \in \mathbb{C}^{2\times 2}$ are two nonsingular matrices. We will also show that *any* biholomorphic map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation, i.e., $\phi = \phi_A$ for some nonsingular $A \in \mathbb{C}^{2\times 2}$.

If $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ are three distinct points and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are also distinct, then there exists a unique Möbius transformation ϕ_A with

$$\phi_A(z_j) = w_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Another remarkable geometric property of Möbius transformations: They map any circle and any straight line onto a circle or a straight line.

Special Möbius transformation play some role in applications. The Cayley transform

$$F(z) = \frac{z-i}{z+i}$$

maps the open upper half–plane

$$\mathbb{H} = \{z = x + iy \ : \ y > 0\}$$

bijectively onto the open unit disk $\mathbb{D} = D(0,1)$. The boundary

$$\partial\mathbb{H} = \{z = x \ : \ x \in \mathbb{R}\}$$

is mapped onto $\partial\mathbb{D} \setminus \{1\}$. If one notes that $F(\infty) = 1$ then one obtains the bijection

$$F \ : \ \partial\mathbb{H} \cup \{\infty\} \to \partial\mathbb{D} \ .$$

3.1 The Riemann Sphere

It is often useful to compactify the complex plane $\mathbb{C}$ by formally adding the point ∞. We use the notation

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \ .$$

One can turn $\hat{\mathbb{C}}$ into a topological space by using an identification (this is nothing but a map which is $1-1$ and onto) with the unit sphere in $\mathbb{R}^3$,

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 = 1\} \ .$$

This identification can be established as follows: Let $N = (0, 0, 1) \in S$ denote the north pole of S and let

$$z = x + iy \in \mathbb{C} \ .$$

Draw the straight line in $\mathbb{R}^3$ from N to $(x, y, 0)$. The line intersects the unit sphere S in a unique point, which we call $\Pi(z)$, the stereographic projection of z to S. The map Π maps $\mathbb{C}$ bijectively onto $S \setminus \{N\}$, i.e., the map

$$\Pi \ : \ \begin{cases} \mathbb{C} & \rightarrow \quad S \setminus \{N\} \\ z & \rightarrow \quad \Pi(z) \end{cases}$$

is bijective. As $|z| \to \infty$, one obtains that $\Pi(z) \to N$ on S. It is therefore natural to extend Π as a mapping from $\hat{\mathbb{C}}$ to S by defining

$$\Pi(\infty) = N = (0, 0, 1) \ .$$

In this way, one obtains a bijection

$$\Pi : \hat{\mathbb{C}} \to S \ .$$

The unit sphere S is a metric space if we use the Euclidean distance in $\mathbb{R}^3$ as distance between points in S. We know, then, what it means that a sequence $q_n \in S$ converges to $q \in S$. We know what the open and the closed sets in S are etc. Using the map Π we transform the concept of convergence etc. (or the metric) to $\hat{\mathbb{C}}$. If the point ∞ is not involved, these concepts agree with the standard concepts in $\mathbb{C}$, but they are now extended in a meaningful way to $\hat{\mathbb{C}}$. The space $\hat{\mathbb{C}}$ is called the one–point compactification of $\mathbb{C}$, or the Riemann sphere.

We mention that one can also consider $\hat{\mathbb{C}}$ as a complex manifold by introducing local coordinates near the point $\infty \in \hat{\mathbb{C}}$. If one has done this, then the notion of holomorphy of a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ makes sense. In practice, to discuss holomorphy when ∞ is involved, one uses the mappings $z \to 1/z$ and $w \to 1/w$ to map ∞ to zero.

Holomorphy of functions where ∞ is involved:

We distinguish between three cases:

1) $f(w)$ is complex valued in a neighborhood of $w = \infty$.

2) $f(w_0) = \infty$ for some $w_0 \in \mathbb{C}$.

3) $f(\infty) = \infty$.

Case 1) Let us first define what it means that a complex–valued function $f(w)$ is holomorphic at $w = \infty$. Let $\Omega \subset \hat{\mathbb{C}}$ denote an open set with $\infty \in \Omega$ and let $f : \Omega \to \mathbb{C}$ denote a map. Since Ω is an open set there exists $\varepsilon > 0$ so that $\frac{1}{z} \in \Omega$ for $0 < |z| < \varepsilon$. Define

$$F(z) = \begin{cases} f(1/z) & \text{for} \quad 0 < |z| < \varepsilon \ , \\ f(\infty) & \text{for} \quad z = 0 \ . \end{cases}$$

Then, by definition, the function $f(w)$ is holomorphic at $w = \infty$ if and only if the function $F(z)$ is holomorphic at $z = 0$. If $F(z)$ is holomorphic at $z = 0$ then

$$F(z) = \sum_{j=0}^{\infty} a_j z^j \quad \text{for} \quad |z| < \varepsilon$$

and

$$f(w) = \begin{cases} \sum_{j=0}^{\infty} a_j w^{-j} & \text{for} \quad \frac{1}{\varepsilon} < |w| < \infty \ , \\ f(\infty) & \text{for} \quad w = \infty \ . \end{cases}$$

Here $a_0 = F(0) = f(\infty)$.

Example 1: Let

$$f(w) = \frac{w}{w+3} \quad \text{for} \quad |w| > 3, \quad f(\infty) = 1 \ .$$

We have

$$F(z) = \frac{1/z}{1/z+3} = \frac{1}{1+3z} \quad \text{for} \quad 0 < |z| < \frac{1}{3} \quad \text{and} \quad F(0) = f(\infty) = 1 \ .$$

The function $F(z)$ is holomorphic at $z = 0$. Thus, by definition, $f(w)$ is holomorphic at $w = \infty$.

Case 2) Holomorphy of a function $f(w)$ with $f(w_0) = \infty$ for some $w_0 \in \mathbb{C}$: Let $\Omega \subset \mathbb{C}$ denote an open set and let $w_0 \in \Omega$. Let

$$f(w) \in \mathbb{C} \quad \text{for} \quad w \in \Omega \setminus \{w_0\}$$

and let $f(w_0) = \infty$. Assume that $f(w)$ is continuous at w_0. Then, for $0 < |w - w_0| < \varepsilon$, we have $f(w) \neq 0$ and set

$$F(w) = \begin{cases} 1/f(w) & \text{for} \quad 0 < |w - w_0| < \varepsilon \\ 0 & \text{for} \quad w = w_0 \end{cases}$$

If $F(w)$ is holomorphic at w_0 then, by definition, $f(w)$ is holomorphic at w_0.

Example 2: Let

$$f(w) = \frac{1}{w-3} \quad \text{for} \quad w \in \mathbb{C} \setminus \{3\}, \quad f(3) = \infty \ .$$

We have

$$F(w) = w - 3 \quad \text{for} \quad w \in \mathbb{C} \setminus \{3\}, \quad F(3) = 0 \ .$$

Clearly, the function $F(w)$ is holomorphic at $w_0 = 3$. Thus, by definition, $f(w)$ is holomorphic at $w_0 = 3$.

Case 3) Holomorphy of a function $f(w)$ with $f(\infty) = \infty$: Let $\Omega \subset \hat{\mathbb{C}}$ denote an open set with $\infty \in \Omega$ and let

$$f(w) \in \mathbb{C} \quad \text{for} \quad w \in \Omega \setminus \{\infty\}, \quad f(\infty) = \infty \ .$$

Assume that $f(w)$ is continuous at $w = \infty$. Then, for $\frac{1}{\varepsilon} < |w| < \infty$, we have $f(w) \neq 0$ and set

$$F(z) = \frac{1}{f(1/z)} \quad \text{for} \quad 0 < |z| < \varepsilon, \quad F(0) = 0 \ .$$

If $F(z)$ is holomorphic at $z_0 = 0$ then, by definition, $f(w)$ is holomorphic at $w = \infty$.

Example 3: Let

$$f(w) = \sum_{j=0}^{n} a_j w^j \quad \text{where} \quad n \geq 1, \quad a_n \neq 0 \ .$$

Thus $f(w)$ is a non–constant polynomial. Since $|f(w)| \to \infty$ as $|w| \to \infty$ we set $f(\infty) = \infty$. We have

$$f(1/z) = \frac{1}{z^n} \Big(a_n + a_{n-1} z + \ldots + a_0 z^n \Big) \neq 0 \quad \text{for} \quad 0 < |z| < \varepsilon$$

and

$$F(z) = \frac{z^n}{a_n + a_{n-1} z + \ldots + a_0 z^n} \quad \text{for} \quad 0 < |z| < \varepsilon, \quad F(0) = 0 \ .$$

The function $F(z)$ is holomorphic at $z = 0$. Thus, by definition, the polynomial $f(w)$ is holomorphic at $w = \infty$ if one sets $f(\infty) = \infty$.

A simple example is

$$f(w) = \begin{cases} w^2 + 1 & \text{for} \quad w \in \mathbb{C} \\ \infty & \text{for} \quad w = \infty \end{cases}$$

One obtains that

$$f(1/z) = \begin{cases} \frac{1}{z^2} + 1 & \text{for} \quad z \in \mathbb{C}, \ z \neq 0 \\ \infty & \text{for} \quad z = 0 \end{cases}$$

and

$$F(z) = \frac{1}{f(1/z)} = \begin{cases} \frac{z^2}{z^2+1} & \text{for} \quad 0 < |z| < 1 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

Clearly, $F(z)$ is holomorphic at $z = 0$, thus $f(w)$ is holomorphic at $w = \infty$.

3.2 Linear Fractional Transformations

A linear fractional transformation is also called a Möbius transformation.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad det(A) \neq 0 \ ,$$

denote a nonsingular matrix in $\mathbb{C}^{2\times 2}$. Then A determines the linear fractional transformation

$$\phi_A(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \ . \tag{3.1}$$

(If one would allow $det(A) = 0$ then the transformation $\phi_A(z)$ would be constant, which is an uninteresting transformation.) Note that A and $qA, q \neq 0$, determine the same transformation, $\phi_A = \phi_{qA}$.

It will be convenient to consider ϕ_A as a bijection of $\hat{\mathbb{C}}$ onto itself. Then one obtains that ϕ_A is an automorphism of $\hat{\mathbb{C}}$.

Case 1: $c = 0$. In this case $d \neq 0$ and $a \neq 0$ and

$$\phi_A(z) = \frac{a}{d}\, z + \frac{b}{d}\ .$$

The transformation $\phi_A : \mathbb{C} \to \mathbb{C}$ is $1-1$ and onto. As $z_n \to \infty$, we have $\phi_A(z_n) \to \infty$. Therefore, we set $\phi_A(\infty) = \infty$ and obtain a continuous bijection of $\hat{\mathbb{C}}$. If one considers $\hat{\mathbb{C}}$ as a one–dimensional complex manifold, then this bijection is holomorphic.

Case 2: $c \neq 0$. In this case, if $z_0 = -d/c$, then $\phi_A(z_0)$ is not defined as a complex number. If $z_n \to z_0 = -d/c$ then $\phi_A(z_n) \to \infty$. This holds since

$$a(-d/c) + b = -\frac{1}{c}(ad - bc) \neq 0\ .$$

Therefore, we set

$$\phi_A(-d/c) = \infty\ . \tag{3.2}$$

In this way, ϕ_A is continuously extended to $z_0 = -d/c$. Also, if $z_n \to \infty$, then

$$\phi_A(z_n) \to a/c\ .$$

Therefore, we set

$$\phi_A(\infty) = a/c\ . \tag{3.3}$$

(Note: If $z \in \mathbb{C}$ then $(az+b)/(cz+d) \neq a/c$.) If one uses the equations (3.2) and (3.3) then $\phi_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is defined as a continuous map, which is holomorphic.

We claim that, with the additional definitions, (3.2) and (3.3), the map $\phi_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a bijection and its inverse is $\phi_{A^{-1}}$. This claim follows from the following important lemma.

Lemma 3.1 *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

denote nonsingular matrices in $\mathbb{C}^{2\times 2}$ *and let* C *denote their product,*

$$C = AB = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}\ .$$

Then we have

$$\phi_C = \phi_{AB} = \phi_A \circ \phi_B\ .$$

Proof: In all cases where the point ∞ is not involved we have

$$\phi_B(z) = \frac{\alpha z + \beta}{\gamma z + \delta} =: w$$

and

$$\begin{aligned}
\phi_A(w) &= \frac{aw+b}{cw+d} \\
&= \frac{a(\alpha z+\beta)+b(\gamma z+\delta)}{c(\alpha z+\beta)+d(\gamma z+\delta)} \\
&= \frac{(a\alpha+b\gamma)z+a\beta+b\delta}{(c\alpha+d\gamma)z+c\beta+d\delta} \\
&= \phi_C(z)
\end{aligned}$$

This proves that

$$\phi_{AB}(z) = \phi_A \circ \phi_B(z)$$

when ∞ is not involved. By continuous extension, the equation

$$\phi_{AB}(z) = \phi_A \circ \phi_B(z) \quad \text{for all} \quad z \in \hat{\mathbb{C}}$$

follows. $\diamond$

3.3 The Automorphisms of the Riemann Sphere

Let M denote the group of all Möbius transformations $\phi_A(z)$ where $A \in \mathbb{C}^{2\times 2}, det(A) \neq 0$. With $GL(2,\mathbb{C})$ one denotes the group of all nonsingular matrices in $\mathbb{C}^{2\times 2}$. The previous lemma says that the map

$$\Phi : \begin{cases} GL(2,\mathbb{C}) & \mapsto & M \\ A & \to & \phi_A(z) \end{cases} \tag{3.4}$$

is a group homomorphism. Clearly, Φ is onto.

Lemma 3.2 *We have*

$$M = Aut(\hat{\mathbb{C}}) \ ,$$

thus every automorphism of $\hat{\mathbb{C}}$ *is a Möbius transformation.*

Proof: Let $f \in Aut(\hat{\mathbb{C}})$. a) Assume first that $f(\infty) = \infty$. In this case, the restriction of f to $\mathbb{C}$ is an automorphism of $\mathbb{C}$. We have shown in Math 561 that

$$f(z) = az + b, \quad a \neq 0 \ .$$

In particular, $f \in M$.

b) Let $f(\infty) = b, b \in \mathbb{C}$. Consider the Möbius transformation

$$\phi(z) = \frac{1}{z-b}$$

and define $g = \phi \circ f$. Since g is an automorphism of $\hat{\mathbb{C}}$ and $g(\infty) = \infty$, we have $g \in M$ and then $f = \phi^{-1} \circ g \in M$. $\diamond$

Thus we have shown that

$$\Phi : \left\{ \begin{array}{ccc} GL(2,\mathbb{C}) & \mapsto & Aut(\hat{\mathbb{C}}) \\ A & \to & \phi_A(z) \end{array} \right. \tag{3.5}$$

is a group epimorphism. (A homomorphism, which is onto, is called an epimorphism.)

By definition, the kernel $ker(\Phi)$ of the epimorphism Φ defined in (3.5) consists of all matrices $A \in GL(2,\mathbb{C})$ for which the Möbius transformation $\phi_A(z)$ is the unit element of the group $Aut(\hat{\mathbb{C}})$, i.e., $A \in ker(\Phi)$ if and only if $\phi_A(z) \equiv z$.

By a simple result of group theory, one obtains that $Aut(\hat{\mathbb{C}})$ is isomorphic to the quotient group

$$GL(2,\mathbb{C})/ker(\Phi) \ .$$

It is not difficult to show that $\phi_A(z) \equiv z$ if and only if $A = aI$ for some $a \in \mathbb{C}, a \neq 0$. Thus,

$$ker(\Phi) = \{aI \ : \ a \in \mathbb{C}, \ a \neq 0\} \ .$$

If one identifies two matrices $A, B \in GL(2,\mathbb{C})$ if and only if $A = qB$ for some $q \in \mathbb{C}$, then one obtains the group $GL(2,\mathbb{C})/ker(\Phi)$. This group is isomorphic to $Aut(\hat{\mathbb{C}})$.

With

$$SL(2,\mathbb{C}) = \{A \in GL(2,\mathbb{C}) \ : \ det(A) = 1\}$$

one denotes the special linear group of all complex 2×2 matrices with determinant 1. Then

$$\Phi_S : \left\{ \begin{array}{ccc} SL(2,\mathbb{C}) & \mapsto & Aut(\hat{\mathbb{C}}) \\ A & \to & \phi_A(z) \end{array} \right. \tag{3.6}$$

is an epimorphism and

$$ker(\Phi_S) = \{I, -I\} \ .$$

Therefore, $Aut(\hat{\mathbb{C}})$ is isomorphic to

$$SL(2,\mathbb{C})/\{I, -I\} \ .$$

3.4 Möbius Transformation are Determined by Three Point–Values

In this section let

$$q_1 = 1, \quad q_2 = 0, \quad q_3 = \infty \ .$$

Theorem 3.1 *Let z_1, z_2, z_3 denote three distinct point in $\hat{\mathbb{C}}$ and let w_1, w_2, w_3 also denote three distinct point in $\hat{\mathbb{C}}$. There exists a unique Möbius transformation $\phi(z)$ with*

$$\phi(z_j) = w_j \quad for \quad j = 1, 2, 3 \ .$$

Proof: 1. Existence of ϕ: Define

$$S(z) := k\,\frac{z - z_2}{z - z_3} \quad \text{with} \quad k = \frac{z_1 - z_3}{z_1 - z_2} \ . \tag{3.7}$$

It is easy to check that

$$S(z_1) = 1, \quad S(z_2) = 0, \quad S(z_3) = \infty \ , \tag{3.8}$$

thus

$$S(z_j) = q_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Similarly, define

$$T(w) := h \, \frac{w - w_2}{w - w_3} \quad \text{with} \quad h = \frac{w_1 - w_3}{w_1 - w_2} \tag{3.9}$$

and note that

$$T(w_1) = 1, \quad T(w_2) = 0, \quad T(w_3) = \infty \ ,$$

thus

$$T(w_j) = q_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Therefore, if we define

$$\phi(z) = T^{-1}(S(z))$$

then

$$\phi(z_j) = w_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Uniqueness of ϕ: a) Assume that

$$\Phi(z) = \frac{az + b}{cz + d}$$

has the fixed points q_j for $j = 1, 2, 3$.

We will prove that $\Phi = id$, i.e., $\Phi(z) \equiv z$. First, obtain that $0 = \Phi(0) = b/d$, thus $b = 0$. From

$$\Phi(z) = \frac{az}{cz + d}$$

and $\Phi(\infty) = \infty$ obtain that $a \neq 0$ and $c = 0$. Thus $\Phi(z) = az/d$. The equation

$$1 = \Phi(1) = \frac{a}{d}$$

yields that $a = d$, thus $\Phi(z) \equiv z$.

b) Let $f(z)$ denote a Möbius transformation with three distinct fixed points z_1, z_2, z_3. We will prove that $f = id$. Let $S(z)$ denote the Möbius transformation (3.7); thus (3.8) holds. We set

$$\Phi(z) = S \circ f \circ S^{-1}(z) \ .$$

From $f(z_j) = z_j$ and $S(z_j) = q_j$ obtain that

$$\Phi(q_j) = S \circ f(z_j) = S(z_j) = q_j \quad \text{for} \quad j - 1, 2, 3 \ .$$

Our previous argument then implies that $\Phi = id$. Therefore, $f = id$.

c) Assume that the Möbius transformations $\phi_1(z)$ and $\phi_2(z)$ both satisfy the condition

$$\phi(z_j) = w_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Set

$$f(z) = \phi_1^{-1}(\phi_2(z))$$

and obtain that

$$f(z_j) = \phi_1^{-1}(w_j) = z_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

Our previous argument yields that $f = id$, thus $\phi_1 = \phi_2$. This proves uniqueness of the Möbius transformation ϕ satisfying

$$\phi(z_j) = w_j \quad \text{for} \quad j = 1, 2, 3 \ .$$

⋄

3.5 A Remarkable Geometric Property of Linear Fractional Transformations

Let $\mathcal{C}$ denote a circle or a straight line in the z–plane. We claim that, if $\phi_A(z)$ is any linear fractional transformation, then the image of $\mathcal{C}$ under the transformation

$$z \to \phi_A(z) = w$$

is a circle or a straight line in the w–plane.

If $c \neq 0$ then

$$\begin{aligned} \phi_A(z) &= \frac{az+b}{cz+d} \\ &= \frac{a(z+d/c)+b-ad/c}{c(z+d/c)} \\ &= \frac{a}{c} + \frac{b-ad/c}{cz+d} \end{aligned}$$

Therefore, every Möbius transformation is a composition of transformations of the form

$$z \to \alpha z, \quad z \to z + \beta, \quad z \to \frac{1}{z} \ ,$$

and since the statement is easily shown for transformations $z \to \alpha z$ and $z \to z + \beta$, we consider the transformation $z \to 1/z$,

$$z = x + iy \to \frac{1}{z} = w = u + iv \ .$$

3.5.1 Analytical Description of Circles

The circle $\mathcal{C} = \mathcal{C}(z_0, r)$ centered at z_0 with radius $r > 0$ has the equation

$$|z - z_0|^2 = r^2 \ ,$$

or,

$$z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + A = 0$$

with

$$A = z_0\bar{z}_0 - r^2, \quad r^2 = z_0\bar{z}_0 - A \ .$$

One obtains:

Lemma 3.3 *Let $z_0 \in \mathbb{C}$ and let $A \in \mathbb{R}$. Then the equation*

$$z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + A = 0$$

describes a circle if and only if

$$z_0\bar{z}_0 - A > 0 \ . \tag{3.10}$$

Assuming (3.10) to hold, the circle is centered at z_0 and has radius $r = \sqrt{z_0\bar{z}_0 - A}$. The circle passes through the point $z = 0$ if and only if $A = 0$.

3.5.2 Circles under Reciprocation

Let $\mathcal{C} = \mathcal{C}(z_0, r)$ denote the circle centered at z_0 with radius $r > 0$. Set $A = z_0\bar{z}_0 - r^2$.

Case 1: The circle does not pass through $z = 0$, thus $A \neq 0$. We apply the transformation

$$T(z) = \frac{1}{z} = w$$

to the points of $\mathcal{C}$. The points $w \in T(\mathcal{C})$ satisfy

$$\frac{1}{w\bar{w}} - \frac{\bar{z}_0}{w} - \frac{z_0}{\bar{w}} + A = 0 \ ,$$

or,

$$1 - \bar{w}\bar{z}_0 - wz_0 + Aw\bar{w} = 0 \ ,$$

or,

$$w\bar{w} - w\,\frac{z_0}{A} - \bar{w}\,\frac{\bar{z}_0}{A} + \frac{1}{A} = 0 \ .$$

The last equation has the form

$$w\bar{w} - w\bar{w}_0 - \bar{w}w_0 + B = 0$$

with

$$w_0 = \frac{\bar{z}_0}{A}, \quad B = \frac{1}{A} \ .$$

We have

$$\begin{aligned} w_0\bar{w}_0 - B &= \frac{z_0\bar{z}_0}{A^2} - \frac{1}{A} \\ &= \frac{1}{A^2}(z_0\bar{z}_0 - A) \\ &> 0 \ . \end{aligned}$$

Using the previous lemma, we obtain that $T(\mathcal{C})$ is the circle centered at

$$w_0 = \frac{\bar{z}_0}{A}$$

with radius

$$R = \frac{r}{|A|} \ .$$

Case 2: The circle passes through $z = 0$, i.e., $A = 0$. We apply the transformation

$$T(z) = \frac{1}{z} = w$$

to the points of $\mathcal{C} \setminus \{0\}$. The points $w = T(z)$ satisfy

$$\frac{1}{w\bar{w}} - \frac{\bar{z}_0}{w} - \frac{z_0}{\bar{w}} = 0 \ ,$$

or,

$$1 - \bar{w}\bar{z}_0 - wz_0 = 0 \ .$$

If

$$w = x + iy, \quad z_0 = a + ib \ ,$$

(with real x, y, a, b) then the above equation reads

$$1 - (x - iy)(a - ib) - (x + iy)(a + ib) = 0 \ ,$$

or,

$$1 - 2ax + 2by = 0 \ .$$

We obtain that $T(\mathcal{C} \setminus \{0\})$ is a straight line. If we set $T(0) = \infty$, then $T(\mathcal{C})$ is a straight line together with $w = \infty$.

Summary: If $\mathcal{C}$ is a circle passing through the origin $z = 0$ then the map $z \to 1/z$ maps $\mathcal{C} \setminus \{0\}$ to a straight line and maps $z = 0$ to ∞.

3.5.3 Straight Lines under Reciprocation

Let α, β, γ be real numbers, $(\alpha, \beta) \neq (0, 0)$. Then the equation

$$\alpha x + \beta y + \gamma = 0$$

describes a straight line $\mathcal{L}$.

Case 1: The line $\mathcal{L}$ does not pass through $z = 0$, i.e., $\gamma \neq 0$. We rewrite the above equation as

$$1 + \frac{\alpha}{\gamma} x + \frac{\beta}{\gamma} y = 0 \ ,$$

or,

$$1 - 2ax + 2by = 0$$

with

$$-2a = \frac{\alpha}{\gamma}, \quad 2b = \frac{\beta}{\gamma} \ .$$

We can also write the equation for $\mathcal{L}$ as

$$1 - (x - iy)(a - ib) - (x + iy)(a + ib) = 0 \ .$$

Setting

$$z_0 = a + ib$$

we obtain that the equation for $\mathcal{L}$ is

$$1 - \bar{z}\bar{z}_0 - zz_0 = 0 \quad \text{with} \quad z = x + iy \ .$$

It is then clear that $T(\mathcal{L})$ consists of the points w with

$$1 - \frac{\bar{z}_0}{\bar{w}} - \frac{z_0}{w} = 0 \ ,$$

or,

$$w\bar{w} - w\bar{z}_0 - \bar{w}z_0 = 0 \ .$$

Thus, $T(\mathcal{L})$ is a circle, centered at z_0, that passes through $w = 0$. The radius of the circle is $r = |z_0|$, of course.

Case 2: The line $\mathcal{L}$ passes through $z = 0$, i.e., $\gamma = 0$. We could proceed as above, but here it is simpler to use the parametric description

$$z = te^{i\theta}, \quad t \in \mathbb{R} \ ,$$

of $\mathcal{L}$. The points of $T(\mathcal{L})$ are

$$w = \frac{1}{t}e^{-i\theta} \quad \text{for} \quad t \neq 0 \ .$$

If we set $1/0 = \infty$ and $1/\infty = 0$ then we obtain that $T(\mathcal{L}) \setminus \{\infty\}$ is another straight line through the origin.

3.6 Example: The Cayley Transform in the Complex Plane

Consider the Möbius transforms

$$F(z) = \frac{z - i}{z + i} = \phi_A(z) \quad \text{where} \quad A = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \ ,$$

and

$$G(w) = i\,\frac{1 + w}{1 - w} = \phi_B(w) \quad \text{where} \quad B = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \ .$$

The map F is called the Cayley transform.

We have $det(A) = 2i$ and

$$A^{-1} = \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = \frac{1}{2i} B \; .$$

We note that B is a scalar multiple of A^{-1} and, therefore, G is the inverse of F if F and G are considered as functions on $\hat{\mathbb{C}}$. Here we use the extensions

$$F(-i) = \infty, \quad F(\infty) = 1, \quad G(1) = \infty, \quad G(\infty) = -i \; .$$

Let

$$\mathbb{H} = \{z = x + iy \; : \; y > 0\}$$

denote the open upper half–plane and recall that $\mathbb{D}$ denotes the open unit disk.

Lemma 3.4 *The Cayley transform*

$$F(z) = \frac{z-i}{z+i}$$

maps the open upper half–plane $\mathbb{H}$ *onto the open unit disk* $\mathbb{D}$*. The map*

$$G(w) = i\,\frac{1+w}{1-w}$$

is its back transform. The boundary of $\mathbb{H}$ *is the real line,*

$$\partial\mathbb{H} = \{z = x \; : \; x \in \mathbb{R}\} \; .$$

The map F *maps* $\partial\mathbb{H} = \mathbb{R}$ *bijectively onto* $\partial\mathbb{D} \setminus \{1\}$*.*

Proof: If $z \in \mathbb{H}$ then the distance of z from i is strictly smaller than the distance of z from $-i$. Therefore, $|F(z)| < 1$ for $z \in \mathbb{H}$. Similarly, if $z = x$ is real then $|F(z)| = 1$, and if $z = x + iy$ with $y < 0$ then $|F(z)| > 1$. Since $F(\infty) = 1$ and $F(-i) = \infty$ and since $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a bijection, it follows that $F : \mathbb{H} \to \mathbb{D}$ is a bijection. Since G inverts F as a function from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, the transformation G also inverts $F : \mathbb{H} \to \mathbb{D}$. The behavior of F and G on the boundaries of $\mathbb{H}$ and $\mathbb{D}$ is then clear. $\diamond$

Remark: Let $\bar{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ denote the closed upper half–plane. We have seen that F maps $\bar{\mathbb{H}}$ bijectively onto $\bar{\mathbb{D}} \setminus \{1\}$ and the inverse map is G. It is clear that one cannot obtain a continuous bijection between $\bar{\mathbb{H}}$ and $\bar{\mathbb{D}}$ since $\bar{\mathbb{D}}$ is compact and $\bar{\mathbb{H}}$ is not compact. Any continuous image of $\bar{\mathbb{D}}$ is also compact.

However, if we use the topology of $\hat{\mathbb{C}}$, then $\bar{\mathbb{H}} \cup \{\infty\}$ is a compact compact subset of $\hat{\mathbb{C}}$ and

$$F : \bar{\mathbb{H}} \cup \{\infty\} \to \bar{\mathbb{D}}$$

is an isomorphism between two compact sets.

3.7 The Cayley Transform of a Matrix

We have shown that

$$F : \mathbb{C} \setminus \{-i\} \to \mathbb{C} \setminus \{1\}, \quad F(z) = \frac{z-i}{z+i} = (z-i)(z+i)^{-1} ,$$

is 1-1 and onto with inverse

$$G : \mathbb{C} \setminus \{1\} \to \mathbb{C} \setminus \{-i\}, \quad G(w) = i\,\frac{1+w}{1-w} .$$

We have also shown that

$$F(\mathbb{R}) = \partial\mathbb{D} \setminus \{1\} . \tag{3.11}$$

Definition: Let $A \in \mathbb{C}^{n\times n}$ and assume that $-i \notin \sigma(A)$. Then

$$F(A) = (A - iI)(A + iI)^{-1}$$

is called the Cayley transform of A.

In analogy to (3.11) we have the following result:

Lemma 3.5 *If $A = A^*$ then $V := F(A)$ is a unitary matrix, i.e., $V^*V = I$. Also, $V = F(A) \neq I$.*

Proof: We have

$$V^* = (A - iI)^{-1}(A + iI) = (A + iI)(A - iI)^{-1} ,$$

thus

$$V^*V = (A + iI)(A + iI)^{-1} = I .$$

Suppose that

$$V = F(A) = (A - iI)(A + iI)^{-1} = I .$$

Then one obtains that

$$A - iI = A + iI ,$$

a contradiction. $\diamond$

Remark 1: On unitary matrices. For $u, v \in \mathbb{C}^n$ denote the Euclidean inner product by

$$\langle u, v\rangle = \sum_{j=1}^{n} \overline{u}_j v_j$$

and let

$$|u| = \sqrt{\langle u, u\rangle}$$

denote the Euclidian norm. For any $B \in \mathbb{C}^{n\times n}$ it holds that

$$\langle Bu, v\rangle = \langle u, B^*v\rangle .$$

Therefore, if V is unitary then

$$|Vu|^2 = \langle Vu, Vu \rangle = \langle u, V^* Vu \rangle = |u|^2 \; .$$

One obtains that

$$|Vu| = |u| \quad \text{for all} \quad u \in \mathbb{C}^n$$

if V is unitary. In particular, $|V| = 1$.

Remark 2: The Cayley transform

$$V = F(A) = (A - iI)(A + iI)^{-1}$$

of an operator A plays a role in functional analysis. One can use it to transform even unbounded Hermitian operators A to unitary operators $V = F(A)$. Unitary operators are obviously bounded, and it is often easier to study them. Then the back transform

$$V = F(A) \to A = i(I + V)(I - V)^{-1}$$

can give information about the unbounded Hermitian operator A. This is used to study spectral theory of unbounded operators.

4 The Riemann Mapping Theorem

Summary: If $U \subset \mathbb{C}$ is an open simply connected set and $U \neq \mathbb{C}$, then there exists a biholomorphic map $f : U \to \mathbb{D} = D(0,1)$. If $P \in U$ is any point then one can require that $f(P) = 0$ and $f'(P) > 0$. These conditions make the biholomorphic map $f : U \to \mathbb{D}$ unique. This is Riemann's mapping theorem.

Our proof uses the Arzela–Ascoli theorem of real analysis, which leads to Montel's theorem of complex analysis.

4.1 Review of $Aut(\mathbb{D})$

Recall that $\mathbb{D} = D(0,1)$ denotes the open unit disk and recall that we know the group $Aut(\mathbb{D})$: First, if $|c| < 1$ then the Möbius transformation

$$\phi_c(z) = \frac{z-c}{1-\bar{c}z}, \quad |z| < 1 \ ,$$

is an automorphism of $\mathbb{D}$ with inverse ϕ_{-c}. Second, if $|\alpha| = 1$ then the rotation

$$R_\alpha(z) = \alpha z, \quad |z| < 1 \ ,$$

is an automorphism of $\mathbb{D}$ with inverse $R_{\bar{\alpha}}$. Third, if $f \in Aut(\mathbb{D})$ is arbitrary and $b = f(0)$ then

$$f = \phi_{-b} \circ R_\alpha$$

for some α with $|\alpha| = 1$. In particular, if $f \in Aut(\mathcal{D})$ and $f(0) = 0$, then f is a rotation.

4.2 Statement and Outline of the Proof

The Riemann mapping theorem is the following remarkable result:

Theorem 4.1 *Let $U \subset \mathbb{C}$ be open and simply connected and let $U \neq \mathbb{C}$. Then there exists a biholomorphic map $f : U \to \mathbb{D}$.*

A sharper result, containing a uniqueness statement, is formulated in Theorem 4.2. If Theorem 4.1 is known, then we can prove Theorem 4.2 rather easily using our knowledge of $Aut(\mathbb{D})$.

Theorem 4.2 *Let $U \subset \mathbb{C}$ be open and simply connected and let $U \neq \mathbb{C}$. Let $P \in U$ be any fixed point in U. Then there exists a unique biholomorphic map $f : U \to \mathbb{D}$ satisfying*

$$f(P) = 0, \quad f'(P) > 0 \ . \tag{4.1}$$

Proof: Existence of f: By Theorem 4.1 there exists a biholomorphic $F : U \to \mathbb{D}$. Set $c = F(P)$ and recall that

$$\phi_c(z) = \frac{z-c}{1-\bar{c}z}, \quad z \in \mathbb{D} \ ,$$

is an automorphism of $\mathbb{D}$ with $\phi_c(c) = 0$. Set

$$g = \phi_c \circ F \ .$$

Then $g : U \to \mathbb{D}$ is biholomorphic and $g(P) = \phi_c(F(P)) = \phi_c(c) = 0, g'(P) \neq 0$. Write

$$g'(P) = re^{i\gamma} \quad \text{where} \quad r > 0 \quad \text{and} \quad \gamma \in \mathbb{R} \ ,$$

and set

$$\alpha = e^{-i\gamma} \ .$$

Define $f = R_\alpha \circ g$, thus

$$f(z) = \alpha g(z), \quad f'(z) = \alpha g'(z) \ .$$

We have $f(P) = \alpha g(P) = 0$ and

$$f'(P) = \alpha g'(P) = \alpha \bar{\alpha} r = r > 0 \ .$$

The function

$$f = R_\alpha \circ \phi_c \circ F : U \to \mathbb{D}$$

is biholomorphic and satisfies (4.1).

Uniqueness of f: Suppose that f_1 and f_2 satisfy the conditions of the theorem and set $h = f_1 \circ f_2^{-1}$. Then $h \in Aut(\mathbb{D})$ and $h(0) = 0$. It follows that h is a rotation, $h(z) \equiv \alpha z, |\alpha| = 1$. From

$$h \circ f_2 = f_1, \quad f_1(P) = f_2(P) = 0 \ ,$$

we obtain that

$$h'(0) f_2'(P) = f_1'(P) \ .$$

Since $f_j'(P) > 0$ and $|\alpha| = 1$ it follows that $h'(0) = \alpha = 1$, yielding that $f_1 = f_2$. $\diamond$

Outline of the Proof of Theorem 4.1:

a) Fix any point $P \in U$ and let the set $\mathcal{F}$ consist of all functions $f : U \to \mathbb{D}$ that have the following properties:

1) $f \in H(U)$;
2) f is $1-1$;
3) $f(P) = 0$.

We will prove by an explicit construction that $\mathcal{F}$ is not empty.

b) If $f \in \mathcal{F}$ then $|f'(P)| > 0$ since, by assumption, f is $1-1$. Using Cauchy's inequality, it is easy to show that

$$s := \sup\{|f'(P)| \ : \ f \in \mathcal{F}\} \tag{4.2}$$

is finite.

c) Using Montel's theorem (see Section 4.4) one obtains the existence of a function $f \in H(U)$ with

$$f(U) \subset \mathbb{D}, \quad f(P) = 0, \quad s = |f'(P)| \ .$$

The function f is constructed as the locally uniform limit of a sequence of functions in $\mathcal{F}$.

d) Using Hurwitz's theorem, it follows that the function f obtained in c) is $1-1$. Therefore, $f \in \mathcal{F}$. In other words, the supremum defining the number s in (4.2) is in fact a maximum.

e) An argument using the automorphisms of $\mathbb{D}$ shows that f maps U *onto* $\mathbb{D}$. If f would not be onto, then one could construct a map $g \in \mathcal{F}$ with $|g'(P)| > |f'(P)| = s$, in contradiction to the definition of s.

4.3 The Theorem of Arzela–Ascoli

The Theorem of Arzela–Ascoli is an important result of real analysis. Montel's Theorem, which we consider in the next section, is a complex version of the Theorem of Arzela–Ascoli.

Let $\Omega \subset \mathbb{R}^s$ be a compact set. With $C(\Omega)$ we denote the linear space of all continuous functions $u : \Omega \to \mathbb{R}$. Let

$$|u|_\infty = \max_{x \in \Omega} |u(x)|$$

denote the maximum norm of $u \in C(\Omega)$. We know that $C(\Omega)$ with $|\cdot|_\infty$ is a Banach space.

We will need the following version of the Arzela–Ascoli Theorem.

Theorem 4.3 *Let Ω be a compact subset of $\mathbb{R}^s$. Let $u_n \in C(\Omega)$ denote a sequence of functions with the following two properties:*

1) For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $n \in \mathbb{N}$:

$$|u_n(x) - u_n(y)| < \varepsilon \quad \text{if} \quad |x-y| < \delta, \quad x, y \in \Omega \ .$$

(It is important that δ does not depend on n. This property is called equicontinuity of the sequence u_n.)

2) There exists a constant $C > 0$ with

$$|u_n|_\infty \le C \quad \text{for all} \quad n \in \mathbb{N} \ .$$

(This property is called uniform boundedness of the sequence u_n.)

Under these assumptions, there exists a subsequence u_{n_j} and a function $u \in C(\Omega)$ with

$$|u_{n_j} - u|_\infty \to 0 \quad \text{as} \quad n_j \to \infty \ .$$

Proof: 1. Using a diagonal sequence argument, we will show that the sequence u_n has a subsequence u_{j_k} which is a Cauchy sequence in the Banach space $\Big(C(\Omega), |\cdot|_\infty\Big)$.

2. A set $F_\varepsilon \subset \Omega$ is called an ε–net for Ω if for every $y \in \Omega$ there exists $x \in F_\varepsilon$ with $|x-y| < \varepsilon$. Since Ω is bounded, it is easy to show that for every $\varepsilon > 0$ there exists a *finite* ε–net for Ω. To see this, cover the set Ω with finitely many boxes of diameter ε and, if a box has an intersection with Ω, choose one point in the intersection as an element in the ε–net.

For every $n \in \mathbb{N}$ let $F_{1/n}$ denote a finite $\frac{1}{n}$–net for Ω and let

$$F = \cup_n F_{1/n} \ .$$

We enumerate the set F by first listing the points in F_1, then the points in $F_{1/2}$, then the points in $F_{1/3}$ etc:

$$F = \{x_1, x_2, \ldots\} \ .$$

We will need the following property of F, which is more special than density of F in Ω: If $\delta > 0$ is given, then there is a finite integer $K = K(\delta)$ so that the points

$$x_1, \ldots, x_K \in F$$

form a δ–net for Ω. This is clear from the construction of F.

3. In the following, $\mathbb{N}_1, \mathbb{N}_2$ etc. denote infinite subsets of $\mathbb{N}$. Since $u_n(x_1)$ is bounded, there exists a set $\mathbb{N}_1 \subset \mathbb{N}$ so that

$$u_n(x_1), \quad n \in \mathbb{N}_1 \ ,$$

is a convergent sequence of real numbers,

$$u_n(x_1) \to a_1, \quad n \in \mathbb{N}_1 \ .$$

Since $u_n(x_2), n \in \mathbb{N}_1$, is bounded, there is $\mathbb{N}_2 \subset \mathbb{N}_1$ so that

$$u_n(x_2), \quad n \in \mathbb{N}_2 \ ,$$

is a convergent sequence of real numbers,

$$u_n(x_2) \to a_2, \quad n \in \mathbb{N}_2 \ .$$

Repeat this construction inductively. For every $k \in \mathbb{N}$ obtain a set $\mathbb{N}_k$ with

$$\mathbb{N}_k \subset \mathbb{N}_{k-1} \subset \ldots \subset \mathbb{N}_1 \subset \mathbb{N}$$

so that

$$u_n(x_k), \quad n \in \mathbb{N}_k \ ,$$

is a convergent sequence of real numbers,

$$u_n(x_k) \to a_k, \quad n \in \mathbb{N}_k \ .$$

We now use a *diagonal sequence* argument: Let

$$\mathbb{N}_k = \{n_1^{(k)} < n_2^{(k)} < \ldots\}$$

and consider the diagonal sequence

$$j_k := n_k^{(k)}, \quad k = 1, 2, \ldots$$

The diagonal sequence $\mathbf{n}_\mathbf{k}^{(\mathbf{k})}$:

$$\begin{array}{lcl}
\mathbb{N}_1 & : & \mathbf{n}_\mathbf{1}^{(\mathbf{1})} \ n_2^{(1)} \ n_3^{(1)} \ \ldots \\
\mathbb{N}_2 & : & n_1^{(2)} \ \mathbf{n}_\mathbf{2}^{(\mathbf{2})} \ n_3^{(2)} \ \ldots \\
\mathbb{N}_3 & : & n_1^{(3)} \ n_2^{(3)} \ \mathbf{n}_\mathbf{3}^{(\mathbf{3})} \ \ldots \\
\mathbb{N}_4 & : & \cdots \ \cdots \ \cdots \ \mathbf{n}_\mathbf{4}^{(\mathbf{4})}
\end{array}$$

4. We claim that

$$u_{j_k}(x_\nu) \to a_\nu \quad \text{as} \quad j_k \to \infty \ ,$$

for every fixed $x_\nu \in F$. This is clear since the tail of the sequence

$$j_1 < j_2 < \ldots < j_\nu < j_{\nu+1} < \ldots$$

is a subsequence of $\mathbb{N}_\nu$ and

$$u_n(x_\nu) \to a_\nu \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{N}_\nu \ .$$

To summarize, we have shown that the sequence $u_{j_k}(x_\nu)$ (where $j_1 < j_2 < j_3 < \ldots$) converges for every $x_\nu \in F$ as $j_k \to \infty$. In particular, given any $x_\nu \in F$ and any $\varepsilon > 0$ there exists $N(x_\nu, \varepsilon)$ with

$$|u_{j_k}(x_\nu) - u_{j_l}(x_\nu)| < \frac{1}{3}\,\varepsilon \quad \text{for} \quad k, l \geq N(x_\nu, \varepsilon) \ .$$

So far, we have only used that the sequence of functions $u_n(x)$ is bounded for every $x \in \Omega$.

5. We now show that the sequence of functions $u_{j_k}(x)$ is a Cauchy sequence in $C(\Omega)$ w.r.t. $|\cdot|_\infty$. Let $\varepsilon > 0$ be given. By assumption, there exists $\delta = \delta(\varepsilon) > 0$ (independent of n) with

$$|u_n(x) - u_n(y)| < \frac{1}{3}\,\varepsilon \quad \text{if} \quad |x - y| < \delta \quad (x, y \in \Omega) \ .$$

In the following, $\delta > 0$ is chosen so that the above estimate holds. If $y \in \Omega$ and $|y - x_\nu| < \delta$ and $k, l \geq N(x_\nu, \varepsilon)$ then we have

$$\begin{aligned} |u_{j_k}(y) - u_{j_l}(y)| &\leq |u_{j_k}(y) - u_{j_k}(x_\nu)| + |u_{j_k}(x_\nu) - u_{j_l}(x_\nu)| + |u_{j_l}(x_\nu) - u_{j_l}(y)| \\ &\leq \frac{1}{3}\,\varepsilon + \frac{1}{3}\,\varepsilon + \frac{1}{3}\,\varepsilon \\ &= \varepsilon \ . \end{aligned}$$

If $\varepsilon > 0$ is given, then there are finitely many points $x_1, \ldots, x_K \in F$ so that for every $y \in \Omega$ there exists x_ν with $1 \leq \nu \leq K$ and $|y - x_\nu| < \delta$. If we set

$$N(\varepsilon) = \max_{1 \leq \nu \leq K} N(x_\nu, \varepsilon)$$

then we have for $k, l \geq N(\varepsilon)$:

$$\max_y |u_{j_k}(y) - u_{j_l}(y)| < \varepsilon \ .$$

This proves the theorem. $\diamond$

Extension: It is clear that the theorem generalizes to sequences of functions $f_n : \Omega \to \mathbb{R}^k$ for any finite k.

4.4 Montel's Theorem

Montel's theorem is a result about families of functions $f \in H(U)$ which are uniformly bounded on compact subsets of U.

We will use the following notation: If $K \subset \mathbb{C}$ is a compact set and $g \in C(K)$, then

$$|g|_K = \max\{|g(z)| \ : \ z \in K\}$$

denotes the maximum norm of g.

We recall:

Theorem 4.4 *(Morera) Let $U \subset \mathbb{C}$ denote an open set and let $f \in C(U)$. If*

$$\int_\Gamma f(z)\,dz = 0$$

for every closed triangle Γ in U then f is holomorphic on U.

Morera's Theorem is often used as follows: Let $f_n \in H(U)$ and let $f \in C(U)$. If $|f_n - f|_K \to 0$ as $n \to \infty$ for every compact subset K of U, then $f \in H(U)$.

Theorem 4.5 *(Montel) Let $U \subset \mathbb{C}$ denote an open set and let $\mathcal{F}$ denote a set of functions $f \in H(U)$, i.e., $\mathcal{F} \subset H(U)$. Assume that for every compact subset K of U there exists a constant C_K with*

$$|f(z)| \le C_K \quad \text{for all} \quad z \in K \quad \text{and} \qquad \text{for all} \quad f \in \mathcal{F} \ .$$

Then, if $f_n \in \mathcal{F}$ is a sequence, there exists a subsequence f_{n_j} and a function $f \in H(U)$ so that f_{n_j} converges locally uniformly to f. (The limit function f may or may not belong to the set $\mathcal{F}$.)

We will only need the following simpler version of the theorem.

Theorem 4.6 *(Montel's theorem for a sequence, simple version) Let $U \subset \mathbb{C}$ be an open set and let $f_n \in H(U)$ be a bounded sequence, i.e., there exists a constant M with*

$$|f_n(z)| \le M \quad \text{for all} \quad z \in U \quad \text{and} \qquad \text{for all} \quad n \in \mathbb{N} \ .$$

Then there exists a function $f \in H(U)$ and a subsequence f_{n_j} of f_n with $f_{n_j} \to f$ locally uniformly in U. This means: If K is any compact subset of U and if $\varepsilon > 0$, then there exists an integer $J(K, \varepsilon)$ with

$$\max_{z \in K} |f_{n_j}(z) - f(z)| < \varepsilon \quad \text{for} \quad j \ge J(K, \varepsilon) \ .$$

Essentially, Montel's theorem follows from the Cauchy inequalities, Morera's theorem and the Arzela–Ascoli theorem. We now give a detailed proof. See Figure 4.1 for sets referred to in the proof.

1. First, let K be any compact subset of U. We assume that $U^c = \mathbb{C} \setminus U$ is not empty. (If $U = \mathbb{C}$ then every f_n is constant and the claim follows directly from the Bolzano–Weierstrass Theorem. The Bolzano–Weierstrass Theorem says that every bounded sequence in the finite dimensional space $\mathbb{R}^s$ has a convergent subsequence.)

Set

$$\begin{aligned} \delta \ &:= \ \inf_{z \in U^c} \min_{w \in K} |z - w| \\ &= \ \min_{z \in U^c} \min_{w \in K} |z - w| \ . \end{aligned}$$

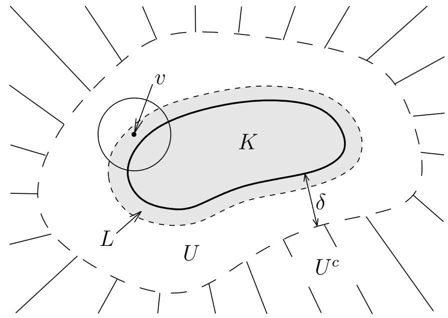

Figure 4.1: Sets in the proof of Montel's theorem

(Here the infimum equals the minimum since U^c is closed, and large $z \in U^c$ play no role in determining δ.) We have $\delta > 0$.

2. We want to show that the sequence f_n is equicontinuous on K. For $z \in \mathbb{C}$ let

$$dist(z, K) = \min_{w \in K} |z - w| \ .$$

Define

$$L = \{z \in \mathbb{C} \ : \ dist(z, K) \leq \delta/4\} \ .$$

Clearly, $K \subset L \subset U$. If $v \in L$ then $\bar{D}(v, \delta/2) \subset U$. By Cauchy's inequality we have for all $v \in L$:

$$|f_n'(v)| \leq \frac{M}{\delta/2} =: C_1 \ .$$

If $z, w \in K$ and $|z - w| \leq \delta/4$ then the line segment

$$\gamma(t) = tz + (1-t)w, \quad 0 \leq t \leq 1 \ ,$$

lies in L since $|\gamma(t) - w| \leq \delta/4$. We then have for $z, w \in K$ with $|z - w| \leq \delta/4$:

$$\begin{aligned} f_n(z) - f_n(w) &= f_n(\gamma(1)) - f_n(\gamma(0)) \\ &= \int_0^1 f_n'(\gamma(t))\gamma'(t)\, dt \\ &= \int_0^1 f_n'(\gamma(t))\, dt\, (z - w) \end{aligned}$$

Therefore,

$$|f_n(z) - f_n(w)| \leq C_1 |z - w| \quad \text{for} \quad z, w \in K \quad \text{if} \quad |z - w| \leq \frac{\delta}{4} \ .$$

It is clear that this estimate implies equicontinuity of the sequence f_n on K. Thus we can apply the Arzela–Ascoli Theorem to the sequence f_n on any compact subset K of U.

3. Given $K \subset U$ as above, there exists a subsequence f_{n_j} and a function $f \in C(K)$ with $\max_{x \in K} |f_{n_j}(x) - f(x)| \to 0$.

4. We now choose a sequence of compact sets K_ν in U as follows. For $\nu = 1, 2, \dots$ let

$$K_\nu = \Big\{ z \in U \; : \; dist(z, U^c) \ge 1/\nu \Big\} \cap \bar{D}(0,\nu) \; .$$

We may assume that K_1 is not empty. It is clear that

$$K_1 \subset K_2 \subset \ldots \subset U \quad \text{and} \quad \cup_\nu \, K_\nu = U \; .$$

(Let $z \in U$ and let $\varepsilon := dist(z, U^c)$. There exists $\nu \in \mathbb{N}$ with $z \in \bar{D}(0,\nu)$ and $\varepsilon > 1/\nu$; thus $z \in K_\nu$.)

Furthermore, if K is any compact subset of U then there exists a sufficiently large ν with

$$K \subset K_\nu \; .$$

The sequence f_n has a subsequence $f_n, n \in \mathbb{N}_1$, which converges uniformly on K_1 to some $f \in C(K_1)$. Then consider the sequence $f_n, n \in \mathbb{N}_1$, on $K_2 \supset K_1$. There exists a subsequence $f_n, n \in \mathbb{N}_2 \subset \mathbb{N}_1$, converging uniformly to some $g \in C(K_2)$. However, g equals f on K_1, and we may denote g by f. We repeat the argument and obtain: For every ν there exists $\mathbb{N}_\nu \subset \mathbb{N}_{\nu-1}$ so that $f_n, n \in \mathbb{N}_\nu$, converges uniformly on K_ν to $f \in C(K_\nu)$. Since

$$\cup_\nu K_\nu = U$$

this process defines a continuous function f on U.

5. Let

$$\mathbb{N}_k = \{n_1^{(k)}, n_2^{(k)}, n_3^{(k)}, \ldots\} \quad \text{where} \quad n_1^{(k)} < n_2^{(k)} < n_3^{(k)} < \ldots$$

and let

$$j_k = n_k^{(k)}, \quad k = 1, 2, 3, \ldots$$

In this way we obtain the subsequence

$$f_{j_k}, \quad k = 1, 2, 3, \ldots$$

of the sequence f_n. A tail of the sequence

$$f_{j_k}, \quad k = 1, 2, 3, \ldots$$

is a subsequence of $f_n, n \in \mathbb{N}_\nu$ for every fixed ν. Therefore, for every fixed ν, the subsequence f_{j_k} converges uniformly on K_ν to f.

6. Finally, if K is an arbitrary compact subset of U then $K \subset K_\nu$ for some large ν. Therefore, f_{j_k} converges uniformly on K to f. The limit function f is holomorphic on U by Morera's theorem. $\diamond$

4.5 Auxiliary Results on Logarithms and Square Roots

Lemma 4.1 *Let $U \subset \mathbb{C}$ be open and simply connected; let $f \in H(U)$. If $f(z) \neq 0$ for all $z \in U$ then there exists a function $h \in H(U)$ with*

$$e^{h(z)} = f(z), \quad z \in U \; .$$

Proof: Motivation: Suppose that $e^{h(z)} = f(z)$. Then we have

$$f'(z) = h'(z)f(z) \ ,$$

thus

$$h'(z) = f'(z)/f(z) \ .$$

This motivates to construct $h(z)$ as a function with $h' = f'/f$.

Fix $z_0 \in U$ and let Γ_z denote a curve in U from z_0 to z. Fix $c \in \mathbb{C}$ with

$$f(z_0) = e^c$$

and define

$$h(z) = c + \int_{\Gamma_z} \frac{f'(w)}{f(w)}\, dw, \quad z \in U \ .$$

Then we have $h'(z) = f'(z)/f(z)$ and $e^{h(z_0)} = e^c = f(z_0)$. Consider the function

$$g(z) = f(z)e^{-h(z)}, z \in U \ .$$

We have

$$g(z_0) = 1$$

and

$$\begin{aligned} g'(z) &= f'(z)e^{-h(z)} - f(z)e^{-h(z)}\, h'(z) \\ &= e^{-h(z)}\Big(f'(z) - f(z)h'(z)\Big) \\ &= 0 \end{aligned}$$

It follows that $g(z) \equiv 1$, proving the lemma. $\diamond$

Remark: Since $e^{h(z)} = f(z)$ we may consider the function $h(z)$ as a complex logarithm of $f(z)$,

$$h(z) = \log(f(z)) \ ,$$

where $\log(w)$ is any inverse of the exponential function defined on the range of $f(z)$.

Lemma 4.2 *Let $U \subset \mathbb{C}$ be open and simply connected; let $f \in H(U)$. If $f(z) \neq 0$ for all $z \in U$ then there exists a function $g \in H(U)$ with*

$$f(z) = (g(z))^2, \quad z \in U \ .$$

Proof: Using the previous lemma, we can write

$$f(z) = e^{h(z)}$$

and define

$$g(z) = e^{h(z)/2} \ .$$

$\diamond$

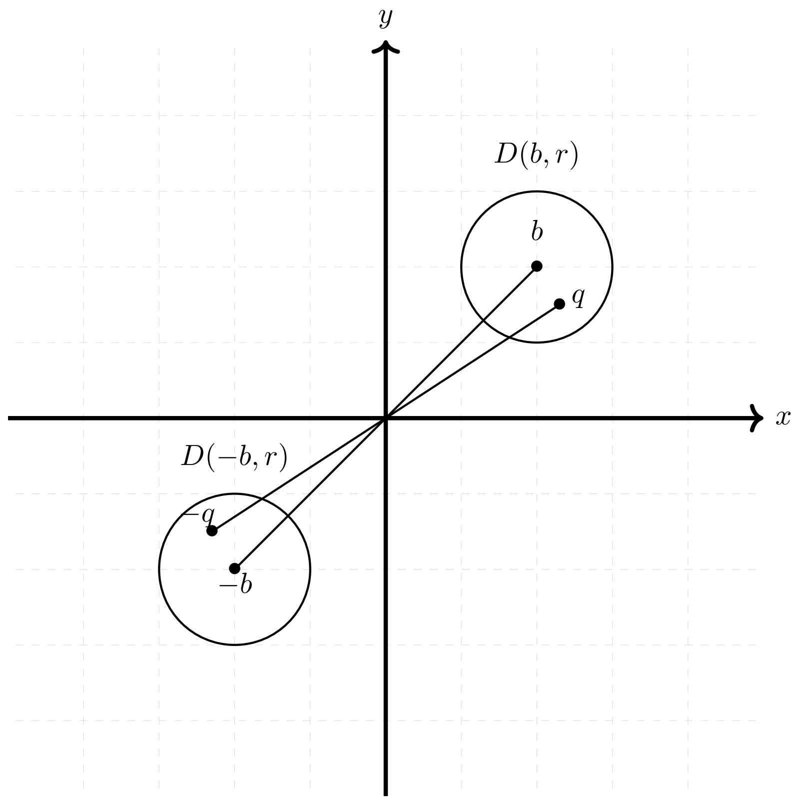

Figure 4.2: The disks $D(\pm b, r)$ and the points $\pm q$

4.6 Construction of a Map in $\mathcal{F}$

Lemma 4.3 *Let $U \subset \mathbb{C}$ be open and simply connected. Further, let $U \neq \mathbb{C}$. Let $P \in U$ be arbitrary, but fixed. Then there exists $f \in H(U)$ with $f(U) \subset \mathbb{D}$, f is 1–1, and $f(P) = 0$.*

Proof: If U is bounded then $f(z)$ can be taken as a function of the form $f(z) = az + b$. We treat this simple case first in Case a. In Case b we treat the general case.

Case a) Let U be bounded. Choose $R > 0$ so that $U \subset D(P, R)$ and set

$$f(z) = \frac{z - P}{R} \; .$$

Clearly, f is $1 - 1$ and $f(P) = 0$. Also, if $z \in U \subset D(P, R)$ then $|z - P| < R$, thus $|f(z)| < 1$. This shows that f maps U into $\mathbb{D}$.

Case b) Let U be an arbitrary open, simply connected subset of $\mathbb{C}$, but $U \neq \mathbb{C}$. Let $Q \in \mathbb{C} \setminus U$ and define

$$\phi(z) = z - Q, \quad z \in U \; .$$

Then $\phi \in H(U)$ is $1 - 1$ and $\phi(z) \neq 0$ for all $z \in U$. By Lemma 4.2 there exists $h \in H(U)$ with

$$h^2(z) = \phi(z), \quad z \in U \; .$$

We will show that there exists an open disk $D(-b, r)$ which is contained in the complement of $h(U)$:

$$D(-b,r) \subset \Big(\mathbb{C} \setminus h(U) \Big) \; .$$

Once this is done, the construction of f will be easy.

Claim: If $q \in h(U), q \neq 0$, then $-q \notin h(U)$.
Proof: Suppose that $q \neq 0, q \in h(U), -q \in h(U)$. There exists $z_1, z_2 \in U$ with

$$h(z_1) = q, \quad h(z_2) = -q, \quad z_1 \neq z_2 \; .$$

Obtain that

$$\phi(z_1) = h^2(z_1) = q^2 = h^2(z_2) = \phi(z_2) \; ,$$

which contradicts that ϕ is 1-1.

Choose any $b \in h(U), b \neq 0$. Since $h(U)$ is open, there exists $0 < r < |b|$ with $D(b,r) \subset h(U)$. If $q \in D(b,r)$ then $q \in h(U), q \neq 0$, thus $-q \notin h(U)$. If $-q \in D(-b,r)$ then $q \in D(b,r), q \neq 0$, thus $-q \notin h(U)$.

The above argument implies that

$$D(-b,r) \subset \Big(\mathbb{C} \setminus h(U) \Big) \; .$$

In other words,

$$|h(z) + b| \geq r \quad \text{for all} \quad z \in U \; .$$

Define

$$f_1(z) = \frac{r}{2(h(z)+b)}, \quad z \in U \; .$$

Then $f_1 \in H(U)$ is $1-1$ and $|f_1(z)| \leq \frac{1}{2} < 1$. Set $a := f_1(P)$ and recall that

$$\phi_a(w) = \frac{w-a}{1-\bar{a}w}$$

is an automorphisms of $\mathbb{D}$ with $\phi_a(a) = 0$. The function $f = \phi_a \circ f_1$ satisfies $f(P) = 0$. To summarize, we have constructed a function $f \in H(U)$ which is $1-1$ and satisfies $f(U) \subset \mathbb{D}$ and $f(P) = 0$. $\diamond$

4.7 A Bound of $|f'(P)|$ for all $f \in \mathcal{F}$

Let $f \in \mathcal{F}$, i.e., $f \in H(U), f(U) \subset \mathbb{D}, f(P) = 0$, and f is one–to–one. There is $r > 0$ with $\bar{D}(P,r) \subset U$. Let $\gamma(t) = P + re^{it}$ for $0 \leq t \leq 2\pi$. We know that for $z \in D(P,r)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \, dw \; .$$

Differentiation yields that

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^2} \, dw \; .$$

For $z = P$ one obtains the Cauchy inequality

$$
\begin{aligned}
|f'(P)| &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(P+re^{it})|}{r^2}\, r\, dt \\
&\le \frac{1}{r}\ .
\end{aligned}
$$

Here we have used the bound $|f(w)| \le 1$, which follows from $f(U) \subset \mathbb{D}$.

We have shown:

Lemma 4.4 *The number s defined by*

$$s = \sup\{|f'(P)| \ : \ f \in \mathcal{F}\}$$

satisfies $0 < s \le \frac{1}{r}$ if $\bar{D}(P, r) \subset U$.

4.8 Application of the Theorems of Montel and Hurwitz

There exists a sequence of functions $f_n \in \mathcal{F}$ with

$$s \ge |f_n'(P)| \ge s - \frac{1}{n}, \quad n = 1, 2, \ldots$$

Since $f_n(U) \subset \mathbb{D}$ the functions f_n are uniformly bounded on U. By Montel's theorem (simple version), there exists a subsequence f_{n_j} and a function $f \in H(U)$ so that f_{n_j} converges locally uniformly to f on U. In particular, $f(P) = 0$ and $|f'(P)| = s$. Also, $|f(z)| \le 1$ for all $z \in U$. If $|f(z)| = 1$ for some $z \in U$ then f is constant, contradicting $|f'(P)| = s > 0$. Therefore, $f(U) \subset \mathbb{D}$. (Another argument showing that $f(U) \subset \mathbb{D}$: The set $f(U)$ is open unless f is constant.)

We now claim that f is $1-1$. To prove this we will use Hurwitz Theorem:

Theorem 4.7 *(Hurwitz) Let $V \subset \mathbb{C}$ be open and connected. Let $h_j, h \in H(V)$ and assume that the sequence $h_j(z)$ converges locally uniformly to $h(z)$ in V. If*

$$h_j(z) \ne 0 \quad \textit{for all} \quad z \in V \quad \textit{and} \qquad \textit{for all} \quad j = 1, 2, \ldots$$

then either $h(z) \equiv 0$ or

$$h(z) \ne 0 \quad \textit{for all} \quad z \in V\ .$$

Proof: We first recall that locally uniform convergence in V of the sequence $h_j(z)$ to $h(z)$ implies locally uniform convergence in V of $h_j'(z)$ to $h'(z)$.

Suppose that $P \in V$ is an isolated zero of $h(z)$. There exists $\varepsilon > 0$ so that

$$h(z) \ne 0 \quad \text{for} \quad 0 < |z - P| \le \varepsilon\ .$$

If Γ denotes the boundary curve of the disk $D(P, \varepsilon)$ then the positive integer

$$\frac{1}{2\pi i}\int_\Gamma \frac{h'(z)}{h(z)}\, dz$$

is the order of the zero P of $h(z)$. On the other hand,

$$\frac{1}{2\pi i}\int_\Gamma \frac{h_j'(z)}{h_j(z)}\, dz = 0 \quad \text{for all} \quad j = 1, 2, \ldots$$

One obtains a contradiction as $j \to \infty$. $\diamond$

Recall that $f_n(z)$ denotes a sequence of 1–1 functions with

$$f_n \in H(U), \quad f_n(U) \subset \mathbb{D}, \quad f_n(P) = 0$$

and

$$s \ge |f_n'(P)| \ge s - \frac{1}{n}, \quad n = 1, 2, \ldots$$

By Montel's Theorem there exists a subsequence f_{n_j} and $f \in H(U)$ so that f_{n_j} converges locally uniformly to f in U. One obtains that $|f'(P)| = s > 0$.

We claim that f is 1–1 on U. Suppose not. Then there exist two points $z_1, z_2 \in U$ with

$$f(z_1) = f(z_2), \quad z_1 \ne z_2 \ .$$

Consider the sequence of functions

$$h_j(z) = f_{n_j}(z) - f_{n_j}(z_2), \quad z \in V = U \setminus \{z_2\} \ .$$

The functions $h_j \in H(V)$ converge locally uniformly on V to $f(z) - f(z_2)$. Also, the functions $h_j(z)$ have no zero in V since they are 1–1 on U. By Hurwitz's theorem, the limit function $f(z) - f(z_2)$ either is identically zero on V or has no zero in V. Since $f(z)$ is not constant, we conclude that $f(z) - f(z_2)$ has no zero in V, contradicting the assumption $f(z_1) = f(z_2)$.

4.9 Proof That f is Onto

We have shown that there exists a function $f \in \mathcal{F}$ with

$$s = |f'(P)| \ge |g'(P)| \quad \text{for all} \quad g \in \mathcal{F}, \quad s > 0 \ . \tag{4.3}$$

By definition of the set $\mathcal{F}$ we have

$$f \in H(U), \quad f(U) \subset \mathbb{D}, \quad f \text{ is one-to-one}, \quad f(P) = 0 \ .$$

We claim that if $f \in \mathcal{F}$ satisfies (4.3) then $f : U \to \mathbb{D}$ is onto. This claim then completes the proof of the Riemann Mapping Theorem.

We will show:

Lemma 4.5 *Let $f \in \mathcal{F}$ and assume that $f : U \to \mathbb{D}$ is not onto $\mathbb{D}$. Then we can construct $g \in \mathcal{F}$ with*

$$|g'(P)| > |f'(P)| \ .$$

Proof: With $S : \mathbb{D} \to \mathbb{D}$ we denote the squaring function, i.e., $S(v) = v^2, v \in \mathbb{D}$. Also, we recall that for $|c| < 1$ the function

$$\phi_c(w) = \frac{w - c}{1 - \bar{c}w}, \quad |w| < 1 \ ,$$

is an automorphism of $\mathbb{D}$. Since $f : U \to \mathbb{D}$ is not onto, we can choose

$$a \in \mathbb{D} \setminus f(U)$$

and consider

$$\begin{aligned}\phi(z) &:= \phi_a \circ f(z) \\ &= \frac{f(z)-a}{1-\bar{a}f(z)}, \quad z \in U \ .\end{aligned}$$

Since $a \notin f(U)$ we have $\phi(z) \neq 0$ for all $z \in U$. Also, $\phi \in H(U)$ is $1-1$ and $\phi(U) \subset \mathbb{D}$. By Lemma 4.2 there exists a function $\psi \in H(U)$ which is $1-1$ and satisfies

$$\phi(z) = \psi(z)\psi(z) = S \circ \psi(z) \quad \text{for all} \quad z \in U, \quad \psi(U) \subset \mathbb{D} \ .$$

Define

$$g(z) = \phi_{\psi(P)}(\psi(z)), \quad z \in U \ .$$

Then $g \in H(U)$, g is $1-1$, $g(U) \subset \mathbb{D}$, and $g(P) = 0$. Therefore, $g \in \mathcal{F}$. Also, if we abbreviate $b := -\psi(P)$, then $g = \phi_{-b} \circ \psi$, thus $\psi = \phi_b \circ g$.

We have

$$\begin{aligned} f &= \phi_{-a} \circ \phi \\ &= \phi_{-a} \circ S \circ \psi \\ &= \phi_{-a} \circ S \circ \phi_b \circ g \end{aligned}$$

We now set

$$h := \phi_{-a} \circ S \circ \phi_b \ .$$

Then the above equation for f says that

$$f(z) = h(g(z)), \quad z \in U \ .$$

We have $h \in H(\mathbb{D}), h(\mathbb{D}) \subset \mathbb{D}$, and

$$0 = f(P) = h(g(P)) = h(0) \ .$$

By Schwarz Lemma, we conclude that

$$|h'(0)| < 1$$

unless h is a rotation, $h = R_\alpha$. Recall that $h = \phi_{-a} \circ S \circ \phi_b$. If h would be a rotation, $h = R_\alpha$, then the squaring function S would be an automorphism of $\mathbb{D}$, which is obviously not the case. It follows that $|h'(0)| < 1$. Since

$$f(z) = h(g(z)), \quad z \in U \ ,$$

we have

$$f'(P) = h'(0)g'(P) \ ,$$

thus

$$|f'(P)| < |g'(P)| \ .$$

This proves the lemma and completes the proof of Riemann's Mapping Theorem. $\diamond$.

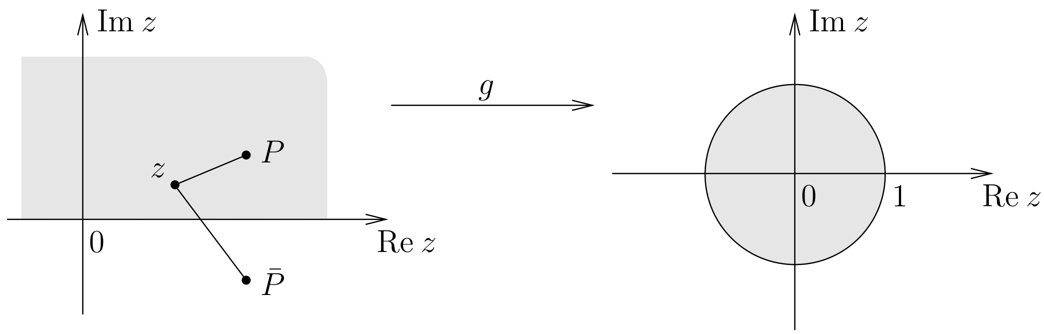

Figure 4.3: Biholomorphic mapping from upper half–plane onto unit disk

4.10 Examples of Biholomorphic Mappings

Recall that $\mathbb{D}$ denotes the open unit disk and $\mathbb{H}$ denotes the open upper half–plane.

Example 1: Let $P = a + ib$ where $a, b \in \mathbb{R}, b > 0$; thus P is a point in $\mathbb{H}$. Then $\bar{P} = a - ib$ is the reflection of P w.r.t. the real axis. The function

$$g(z) = \frac{z - P}{z - \bar{P}}, \quad z \neq \bar{P} \ ,$$

maps $\mathbb{H}$ onto $\mathbb{D}$, maps the real line onto $\partial\mathbb{D} \setminus \{1\}$, and maps $\mathbb{C} \setminus \bar{\mathbb{H}}$ onto $\mathbb{C} \setminus \bar{\mathbb{D}}$. In particular, g is a biholomorphic mapping from $\mathbb{H}$ onto $\mathbb{D}$ with $g(P) = 0$. If you want to find a biholomorphic mapping $f : \mathbb{H} \to \mathbb{D}$ with $f(P) = 0$ and $f'(P) > 0$ then set

$$f(z) = e^{i\theta} g(z)$$

and determine θ so that $f'(P) > 0$.

Example 2: Let H denote an open half–plane in $\mathbb{C}$. The boundary of H is a straight line, $L = \partial H$. Let $P \in H$ and let $Q \in H^c$ denote the reflection of P w.r.t. L. Then

$$g(z) = \frac{z - P}{z - Q}, \quad z \in H,$$

maps H biholomorphically onto $\mathbb{D}$.

5 An Introduction to Schwarz–Christoffel Formulas

Schwarz–Christoffel formulas give biholomorphic maps $w = w(z)$ from the open upper half–plane $\mathbb{H}$ onto regions enclosed by polygons. The maps $w(z)$ can be extended continuously to the real axis, i.e., to the boundary of $\mathbb{H}$. The real axis (and the point $z = \infty$) is then mapped to the polygon.

The maps $w(z)$ are given in terms of integrals. Similar integrals also appear in the theory of elliptic functions.

We treat an example of a Schwarz–Christoffel map, which is Example 8, p. 350, from [Hahn, Epstein]. A more comprehensive treatment of Schwarz–Christoffel maps is given in [Hille] and in [Stein, Sharkarchi].

A clear understanding of general power functions $z \to z^\alpha$ is helpful and will be reviewed first.

5.1 General Power Functions

Let

$$U_0 = \mathbb{C} \setminus \{z = iy \ : \ y \le 0\}$$

denote a slit plane. Precisely, U_0 is the complex plane $\mathbb{C}$ with the negative imaginary axis, including $z = 0$, removed. On U_0 we define the logarithm $\log_0(z)$ as a holomorphic function as follows: If $z \in U_0$ then there is a unique argument

$$\arg_0(z) = \theta \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

so that

$$z = |z| e^{i\theta} = e^{\ln|z| + i\theta} \ .$$

We then set

$$\log_0(z) = \ln|z| + i\theta, \quad z \in U_0 \ ,$$

and call $\log_0(z)$ the principle branch of the logarithm on U_0. This function $\log_0(z)$ is holomorphic on U_0 and extends the natural logarithm $\ln r = \int_1^r dx/x$, defined for positive real numbers r, to the slit plane U_0.

General Powers on U_0. Let $z \in U_0$ and let $\alpha \in \mathbb{C}$. We then set

$$z^\alpha = e^{\alpha \log_0(z)} = e^{\alpha \ln|z|} e^{i\alpha\theta} \quad \text{where} \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \ . \tag{5.1}$$

If α is real and z is real and positive then the definition of z^α by (5.1) agrees with the usual definition of z^α in real analysis and $z^\alpha > 0$. For fixed $\alpha \in \mathbb{R}$, the function $z \to z^\alpha$ given in (5.1) extends the real analysis function $x \to x^\alpha$, defined for real positive x, to the slit plane U_0 and this function $z \to z^\alpha$ is holomorphic on U_0. We call the function given in (5.1) the principle branch of the power function z^α on U_0.

Lemma 5.1 *Let $z \in U_0$ and let $\alpha \in \mathbb{R}$. Then we have*

$$|z^\alpha| = |z|^\alpha \ .$$

Proof: From

$$z = |z|e^{i\theta} = e^{\ln|z|}e^{i\theta}$$

we obtain

$$z^\alpha = e^{\alpha \ln|z|}e^{i\alpha\theta} \ ,$$

thus

$$|z^\alpha| = e^{\alpha \ln|z|} = |z|^\alpha \ .$$

$\diamond$

The following simple examples show that one has to be careful when computing powers of powers.

1) Since $-1 = e^{\pi i}$ we have

$$(-1)^{(-1)} = (e^{\pi i})^{-1} = e^{-\pi i} = -1 = e^{\pi i} \ .$$

2) Therefore,

$$\left((-1)^{(-1)}\right)^{2/3} = (e^{\pi i})^{2/3} = e^{2\pi i/3} \ .$$

3) On the other hand

$$(-1)^{-2/3} = e^{-2\pi i/3} \ .$$

This shows that

$$\left((-1)^{(-1)}\right)^{2/3} \neq (-1)^{-2/3} \ .$$

We see that, in general,

$$(z^{-1})^\alpha \neq z^{-\alpha} \ .$$

Find the error:

$$\begin{aligned} i &= e^{\pi i/2} \\ &= \left(e^{\pi i}\right)^{1/2} \\ &= (-1)^{1/2} \\ &= \left(e^{-\pi i}\right)^{1/2} \\ &= e^{-\pi i/2} \\ &= -i \end{aligned}$$

5.2 An Example of a Schwarz–Christoffel Map

The following example is Example 8, p. 350, from [Hahn, Epstein]. It gives a good introduction to Schwarz–Christoffel formulas.

For real β let L_β denote the half–line

$$L_\beta = \{z = \beta + iy \ : \ y \le 0\}$$

and let

$$V := \mathbb{C} \setminus \{L_{-1} \cup L_0 \cup L_1\} \ ,$$

i.e., V is the plane $\mathbb{C}$ with three half–lines removed.

Consider the holomorphic function $g : V \to \mathbb{C}$ given by

$$g(\zeta) = e^{2\pi i/3} \cdot (\zeta+1)^{-5/6} \cdot \zeta^{-1/2} \cdot (\zeta-1)^{-2/3}, \quad \zeta \in V \ .$$

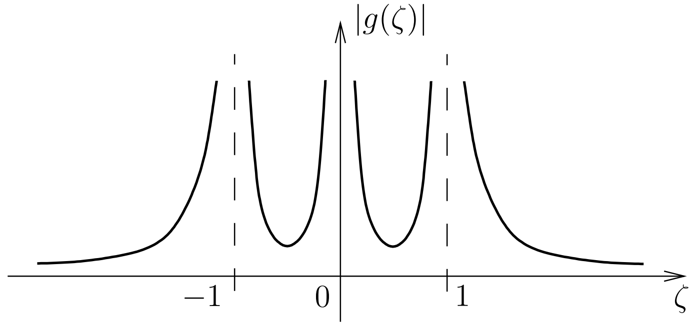

Figure 5.1: $|g(\zeta)| = |\zeta+1|^{-5/6}|\zeta|^{-1/2}|1-\zeta|^{-2/3}$

Figure 5.1 shows a rough graph of the function

$$|g(\zeta)| = |\zeta+1|^{-5/6} \cdot |\zeta|^{-1/2} \cdot |\zeta-1|^{-2/3}$$

for real values of ζ.

The domain of definition of g is the plane $\mathbb{C}$ with three half–lines removed. By removing the three half–lines, we can use the principle branches of the power functions on U_0 to evaluate the factors of $g(\zeta)$. It is then clear that $g \in H(V)$.

For the following, note that the domain V is simply connected. If $z \in V$ then let Γ_z denote a curve in V from 0 to z and set

$$w(z) = \int_{\Gamma_z} g(\zeta)\, d\zeta \ .$$

Since the integral does not depend on the choice of the curve Γ_z, but only depends on z, we write

$$\int_0^z = \int_{\Gamma_z} \ .$$

(Curve–independence of the integral holds though the starting point of Γ_z, the point $z_0 = 0$, lies on the boundary of V, but does not lie inside V. Note, however, that the singularity $\zeta^{-1/2}$ of $g(\zeta)$ at $\zeta = 0$ is integrable.) Note that the function $w(z)$ is holomorphic on V and continuous on

$$V \cup \{-1, 0, 1\} \; .$$

Here it is important that that the exponents

$$-\frac{5}{6}, \quad -\frac{1}{2}, \quad -\frac{2}{3}$$

of the factors of $g(\zeta)$ are all strictly larger than -1.

We will now discuss the function $w(z)$ for z on the real axis. We note that the singularities of the function $g(\zeta)$ are integrable and that $|g(\zeta)|$ decays like $|\zeta|^{-2}$ as $|\zeta| \to \infty$. Decay like the power $|\zeta|^{-2}$ occurs since

$$\frac{5}{6} + \frac{1}{2} + \frac{2}{3} = 2 \; . \tag{5.2}$$

Considering the arguments of the factors of $g(\zeta)$, we have for real ζ:

$$g(\zeta) = |g(\zeta)| \cdot \begin{cases} e^{2\pi i/3} & \text{for} \quad \zeta > 1 \\ 1 & \text{for} \quad 0 < \zeta < 1 \\ e^{-\pi i/2} = -i & \text{for} \quad -1 < \zeta < 0 \\ e^{2\pi i/3} & \text{for} \quad \zeta < -1 \end{cases} \tag{5.3}$$

To obtain (5.3) for $0 < \zeta < 1$ note that $\zeta + 1 > 0$ and $\zeta > 0$, but $\zeta - 1 < 0$. Therefore,

$$\zeta - 1 = |\zeta - 1| e^{\pi i} \quad \text{and} \quad (\zeta - 1)^{-2/3} = |\zeta - 1| e^{-2\pi i/3} \; .$$

For $-1 < \zeta < 0$ the equation (5.3) follows similarly. To obtain (5.3) for $\zeta < -1$ note that

$$-ie^{-5\pi i/6} = e^{-\pi i(\frac{1}{2}+\frac{5}{6})} = e^{2\pi i/3}$$

since (see (5.2))

$$-\frac{1}{2} - \frac{5}{6} = \frac{2}{3} - 2 \; .$$

We now consider the straight lines $\Gamma_0, \ldots, \Gamma_3$ on the real axis of the z–plane (see Figure 5.2) and the image of Γ_j under the map $z \to w(z) = \int_0^z g(\zeta)\, d\zeta$.

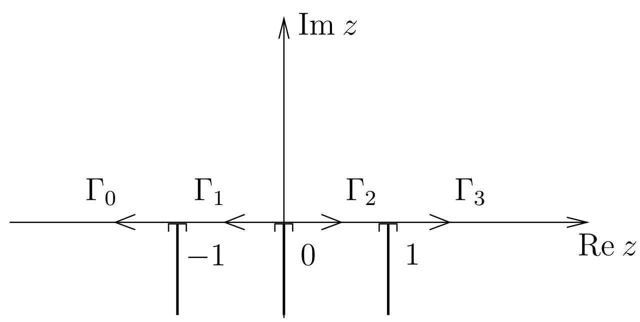

Figure 5.2: Path of integration for $g(\zeta)$

We start the consideration by determining $w(z)$ for $z \in \Gamma_2$. Note that $g(z) = |g(z)|$ for $z \in \Gamma_2$. When z moves from $z = 0$ to $z = 1$ along Γ_2, then the point $w(z)$ moves from $w(0) = 0$ to the finite positive value

$$w(1) = \int_0^1 |g(\zeta)| \, d\zeta$$

along the positive real axis.

Next consider $w(z)$ for $z \in \Gamma_1$ and note that $g(z) = -i|g(z)|$ for $z \in \Gamma_1$. We have

$$w(z) = -\int_z^0 g(\zeta) \, d\zeta = i \int_z^0 |g(\zeta)| \, d\zeta \quad \text{for} \quad -1 \le z \le 0 \; .$$

Therefore, if z moves from $z = 0$ to $z = -1$ along the real axis, then $w(z)$ moves from $w(0) = 0$ to a finite value $w(-1)$ up the imaginary axis.

Next consider $z > 1$. We have

$$w(z) = w(1) + \int_1^z g(\zeta) \, d\zeta = w(1) + e^{2\pi i/3} \int_1^z |g(\zeta)| \, d\zeta \; .$$

The point $w(z)$ moves along the hypotenuse of the triangle in Figure 5.3. Note that $2\pi/3$ corresponds to 120 degrees. Therefore, the angle of the triangle at $w(1)$ is 60 degrees.

Since $|g(\zeta)|$ decays like $|\zeta|^{-2}$ as $\zeta \to \infty$, the point $w(z)$ approaches a finite limit as $z \to \infty$ along the real axis:

$$a_\infty := \lim_{z \to \infty} w(z) = w(1) + e^{2\pi i/3} \int_1^\infty |g(\zeta)| \, d\zeta \; .$$

Similarly, if $z < -1$, then

$$w(z) = w(-1) - \int_z^{-1} g(\zeta) \, d\zeta = w(-1) - e^{2\pi i/3} \int_z^{-1} |g(\zeta)| \, d\zeta \; .$$

One obtains that $w(z)$ moves along a direction parallel to the direction from 0 to $-e^{2\pi i/3}$. Again, as $z \to -\infty$ along the real axis, the point $w(z)$ approaches a finite limit:

$$a_{-\infty} := \lim_{z \to -\infty} w(z) = w(-1) - e^{2\pi i/3} \int_{-\infty}^{-1} |g(\zeta)| \, d\zeta \; .$$

To complete the picture, it is important to understand that

$$a_{-\infty} = a_\infty \; . \tag{5.4}$$

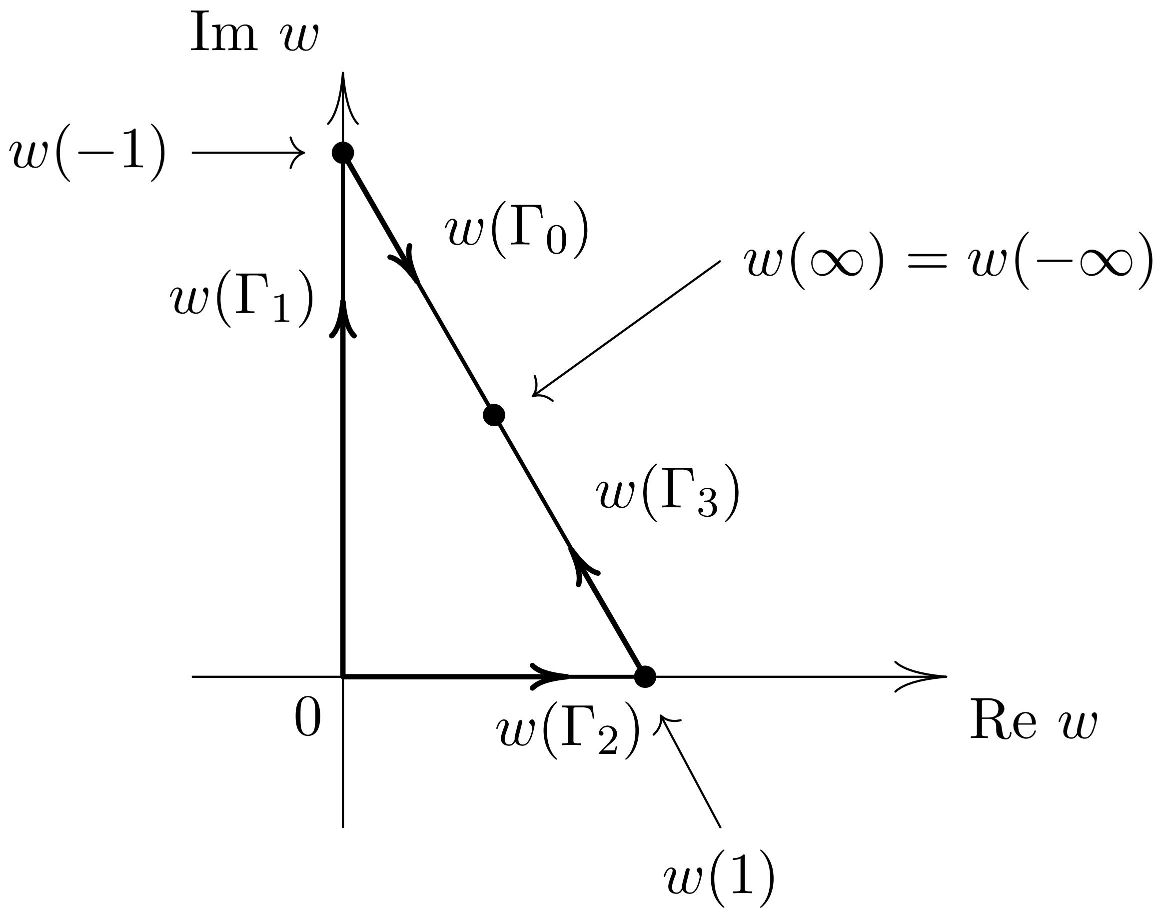

Figure 5.3: $w(z) = \int_0^z g(\zeta)\, d\zeta$

To show this, consider the curve from $z = -R$ to $z = R$ along the real axis completed by the semi–circle $z(\theta) = Re^{i\theta}, 0 \le \theta \le \pi$. Denote this closed curve by $\mathcal{C}_R$ and denote the semicircle by Γ_R.

By Cauchy's theorem,

$$\int_{-R}^{R} g(\zeta)\, d\zeta + \int_{\Gamma_R} g(\zeta)\, d\zeta = \int_{\mathcal{C}_R} g(\zeta)\, d\zeta = 0 \ .$$

(To be completely correct, the straight line along the real axis should avoid the singularities of $g(\zeta)$ at $\zeta = -1, 0, 1$ and one should consider a corresponding curve $\mathcal{C}_{R,\varepsilon}$. However, since the singularities of g are integrable, one can let $\varepsilon \to 0$ without difficulty.)

Since $|g(\zeta)| \le C|\zeta|^{-2}$ for large $|\zeta|$, the integral along the semicircle is bounded by C/R for large R.

We have

$$w(R) = \int_0^R g(\zeta)\, d\zeta \quad \text{and} \quad w(-R) = -\int_{-R}^0 g(\zeta)\, d\zeta \ .$$

Therefore,

$$w(R) - w(-R) = \int_{-R}^{R} g(\zeta)\, d\zeta = -\int_{\Gamma_R} g(\zeta)\, d\zeta \ .$$

It follows that

$$|w(R) - w(-R)| \le C/R \ ,$$

for large R. As $R \to \infty$ we obtain that $a_\infty = a_{-\infty}$.

These considerations prove that the image of the real axis under the map

$$z \to w(z) = \int_0^z g(\zeta)\, d\zeta$$

is the triangle shown in Figure 5.3 where the point $a_\infty = a_{-\infty}$ is removed from the hypotenuse.

Another good example of a Schwarz–Christoffel map is

$$w(z) = \int_0^z (1-\zeta^2)^{-1/2}\, d\zeta = \int_0^z (1+\zeta)^{-1/2}(1-\zeta)^{-1/2}\, d\zeta \ .$$

Here one should note that

$$\int_{-1}^{1} \frac{d\zeta}{\sqrt{1-\zeta^2}} = \arcsin(1) - \arcsin(-1) = \pi \ .$$

5.3 General Schwarz–Christoffel Integrals

Let $n \in \mathbb{N}, n \ge 3$, and let s_j and t_j denote real numbers for $j = 1, 2, \ldots, n$. Assume that

$$s_j < 1 \quad \text{for} \quad j = 1, 2, \ldots, n \quad \text{and} \quad \sum_{j=1}^{n} s_j = 2$$

and

$$t_1 < t_2 < \ldots < t_n \ .$$

Set

$$V = \mathbb{C} \setminus \{L_{t_1} \cup \ldots \cup L_{t_n}\} \quad \text{and} \quad \hat{V} = V \cup \{t_1, \ldots, t_n\} \ .$$

Consider

$$g(\zeta) = (\zeta - t_1)^{-s_1}(\zeta - t_2)^{-s_2} \cdots (\zeta - t_n)^{-s_n} \quad \text{for} \quad \zeta \in V \ .$$

For constants $K, A \in \mathbb{C}$ the function

$$w(z) = K \int_{t_1}^{z} g(\zeta)\, d\zeta + A, \quad z \in \hat{V} \ ,$$

is called a Schwarz–Christoffel integral. For simplicity, we will assume that $K = 1$ and $A = 0$.

Consider

$$w(z) = \int_{t_1}^{z} g(\zeta)\, d\zeta \quad \text{for} \quad z \in \mathbb{R} \ .$$

Set

$$B_j = w(t_j) \quad \text{for} \quad j = 1, 2, \ldots, n \ .$$

Clearly, $B_1 = 0$. Under appropriate assumptions, the points $B_1, B_2, \ldots, B_n$ form the corners of a polygon. The interior angle at B_j is $\alpha_j = \pi(1 - s_j)$. The sum of the interior angles is

$$\sum_{j=1}^n \alpha_j = \pi(n-2)$$

since, by assumption, $\sum_{j=1}^n \beta_j = 2$.

Let

$$\zeta < t_j \quad \text{and} \quad k \geq j \ ,$$

thus $\zeta < t_k$. We have

$$(\zeta - t_k)^{-s_k} = |\zeta - t_k| \, e^{-\pi i s_k} \ .$$

Therefore, for $t_{j-1} < \zeta < t_j$:

$$\begin{aligned} g(\zeta) &= |g(\zeta)| e^{-\pi i (s_j + s_{j+1} + \ldots + s_n)} \\ &= |g(\zeta)| e^{\pi i (2 - s_j - s_{j+1} - \ldots - s_n)} \\ &= |g(\zeta)| e^{\pi i (s_1 + \ldots + s_{j-1})} \end{aligned}$$

We have used that $e^{2\pi i} = 1$ and $\sum_{j=1}^n s_j = 2$, thus

$$2 - (s_j + s_{j+1} + \ldots + s_n) = s_1 + \ldots s_{j-1} \ .$$

Consider $w(z)$ for $t_1 \leq z \leq t_2$. Using the equation

$$g(\zeta) = |g(\zeta)| e^{s_1 + \ldots + s_{j-1}} \quad \text{for} \quad t_{j-1} < \zeta < t_j$$

for $j = 2$ we have

$$g(\zeta) = |g(\zeta)| \, e^{\pi i s_1} \quad \text{for} \quad t_1 < \zeta < t_2 \ .$$

Therefore,

$$w(z) = e^{\pi i s_1} \int_{t_1}^z |g(\zeta)| \, d\zeta \quad \text{for} \quad t_1 \leq z \leq t_2 \ .$$

For $t_2 \leq z \leq t_3$ we have

$$\begin{aligned} w(z) &= B_2 + \int_{t_2}^z g(\zeta) \, d\zeta \\ &= B_2 + e^{\pi i (s_1 + s_2)} \int_{t_2}^z |g(\zeta)| \, d\zeta \end{aligned}$$

Consider the line from B_1 to B_2 and extend the line as a straight line. Also, consider the line from B_2 to B_3. The angle between the lines at B_2 equals πs_2. If the points B_1, B_2, B_3 belong to a polygon, then the interior angle at B_2 is $\alpha_2 = \pi - \pi s_2$.

The process can be continued. One obtains that, under appropriate assumptions, the points $B_1 = 0, B_2, B_3, \ldots, B_n$ form the corners of an n–gone. The interior angle at B_j is $\alpha_j = \pi(1 - s_j)$.

The following arguments shown that the point B_n lies on the negative real axis, i.e., to the left of the point $B_1 = 0$.

Set

$$a_\infty = \int_{t_1}^{\infty} g(\zeta)\, d\zeta \quad \text{and} \quad a_{-\infty} = \int_{t_1}^{-\infty} g(\zeta)\, d\zeta \ .$$

Since

$$g(\zeta) = |g(\zeta)| \quad \text{for} \quad \zeta > t_n \quad \text{and} \qquad \text{for} \quad \zeta < t_1$$

we have

$$a_\infty = B_n + \int_{t_n}^{\infty} |g(\zeta)|\, d\zeta > B_n$$

and

$$a_{-\infty} = \int_{t_1}^{-\infty} |g(\zeta)|\, d\zeta = -\int_{-\infty}^{t_1} |g(\zeta)|\, d\zeta < 0 \ .$$

By the same arguments as in the example one obtains that $a_\infty = a_{-\infty}$. It follows that

$$B_n < a_\infty = a_{-\infty} < 0 = B_1 \ .$$

6 Meromorphic Functions with Prescribed Poles

6.1 Meromorphic Functions on $\mathbb{C}$

Recall that a function f is called meromorphic on $\mathbb{C}$ if there is a finite or denumerable set $\mathcal{D} = \{b_1, b_2, \ldots\} \subset \mathbb{C}$ so that

(a) f is holomorphic on $\mathbb{C} \setminus \mathcal{D}$;

(b) every point $b_k \in \mathcal{D}$ is a pole of f.

The set $\mathcal{D}$ is finite or denumerable and does not have an accumulation point in $\mathbb{C}$. (Suppose b is an accumulation point of $\mathcal{D}$. If $b \in \mathcal{D}$ then the point b is an accumulation point of poles and cannot be a pole itself. If $b \in \mathbb{C} \setminus \mathcal{D}$ then f is holomorphic in b, and poles of f cannot accumulated at b.)

If f has infinitely many poles b_k then we will order them so that

$$|b_1| \le |b_2| \le \ldots \quad \text{and} \quad |b_k| \to \infty \quad \text{as} \quad k \to \infty \; . \tag{6.1}$$

If f is meromorphic on $\mathbb{C}$ we write $f \in M(\mathbb{C})$.

Example 1: $f(z) = \frac{1}{\sin(\pi z)}$. It is easy to show that the pole set of f is the set $\mathbb{Z}$ of all integers j. Also, for the function $h(z) = \sin(\pi z)$ we have $h'(j) = \pi \cos(\pi j) = \pi(-1)^j$. This yields that

$$Res\Big(\frac{1}{\sin(\pi z)}, z = j\Big) = \frac{(-1)^j}{\pi} \; .$$

Example 2: Every rational function $f(z) = p(z)/q(z)$ is meromorphic on $\mathbb{C}$ with finitely many poles.

Example 3: The function $f(z) = e^{1/z}$ is not meromorphic since $z = 0$ is not a pole, but an essential singularity.

Let $f \in M(\mathbb{C})$ and let b_k be a pole of f of order m_k. The Laurent expansion of f at b_k has the form

$$\begin{aligned} f(z) &= \sum_{j=-m_k}^{\infty} \alpha_{jk} \, (z - b_k)^j \\ &= P_k((z - b_k)^{-1}) + g_k(z) \end{aligned}$$

where

$$P_k(w) = \sum_{j=1}^{m_k} \alpha_{-j,k} w^j$$

is a polynomial without constant term and $g_k \in H(D(b_k, r_k))$. Here

$$r_k = dist\Big(b_k, \mathcal{D} \setminus \{b_k\}\Big) \; .$$

The function

$$P_k((z - b_k)^{-1}), \quad z \in \mathbb{C} \setminus \{b_k\} \; ,$$

is called the singular part of $f(z)$ at b_k.

6.2 The Mittag–Leffler Theorem

Notation: If $K \subset \mathbb{C}$ is compact and $f \in C(K)$ then let

$$|f|_K = \max\{|f(z)| \ : \ z \in K\} \ .$$

Definition: Let $U \subset \mathbb{C}$ be open and let $f_k \in H(U)$. The series

$$\sum_{k=1}^{\infty} f_k(z) \tag{6.2}$$

converges normally in U if for every compact set $K \subset U$ the series

$$\sum_{k=1}^{\infty} |f_k|_K$$

converges.

Thus, normal convergence of a series (6.2) implies that the sequence of partial sums,

$$s_n(z) = \sum_{k=1}^{n} f_k(z)$$

converges locally uniformly on U to a unique limit, $f \in H(U)$. As usual, the limit is denoted by

$$f(z) = \sum_{k=1}^{\infty} f_k(z) \ .$$

Also, note that in case of normal convergence of (6.2), the series $\sum_{k=1}^{\infty} f_k(z)$ converges absolutely at every point $z \in U$. This implies that the terms of the series can be reordered without changing the value of the limit. In other words, the ordering of the terms $f_k(z)$ is not important.

We also recall that normal convergence of the series

$$f(z) = \sum_{k=1}^{\infty} f_k(z)$$

in U implies normal convergence of

$$f'(z) = \sum_{k=1}^{\infty} f'_k(z)$$

in U.

Theorem 6.1 *(Mittag–Leffler) Let $b_1, b_2, \ldots$ denote an infinite sequence of distinct points in $\mathbb{C}$ without accumulation point in $\mathbb{C}$ and set $\mathcal{D} = \{b_1, b_2, \ldots\}$. Further, let $P_k(w)$ denote a sequence of polynomials without constant terms, i.e., $P_k(0) = 0$ for every k. Then there is a function $f \in M(\mathbb{C})$ so that:*

(a) the pole set of f equals $\mathcal{D}$;

(b) the singular part of f at b_k equals $P_k((z - b_k)^{-1})$.

Remark: We cannot simply define

$$f(z) = \sum_{k=1}^{\infty} P_k((z-b_k)^{-1})$$

since, in general, the series does not converge. For example, let $b_k = k$ and $P_k(w) = w$. Then the above series is

$$\sum_{k=1}^{\infty} \frac{1}{z-k} \ .$$

This series diverges for every z.

Proof of Theorem 6.1: We first assume that $z = 0$ is not an element of $\mathcal{D}$. Then we may assume $0 < |b_1| \le |b_2| \le \ldots$ and $|b_k| \to \infty$ as $k \to \infty$. Let $\sum_{k=1}^{\infty} c_k$ denote a convergent series of positive numbers, $c_k > 0$. For example, $c_k = 1/k^2$.

The function

$$z \to P_k((z-b_k)^{-1})$$

is singular only at $z = b_k$. Thus we can write

$$P_k((z-b_k)^{-1}) = \sum_{j=0}^{\infty} a_{jk} z^j, \quad |z| < |b_k| \ .$$

Let

$$Q_k(z) = \sum_{j=0}^{n_k} a_{jk} z^j$$

where the integer n_k is chosen so large that

$$\max_{|z| \le |b_k|/2} \Big| P_k((z-b_k)^{-1}) - Q_k(z) \Big| \le c_k \ . \tag{6.3}$$

We claim that the series

$$f(z) := \sum_{k=1}^{\infty} \Big(P_k((z-b_k)^{-1}) - Q_k(z) \Big), \quad z \in \Big(\mathbb{C} \setminus \mathcal{D} \Big) =: U \ ,$$

converges normally in U and the limit f is a meromorphic function with the desired properties.

Set

$$f_k(z) = P_k((z-b_k)^{-1}) - Q_k(z), \quad z \in \mathbb{C} \setminus \{b_k\} \ .$$

Let $K \subset U$ be any compact set. There exists $R > 0$ with

$$K \subset \bar{D}(0,R) \ .$$

Since $|b_k| \to \infty$ there exists a positive integer $k_0(R)$ with

$$|b_k| \ge 2R \quad \text{for} \quad k \ge k_0(R) \ .$$

If $k \ge k_0(R)$ then, by (6.3),

$$|f_k|_K \le |f_k|_{\bar{D}(0,R)} \le c_k \ .$$

Therefore,

$$\sum_{k=1}^{\infty} |f_k|_K < \infty \ .$$

This proves normal convergence of the series

$$f(z) := \sum_{k=1}^{\infty} f_k(z) \quad \text{in} \quad U \ .$$

We must show that the pole set of f is precisely $\mathcal{D}$ and that

$$P_k((z-b_k)^{-1})$$

is the singular part of f at b_k. To this end, fix any $R > 0$. For $z \in U \cap D(0,R)$ we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{k_0(R)} f_k(z) + \sum_{k > k_0(R)} f_k(z) \\ &=: f^I(z) + f^{II}(z) \ . \end{aligned}$$

Note that the decomposition $f = f^I + f^{II}$ depends on R. The first part, $f^I(z)$, is meromorphic in $\mathbb{C}$. In fact, f^I is a rational function. In the disk $D(0,R)$ the function f^I has poles precisely at those b_k which lie in $D(0,R)$ and at each b_k the singular part of f^I is $P_k((z-b_k)^{-1})$. The second function, $f^{II}(z)$, is holomorphic in $D(0,R)$ because every function $f_k(z), k > k_0(R)$, is holomorphic in $D(0,2R)$ and the series defining f^{II} converges uniformly on $\bar{D}(0,R)$.

Since $R > 0$ was arbitrary, it is shown that f has the desired properties.

So far we have assumed that $b_1 \neq 0$. If $b_1 = 0$ we just add the term

$$P_1(1/z)$$

to the constructed function.

Remark: An important technical point of the proof is that the decomposition, $f = f^I + f^{II}$, depends on R, and we consider it in $D(0,R)$. In this way, for each finite R, one only considers functions with finitely poles in $D(0,R)$.

6.3 Example 4

As before, let $\mathcal{D} = \{b_1, b_2, \ldots\}$ denote an infinite set in $\mathbb{C}$ without accumulation point in $\mathbb{C}$. For simplicity, let $0 < |b_1| \le |b_2| \le \ldots$

Assume that $P_k(w) = w$ for all k, i.e., we want to construct a meromorphic functions with simple poles and residue 1 at each b_k. Assume that

$$\sum_k \frac{1}{|b_k|} = \infty, \quad \sum_k \frac{1}{|b_k|^2} < \infty \ ,$$

where the sum is taken over all $b_k \neq 0$.

For $b_k \neq 0$ we expand $P_k((z-b_k)^{-1})$ about $z = 0$:

$$P_k((z-b_k)^{-1}) = \frac{1}{z-b_k} = -\frac{1}{b_k} + \mathcal{O}(z) \ .$$

If we take

$$Q_k(z) \equiv -\frac{1}{b_k}$$

we obtain

$$\begin{aligned} f_k(z) &= \frac{1}{z-b_k} - Q_k(z) \\ &= \frac{1}{z-b_k} + \frac{1}{b_k} \\ &= \frac{z}{(z-b_k)b_k} \end{aligned}$$

If $|z| \leq R$ and $|b_k| \geq 2R$ then we have

$$|z-b_k| \geq |b_k| - |z| \geq \frac{1}{2}|b_k| \ ,$$

and, therefore,

$$|f_k|_{\bar{D}(0,R)} \leq \frac{2R}{|b_k|^2} \ .$$

It follows that the series

$$f(z) = \sum_{k=1}^{\infty}\Big(\frac{1}{z-b_k} + \frac{1}{b_k}\Big), \quad z \in \mathbb{C} \setminus \mathcal{D} =: U \ , \tag{6.4}$$

converges normally in U to a meromorphic function $f(z)$. The function $f(z)$ has a simple pole at each b_k with residue 1; the function $f(z)$ has no other poles. We also know that we can differentiate (6.4) arbitrarily often term by term.

Now assume that we want to construct a meromorphic function $f(z)$ with poles precisely at the integers and residue 1 at each integer. According to the above, such a function is

$$f(z) = \frac{1}{z} + \sum_{n\neq 0}\Big(\frac{1}{z-n} + \frac{1}{n}\Big) \ .$$

6.4 Example 5

Determine a meromorphic function $f(z)$ with a simple pole at $b_k = \sqrt{k}$ for $k = 1, 2, \ldots$ and $Res(f, z = b_k) = 1$.

The singular part at $b_k = \sqrt{k}$ is

$$s_k(z) = \frac{1}{z-\sqrt{k}} \ .$$

Write $s_k(z)$ as a power series centered at $z = 0$. For $|z| < \sqrt{k}$ we have

$$\begin{aligned} s_k(z) &= \frac{1}{-\sqrt{k}} \cdot \frac{1}{1 - z/\sqrt{k}} \\ &= \frac{1}{-\sqrt{k}} \sum_{j=0}^{\infty} \Big(\frac{z}{\sqrt{k}}\Big)^j \\ &= \frac{1}{-\sqrt{k}} \Big(1 + \frac{z}{\sqrt{k}} + \frac{z^2}{k} + \dots\Big) \end{aligned}$$

We will use

$$Q_k(z) = \frac{1}{-\sqrt{k}} \Big(1 + \frac{z}{\sqrt{k}}\Big)$$

to approximate $s_k(z)$ for $|z| \le R$ and $\sqrt{k} \ge 2R$. This leads to

$$f_k(z) = \frac{1}{z - \sqrt{k}} + \frac{1}{\sqrt{k}} + \frac{z}{k}$$

and the meromorphic function

$$f(z) = \sum_{k=1}^{\infty} f_k(z) \ .$$

Let us prove that the above series converges normally in $\mathbb{C} \setminus \mathcal{D}$ where $\mathcal{D}$ is the set of poles. Fix any $R > 0$ and let $|z| \le R$, $\sqrt{k} \ge 2R$, thus

$$\frac{|z|}{\sqrt{k}} \le \frac{1}{2} \ .$$

A simple result about the geometric sum:

Lemma 6.1 *For $|\varepsilon| \le \frac{1}{2}$ we have*

$$\Big|\frac{1}{1-\varepsilon} - (1+\varepsilon)\Big| \le 2|\varepsilon|^2 \ .$$

Proof: We have

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2\Big(1 + \varepsilon + \varepsilon^2 + \dots\Big)$$

thus

$$\Big|\frac{1}{1-\varepsilon} - 1 - \varepsilon\Big| = \frac{|\varepsilon|^2}{|1-\varepsilon|} \le 2|\varepsilon|^2 \ .$$

$\diamond$

In the following, let $R > 0$ and let $|z| \le R$, $\sqrt{k} \ge 2R$, thus

$$\frac{|z|}{\sqrt{k}} \le \frac{1}{2} \ .$$

We use the Lemma with $\varepsilon = z/\sqrt{k}$ to obtain that

$$\begin{aligned}
\left|\frac{1}{z-\sqrt{k}}+\frac{1}{\sqrt{k}}+\frac{z}{k}\right| &= \frac{1}{\sqrt{k}}\left|\frac{1}{1-z/\sqrt{k}}+1+\frac{z}{\sqrt{k}}\right| \\
&\leq \frac{1}{\sqrt{k}}\,\frac{2|z|^2}{k} \\
&\leq \frac{2R^2}{k^{3/2}}
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ converges, the meromorphic function

$$f(z)=\sum_{k=1}^{\infty}\Big(\frac{1}{z-\sqrt{k}}+\frac{1}{\sqrt{k}}+\frac{z}{k}\Big)$$

has the required properties.

7 Infinite Products

7.1 Infinite Products of Complex Numbers

Let $q_1, q_2, \dots$ denote a sequence of complex numbers. We want to define what it means that the infinite product

$$\Pi_{j=1}^\infty \, q_j$$

converges and what the value of this product is if it converges. Naively, and in analogy to series, one considers the sequence p_n of finite products,

$$p_n = \Pi_{j=1}^n \, q_j \quad \text{for} \quad n = 1, 2, \dots$$

and calls the infinite product convergent to p if $p_n \to p$. This leads to Definition 7.1 below. Most authors define convergence of an infinite product more restrictively. Using the more restrictive definition, the theorems about infinite products become simpler to formulate.

7.1.1 Two Definitions of Convergence

Let

$$q_j, \quad j = 1, 2, \dots$$

denote a sequence of complex numbers. We form the finite products

$$p_n = \Pi_{j=1}^n \, q_j$$

for $n = 1, 2, \dots$ In correspondence with infinite series, we define:

Definition 7.1: *The infinite product*

$$\Pi_{j=1}^\infty \, q_j$$

converges (in the simple sense) if the sequence of partial products p_n converges as $n \to \infty$. If $p_n \to p$ as $n \to \infty$ then we write

$$\Pi_{j=1}^\infty \, q_j = p \quad (\textit{in the simple sense})$$

and call p the value of the infinite product.

This definition is used in [Stein, Sharkarchi].

If one of the factors q_j in the infinite product is zero then, using the above definition, the infinite product $\Pi_j q_j$ always converges to zero (in the simple sense). Other authors prefer a more restrictive definition of convergence for infinite products. The following definition is used by [Greene, Krantz] and others.

Definition 7.2: *The infinite product*

$$\Pi_{j=1}^\infty \, q_j$$

converges if the following holds:

1) At most finitely many of the q_j are equal to zero.

2) If $N_0 > 0$ is so large that $q_j \neq 0$ for $j > N_0$ then

$$\lim_{N\to\infty} \Pi_{j=N_0+1}^{N}\; q_j$$

exists and is non–zero. If these two conditions are met, then the value of the infinite product is the number

$$\left(\Pi_{j=1}^{N_0}\; q_j\right) \cdot \left(\lim_{N\to\infty} \Pi_{j=N_0+1}^{N}\; q_j\right) \; .$$

It is easy to show that the value of the product does not depend on the choice of N_0. Here we assume that $q_j \neq 0$ for $j > N_0$.

The difference between the two definitions is not profound. Like most authors, we prefer here to work with Definition 7.2. Then the formulation of many theorems becomes simpler. For example, below we will consider products of holomorphic functions,

$$p(z) = \Pi_{j=1}^{\infty}\; (1 + a_j(z)), \quad a_j \in H(U) \; .$$

Then, using Definition 7.2, the product $p(z)$ is zero at some $z = z_0$ if and only if at least one factor $1 + a_j(z)$ is zero at $z = z_0$.

7.1.2 Examples

Example 1: According to Definition 7.1, the infinite product

$$\Pi_{j=0}^{\infty}\; j = 0 \cdot 1 \cdot 2 \cdot \ldots$$

converges to zero (in the simple sense). According to Definition 7.2, this product diverges because

$$\Pi_{j=1}^{n}\; j = n!$$

does not converge to a finite limit as $n \to \infty$.

Example 2: According to Definition 7.1, the infinite product

$$\Pi_{j=1}^{\infty}\; \left(\frac{1}{2}\right)^j$$

converges to zero (in the simple sense). According to Definition 7.2, this product diverges because

$$\Pi_{j=1}^{n}\; \left(\frac{1}{2}\right)^j \to 0 \quad \text{as} \quad n \to \infty \; ,$$

but no factor is zero.

Example 3: Let

$$q_j = 1 - \frac{1}{j^2}, \quad j = 2, 3, \ldots$$

Consider the infinite product

$$\Pi_{j=2}^{\infty}\; \left(1 - \frac{1}{j^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots \tag{7.1}$$

Here the partial products are

$$\begin{aligned} p_n &= \frac{1}{2^2}(2^2-1)\,\frac{1}{3^2}(3^2-1)\ldots\frac{1}{n^2}(n^2-1) \\ &= \frac{1}{n!n!}\,(1\cdot 3)(2\cdot 4)\ldots((n-1)(n+1)) \\ &= \frac{1}{n!n!}\,(n-1)!\,\frac{1}{2}(n+1)! \\ &= \frac{1}{2}\,\frac{n+1}{n} \end{aligned}$$

It follows that $p_n \to \frac{1}{2}$. The infinite product (7.1) converges to $\frac{1}{2}$, using either definition.

Example 4: Let

$$q_j = 1 - \frac{1}{j}, \quad j = 2, 3, \ldots$$

Consider the infinite product

$$\Pi_{j=2}^{\infty}\left(1-\frac{1}{j}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots \tag{7.2}$$

Here the partial products are

$$\begin{aligned} p_n &= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right) \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\ldots\frac{n-1}{n} \\ &= \frac{1}{n}\,. \end{aligned}$$

It follows that $p_n \to 0$. According to Definition 7.1, the infinite product (7.2) converges to zero, in the simple sense. According to Definition 7.2, the infinite product diverges since $p_n \to 0$, but no factor is zero.

Example 5: We claim that the infinite product

$$\Pi_{j=1}^{\infty}\left(1+\frac{1}{j}\right)$$

also diverges. We have

$$\begin{aligned} p_n &= \Pi_{j=1}^{n}\left(1+\frac{1}{j}\right) \\ &= \Pi_{j=1}^{n}\left(\frac{j+1}{j}\right) \\ &= \frac{2}{1}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{n+1}{n} \\ &= n+1 \end{aligned}$$

Since $p_n \to \infty$ as $n \to \infty$ the infinite product diverges.

Example 6: According to [Remmert, Classical Topics in Complex Function Theory], infinite products first appeared in 1579 in the work of F. Vieta. He gave the formula

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$

Let us understand the result. The infinite product can be written as

$$\Pi_{j=2}^{\infty}\, q_j$$

with

$$q_2 = \sqrt{\frac{1}{2}}$$

and

$$q_{j+1} = \sqrt{\frac{1}{2} + \frac{1}{2}\, q_j} \quad \text{for} \quad j = 2, 3, \ldots$$

Thus, the q_j obey the recursion

$$q_{j+1}^2 = \frac{1}{2} + \frac{1}{2}\, q_j \quad \text{for} \quad j \geq 2 \quad \text{and} \quad q_2 = \sqrt{\frac{1}{2}} = \cos(\pi/4)\ .$$

Recall that

$$\begin{aligned} \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 2\cos^2\alpha - 1\ , \end{aligned}$$

thus

$$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\,\cos 2\alpha\ .$$

With $\beta = 2\alpha$ we write this as

$$\cos^2\frac{\beta}{2} = \frac{1}{2} + \frac{1}{2}\,\cos\beta\ .$$

Since

$$q_2 = \sqrt{\frac{1}{2}} = \cos(\pi/2^2)$$

and

$$q_{j+1}^2 = \frac{1}{2} + \frac{1}{2}\, q_j \quad \text{for} \quad j \geq 2$$

it follows that

$$q_j = \cos(\pi/2^j), \quad j = 2, 3, \ldots$$

This formula for q_j makes the determination of the infinite product manageable.

Recall that

$$\sin 2\alpha = 2 \sin\alpha \cos\alpha \ ,$$

thus

$$\cos\alpha = \frac{1}{2} \frac{\sin 2\alpha}{\sin\alpha} \ .$$

We then obtain that

$$q_j = \cos(\pi/2^j) = \frac{1}{2} \frac{\sin(\pi/2^{j-1})}{\sin(\pi/2^j)} \ ,$$

thus

$$\begin{aligned} q_2\, q_3 \cdots q_n &= \frac{1}{2^{n-1}} \cdot \frac{\sin(\pi/2)}{\sin(\pi/2^2)} \cdot \frac{\sin(\pi/2^2)}{\sin(\pi/2^3)} \cdot \ldots \cdot \frac{\sin(\pi/2^{n-1})}{\sin(\pi/2^n)} \\ &= \frac{1}{2^{n-1}} \cdot \frac{\sin(\pi/2)}{\sin(\pi/2^n)} \\ &= \frac{2}{\pi} \cdot \frac{\pi/2^n}{\sin(\pi/2^n)} \end{aligned}$$

Since

$$\frac{t}{\sin t} \to 1 \quad \text{as} \quad t \to 0$$

it follows that the partial products

$$p_n = q_2\, q_3 \cdots q_n$$

converge to $2/\pi$ as $n \to \infty$. Using Definition 7.1 or 7.2, Vieta's product converges to $2/\pi$.

7.2 Infinite Products of Numbers: Convergence Theory

Recall the Cauchy convergence criterion for a sequence p_n of complex numbers: The sequence p_n converges if and only if for every $\varepsilon > 0$ there exists $J = J(\varepsilon) \in \mathbb{N}$ so that

$$|p_n - p_m| < \varepsilon \quad \text{for} \quad n > m \geq J(\varepsilon) \ .$$

Lemma 7.1 *(Cauchy Criterion) The infinite product*

$$\Pi_{j=1}^{\infty} q_j \tag{7.3}$$

converges in the sense of Definition 7.2 if and only if for all $\varepsilon > 0$ there exists $J = J(\varepsilon) \in \mathbb{N}$ so that

$$\Big| \Pi_{j=m+1}^{n}\, q_j - 1 \Big| < \varepsilon \quad \textit{for} \quad n > m \geq J(\varepsilon) \ . \tag{7.4}$$

Proof: 1) Assume that (7.3) converges. There exists $J_0 \in \mathbb{N}$ so that $q_j \neq 0$ for $j > J_0$. Set

$$p_n = \Pi_{j=J_0+1}^{n} q_j \quad \text{for} \quad n > J_0 \ .$$

By assumption,

$$p_n \to L \quad \text{as} \quad n \to \infty, \quad L \neq 0 \ .$$

For $n > m > J_0$ we have

$$\frac{p_n}{p_m} = \Pi_{j=m+1}^n q_j \ .$$

If $\varepsilon_0 > 0$ is given, then there exists $K(\varepsilon_0) > J_0$ with

$$|p_n - L| < \varepsilon_0 \quad \text{and} \quad |p_m - L| < \varepsilon_0 \quad \text{for} \quad n > m \geq K(\varepsilon_0) \ .$$

We may assume that $0 < \varepsilon_0 < |L|/2$.

Set

$$\eta_n = p_n - L \quad \text{and} \quad \eta_m = p_m - L$$

and obtain that

$$\frac{p_n}{p_m} = \frac{L + \eta_n}{L + \eta_m} = 1 + \frac{\eta_n - \eta_m}{L + \eta_m} \ ,$$

thus

$$\Big| \frac{p_n}{p_m} - 1 \Big| \leq \frac{4\varepsilon_0}{|L|} \leq \varepsilon$$

if

$$0 < \varepsilon_0 \leq \frac{\varepsilon |L|}{4} \quad \text{and} \quad 0 < \varepsilon_0 < |L|/2 \ .$$

This proves (7.4).

2) Conversely, assume that for all $\varepsilon > 0$ there exists $J(\varepsilon) \in \mathbb{N}$ so that (7.4) holds. First, let $J_0 = J(1/2)$. Then the estimate (7.4) with $\varepsilon = 1/2$ implies that $q_j \neq 0$ for $j > J_0$. Set

$$p_n = \Pi_{j=J_0+1}^n q_j \quad \text{for} \quad n > J_0 \ .$$

We must show that the sequence p_n converges to a non–zero limit. We first show that the sequence p_n is bounded. We have

$$\Big| \frac{p_n}{p_m} - 1 \Big| \leq \frac{1}{2} \quad \text{for} \quad n > m > J_0 \ .$$

Therefore,

$$|p_n - p_m| \leq \frac{1}{2} |p_m| \quad \text{for} \quad n > m > J_0 \ .$$

If we fix $m = J_0 + 1$ we obtain that $|p_n|$ is bounded, $|p_n| \leq C$ for all $n > J_0$.

Let $\varepsilon > 0$ be given and let

$$\Big| \frac{p_n}{p_m} - 1 \Big| = \Big| \Pi_{j=m+1}^n q_j - 1 \Big| < \varepsilon \quad \text{for} \quad n > m \geq J(\varepsilon) > J_0 \ .$$

This estimate implies that

$$|p_n - p_m| = \left|\frac{p_n}{p_m} - 1\right||p_m| < \varepsilon C \quad \text{for} \quad n > m \ge J(\varepsilon) > J_0 \ .$$

Therefore, the sequence p_n converges, $p_n \to L$.

The estimate $|p_n - p_m| \le \frac{1}{2}\,|p_m|$ proved above for $m = J_0 + 1$ and all large n implies that $|L - p_m| \le \frac{1}{2}\,|p_m|$, thus $L \ne 0$ since $p_m \ne 0$. $\diamond$

Using the notation of the above proof, we have

$$|q_{j+1} - 1| = \left|\frac{p_{j+1}}{p_j} - 1\right| < \varepsilon \quad \text{for} \quad j \ge J(\varepsilon) \ .$$

This proves the following:

Lemma 7.2 *If*

$$\Pi_{j=1}^{\infty}\, q_j$$

converges, then

$$q_j \to 1 \quad \textit{as} \quad j \to \infty \ .$$

The previous lemma suggests to write the factors q_j in an infinite product

$$\Pi_{j=1}^{\infty} q_j$$

in the form

$$q_j = 1 + a_j \ .$$

If the infinite product $\Pi(1 + a_j)$ converges, then $a_j \to 0$. The converse is not true, of course, as the example

$$\begin{aligned}
\Pi_{j=2}^{\infty}\Big(1 - \frac{1}{j}\Big) &= \Big(1 - \frac{1}{2}\Big)\Big(1 - \frac{1}{3}\Big)\ldots\Big(1 - \frac{1}{n}\Big) \\
&= \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{1-n}{n} \\
&= \frac{1}{n}
\end{aligned}$$

shows.

We will use the following simple estimates for the real exponential function.

Lemma 7.3 *We have*

$$1 + x \le e^x \quad \textit{for} \quad x \ge 0 \tag{7.5}$$

and

$$e^{x/2} \le 1 + x \quad \textit{for} \quad 0 \le x \le 2 \ . \tag{7.6}$$

Proof: The second assertion follows from $e < 3$ and convexity of the exponential function. $\diamond$

The following theorem relates the convergence of an infinite product to the absolute convergence of a series.

Theorem 7.1 *Let a_j denote a sequence of complex numbers. Then $\sum_{j=1}^\infty |a_j| < \infty$ if and only if*

$$\Pi_{j=1}^\infty (1 + |a_j|)$$

converges.

Proof: a) First assume that

$$S := \sum_{j=1}^\infty |a_j| < \infty \ .$$

Set

$$p_n := \Pi_{j=1}^n (1 + |a_j|)$$

and obtain that

$$p_n \le e^{|a_1| + \ldots + |a_n|} \le e^S < \infty \ .$$

Because of

$$1 \le p_1 \le p_2 \le \ldots \le p_n \le e^S < \infty \quad \text{for all} \quad n \in \mathbb{N} \ ,$$

convergence

$$p_n \to p, \quad p \ge 1 \ ,$$

follows.

b) Assume that

$$\Pi_{j=1}^\infty (1 + |a_j|)$$

converges. Since $|a_j| \to 0$ (by Lemma 7.2) we have

$$|a_j| \le 2 \quad \text{for} \quad j > J \ ,$$

thus

$$e^{|a_j|/2} \le 1 + |a_j| \quad \text{for} \quad j > J \ .$$

Setting

$$S_1 = \lim_{n \to \infty} \Pi_{j=J+1}^n (1 + |a_j|)$$

we obtain:

$$\exp\Big(\frac{1}{2} \sum_{j=J+1}^n |a_j|\Big) \le \Pi_{j=J+1}^n (1 + |a_j|) \le S_1 < \infty$$

for all $n \geq J+1$. This yields the bound

$$\sum_{j=J+1}^{n} |a_j| \leq 2 \ln S_1 < \infty \quad \text{for} \quad n \geq J+1 \ ,$$

which implies convergence of the series $\sum |a_j|$. $\diamond$

It is less clear how to relate convergence of the series

$$\sum a_j$$

to convergence of the product

$$\Pi(1+a_j) \ .$$

However, convergence of the series $\sum |a_j|$ does imply convergence of the product $\Pi(1+a_j)$. We will formulate this as the next theorem.

Theorem 7.2 *Let a_j be a sequence of complex numbers. If $\sum_{j=1}^{\infty} |a_j| < \infty$ then*

$$\Pi_{j=1}^{\infty}(1+a_j)$$

converges.

The proof uses the following lemma.

Lemma 7.4 *Let $a_1, \ldots, a_n \in \mathbb{C}$ and set*

$$p_n = \Pi_{j=1}^{n}(1+a_j), \quad q_n = \Pi_{j=1}^{n}(1+|a_j|) \ .$$

Then the bound

$$|p_n - 1| \leq q_n - 1$$

holds.

Proof: We have

$$\begin{aligned} p_n &= (1+a_1)\ldots(1+a_n) \\ &= 1 + \sum a_{i_1}\ldots a_{i_r} \end{aligned}$$

where the sum is taken over all indices $i_1, \ldots, i_r$ with

$$1 \leq i_1 < \ldots < i_r \leq n \ .$$

Therefore,

$$\begin{aligned} |p_n - 1| &\leq \sum |a_{i_1}\ldots a_{i_r}| \\ &= q_n - 1 \end{aligned}$$

This proves the lemma. $\diamond$

Proof of Theorem 7.2: By Theorem 7.1 the infinite product

$$\Pi_{j=1}^{\infty}\Big(1+|a_j|\Big)$$

converges. Therefore, by the Cauchy criterion, Lemma 7.1, for all $\varepsilon > 0$ there exists $J(\varepsilon)$ so that

$$\Big|\Pi_{j=m+1}^{n}(1+|a_j|)-1\Big| \le \varepsilon \quad \text{for} \quad n > m \ge J(\varepsilon) \ .$$

We have

$$\begin{aligned} \Big|\Pi_{j=m+1}^{n}(1+a_j)-1\Big| &\le \Big|\Pi_{j=m+1}^{n}(1+|a_j|)-1\Big| \\ &\le \varepsilon \end{aligned}$$

for $n > m \ge J(\varepsilon)$. By the Cauchy criterion, Lemma 7.1, the infinite product $\Pi_{j=1}^{\infty}(1+a_j)$ converges. $\diamond$

7.3 Infinite Products of Functions

Looking carefully at the proofs of the previous section, one obtains *uniformity* of convergence of a product of functions

$$\Pi(1+f_j(z))$$

if the corresponding series $\sum |f_j(z)|$ converges uniformly. We formulate this result in the following lemma.

Lemma 7.5 *Let $K \subset \mathbb{C}$ denote a compact set and let $f_j : K \to \mathbb{C}$ denote a sequence of continuous functions with*

$$\sum_{j=1}^{\infty} |f_j|_K < \infty \quad \textit{where} \quad |f_j|_K := \max_{z\in K} |f_j(z)| \ .$$

Then the sequence of functions

$$p_n(z) = \Pi_{j=1}^{n}(1+f_j(z)), \quad n = 1, 2 \ldots$$

converges uniformly on K.

Proof: Fix $z \in K$ and set

$$\begin{aligned} p_n(z) &= \Pi_{j=1}^{n}\,(1+f_j(z)) \\ q_n(z) &= \Pi_{j=1}^{n}\,(1+|f_j(z)|) \\ Q_n &= \Pi_{j=1}^{n}\,(1+|f_j|_K) \end{aligned}$$

By Theorem 7.1, the sequence of numbers Q_n converges.

We have, for $n > m \ge 1$:

$$
\begin{aligned}
|p_n(z) - p_m(z)| &= \Big|\frac{p_n(z)}{p_m(z)} - 1\Big||p_m(z)| \\
&\le |q_n(z) - q_m(z)| \\
&= |q_m(z)|\Big|\Pi_{j=m+1}^n (1 + |f_j(z)|) - 1\Big| \\
&\le Q_m\Big|\Pi_{j=m+1}^n (1 + |f_j|_K) - 1\Big| \\
&= Q_m\Big|\frac{Q_n}{Q_m} - 1\Big| \\
&= |Q_n - Q_m| \ .
\end{aligned}
$$

Therefore, if $\varepsilon > 0$ is given, there exists $N(\varepsilon)$ with

$$
|p_n(z) - p_m(z)| \le |Q_n - Q_m| < \varepsilon \quad \text{for} \quad n > m \ge N(\varepsilon) \ .
$$

Here $N(\varepsilon)$ does not depend on the point $z \in K$. This proves the lemma. $\diamond$

The following result is the workhorse for convergence of infinite products of holomorphic functions.

Theorem 7.3 *(Main Theorem on Convergence of Infinite Products of Holomorphic Functions) Assume that U is an open subset of $\mathbb{C}$ and let $f_j \in H(U)$ denote a sequence of holomorphic functions on U. Assume that the series $\sum_{j=1}^\infty |f_j|_K$ converges for every compact subset K of U. Then the following holds:*

1) For every $z \in U$ the infinite product

$$
\Pi_{j=1}^\infty \Big(1 + f_j(z)\Big) =: F(z)
$$

converges and defines a function $F \in H(U)$.

2) The sequence of holomorphic functions

$$
F_N(z) = \Pi_{j=1}^N \Big(1 + f_j(z)\Big), \quad z \in U \ ,
$$

converges locally uniformly in U to $F(z)$.

3) For every $z_0 \in U$ the function $F(z)$ has a zero at z_0 if and only if one of the factors $q_j(z) = 1 + f_j(z)$ has a zero at z_0. Furthermore, the multiplicity of z_0 as a zero of $F(z)$ is the finite sum (over j) of the multiplicities of the zero z_0 of the factors $q_j(z)$.

4) If $F(z) \ne 0$ then

$$
\frac{F'(z)}{F(z)} = \sum_{j=1}^\infty \frac{f_j'(z)}{1 + f_j(z)} \ .
$$

Here the series converges locally uniformly in the set

$$
U_0 = \{z \in U \ : \ F(z) \ne 0\} \ .
$$

Proof: 1) and 2): The convergence of the infinite product for each fixed $z \in U$ follows from Theorem 7.2. Fix $z_0 \in U$ and let $K := \bar{D}(z_0, \varepsilon) \subset U$. Let $|f_j|_K \le \frac{1}{2}$ for $j > J$. For $N > J$ write

$$
F_N(z) = \Pi_{j=1}^J (1 + f_j(z)) \cdot \Pi_{j=J+1}^N (1 + f_j(z)) =: F_J(z) \cdot G_N(z) \ . \tag{7.7}
$$

Note that the function $G_N(z)$ has no zero in K since $|f_j|_K \le \frac{1}{2}$ for $j > J$. The first finite product, $F_J(z)$, is a function in $H(U)$. The sequence $G_N(z)$ converges uniformly on K to a function $G \in C(K)$. This follows from the previous lemma. We then have $G \in H(D(z_0, \varepsilon))$. The sequence F_N converges uniformly on K to F. It follows that $F \in H(D(z_0, \varepsilon))$. Since $z_0 \in U$ was arbitrary, we have shown that $F \in H(U)$.

3) In the factorization (7.7) let $N \to \infty$ to obtain that

$$F(z) = F_J(z) \cdot G(z), \quad z \in D(z_0, \varepsilon) \; .$$

Here G is holomorphic and nowhere zero in $D(z_0, \varepsilon)$. This implies 3).

4) For a product of three holomorphic functions $q_j(z)$,

$$Q = q_1 q_2 q_3$$

we have

$$Q' = q_1' q_2 q_3 + q_1 q_2' q_3 + q_1 q_2 q_3' \; .$$

Therefore, if $Q(z) \neq 0$, then

$$\frac{Q'(z)}{Q(z)} = \sum_{j=1}^{3} \frac{q_j'(z)}{q_j(z)} \; .$$

It is clear that this generalizes to any finite product and we have if $F(z) \neq 0$:

$$\frac{F_N'(z)}{F_N(z)} = \sum_{j=1}^{N} \frac{f_j'(z)}{1 + f_j(z)} \; .$$

Letting $N \to \infty$ we obtain that

$$\frac{F'(z)}{F(z)} = \sum_{j=1}^{\infty} \frac{f_j'(z)}{1 + f_j(z)} \; .$$

It is not difficult to prove that the above series converges locally uniformly in U_0. (Homework) $\diamond$

Definition Let $\Omega \subset \mathbb{C}$ be an open set and let $f_j \in H(\Omega)$ for $j = 1, 2, \ldots$. The infinite product

$$\Pi_{j=1}^{\infty} \left(1 + f_j(z)\right)$$

converges normally on Ω if

$$\sum_{j=1}^{\infty} |f_j|_K < \infty$$

for every compact set $K \subset \Omega$.

7.4 Example: The Product Formula for the Sine Function

We recall: [2]

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2-n^2}, \quad z \in \mathbb{C}\setminus\mathbb{Z}\ . \tag{7.8}$$

We want to use this to prove the product formula

$$\frac{1}{\pi}\sin(\pi z) = z\,\Pi_{n=1}^\infty\Big(1-\frac{z^2}{n^2}\Big), \quad z\in\mathbb{C}\ . \tag{7.9}$$

Let

$$f_n(z) = -\frac{z^2}{n^2}, \quad n\in\mathbb{N}\ .$$

If K is a compact subset of $\mathbb{C}$ then $K\subset \bar{D}(0,R)$ for some $R>0$. We then have

$$|f_n|_K \le \frac{R^2}{n^2}\ .$$

Since $\sum n^{-2}<\infty$ we obtain that $\sum |f_n|_K<\infty$ and, by the previous theorem, convergence of the infinite product $\Pi_n(1+f_n(z))$ follows. The function

$$P(z) = z\,\Pi_{n=1}^\infty\Big(1-\frac{z^2}{n^2}\Big)$$

is entire and $P(z)$ has a simple zero at each integer. Clearly, $P(z)$ has no other zeros.

Also, for $z\in\mathbb{C}\setminus\mathbb{Z}$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z}+\sum_{n=1}^\infty\frac{2z}{z^2-n^2} = \pi\cot(\pi z)\ .$$

If we set

$$G(z) = \frac{1}{\pi}\sin(\pi z)$$

then we also have, from calculus,

$$\frac{G'(z)}{G(z)} = \pi\cot(\pi z), \quad z\in\mathbb{C}\setminus\mathbb{Z}\ .$$

The function P/G has a removable singularity at each integer. We have

$$\Big(\frac{P}{G}\Big)' = \frac{P'G-PG'}{G^2} = \frac{P}{G}\cdot\Big(\frac{P'}{P}-\frac{G'}{G}\Big) = 0\ .$$

Therefore, $P/G = c = const.$ Considering

$$\lim_{z\to 0}\frac{P(z)}{z} = 1 = \lim_{z\to 0}\frac{G(z)}{z}\ ,$$

we conclude that $c=1$, i.e., $P(z)\equiv G(z)$. This proves the product formula (7.9).

[2] Using residue calculus for $\int_{\gamma_n}\frac{\cot(\pi\zeta)}{(\zeta-z)^2}\,d\zeta,\ z\in\mathbb{C}\setminus\mathbb{Z}$, where γ_n is a sequence of growing rectangles, we have shown that $\sum_{j=-\infty}^\infty\frac{1}{\pi(j-z)^2} = \frac{\pi}{\sin^2(\pi z)}$. Then the formula (7.8) follows by integration.

7.5 Euler's Constant

We define here Euler's constant γ, also called the Euler–Mascheroni constant. It occurs frequently when one discusses the Γ–function. A numerical value is

$$\gamma = 0.577\,215\,66\ldots$$

It is unknown whether γ is rational or irrational.

Lemma 7.6 *For $n = 1, 2, \ldots$ let*

$$\begin{aligned}\gamma_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)\\ \delta_n &= 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n+1)\end{aligned}$$

We claim that

$$\delta_n < \delta_{n+1} < \gamma_{n+1} < \gamma_n \ .$$

The sequences γ_n and δ_n converge to the same limit γ, called Euler's constant.

Proof: We have

$$\begin{aligned}\gamma_{n+1} - \gamma_n &= \frac{1}{n+1} + \ln(n) - \ln(n+1)\\ &= \frac{1}{n+1} - \int_n^{n+1} \frac{dx}{x} \ .\end{aligned}$$

For $n < x < n+1$ we have

$$\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n} \ ,$$

thus

$$\gamma_{n+1} - \gamma_n < 0 \ .$$

Similarly,

$$\begin{aligned}\delta_{n+1} - \delta_n &= \frac{1}{n+1} + \ln(n+1) - \ln(n+2)\\ &= \frac{1}{n+1} - \int_{n+1}^{n+2} \frac{dx}{x} \ .\end{aligned}$$

For $n+1 < x < n+2$ we have

$$\frac{1}{n+2} < \frac{1}{x} < \frac{1}{n+1} \ ,$$

thus

$$\delta_{n+1} - \delta_n > 0 \ .$$

Also,

$$\gamma_n - \delta_n = \ln(n+1) - \ln(n) = \ln\left(1 + \frac{1}{n}\right) > 0 \ .$$

Since $\gamma_n - \delta_n \to 0$ as $n \to \infty$, the lemma is proved. $\diamond$

7.6 The Gauss Formula for $\Gamma(z)$ and Weierstrass' Product Formula for $1/\Gamma(z)$

Recall that

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad \text{for} \quad \mathrm{Re}\, z > 0 \ .$$

Splitting the integral as

$$\int_0^\infty = \int_0^1 \ldots + \int_1^\infty \ldots =: g(z) + h(z)$$

we obtain $h \in H(\mathbb{C})$ and use the series

$$e^{-t} = \sum_{j=0}^\infty \frac{(-1)^j}{j!}\, t^j$$

to obtain

$$\begin{aligned} g(z) &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} \int_0^1 t^{z-1+j}\, dt \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} \, \frac{1}{z+j} \ . \end{aligned}$$

In this way we obtain a formula for the Γ–function valid for

$$z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} =: U \ ;$$

the formula is

$$\Gamma(z) = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \, \frac{1}{z+j} + \int_1^\infty t^{z-1} e^{-t}\, dt, \quad z \in U \ .$$

It is clear that $\Gamma \in H(U)$. The function Γ has a simple pole at each $n \in \{0, -1, -2, \ldots\}$ and

$$\mathrm{Res}(\Gamma, -j) = \frac{(-1)^j}{j!} \quad \text{for} \quad j = 0, 1, 2, \ldots$$

We also recall the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C} \setminus \mathbb{Z} \ ,$$

which implies that Γ has no zero. Therefore, the function

$$\Delta(z) := \frac{1}{\Gamma(z)}$$

is entire. The function $\Delta(z)$ has a simple zero at each $n \in \{0, -1, -2, \ldots\}$ and has no other zeros. The zeros of $\Delta(z)$ clearly show up in the following product formula.

Theorem 7.4 *(Weierstrass) We have*

$$\Delta(z) = \frac{1}{\Gamma(z)} = e^{\gamma z} \cdot z \cdot \Pi_{j=1}^{\infty} \Big(1 + \frac{z}{j}\Big) e^{-z/j}, \quad z \in \mathbb{C} \ , \tag{7.10}$$

where γ is Euler's constant,

$$\gamma = \lim_{n \to \infty} \gamma_n = 0.577\,21\ldots$$

with

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n) \ .$$

Convergence of the infinite product in (7.10): We have

$$e^{-z/j} = 1 - \frac{z}{j} + \eta_j(z), \quad |\eta_j(z)| \le C(R) j^{-2} \quad \text{for} \quad |z| \le R \ .$$

We then have

$$\Big(1 + \frac{z}{j}\Big) e^{-z/j} = 1 - \frac{z^2}{j^2} + \tilde{\eta}_j(z)$$

with

$$|\tilde{\eta}_j(z)| \le \tilde{C}(R) j^{-2} \quad \text{for} \quad |z| \le R \ .$$

It is then clear that the infinite product in (7.10) converges uniformly for $|z| \le R$, where $R > 0$ is arbitrary. Therefore,

$$Q(z) := e^{\gamma z} \cdot z \cdot \Pi_{j=1}^{\infty} \Big(1 + \frac{z}{j}\Big) e^{-z/j} \tag{7.11}$$

is an entire function. This function has a simple zero at each $n \in \{0, -1, -2, \ldots\}$ and has no other zeros.

We want to prove that $Q(z) = \Delta(z)$. To do this, we will prove the equality for $0 < x = z \le 1$ and then apply the identity theorem.

Recall that $\phi(x) = \ln(\Gamma(x)), x > 0$, satisfies $\phi''(x) > 0$ for $x > 0$. (See Math 561.)

Lemma 7.7 *Let $\phi : (0, \infty) \to \mathbb{R}$ be a C^2 function with $\phi''(x) > 0$ for all $x > 0$. If $0 < a < b < c$ then we have*

$$\frac{\phi(b) - \phi(a)}{b - a} \le \frac{\phi(c) - \phi(b)}{c - b}$$

and

$$\frac{\phi(b) - \phi(a)}{b - a} \le \frac{\phi(c) - \phi(a)}{c - a} \ .$$

Proof: The first estimate follows from the mean value theorem. To prove the second estimate, consider the function

$$h(s) = \frac{\phi(s) - \phi(a)}{s - a} \quad \text{for} \quad s > a \ .$$

Differentiation yields

$$h'(s) = \frac{1}{s-a}\Big(\phi'(s) - \frac{\phi(s) - \phi(a)}{s-a}\Big) \ .$$

Then, using the mean value theorem, we find that $h'(s) \geq 0$ for $s > a$. $\diamond$

Application: We let $\phi(x) = \ln\Gamma(x), x > 0$. Let n be an integer, $n \geq 2$, and let $0 < x \leq 1$. We have

$$0 < n-1 < n < n+x \leq n+1$$

and obtain

$$\frac{\phi(n) - \phi(n-1)}{n-(n-1)} \leq \frac{\phi(n+x) - \phi(n)}{n+x-n} \leq \frac{\phi(n+1) - \phi(n)}{n+1-n} \ .$$

Multiply by $0 < x \leq 1$ to obtain

$$x \ln\Big(\frac{\Gamma(n)}{\Gamma(n-1)}\Big) \leq \ln\Big(\frac{\Gamma(n+x)}{\Gamma(n)}\Big) \leq \ln\Big(\frac{\Gamma(n+1)}{\Gamma(n)}\Big) \ .$$

Therefore,

$$x\ln(n-1) \leq \ln\Big(\frac{\Gamma(n+x)}{\Gamma(n)}\Big) \leq x\ln n \ ,$$

thus

$$(n-1)^x \leq \frac{\Gamma(n+x)}{\Gamma(n)} \leq n^x \ .$$

This yields that

$$(n-1)!\,(n-1)^x \leq \Gamma(n+x) \leq (n-1)!\,n^x \ .$$

Using the functional equation $\Gamma(z+1) = z\Gamma(z)$ we have

$$\Gamma(n+x) = (x+n-1)(x+n-2)\ \cdots\ (x+1)x\,\Gamma(x) \ .$$

This yields the bounds

$$\frac{(n-1)!\,(n-1)^x}{x(x+1)\ldots(x+n-1)} \leq \Gamma(x) \leq \frac{(n-1)!\,n^x}{x(x+1)\ldots(x+n-1)} \ .$$

In the lower bound, we may replace $n-1$ by n. Then we have

$$\frac{n!\,n^x}{x(x+1)\ldots(x+n)} \leq \Gamma(x) \leq \frac{n!\,n^x}{x(x+1)\ldots(x+n)} \cdot \frac{x+n}{n} \ .$$

Denote

$$A_n = \frac{n!\, n^x}{x(x+1)\ldots(x+n)} \ .$$

We have shown that

$$A_n \le \Gamma(x) \le A_n \cdot \frac{x+n}{n} \ .$$

Equivalently,

$$\frac{n}{n+x}\,\Gamma(x) \le A_n \le \Gamma(x) \ .$$

We obtain for $n \to \infty$:

Lemma 7.8 *Let $0 < x \le 1$. Then we have*

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\ldots(x+n)} \ .$$

Define, for $z \in \mathbb{C}$,

$$\begin{aligned} h_n(z) &= \frac{z(z+1)\ldots(z+n)}{n!\, n^z} \\ &= e^{-(\ln n)\, z} \cdot z \cdot \Big(1+\frac{z}{1}\Big)\Big(1+\frac{z}{2}\Big) \ \ldots \ \Big(1+\frac{z}{n}\Big) \\ &= e^{\gamma_n z} \cdot z \cdot \Big(1+\frac{z}{1}\Big)e^{-z/1}\,\Big(1+\frac{z}{2}\Big)e^{-z/2} \ \ldots \ \Big(1+\frac{z}{n}\Big)e^{-z/n} \end{aligned}$$

The above lemma implies that, for $0 < x \le 1$:

$$\Gamma(x) = \lim_{n\to\infty} \frac{1}{h_n(x)} \ . \qquad (7.12)$$

Also,

$$h_n(z) \to Q(z)$$

locally uniformly in $\mathbb{C}$ where $Q(z)$ is defined in (7.11). Therefore,

$$\frac{1}{h_n(z)} \to \frac{1}{Q(z)}$$

locally uniformly in U. By (7.12) we obtain that

$$\Gamma(x) = \frac{1}{Q(x)} \quad \text{for} \quad 0 < x \le 1 \ .$$

The identity theorem then yields that

$$\Gamma(z) = \frac{1}{Q(z)} \quad \text{for} \quad z \in U \ .$$

This proves the Weierstrass' product formula for $\Delta(z) = 1/\Gamma(z)$.

The following result is easy to show:

Lemma 7.9 *Let $K \subset \mathbb{C}$ be compact. Let $a_n, a \in C(K)$ and assume that $|a_n - a|_K \to 0$ as $n \to \infty$. If $a(z) \neq 0$ for all $z \in \mathbb{K}$ then*

$$\Big|\frac{1}{a_n} - \frac{1}{a}\Big|_K \to 0 \quad as \quad n \to \infty \ .$$

One obtains the following Gauss' formula for $\Gamma(z)$:

Theorem 7.5 *(Gauss) We have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\ \cdots\ (z+n)}$$

with locally uniform convergence in $U = \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

7.6.1 A Formula for $(\Gamma'/\Gamma)'$

Let $U = \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and set

$$f_n(z) = n!\, e^{(\ln n)z}\, \Pi_{j=0}^n (z+j)^{-1}, \quad z \in U \ .$$

By Gauss' formula we have $f_n(z) \to \Gamma(z)$ locally uniformly in U. We have

$$\frac{f_n'(z)}{f_n(z)} = \ln n - \sum_{j=0}^n (z+j)^{-1}$$

and

$$\frac{d}{dz}\, \frac{f_n'(z)}{f_n(z)} = \sum_{j=0}^n (z+j)^{-2} \ .$$

For $n \to \infty$ obtain:

$$\frac{d}{dz}\, \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{j=0}^\infty (z+j)^{-2} \ .$$

Recall that we have used the inequality $\phi''(x) > 0$ for $0 < x < \infty$ where $\phi(x) = \ln \Gamma(x)$. We now have shown that

$$\phi''(x) = \sum_{j=0}^\infty \frac{1}{(x+j)^2}, \quad 0 < x < \infty \ .$$

7.7 Entire Functions with Prescribed Zeros

If $a_1, \ldots, a_n \in \mathbb{C}$ then the polynomial

$$p(z) = (a_1 - z)(a_2 - z)\ \cdots\ (a_n - z)$$

vanishes precisely at $a_1, \ldots, a_n$. Now let $a_1, a_2, \ldots$ denote an infinite sequence in $\mathbb{C}$ without accumulation point in $\mathbb{C}$, i.e., $|a_n| \to \infty$. We want to construct an entire function $f(z)$ with $f(z) = 0$ if and only if $z \in \{a_1, a_2, \ldots\}$. We allow repetitions in the a_j. Then, if a number a appears q times among the a_j, we want $f(z)$ to have a zero precisely of order q at a. We will show below how to construct such a function $f(z)$.

Remarks on Non–Uniqueness. If $f(z)$ vanishes precisely at the a_j (to the correct order) then any function

$$g(z) = f(z)e^{h(z)}, \quad h \in H(\mathbb{C}) \ ,$$

has the same property. We show:

Lemma 7.10 *Let a_j be a sequence in $\mathbb{C}$ without accumulation point in $\mathbb{C}$. Let $f(z)$ and $g(z)$ be entire functions that vanish precisely at the a_j. If a appears q times in the sequence, then let $f(z)$ and $g(z)$ have a zero of order q at a. Under these assumptions, there exists $h \in H(\mathbb{C})$ with*

$$g(z) = f(z)e^{h(z)}, \quad z \in \mathbb{C} \ .$$

Proof: The quotient

$$Q(z) = \frac{f(z)}{g(z)}, \quad z \in \mathbb{C} \setminus \{a_1, a_2, \ldots\} \ ,$$

is bounded near every a_j. By Riemann's removability theorem, $Q(z)$ extends to an entire function. Also, $Q(z) \neq 0$ for all z. Therefore, $Q(z)$ has the form $Q(z) = e^{h(z)}$ where $h(z)$ is entire. See Lemma 4.1. $\diamond$

7.7.1 Construction of $f(z)$; Motivation

One might try to construct $f(z)$ as the infinite product

$$(a_1 - z)(a_2 - z) \ \cdots$$

However, since $|a_j| \to \infty$, this product never converges.

Somewhat smarter is the following: Let us assume that

$$a_1 = a_2 = \ldots = a_m = 0 \quad \text{and} \quad a_j \neq 0 \quad \text{for} \quad j > m \ .$$

Then try the infinite product

$$z^m \Big(1 - \frac{z}{a_{m+1}}\Big)\Big(1 - \frac{z}{a_{m+2}}\Big) \cdots \tag{7.13}$$

This works if

$$\sum_{j>m} \frac{1}{|a_j|} < \infty \ .$$

Under this assumption the above infinite product defines an entire function with the desired property.

For example, it suffices that

$$|a_j| \geq c\, j^{1+\varepsilon}, \quad j \geq J \ ,$$

for some $c > 0$ and $\varepsilon > 0$. However, if

$$|a_j| \sim j \quad \text{or} \quad |a_j| \sim \sqrt{j} \ ,$$

then the infinite product (7.13) will typically not converge since $\sum |z|/|a_j|$ diverges if $z \neq 0$.

Weierstrass' idea was to use extra factors in the infinite product (7.13), which ensure convergence, but do not introduce additional zeros.

7.7.2 Weierstrass' Canonical Factors

Let us assume $a_j \neq 0$ for all j, for simplicity. Instead of the infinite product

$$\Big(1 - \frac{z}{a_1}\Big)\Big(1 - \frac{z}{a_2}\Big) \ \cdots \tag{7.14}$$

try

$$E\Big(\frac{z}{a_1}\Big)\, E\Big(\frac{z}{a_2}\Big) \ \cdots \tag{7.15}$$

where $E(z)$ is an entire function of the form

$$E(z) = (1 - z)\phi(z) \ .$$

Here $\phi(z)$ should be suitably chosen, with $\phi(z) \neq 0$ for all z. Then one obtains

$$E\Big(\frac{z}{a_1}\Big)\, E\Big(\frac{z}{a_2}\Big) \ \cdots = \Big(1 - \frac{z}{a_1}\Big)\phi\Big(\frac{z}{a_1}\Big)\Big(1 - \frac{z}{a_2}\Big)\phi\Big(\frac{z}{a_2}\Big) \ \cdots \tag{7.16}$$

The factors $\phi(z/a_j)$ should make the product converge without introducing new zeros. Note that, as $|a_j| \to \infty$, the argument z/a_j of ϕ converges to zero. Therefore, we want to construct $\phi(z)$ with

$$(1 - z)\phi(z) \sim 1 \quad \text{for} \quad |z| < \varepsilon$$

to enhance convergence of the product. However, if we would require $(1 - z)\phi(z) = 1$, then $\phi(z)$ would become singular at $z = 1$, and we do not obtain an entire function.

We want to construct $\phi(z)$ so that

a) $(1 - z)\phi(z) = 1 + \mathcal{O}(z^{k+1})$ for $z \sim 0$;

b) ϕ is entire.

c) $\phi(z) \neq 0$ for all $z \in \mathbb{C}$.

To ensure that $\phi(z)$ never vanishes, let us construct $\phi(z)$ in the form

$$\phi(z) = e^{h(z)} \ .$$

We then have the requirement, for small $|z|$,

$$e^{h(z)} = \phi(z) = \frac{1}{1 - z} + \mathcal{O}(z^{k+1}) \ ,$$

thus,

$$h(z) = \log\Big(\frac{1}{1 - z} + \mathcal{O}(z^{k+1})\Big) = -\log(1 - z) + \mathcal{O}(z^{k+1}) \ .$$

The second equation holds since $\log\left(\frac{1}{1-z} + \varepsilon\right) = -\log(1-z) + \mathcal{O}(\varepsilon)$ for $z \sim 0$. Set $r(z) = -\log(1-z)$. Then we have $r(0) = 0$ and

$$r'(z) = \frac{1}{1-z} = 1 + z + z^2 + \ldots$$

Therefore,

$$r(z) = -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots \; .$$

This leads to the following requirement for $h(z)$:

$$h(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \mathcal{O}(z^{k+1}) \; .$$

Therefore, we set

$$h_k(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^k}{k} \; .$$

This leads to the following definition of the Weierstrass' canonical factors.

Definition: For $k = 0, 1, \ldots$ define $E_k(z)$ by

$$E_0(z) = 1 - z, \quad E_k(z) = (1-z)\, \exp\Big(z + \frac{z^2}{2} + \ldots + \frac{z^k}{k}\Big) \quad \text{for} \quad k \geq 1 \; .$$

Clearly, $E_k(z)$ vanishes only at $z = 1$ and has a simple zero at $z = 1$. Therefore, the function $z \to E_k(z/a_j)$ vanishes to first order at $z = a_j$ and vanishes nowhere else. We will show now that the function $1 - E_k(z)$ vanishes to order $k+1$ at $z = 0$. In this sense, $E_k(z) \sim 1$ to order $k+1$ at $z = 0$.

Lemma 7.11 *There is a constant $C > 0$, independent of k and z, with*

$$|1 - E_k(z)| \leq C|z|^{k+1} \quad \textit{for} \quad |z| \leq \frac{1}{2} \; .$$

One can choose $C = 2(e-1)$.

Proof: In the following let $|z| \leq \frac{1}{2}$. Let

$$h_k(z) = z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \; .$$

We have

$$\begin{aligned} E_k(z) &= (1-z)\exp(h_k(z)) \\ &= \exp\Big(\log(1-z) + h_k(z)\Big) \\ &= e^w \end{aligned}$$

with

$$w = \log(1-z) + h_k(z) \; .$$

Since

$$\log(1-z) = -\sum_{j=1}^\infty \frac{z^j}{j}$$

we have

$$w = -\sum_{j=k+1}^\infty \frac{z^j}{j} \ .$$

Therefore,

$$\begin{aligned}
|w| &\leq |z|^{k+1} \sum_{j=k+1}^\infty \frac{1}{j}\, |z|^{j-k-1} \\
&\leq |z|^{k+1} \Big(1 + \frac{1}{2} + \frac{1}{4} + \ldots \Big) \\
&= 2|z|^{k+1} \ .
\end{aligned}$$

In particular, $|w| \leq 1$ for $|z| \leq \frac{1}{2}$. Thus,

$$\begin{aligned}
|1 - E_k(z)| &= |1 - e^w| \\
&= \Big| w + w^2/2! + w^3/3! + \ldots \Big| \\
&\leq |w| \Big(1 + 1/2! + 1/3! + \ldots \Big) \\
&= |w|(e-1) \\
&\leq 2(e-1)|z|^{k+1}
\end{aligned}$$

$\diamond$

Remark: It will be useful below that the constant C in the previous lemma does not depend on k. This is not completely obvious from the construction of $E_k(z)$, which only yields that $1 - E_k(z) = \mathcal{O}(z^{k+1})$.

Theorem 7.6 *Let a_j denote a sequence of complex numbers with $|a_j| \to \infty$ as $j \to \infty$. Assume that*

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}| \leq \ldots$$

Then the infinite product

$$f(z) = z^m \, \Pi_{j=m+1}^\infty E_j(z/a_j), \quad z \in \mathbb{C} \ ,$$

defines an entire function $f(z)$ which has zeros precisely at the a_j. The multiplicity of a zero a_n of $f(z)$ equals the number of occurrences of a_n in the sequence a_j.

Proof: Assume $m = 0$. Fix $R > 0$ and note that $|a_j| \geq 2R$ for $j \geq j_0(R)$. If $|z| \leq R$ then

$$\Big| \frac{z}{a_j} \Big| \leq \frac{1}{2} \quad \text{for} \quad j \geq j_0(R) \ .$$

Therefore, by Lemma 7.11,

$$|1 - E_j(z/a_j)| \le C \frac{|z|^{j+1}}{|a_j|^{j+1}} \le C\, 2^{-(j+1)} \ .$$

If one writes

$$E_j(z/a_j) = 1 + f_j(z)$$

then

$$\sum_{j=1}^{\infty} |f_j|_{\infty, \bar{D}(0,R)} < \infty \ .$$

The claim now follows from Theorem 7.3. $\diamond$

In the previous theorem, the index j of E_j increases unboundedly. It is of interest to give a condition under which this index can be chosen constant.

Theorem 7.7 *Let a_j denote a sequence of complex numbers with $|a_j| \to \infty$ as $j \to \infty$. Assume that*

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \le |a_{m+2}| \le \ldots$$

Also, assume that

$$\sum_{j=m+1}^{\infty} \frac{1}{|a_j|^s} < \infty$$

for some $s > 0$. If k is an integer with $s \le k+1$ then the infinite product

$$f(z) = z^m \, \Pi_{j=m+1}^{\infty} E_k(z/a_j), \quad z \in \mathbb{C} \ ,$$

defines an entire function $f(z)$ which has zeros precisely at the a_j. The multiplicity of a zero a of $f(z)$ equals the number of occurances of a in the sequence a_j.

Proof: Assume $m = 0$. Fix $R > 0$ and note that $|a_j| \ge 2R$ for $j \ge j_0(R)$. If $|z| \le R$ then

$$\left| \frac{z}{a_j} \right| \le \frac{1}{2} \quad \text{for} \quad j \ge j_0(R) \ .$$

Therefore, by Lemma 7.11,

$$|1 - E_k(z/a_j)| \le C \frac{|z|^{k+1}}{|a_j|^{k+1}} \le C(R,k) \frac{1}{|a_j|^{k+1}} \ .$$

The claim now follows from

$$\sum_{j=m+1}^{\infty} \frac{1}{|a_j|^{k+1}} < \infty$$

and Theorem 7.3. $\diamond$

7.8 Entire Functions with Prescribed Values

Let $a_j \in \mathbb{C}$ denote a sequence with $a_j \neq a_k$ for $j \neq k$ and assume that $|a_j| \to \infty$ as $j \to \infty$. Let $b_j \in \mathbb{C}$. We claim that there exists an entire function $f(z)$ with

$$f(a_j) = b_j \quad \text{for} \quad j = 1, 2, \ldots$$

Using Weierstrass Theorem, there exists an entire function $g(z)$ with

$$g(a_j) = 0 \quad \text{and} \quad g'(z_j) \neq 0 \quad \text{for} \quad j = 1, 2, \ldots$$

We apply Mittag–Leffler's Theorem where $P_j(w)$ is the first degree polynomial

$$P_j(w) = \alpha_j w \; .$$

The constant $\alpha_j \in \mathbb{C}$ will be determined. Let $\mathcal{D} = \{a_1, a_2, \ldots\}$. By Mittag–Leffler there exists $h \in H(\mathbb{C} \setminus \mathcal{D})$ with

$$h(z) = \frac{\alpha_j}{z - a_j} + \tilde{h}(z)$$

where $\tilde{h}(z)$ is holomorphic at $z = a_j$. Set $f(z) = g(z)h(z)$. For $z \sim a_j$ obtain that

$$\begin{aligned} f(z) &= \Big(g'(a_j)(z - a_j) + \mathcal{O}((z - a_j)^2)\Big)\Big(\frac{\alpha_j}{z - a_j} + \mathcal{O}(1)\Big) \\ &= g'(a_j)\alpha_j + \mathcal{O}(z - a_j) \end{aligned}$$

If we choose

$$\alpha_j = \frac{b_j}{g'(a_j)}$$

then we have $f(a_j) = b_j$.

8 The Bernoulli Numbers and Applications

8.1 The Bernoulli Numbers

The function $g(z)$ defined by

$$g(z) = z/(e^z - 1) \quad \text{for} \quad 0 < |z| < 2\pi, \quad g(0) = 1 \ ,$$

is holomorphic in $D(0, 2\pi)$. We write its Taylor series as

$$g(z) = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu, \quad |z| < 2\pi \ , \tag{8.1}$$

where the numbers B_ν are, by definition, the Bernoulli numbers. Since

$$\begin{aligned} g(z) &= \frac{1}{1 + \frac{1}{2}z + \frac{1}{6}z^2 + \ldots} \\ &= 1 - \frac{1}{2}z + \ldots \end{aligned}$$

it follows that

$$B_0 = 1, \quad B_1 = -\frac{1}{2} \ .$$

Lemma 8.1 *The function*

$$h(z) = g(z) + \frac{z}{2}$$

is even. Consequently,

$$B_\nu = 0 \quad \textit{for} \quad \nu \geq 3, \quad \nu \quad \textit{odd} \ .$$

Proof: We must show that

$$g(-z) - \frac{z}{2} = g(z) + \frac{z}{2} \ ,$$

i.e.,

$$g(-z) - g(z) = z \ .$$

With $a = e^z$ we have

$$\begin{aligned} g(-z) - g(z) &= \frac{-z}{e^{-z} - 1} - \frac{z}{e^z - 1} \\ &= z\Big(\frac{-1}{\frac{1}{a} - 1} - \frac{1}{a - 1}\Big) \end{aligned}$$

and

$$\frac{-a}{1 - a} - \frac{1}{a - 1} = \frac{1 - a}{1 - a} = 1 \ .$$

$\diamond$

One can compute the Bernoulli numbers easily using a recursion. We claim:

Lemma 8.2 *For $n \geq 1$ we have*

$$\sum_{\nu=0}^{n} \binom{n+1}{\nu} B_\nu = 0 \ .$$

Proof: We have, for $0 < |z| < 2\pi$:

$$\begin{aligned} 1 &= \frac{e^z - 1}{z} \cdot \frac{z}{e^z - 1} \\ &= \Big(\sum_{\mu=0}^{\infty} \frac{z^\mu}{(\mu+1)!}\Big) \cdot \Big(\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu\Big) \\ &= \sum_{\mu.\nu=0}^{\infty} \frac{B_\nu}{\nu!(\mu+1)!} z^{\mu+\nu} \\ &= \sum_{n=0}^{\infty} \Big(\sum_{\nu=0}^{n} \frac{B_\nu}{\nu!(n+1-\nu)!}\Big) z^n \end{aligned}$$

Since

$$\binom{n+1}{\nu} = \frac{(n+1)!}{\nu!(n+1-\nu)!}$$

the lemma is proved. $\diamond$

Using Pascal's triangle, we can compute the binomial coefficients. Then, using the previous lemma and $B_0 = 1$ we obtain:

For $n = 1$:

$$B_0 + 2B_1 = 0, \quad \text{thus} \quad B_1 = -\frac{1}{2} \ .$$

For $n = 2$:

$$B_0 + 3B_1 + 3B_2 = 0, \quad \text{thus} \quad B_2 = \frac{1}{6} \ .$$

For $n = 3$:

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \quad \text{thus} \quad B_3 = 0 \ .$$

For $n = 4$:

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0, \quad \text{thus} \quad B_4 = -\frac{1}{30} \ .$$

Continuing this process, one obtains the following non–zero Bernoulli numbers:

$$\begin{aligned} B_6 &= \frac{1}{42} \\ B_8 &= -\frac{1}{30} \\ B_{10} &= \frac{5}{66} \\ B_{12} &= -\frac{691}{2730} \\ B_{14} &= \frac{7}{6} \end{aligned}$$

etc.
Remark: The sequence $|B_{2\nu}|$ is unbounded since otherwise the series (8.1) would have a finite radius of convergence. We will see below that $(-1)^{\nu+1}B_{2\nu} > 0$. Thus, the sign pattern observed for B_2 to B_{14} continuous.

8.2 The Taylor Series for $z \cot z$ in Terms of Bernoulli Numbers

Recall that

$$g(w) = \frac{w}{e^w - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} w^\nu .$$

We now express the Taylor series for $z \cot z$ about $z = 0$ in terms of Bernoulli numbers. Note that

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\
\cot z &= i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \\
&= i\,\frac{1 + e^{-2iz}}{1 - e^{-2iz}} \\
&= i\,\frac{1 - e^{-2iz} + 2e^{-2iz}}{1 - e^{-2iz}} \\
&= i\left(1 + \frac{2}{e^{2iz} - 1}\right)
\end{aligned}$$

Therefore,

$$\cot z = i + \frac{1}{z} \cdot \frac{2iz}{e^{2iz} - 1} ,$$

thus

$$\begin{aligned}
z \cot z &= iz + g(2iz) \\
&= iz + 1 - \frac{1}{2}(2iz) + \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!} (2iz)^\nu \\
&= 1 + \sum_{\nu=1}^{\infty} (-1)^\nu \frac{4^\nu}{(2\nu)!} B_{2\nu}\, z^{2\nu} .
\end{aligned}$$

We substitute πz for z and summarize this:

Lemma 8.3 *If B_ν denotes the sequence of the Bernoulli numbers, then we have for $|z| < 1$:*

$$\pi z \cot(\pi z) = 1 + \sum_{\nu=1}^{\infty} (-1)^\nu \frac{(2\pi)^{2\nu}}{(2\nu)!} B_{2\nu}\, z^{2\nu} . \qquad (8.2)$$

8.3 The Mittag–Leffler Expansion of $\pi z \cot(\pi z)$

One can derive the following Mittag–Leffler expansion (see, for example, [Hahn, Epstein, p. 214]). One can also use residue calculus; see Math 561. Note that the function on the left side and the function on the right side of the following formula both have a simple pole at each integer.

$$\pi \cot(\pi z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} \ .$$

Therefore,

$$\pi z \cot(\pi z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 - z^2} \ .$$

Here, for $|z| < 1$:

$$\begin{aligned} \frac{z^2}{n^2 - z^2} &= \frac{(z/n)^2}{1 - (z/n)^2} \\ &= \sum_{m=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} \end{aligned}$$

Therefore,

$$\begin{aligned} \pi z \cot(\pi z) &= 1 - 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} && (8.3) \\ &= 1 - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(\frac{z}{n}\Big)^{2m} && (8.4) \\ &= 1 - 2 \sum_{m=1}^{\infty} \zeta(2m)\, z^{2m} && (8.5) \end{aligned}$$

8.4 The Values of $\zeta(2m)$

Comparing the expressions (8.5) and (8.2), we obtain the following result about the values of the Riemann ζ–function at even integers. This result was already known to Euler in 1734.

Theorem 8.1 *For $m = 1, 2, \ldots$:*

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{2}\, (-1)^{m+1}\, \frac{(2\pi)^{2m}}{(2m)!}\, B_{2m} \ . \qquad (8.6)$$

Remark: Since, clearly, $\zeta(2m) > 0$ we obtain that $(-1)^{m+1} B_{2m} > 0$.

Examples:

For $m = 1$ we have $B_2 = \frac{1}{6}$, thus

$$\zeta(2) = \frac{(2\pi)^2}{2 \cdot 2} \cdot \frac{1}{6} = \frac{\pi^2}{6} \ .$$

For $m = 2$ we have $B_4 = -\frac{1}{30}$, thus

$$\zeta(4) = \frac{(2\pi)^4}{2 \cdot 4!} \cdot \frac{1}{30} = \frac{\pi^4}{90} \ .$$

For $m = 3$ we have $B_6 = \frac{1}{42}$, thus

$$\zeta(6) = \frac{(2\pi)^6}{2 \cdot 6!} \cdot \frac{1}{42} = \frac{\pi^6}{945} \ .$$

For $m = 4$ one obtains

$$\zeta(8) = \frac{\pi^8}{9450} \ .$$

Remark: According to [Temme, p. 6], no nice formula for $\zeta(2m+1)$ seems to be known if m is an integer.

8.5 Sums of Powers and Bernoulli Numbers

It is not difficult to show the following formulae by induction in n:

$$\begin{aligned} \sum_{j=1}^{n-1} j &= \frac{1}{2}\, n^2 - \frac{1}{2}\, n \\ \sum_{j=1}^{n-1} j^2 &= \frac{1}{3}\, n^3 - \frac{1}{2}\, n^2 + \frac{1}{6}\, n \\ \sum_{j=1}^{n-1} j^3 &= \frac{1}{4}\, n^4 - \frac{1}{2}\, n^3 + \frac{1}{4}\, n^2 \end{aligned}$$

The formulae follow the pattern:

$$\sum_{j=1}^{n-1} j^k = \frac{1}{k+1}\, n^{k+1} - \frac{1}{2}\, n^k + \ldots + B_k\, n \ ,$$

but it is not obvious how the general formula should read.

Define the sum

$$S_k(n-1) = \sum_{j=0}^{n-1} j^k$$

where $k = 0, 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$. We claim that, for every fixed integer $k \ge 0$, the sum $S_k(n-1)$ is a polynomial

$$\Phi_k(n)$$

of degree $k+1$ in the variable n and that the coefficients of $\Phi_k(n)$ can be obtained in terms of Bernoulli numbers. Precisely:

Theorem 8.2 *For every integer $k \geq 0$, let Φ_k denote the polynomial*

$$\Phi_k(n) = \frac{1}{k+1} \sum_{\mu=0}^{k} \binom{k+1}{\mu} B_\mu n^{k+1-\mu} .$$

Then we have

$$S_k(n-1) = \Phi_k(n) \quad \textit{for all} \quad n = 1, 2, \cdots .$$

Remark: Writing out a few terms of $\Phi_k(n)$, the theorem says that

$$S_k(n-1) = \frac{1}{k+1} n^{k+1} - \frac{1}{2} n^k + \frac{1}{k+1} \binom{k+1}{2} B_2 n^{k-1} + \cdots + B_k\, n .$$

Proof of Theorem: The trick is to write the finite geometric sum

$$E_n(w) = 1 + e^w + e^{2w} + \cdots + e^{(n-1)w}$$

in two ways and then to compare coefficients. We have

$$\begin{aligned} E_n(w) &= \sum_{j=0}^{n-1} e^{jw} \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{j^k}{k!} w^k \\ &= \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{n-1} j^k \Big) \frac{1}{k!} w^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} S_k(n-1)\, w^k \end{aligned}$$

(Here we have used the convention $0^0 = 1$.)

On the other hand, we have

$$\begin{aligned} E_n(w) &= \frac{e^{nw} - 1}{e^w - 1} \\ &= \frac{w}{e^w - 1} \cdot \frac{e^{nw} - 1}{w} \\ &= \Big(\sum_{\mu=0}^{\infty} \frac{B_\mu}{\mu!} w^\mu \Big) \cdot \Big(\sum_{\lambda=0}^{\infty} \frac{n^{\lambda+1}}{(\lambda+1)!} w^\lambda \Big) \\ &= \sum_{k=0}^{\infty} \Big(\sum_{\mu+\lambda=k} \frac{B_\mu}{\mu!(\lambda+1)!} n^{\lambda+1} \Big) w^k \end{aligned}$$

Comparison yields that

$$
\begin{aligned}
S_k(n-1) &= \sum_{\mu+\lambda=k} \frac{k!}{\mu!(\lambda+1)!}\, B_\mu\, n^{\lambda+1} \\
&= \frac{1}{k+1} \sum_{\mu=0}^{k} \frac{(k+1)!}{\mu!(k+1-\mu)!}\, B_\mu\, n^{k+1-\mu}
\end{aligned}
$$

This proves the claim since

$$
\left(\begin{array}{c} k+1 \\ \mu \end{array} \right) = \frac{(k+1)!}{\mu!(k+1-\mu)!} \; .
$$

$\diamond$

9 The Riemann Zeta–Function

When discussing Riemann's Zeta–Function, it is standard to use Riemann's notation and use the variable $s = \sigma + it$ instead of $z = x + iy$.

9.1 Definition for $\mathrm{Re}\, s > 1$

Let $s = \sigma + it$. If $\sigma > 1$ then the formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{9.1}$$

defines $\zeta(s)$. For every $n = 1, 2, \ldots$ the function

$$s \to n^{-s} = e^{-s \ln n}$$

is entire. Therefore, for $N = 1, 2, \ldots$, the finite sum

$$S_N(s) = \sum_{n=1}^{N} n^{-s}$$

is an entire function. The sequence $S_N(s)$ converges uniformly for

$$\mathrm{Re}\, s \geq 1 + \delta > 1 \ .$$

Thus, (9.1) defines a holomorphic function in the half–plane

$$U_1 = \{s \ : \ \mathrm{Re}\, s > 1\} \ .$$

9.2 Simple Bounds of $\zeta(s)$ for $s > 1$

Let n be a positive integer and let $s > 1$. Then we have

$$(n+1)^{-s} \leq x^{-s} \leq n^{-s} \quad \text{for} \quad n \leq x \leq n+1 \ .$$

Therefore,

$$\sum_{n=1}^{\infty} (n+1)^{-s} \leq \int_1^{\infty} \frac{dx}{x^s} \leq \sum_{n=1}^{\infty} n^{-s} \ .$$

This says that

$$\zeta(s) - 1 \leq \frac{1}{s-1} \leq \zeta(s) \quad \text{for} \quad s > 1 \ .$$

In other words,

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1} \quad \text{for} \quad s > 1 \ . \tag{9.2}$$

The lower bound $\zeta(s) > 1$ for $s > 1$ is also obvious.

9.3 Meromorphic Continuation of $\zeta(s)$ to $\mathrm{Re}\, s > 0$

Let

$$U_0 = \{s \ : \ \mathrm{Re}\, s > 0\} \ .$$

We will show that $\zeta(s)$ can be continued as a holomorphic function defined in $U_0 \setminus \{1\}$. Also, $\zeta(s)$ has a simple pole at $s = 1$ with residue equal to 1. It turns out that $\zeta(s) - 1/(s-1)$ is in fact an entire function, but we will not show this here since our analytic continuation is only valid for $\mathrm{Re}\, s > 0$.

Notation: For every $x \in \mathbb{R}$ there exists a unique integer n with $n \le x < n+1$. We then write

$$\{x\} = x - n, \quad x = n + \{x\} = [x] + \{x\}$$

and call $\{x\}$ the fractional part and $[x] = n$ the integer part of x.

We use the function $[x]$ to derive an integral representation for $\zeta(s)$. The following holds for all $s \in \mathbb{C}$:

$$\begin{aligned}
\sum_{n=1}^{N} n\Big(n^{-s} - (n+1)^{-s}\Big) &= 1 - 2^{-s} + 2\Big(2^{-s} - 3^{-s}\Big) + \ldots + N\Big(N^{-s} - (N+1)^{-s}\Big) \\
&= 1 + 2^{-s} + 3^{-s} + \ldots + N^{-s} \ - \ N(N+1)^{-s}
\end{aligned}$$

Here, for all $s \in \mathbb{C}$:

$$\begin{aligned}
n^{-s} - (n+1)^{-s} &= -x^{-s}\Big|_{x=n}^{x=n+1} \\
&= s \int_n^{n+1} x^{-s-1}\, dx
\end{aligned}$$

Multiply by n to obtain that

$$n\Big(n^{-s} - (n+1)^{-s}\Big) = s \int_n^{n+1} [x] x^{-s-1}\, dx \ .$$

Sum over n from $n = 1$ to $n = N$ to obtain:

$$\begin{aligned}
s \sum_{n=1}^{N} \int_n^{n+1} [x] x^{-s-1}\, dx &= \sum_{n=1}^{N} n\Big(n^{-s} - (n+1)^{-s}\Big) \\
&= \sum_{n=1}^{N} n^{-s} - N(N+1)^{-s}
\end{aligned}$$

Thus we have for all $s \in \mathbb{C}$ and all $N = 1, 2, \ldots$:

$$s \int_1^{N+1} [x] x^{-s-1}\, dx = \sum_{n=1}^{N} n^{-s} - N(N+1)^{-s} \ .$$

In this equation we now assume $\operatorname{Re} s > 1$ and let $N \to \infty$. This yields the following integral representation of the Zeta–function

$$s \int_1^\infty [x] x^{-s-1}\, dx = \zeta(s) \quad \text{for} \quad \operatorname{Re} s > 1 \; .$$

If we replace the integer part of x, denoted by $[x]$, by x itself, then we can evaluate the integral. For $\operatorname{Re} s > 0$:

$$s \int_1^\infty x \cdot x^{-s-1}\, dx = s \int_1^\infty x^{-s}\, dx = \frac{s}{s-1} \; .$$

Taking the difference between the last two equations and noting that $x - [x] = \{x\}$ we have

$$s \int_1^\infty (x - [x]) x^{-s-1}\, dx = \frac{s}{s-1} - \zeta(s) \; ,$$

thus

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1}\, dx \quad \text{for} \quad \operatorname{Re} s > 1 \; . \tag{9.3}$$

Here the function $s/(s-1)$ is holomorphic in $\mathbb{C} \setminus \{1\}$. In (9.3), we have identified the singular behavior of $\zeta(s)$ at $s = 1$.

Since $\{x\}$ is bounded, the integral

$$H(s) := \int_1^\infty \{x\} x^{-s-1}\, dx$$

defines a holomorphic function of s for $\operatorname{Re} s > 0$. We obtain that

$$\frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1}\, dx \tag{9.4}$$

is holomorphic in $U_0 \setminus \{1\}$ and, by (9.3), agrees with $\zeta(s)$ for $\operatorname{Re} s > 1$. Thus, the holomorphic extension of $\zeta(s)$ to $U_0 \setminus \{1\}$ exists and is given by

$$\begin{aligned} \zeta(s) &= \frac{s}{s-1} - s \int_1^\infty \{x\} x^{-s-1}\, dx \\ &= \frac{1}{s-1} + 1 - s \int_1^\infty \{x\} x^{-s-1}\, dx \\ &= \frac{1}{s-1} + F(s) \end{aligned}$$

with

$$F(s) = 1 - s \int_1^\infty \{x\} x^{-s-1}\, dx \; . \tag{9.5}$$

We have shown:

Lemma 9.1 *The function $\zeta(s)$ is meromorphic in $U_0 = \{s \; : \; \operatorname{Re} s > 0\}$. It has a simple pole at $s = 1$ with residue equal to 1. It has no other poles in U_0.*

For $\text{Re}\, s > 0, s \neq 1$, we have the Laurent expansion centered at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} a_j (s-1)^j$$

where

$$\sum_{j=0}^{\infty} a_j (s-1)^j = F(s) \quad \text{for} \quad \text{Re}\, s > 0 \; .$$

The first term of the series is

$$a_0 = F(1) \; .$$

Therefore the value $F(1)$ is of some interest.

Lemma 9.2 *Let $F(s)$ be defined in (9.5). Then we have*

$$F(1) = \gamma$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{j=1}^n \frac{1}{j} - \ln n \right)$ is Euler's constant.[3]

Proof: Below we will use that

$$\begin{aligned} n \int_n^{n+1} \frac{dx}{x^2} &= -n \cdot \frac{1}{x}\Big|_{x=n}^{x=n+1} \\ &= -\frac{n}{n+1} + 1 \\ &= \frac{1}{n+1} \end{aligned}$$

We have $F(1) = 1 - \lim_{N\to\infty} J_N$ with

$$\begin{aligned} J_N &= \int_1^{N+1} \{x\} x^{-2}\, dx \\ &= \sum_{n=1}^{N} \int_n^{n+1} (x-n) x^{-2}\, dx \\ &= \int_1^{N+1} \frac{dx}{x} - \sum_{n=1}^{N} n \int_n^{n+1} \frac{dx}{x^2} \\ &= \ln(N+1) - \sum_{n=1}^{N} \frac{1}{n+1} \\ &= \ln(N+1) - \sum_{n=2}^{N+1} \frac{1}{n} \\ &= \ln(N+1) - \sum_{n=1}^{N+1} \frac{1}{n} + 1 \end{aligned}$$

[3]It is not known if γ is rational or irrational.

Therefore,

$$F(1) = \lim_{N\to\infty} \Big(\sum_{n=1}^{N+1} \frac{1}{n} - \ln(N+1) \Big) = \gamma \ .$$

◇

This proves for the Laurent expansion of $\zeta(s)$ centered at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{j=1}^{\infty} a_j (s-1)^j \quad \text{for} \quad 0 < |s-1| < 1 \ .$$

9.3.1 A Second Proof of Meromorphic Continuation

For $s \in \mathbb{C}$ set

$$\delta_n(s) = \int_n^{n+1} (n^{-s} - x^{-s})\, ds = n^{-s} - \int_n^{n+1} x^{-s}\, dx \ .$$

If $\mathrm{Re}\, s > 1$ we can sum and obtain

$$\sum_{n=1}^{\infty} \delta_n(s) = \zeta(s) - \int_1^{\infty} x^{-s} = \zeta(s) - \frac{1}{s-1} \ ,$$

thus

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \delta_n(s) \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

We claim that

$$H(s) := \sum_{n=1}^{\infty} \delta_n(s)$$

is holomorphic for $\mathrm{Re}\, s > 0$.

Let $f(x) = x^s, f'(x) = -sx^{-s-1}$. We have

$$\delta_n(s) = \int_n^{n+1} \Big(f(n) - f(x) \Big)\, dx \ .$$

Let $s = \sigma + it, \sigma > 0$. The lemma below gives us the estimate

$$|f(n) - f(x)| \leq \max_{n \leq q \leq x} |f'(q)||x-n| \ ,$$

thus

$$\begin{aligned} |\delta_n(s)| &\leq \max_{n\leq q\leq n+1} |f'(q)| \\ &= |s| \max_{n\leq q\leq n+1} |q^{-\sigma-1}| \\ &\leq \frac{|s|}{n^{\sigma+1}} \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+1}} = \zeta(\sigma+1) < \infty \quad \text{for} \quad \sigma > 0$$

one obtains that $H(s)$ is holomorphic for $\mathrm{Re}\, s > 0$.

Lemma 9.3 *Let $U \subset \mathbb{C}$ be open. Let $f \in H(U)$ and let*

$$\Gamma := \{z(t) = a + t(b-a) \ : \ 0 \le t \le 1\} \subset U \ .$$

Then we have the estimate

$$|f(b) - f(a)| \le \max_{q \in \Gamma} |f'(q)| \ |b-a| \ .$$

Proof: For $0 \le t \le 1$ let

$$\begin{aligned} \phi(t) &= f(a + t(b-a)) \\ \phi'(t) &= f'(a + t(b-a))(b-a) \end{aligned}$$

We have

$$f(b) - f(a) = \phi(1) - \phi(0) = \int_0^1 \phi'(t)\, dt$$

and the estimate follows. $\diamond$

9.4 Analytic Continuation of $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$; the Trivial Zeros of $\zeta(s)$

We claim that

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

This follows from the geometric sum formula

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx} \quad \text{for} \quad x > 0$$

and

$$\begin{aligned} \Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^\infty e^{-t} t^{s-1}\, dt \quad (\text{let } t = nx,\, dt = n dx) \\ &= \sum_{n=1}^{\infty} \int_0^\infty e^{-nx} x^{s-1}\, dx \\ &= \int_0^\infty \sum_{n=1}^{\infty} e^{-nx} x^{s-1}\, dx \\ &= \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \end{aligned}$$

For $\operatorname{Re} s > 1$ we can write the equation as follows:

$$\Gamma(s)\zeta(s) = \int_0^1 \frac{x^{s-1}}{e^x - 1}\, dx + \int_1^\infty \frac{x^{s-1}}{e^x - 1}\, dx =: I_1(s) + I_2(s) \ .$$

Here the second integral, $I_2(s)$, defines an entire function of s. To discuss the first integral, recall that

$$\frac{x}{e^x - 1} = \sum_{m=0}^\infty \frac{B_m}{m!}\, x^m, \quad |x| < 2\pi \ ,$$

where $B_0, B_1, \ldots$ is the sequence of Bernoulli numbers.[4] For $\operatorname{Re} s > 1$ we have

$$\int_0^1 x^{m+s-2}\, dx = \frac{1}{m+s-1}, \quad m = 0, 1, \ldots$$

and one obtains that

$$\begin{aligned} I_1(s) &= \int_0^1 \frac{x^{s-1}}{e^x - 1}\, dx \\ &= \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \int_0^1 x^{m+s-2}\, dx \\ &= \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \frac{1}{s+m-1}, \quad \operatorname{Re} z > 1 \ . \end{aligned}$$

For $m = 0, 1, 2, \ldots$ the functions

$$\frac{1}{s - (1-m)}$$

have poles at the s–values

$$1, 0, -1, -2, -3, \ldots$$

However, since $B_m = 0$ for $m = 3, 5, 7, \ldots$ the functions

$$\frac{B_m}{m!} \cdot \frac{1}{s - (1-m)}$$

have poles at the s–values

$$1, 0, -1, -3, -5, \ldots$$

Since

$$\frac{|B_m|}{m!} \le \frac{1}{2^m} \quad \text{for} \quad m \ge m_0 \ ,$$

it follows that

$$I_1(s) = \sum_{m=0}^\infty \frac{B_m}{m!} \cdot \frac{1}{s - (1-m)}, \quad s \in \mathbb{C} \setminus \{1, 0, -1, -3, -5, \ldots\} \ ,$$

[4] Recall that $B_m = 0$ if m is odd and $m \ge 3$. Also, by Hadamard's formula, $\limsup_{m\to\infty} |B_m/m!|^{1/m} = \frac{1}{2\pi}$.

is an analytic function with simple poles at

$$1, 0, -1, -3, -5, -7, -9, \ldots$$

We have

$$\zeta(s) = \frac{1}{\Gamma(s)}\, I_1(s) + \frac{1}{\Gamma(s)}\, I_2(s) \quad \text{for} \quad \operatorname{Re} s > 1$$

and, since

$$\Delta(s) = \frac{1}{\Gamma(s)}$$

is zero at the s–values

$$0, -1, -2, -3, \ldots$$

the term

$$\frac{1}{\Gamma(s)}\, I_1(s) = \Delta(s) I_1(s)$$

is an analytic function in $\mathbb{C} \setminus \{1\}$. One obtains that

$$\zeta(s) = \Delta(z) I_1(s) + \Delta(s) I_2(s), \quad s \in \mathbb{C} \setminus \{1\}\ ,$$

with

$$\begin{aligned}
\Delta(s) &= \frac{1}{\Gamma(s)}, \quad \Delta(s) \text{ entire} \\
I_1(s) &= \sum_{m=0}^{\infty} \frac{B_m}{m!} \frac{1}{s+m-1}, \quad s \in \mathbb{C} \setminus \{1, 0, -1, -3, -5, \ldots\} \\
I_2(s) &= \int_1^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx, \quad I_2(s) \text{ entire}\ .
\end{aligned}$$

The function $\Delta(s) I_2(s)$ is the product of two entire functions. The function $\Delta(s) I_1(s)$ is analytic in $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. Recall that $\Delta(s) = 1/\Gamma(s)$ has a simple zero at

$$s = 0, -1, -2, -3, -4, \ldots$$

and $I_1(s)$ has a simple pole at

$$s = 1, 0, -1, -3, -5 \ldots$$

Therefore

$$\zeta(s) = \Delta(s) \Big(I_1(s) + I_2(s) \Big)$$

has a simple zero at

$$s = -2, -4, -6, \ldots$$

These zeros are called the trivial zeros of the *zeta*–function.

In the following table, $poles_1$ are the poles of the function

$$s \to \frac{1}{s+m-1}$$

and $poles_2$ are the poles of the function

$$s \to \frac{B_m}{s+m-1} \ .$$

Here $0 \le m \le 5$.

m	B_m	$poles_1$	$poles_2$	zeros of $\zeta(s)$
0	1	1	1	
1	-1/2	0	0	
2	1/6	-1	-1	
3	0	-2	no pole	-2
4	-1/30	-3	-3	
5	0	-4	no pole	-4

9.5 Euler's Product Formula for $\zeta(s)$

Let $p_j, j = 1, 2, \ldots$ denote the sequence of prime numbers, i.e.,

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \ldots$$

Euler's product formula for the ζ–functions says that

$$\zeta(s) = \Pi_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}} \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

Let us first prove convergence of the infinite product.

Lemma 9.4 *The infinite product*

$$P(s) = \Pi_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}} \quad \textit{for} \quad \mathrm{Re}\, s > 1$$

converges and defines a holomorphic function for $\mathrm{Re}\, s > 1$. *The function* $P(s)$ *has no zero in* $U_1 = \{s \ : \ \mathrm{Re}\, s > 1\}$.

Proof: Since $p_j \ge 2$ we have

$$|p_j^{-s}| \le \frac{1}{2} \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

For $|\varepsilon| \le \frac{1}{2}$ we have

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon(1 + \varepsilon + \varepsilon^2 + \ldots) = 1 + R(\varepsilon)$$

with

$$|R(\varepsilon)| \le 2|\varepsilon| \quad \text{for} \quad |\varepsilon| \le \frac{1}{2} \ .$$

Therefore, if $\mathrm{Re}\, s = 1 + \eta > 1$, then

$$\frac{1}{1 - p_j^{-s}} = 1 + f_j(s)$$

with

$$|f_j(s)| \le \frac{2}{|p_j|^{1+\eta}} \le \frac{2}{j^{1+\eta}} \ .$$

Using Theorem 7.3, we obtain convergence of the infinite product

$$P(s) = \Pi_{j=1}^\infty \frac{1}{1 - p_j^{-s}} \quad \text{for} \quad \mathrm{Re}\, s > 1 \ ,$$

the function $P(s)$ is holomorphic for $\mathrm{Re}\, s > 1$ and

$$P(s) \ne 0 \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

$\diamond$

An intuitive, but somewhat imprecise argument for the equation

$$\zeta(s) = \Pi_{j=1}^\infty \frac{1}{1 - p_j^{-s}} \quad \text{for} \quad \mathrm{Re}\, s > 1$$

is the following: We have

$$\frac{1}{1 - p_j^{-s}} = 1 + p_j^{-s} + p_j^{-2s} + \ldots$$

and

$$\begin{aligned} \frac{1}{1 - p_1^{-s}} \cdot \frac{1}{1 - p_2^{-s}} \cdots \quad &= \quad \Big(1 + p_1^{-s} + p_1^{-2s} + \ldots\Big) \cdot \Big(1 + p_2^{-s} + p_2^{-2s} + \ldots\Big) \cdots \\ &= \quad \sum \Big(p_1^{k_1} \cdots p_J^{k_J}\Big)^{-s} \end{aligned}$$

where the sum is taken over all $J = 1, 2, \ldots$ and nonnegative integers $k_1, \ldots, k_J$. By the Fundamental Theorem of Arithmetic, every $n = 1, 2, \ldots$ has a unique representation as a product of prime powers,

$$n = p_1^{k_1} \cdots p_J^{k_J}$$

and, therefore, the above sum equals

$$\sum_{n=1}^\infty n^{-s} = \zeta(s), \quad \mathrm{Re}\, s > 1 \ .$$

The argument is not rigorous since we multiplied infinitely many series somewhat carelessly.

We now give a rigorous argument and fix

$$J, K \in \mathbb{N}, \quad s > 1 \ .$$

We have, for every $j = 1, 2, \ldots$

$$\begin{aligned}\frac{1}{1-p_j^{-s}} &= 1+p_j^{-s}+p_j^{-2s}+\ldots \\ &\geq \sum_{k=0}^{K} p_j^{-ks} \ .\end{aligned}$$

Taking the finite product over $j=1,\ldots,J$ yields that

$$\begin{aligned}\Pi_{j=1}^J \frac{1}{1-p_j^{-s}} &\geq \Pi_{j=1}^J \Big(\sum_{k=0}^K p_j^{-ks}\Big) \\ &= \sum_{0\leq k_\nu \leq K} \Big(p_1^{k_1}\cdots p_J^{k_J}\Big)^{-s} \\ &= \sum_{n\in S(J,K)} n^{-s}\end{aligned}$$

Here $S(J,K)$ is the set of all positive integers n of the form

$$n=p_1^{k_1}\cdots p_J^{k_J} \quad \text{with} \quad 0\leq k_\nu \leq K \quad \text{for} \quad \nu=1,\ldots,J \ .$$

Since the above product is finite, we can let $K\to\infty$ and obtain that

$$\Pi_{j=1}^J \frac{1}{1-p_j^{-s}} = \sum_{n\in S(J)} n^{-s}$$

where $S(J)$ consists of all positive integers n of the form

$$n=p_1^{k_1}\cdots p_J^{k_J} \quad \text{with} \quad k_\nu \in \{0,1,\ldots\} \quad \text{for} \quad \nu=1,\ldots,J \ .$$

If $1\leq n\leq J$ then $n\in S(J)$. Therefore,

$$\sum_{n=1}^J n^{-s} \leq \sum_{n\in S(J)} n^{-s} \leq \zeta(s), \quad s>1 \ .$$

Taking the limit as $J\to\infty$ and recalling that

$$\Pi_{j=1}^J \frac{1}{1-p_j^{-s}} = \sum_{n\in S(J)} n^{-s}$$

we have shown that

$$\Pi_{j=1}^\infty \frac{1}{1-p_j^{-s}} = \sum_{n=1}^\infty n^{-s} = \zeta(s), \quad s>1 \ .$$

Since $P(s)$ and $\zeta(s)$ are holomorphic for $\mathrm{Re}\, s>1$ we have shown:

Theorem 9.1 *For all $s\in\mathbb{C}$ with $\mathrm{Re}\, s>1$ we have*

$$\Pi_{j=1}^\infty \frac{1}{1-p_j^{-s}} = \sum_{n=1}^\infty n^{-s} = \zeta(s) \ .$$

The function $\zeta(s)$ is zero–free for $\mathrm{Re}\, s>1$.

9.6 The Sum $\sum \frac{1}{p}$ and the Prime Number Theorem

Theorem 9.2 *(Euler) If $p_1, p_2, \ldots$ denotes the sequence of primes, then*

$$\sum_{j=1}^{\infty} \frac{1}{p_j} = \infty \ .$$

Proof: Suppose that

$$\sum_{j=1}^{\infty} \frac{1}{p_j} =: L < \infty \ .$$

Then let $s > 1$ and recall that

$$\frac{1}{1-p_j^{-s}} = 1 + f_j(s) \quad \text{with} \quad 0 < f_j(s) \le 2p_j^{-s} \le \frac{2}{p_j} \ .$$

Therefore, for any finite J,

$$\begin{aligned}
\Pi_{j=1}^{J} \frac{1}{1-p_j^{-s}} &\le \Pi_{j=1}^{J} \Big(1 + \frac{2}{p_j} \Big) \\
&\le \Pi_{j=1}^{J} e^{2/p_j} \\
&= \exp \Big(2(\frac{1}{p_1} + \ldots + \frac{1}{p_J}) \Big) \\
&\le e^{2L}
\end{aligned}$$

In this estimate we can let $J \to \infty$ and obtain that

$$\zeta(s) \le e^{2L} \quad \text{for} \quad s > 1 \ .$$

However, we know that

$$\frac{1}{s-1} \le \zeta(s) \quad \text{for} \quad s > 1$$

and obtain a contradiction as $s \to 1+$. $\diamond$

It is clear that $p_j >> j$ for all large j. One can ask how fast, in comparison with j, the sequence p_j grows as $j \to \infty$. As a consequence of the previous theorem, the following Corollary says that p_j cannot grow as fast as $j^{1+\varepsilon}$ if $\varepsilon > 0$.

Corollary: *If $\varepsilon > 0$ then numbers $c_\varepsilon > 0$ and $J \in \mathbb{N}$ with*

$$c_\varepsilon j^{1+\varepsilon} \le p_j \quad \textit{for} \quad j \ge J$$

do not exist.

Proof: Otherwise one would obtain

$$c_\varepsilon \frac{1}{p_j} \le \frac{1}{j^{1+\varepsilon}} \quad \text{for} \quad j \ge J \ ,$$

in contradiction to the previous theorem. $\diamond$

Instead of comparing p_j with $j^{1+\varepsilon}$, it is common to introduce the function

$$\pi(x) = \#\{p_j \ : \ p_j \le x\}, \quad x \ge 1 \ ,$$

and to compare the growth of $\pi(x)$ with the growth of x as $x \to \infty$. Here $\pi(x)$ is the number of primes $\le x$.

Roughly speaking, Theorem 9.2 says that the sequence p_j of primes grows rather slowly, slower than the sequence

$$j^{1+\varepsilon}, \quad j = 1, 2, \ldots$$

for any $\varepsilon > 0$ since

$$\sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon}} < \infty \ .$$

Since the sequence p_j grows only 'a little' faster than the sequence j (there are 'many' primes), the function $\pi(x)$ goes to infinity only 'a little' slower than x. Thus, one expects that

$$\frac{\pi(x)}{x}$$

goes to zero as $x \to \infty$, but only quite slowly. In fact, one can prove that

$$\frac{\pi(x)}{x} \sim \frac{1}{\ln x} \quad \text{as} \quad x \to \infty \ .$$

Precisely:

Theorem 9.3 *(The Prime Number Theorem)*

$$\lim_{x \to \infty} \frac{\ln(x)\pi(x)}{x} = 1 \ .$$

9.7 Auxiliary Results about Fourier Transformation: Poisson's Summation Formula

Remarks: Laurent Schwartz (1915–2002, French) introduced the Schwartz space $S = S(\mathbb{R}^n, \mathbb{C})$ and its dual S'. The dual S' is the space of tempered distributions. We will only use the Fourier transform on the space $S = S(\mathbb{R}, \mathbb{C})$.

Results in this section will be used to prove a functional equation for the θ–function in the next section. The functional equation for the θ–function will be used to prove the functional equation for $\zeta(s)$ in Section 9.9.

The Schwartz space $S(\mathbb{R})$ consists of all C^∞– functions $f : \mathbb{R} \to \mathbb{C}$ for which all derivatives decay rapidly. More precisely, for all $k = 0, 1, 2, \ldots$ and all $j = 0, 1, 2, \ldots$ there exists a constant C_{jk} so that

$$|f^{(k)}(x)x^j| \le C_{jk} \quad \text{for all} \quad x \in \mathbb{R} \ .$$

For $f \in S(\mathbb{R})$ the Fourier transform is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R} \ .$$

The Fourier inversion formula holds:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi, \quad x \in \mathbb{R} \; .$$

Let $f \in S(\mathbb{R})$. The function

$$F_1(x) := \sum_{n=-\infty}^{\infty} f(x+n), \quad x \in \mathbb{R} \; ,$$

is called the periodization of the function $f(x)$. Clearly, $F_1(x+1) \equiv F_1(x)$. It is not difficult to prove that $F_1 \in C^\infty$.

Another way to obtain a 1–periodic function from $f(x)$ is to consider

$$F_2(x) := \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}, \quad x \in \mathbb{R} \; .$$

The following lemma says that $F_1(x) \equiv F_2(x)$.

Lemma 9.5 *For all $f \in S(\mathbb{R})$ we have*

$$F_1(x) := \sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} =: F_2(x), \quad x \in \mathbb{R} \; . \tag{9.6}$$

Proof: For all $m \in \mathbb{Z}$:

$$\begin{aligned}
\int_0^1 F_1(x) e^{-2\pi i m x}\, dx &= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m x}\, dx \quad (\text{let}\, x+n=y, \text{ use that}\, e^{2\pi i m n}=1) \\
&= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(y) e^{-2\pi i m y}\, dy \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y}\, dy \\
&= \hat{f}(m)
\end{aligned}$$

For the function $F_2(x)$ we have

$$\begin{aligned}
\int_0^1 F_2(x) e^{-2\pi i m x}\, dx &= \sum_{n=-\infty}^{\infty} \hat{f}(n) \int_0^1 e^{2\pi i x(n-m)}\, dx \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)\, \delta_{nm} \\
&= \hat{f}(m)
\end{aligned}$$

Therefore,

$$\int_0^1 \Big(F_1(x) - F_2(x) \Big) e^{-2\pi i m x}\, dx = 0 \quad \text{for all} \quad m \in \mathbb{Z} \; .$$

This implies that $F_1(x) \equiv F_2(x)$. $\diamond$

If one sets $x = 0$ in formula (9.6) one obtains **Poisson's Summation Formula** for $f \in S(\mathbb{R})$:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \tag{9.7}$$

Fourier Transform of a Gaussian: Consider the Gaussian $f(x) = e^{-\pi x^2}$ with Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x}\, dx, \quad \xi \in \mathbb{R} \ .$$

We have

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx =: J$$

where (by Fubini)

$$\begin{aligned} J^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\, dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r\, dr d\phi \\ &= \int_0^{\infty} e^{-\pi r^2} 2\pi r\, dr \\ &= \int_0^{\infty} e^{-q}\, dq \\ &= 1 \end{aligned}$$

Therefore, $\hat{f}(0) = J = 1$.

Also, using integration by parts,

$$\begin{aligned} \hat{f}'(\xi) &= \int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi i x) e^{-2\pi i \xi x}\, dx \\ &= i\int_{-\infty}^{\infty} \frac{d}{dx}\Big(e^{-\pi x^2}\Big) e^{-2\pi i \xi x}\, dx \\ &= -2\pi\xi \hat{f}(\xi) \end{aligned}$$

Thus, the function $\hat{f}(\xi)$ solves the initial values problem

$$\hat{f}'(\xi) = -2\pi\xi\hat{f}(\xi), \quad \hat{f}(0) = 1 \ .$$

Since the Gaussian $h(\xi) = e^{-\pi\xi^2}$ solves the same initial value problem and since the solution of the initial value problem is unique, it follows that

$$\hat{f}(\xi) = e^{-\pi\xi^2} \ .$$

Thus, the Gaussian $f(x) = e^{-\pi x^2}$ has the Fourier transform $\hat{f}(\xi) = e^{-\pi\xi^2}$.

Let $t > 0$ be fixed and consider the Gaussian

$$f(x) = e^{-\pi t x^2} \ .$$

We have

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-\pi t x^2} e^{-2\pi i \xi x}\, dx \quad (\text{let } \sqrt{t}\, x = y) \\ &= \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\pi y^2} e^{-2\pi i (\xi/\sqrt{t})\, y}\, dy \\ &= \frac{1}{\sqrt{t}}\, e^{-\pi \xi^2/t} \end{aligned}$$

Therefore, for $t > 0$ the Gaussian $f(x) = e^{-\pi t x^2}$ has the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{t}}\, e^{-\pi \xi^2/t} \ . \tag{9.8}$$

9.8 A Functional Equation for the Theta–Function

Remarks: The Jacobi theta– function

$$\Theta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} e^{2\pi i n z}$$

is defined for $z \in \mathbb{C}$ and τ in the upper half–plane. The function is 1–periodic in z and plays a role in the theory of elliptic functions. If $z = 0$ and $\tau = it, t > 0$, one obtains

$$\Theta(0, it) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \ .$$

This function is also called a theta–function.

For $t > 0$ we define the (special) theta–function by

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \ .$$

If $t > 0$ is fixed and

$$f(x) = e^{-\pi x^2 t}, \quad x \in \mathbb{R} \ ,$$

then

$$\hat{f}(\xi) = \frac{1}{\sqrt{t}}\, e^{-\pi \xi^2/t} \ .$$

Note that the Gaussian $f(x)$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R})$. Poisson's summation formula applied to the function $f(x) = e^{-\pi x^2 t}$ (for fixed $t > 0$) yields that

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} \ ,$$

thus

$$\theta(t) = \frac{1}{\sqrt{t}}\,\theta(1/t) \quad \text{for} \quad t > 0\ . \tag{9.9}$$

Equation (9.9) is called the functional equation for the theta–function.

For the application to the Zeta–Function we will need the following modification: Set

$$q(u) = \sum_{n=1}^{\infty} e^{-\pi n^2 u} \quad \text{for} \quad u > 0\ .$$

Then we have

$$\begin{aligned}
\theta(u) &= 2q(u) + 1 \\
q(u) &= \frac{1}{2}\theta(u) - \frac{1}{2} \\
&= \frac{1}{2\sqrt{u}}\,\theta(1/u) - \frac{1}{2} \\
&= \frac{1}{2\sqrt{u}}\Big(2q(1/u) + 1\Big) - \frac{1}{2}
\end{aligned}$$

Therefore,

$$q(u) = \frac{1}{\sqrt{u}}\,q(1/u) + \frac{1}{2\sqrt{u}} - \frac{1}{2} \quad \text{for} \quad u > 0\ . \tag{9.10}$$

9.9 The Functional Equation for the Zeta–Function

In the formula

$$\Gamma(s/2) = \int_0^\infty e^{-t}\, t^{s/2}\,\frac{dt}{t} \quad \text{for} \quad s > 0$$

use the linear substitution

$$t = \pi n^2 u$$

to obtain that

$$\pi^{-s/2}\Gamma(s/2) n^{-s} = \int_0^\infty e^{-\pi n^2 u}\, u^{s/2}\,\frac{du}{u}\ .$$

Then summation over n yields for $s > 1$:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty q(u) u^{s/2}\,\frac{du}{u}\ .$$

Write the integral as

$$\begin{aligned}
Int(s) &= \int_0^1 q(u) u^{s/2}\,\frac{du}{u} + \int_1^\infty q(u) u^{s/2}\,\frac{du}{u} \\
&=: Int_1(s) + Int_2(s)
\end{aligned}$$

and then use equation (9.10) in $Int_1(s)$ to obtain that

$$Int_1(s) = \int_0^1 u^{(s-1)/2} q(1/u)\, \frac{du}{u} + \frac{1}{2} \int_0^1 \Big(u^{(s-3)/2} - u^{(s-2)/2} \Big)\, du =: Int_3(s) + Int_4(s) \ .$$

The last integral equals

$$Int_4(s) = \frac{1}{s-1} - \frac{1}{s} \ .$$

In $Int_3(s)$ use the substitution

$$u = \frac{1}{v}, \quad \frac{du}{u} = -\frac{dv}{v}$$

to obtain that

$$Int_3(s) = \int_1^\infty v^{(1-s)/2} q(v)\, \frac{dv}{v}$$

To summarize, one obtains that for $s > 1$:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \Big(u^{(1-s)/2} + u^{s/2} \Big) q(u)\, \frac{du}{u} \ . \tag{9.11}$$

Both sides of the above equation determine holomorphic functions in $\mathbb{C} \setminus \{0,1\}$. Therefore, equation (9.11) holds for all $s \in \mathbb{C} \setminus \{0,1\}$. It is also obvious that the right–hand side of equation (9.11) remains unchanged if one replaces s by $1-s$, i.e.,

$$rhs(1-s) = rhs(s) \ .$$

This proves that the function

$$h(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

satisfies

$$h(1-s) = h(s), \quad s \in \mathbb{C} \setminus \{0,1\} \ .$$

9.10 The Order of Growth of $\xi(s)$

Set

$$\xi(s) = \frac{1}{2}\, s(s-1)h(s) = \frac{1}{2}\, s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \text{for} \quad s \in \mathbb{C} \ .$$

The function $\xi(s)$ is entire and satisfies

$$\xi(s) = \xi(1-s) \quad \text{for} \quad s \in \mathbb{C} \ .$$

We will prove that $\xi(s)$ has the growth order 1.

An implication is the following: Set

$$g(w) = \xi(\frac{1}{2} + w), \quad w \in \mathbb{C} \ .$$

If $s = \frac{1}{2} + w$ then $1 - s = \frac{1}{2} - w$ and one obtains that

$$g(w) = g(-w), \quad w \in \mathbb{C} \ .$$

Therefore, if

$$g(w) = \sum_{n=0}^{\infty} g_n w^n$$

then $g_n = 0$ if n is odd. Thus,

$$g(w) = g_0 + g_2 w^2 + g_4 w^4 + \ldots$$

Set

$$G(v) = g_0 + g_2 v + g_4 v^2 + \ldots$$

thus

$$G(w^2) = g(w) \ .$$

In other words,

$$G(v) = g(w) \quad \text{if} \quad v = w^2 \ .$$

Since $g(w)$ has growth order one, an estimate of the form

$$|g(w)| \le A e^{B|w|^{1+\varepsilon}} \quad \text{for all} \quad w \in \mathbb{C}$$

holds for all $\varepsilon > 0$. If $v = w^2$ then one obtains that

$$|G(v)| = |g(w)| \le A e^{B|w|^{1+\varepsilon}} = A e^{B|v|^{(1+\varepsilon)/2}} \ .$$

This implies that the function $G(v)$ has the growth order $\frac{1}{2}$. By Hadamard's Theorem, the entire function $G(v)$ has infinitely many zeros. Therefore, $g(w)$ has infinitely many zeros and $\xi(s)$ has infinitely many zeros. We also know that $\xi(s)$ has no zero s with $\mathrm{Re}\, s \ge 1$ or $\mathrm{Re}\, s \le 0$. It follows that $\xi(s)$ has infinitely many zeros s with $0 < \mathrm{Re}\, s < 1$. Every zero of $\xi(s)$ is also a zero of $\zeta(s)$. It follows that $\zeta(s)$ has infinitely many zeros in the critical strip $0 < \mathrm{Re}\, s < 1$.

It remains to prove:

Theorem 9.4 *The entire function*

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \textit{for} \quad s \in \mathbb{C}$$

has growth order 1.

Proof: a) We first show that the order of growth of $\xi(s)$ cannot be less than 1. Suppose that

$$\xi(x) \le A e^{Bx} \quad \text{for} \quad x \in \mathbb{R}, \quad x \ge x_0 \ .$$

This yields that

$$\Gamma(x/2) \le A e^{Bx} \pi^{x/2} \frac{1}{\zeta(x)} \quad \text{for} \quad x \ge x_0 \ .$$

Since $\zeta(x) \geq 1$ one obtains that

$$\Gamma(x) \leq Ae^{\beta x} \quad \text{for} \quad x \geq x_1 \tag{9.12}$$

for some $\beta > 0$. However, by Stirling's formula,

$$\Gamma(x+1) = \Big(\frac{x}{e}\Big)^x \sqrt{2\pi x}\,\Big(1 + \mathcal{O}(1/x)\Big) \quad \text{as} \quad x \to \infty \; .$$

This implies that

$$\Gamma(x+1) \geq \Big(\frac{x}{e}\Big)^x \quad \text{for} \quad x \geq x_1 \; ,$$

thus

$$x^x \leq e^x \Gamma(x+1) = e^x x \Gamma(x) \quad \text{for} \quad x \geq x_1 \; .$$

If the estimate (9.12) would hold then one would obtain that

$$x^x \leq e^{\gamma x} \quad \text{for} \quad x \geq x_2$$

for some $\gamma > 0$. This would yield

$$x \ln x \leq \gamma x \quad \text{for} \quad x \geq x_2 \; ,$$

a contradiction.

b) We have

$$\xi(s) = g\Big(s - \frac{1}{2}\Big) = \sum_{n=0}^\infty g_{2n}\Big(s - \frac{1}{2}\Big)^{2n} \; .$$

We will prove in part c) that $g_{2n} > 0$ for all $n = 0, 1, \ldots$.

Let's assume that $g_{2n} > 0$ for all n. In the following, let

$$s \in \mathbb{C} \quad \text{and} \quad R = \Big|s - \frac{1}{2}\Big| \; .$$

We will assume that $s_0 > 0$ and $R_0 > 0$ are sufficiently large. We have

$$\begin{aligned} |\xi(s)| &\leq \sum_{n=0}^\infty g_{2n}\Big|s - \frac{1}{2}\Big|^{2n} \\ &= \sum_{n=0}^\infty g_{2n} R^{2n} \\ &= g(R) \\ &= \xi\Big(\frac{1}{2} + R\Big) \end{aligned}$$

The last equation holds since $\xi(q) = g\Big(q - \frac{1}{2}\Big)$ for all $q \in \mathbb{C}$.

If $R \geq 1$ then $\frac{1}{2} + R \leq 2R$ and

$$\begin{aligned}
|\xi(s)| &\le \xi\Big(\frac{1}{2}+R\Big)\\
&\le \xi(2R)\\
&= R(2R-1)\pi^{-R}\Gamma(R)\zeta(2R)\\
&\le \Gamma(R+1)
\end{aligned}$$

for $R \ge R_0$. Using Stirling's formula,

$$\Gamma(R+1) = \Big(\frac{R}{e}\Big)^R \sqrt{2\pi R}\Big(1+\mathcal{O}(1/R)\Big) \quad \text{as} \quad R\to\infty$$

one obtains that

$$|\xi(s)| \le \Gamma(R+1) \le R^R \quad \text{for} \quad R \ge R_0 \ .$$

Since $R = |s-\frac{1}{2}|$ one obtains that

$$R \le |s| + \frac{1}{2} \le 2|s| \quad \text{for} \quad |s| \ge 1 \ .$$

Therefore,

$$|\xi(s)| \le (2|s|)^{2|s|} = e^{\ln(2|s|)\,2|s|} \quad \text{for} \quad |s| \ge s_0 \ .$$

For all $\varepsilon > 0$ there exists $B_\varepsilon > 0$ so that

$$2\ln(2|s|)\,|s| \le B_\varepsilon |s|^{1+\varepsilon} \quad \text{for} \quad |s| \ge s_0 \ .$$

Together with part a) of the proof, it follows that the growth order of $\xi(s)$ equals 1.

c) It remains to prove that $g_{2n} > 0$ for $n = 0,1,2,\ldots$ We first prove two auxiliary results about the function

$$q(t) = \sum_{j=1}^{\infty} e^{-\pi j^2 t}, \quad t > 0 \ .$$

Lemma 9.6 *We have*

$$\frac{1}{2} + q(1) + 4q'(1) = 0 \tag{9.13}$$

and

$$\frac{d}{dt}\Big(t^{3/2}q'(t)\Big) > 0 \quad \textit{for} \quad t \ge 1 \ . \tag{9.14}$$

Proof: 1) Differentiating the functional equation

$$2q(t)+1 = t^{-1/2}\Big(2q(1/t)+1\Big)$$

one obtains

$$2q'(t) = -\frac{1}{2}\,t^{-3/2}\Big(2q(1/t)+1\Big) + 2t^{-1/2}\,q'(1/t)\,(-t^{-2}) \ .$$

Evaluation at $t=1$ yields that

$$2q'(1) = -\frac{1}{2}\Big(2q(1)+1\Big) - 2q'(1) \ .$$

Equation (9.13) follows.

2) We have

$$q'(t) = -\pi \sum_{j=1}^\infty j^2 e^{-\pi j^2 t}$$

and

$$\begin{aligned}
\frac{d}{dt}\Big(t^{3/2}q'(t)\Big) &= -\pi \sum_{j=1}^\infty j^2 \, \frac{d}{dt}\Big(t^{3/2}\, e^{-\pi j^2 t}\Big) \\
&= -\pi \sum_{j=1}^\infty j^2 \Big(- \pi j^2 t^{3/2} + \frac{3}{2}\, t^{1/2}\Big) e^{-\pi j^2 t} \\
&= \pi \sum_{j=1}^\infty j^2 t^{1/2} \Big(\pi j^2 t - \frac{3}{2}\Big) e^{-\pi j^2 t}
\end{aligned}$$

Since $\pi > 3/2$ and $t \ge 1$ is assumed it follows that (9.14) holds. $\diamond$

We continue the proof of Theorem 9.4, part c).

Recall formula (9.11):

$$h(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty q(t)\Big(t^{s/2} + t^{(1-s)/2}\Big)\,\frac{dt}{t} \ .$$

Since

$$\frac{1}{2}\,s(s-1)\Big(\frac{1}{s-1} - \frac{1}{s}\Big) = \frac{1}{2}\,\Big(s-(s-1)\Big) = \frac{1}{2}$$

one obtains the following equation for $\xi(s) = \frac{1}{2}\,s(s-1)h(s)$:

$$\xi(s) = \frac{1}{2} - \frac{1}{2}\,s(1-s)\int_1^\infty q(t)\Big(t^{s/2-1} + t^{(1-s)/2-1}\Big)\,dt \ .$$

For $t \ge 1$ set

$$\begin{aligned}
T_1(t) &= q(t)\Big(t^{s/2-1} + t^{(1-s)/2-1}\Big) \\
T_2(t) &= q'(t)\Big(\frac{t^{s/2}}{s/2} + \frac{t^{(1-s)/2}}{(1-s)/2}\Big) \\
T_3(t) &= q(t)\Big(\frac{t^{s/2}}{s/2} + \frac{t^{(1-s)/2}}{(1-s)/2}\Big)
\end{aligned}$$

(The functions $T_j(t)$ also depend on s, of course.) The product rule of differentiation shows that

$$\frac{d}{dt}\,T_3(t) = T_1(t) + T_2(t) \ ,$$

thus

$$T_1(t) = T_3'(t) - T_2(t) \ .$$

Since $q(t)$ decays exponentially as $t \to \infty$ one obtains that

$$\int_1^\infty T_1(t)\,dt = -q(1)\Big(\frac{2}{s} + \frac{2}{1-s}\Big) - \int_1^\infty T_2(t)\,dy \ .$$

Since

$$\frac{1}{2}\,s(1-s)\Big(\frac{2}{s} + \frac{2}{1-s}\Big) = 1 - s + s = 1$$

and

$$\frac{1}{2}\,s(1-s)\Big(\frac{t^{s/2}}{s/2} + \frac{t^{(1-s)/2}}{(1-s)/2}\Big) = (1-s)t^{s/2} + st^{(1-s)/2}$$

one obtains that

$$\begin{aligned}
\xi(s) &= \frac{1}{2} - \frac{1}{2}\,s(1-s)\int_1^\infty T_1(t)\,dt \\
&= \frac{1}{2} + q(1) + \frac{1}{2}\,s(1-s)\int_1^\infty T_2(t)\,dt \\
&= \frac{1}{2} + q(1) + \int_1^\infty q'(t)\Big((1-s)t^{s/2} + st^{(1-s)/2}\Big)\,dt \\
&= \frac{1}{2} + q(1) + \int_1^\infty q'(t)t^{3/2}\Big((1-s)t^{(s-1)/2-1} + st^{-s/2-1}\Big)\,dt
\end{aligned}$$

Set

$$\begin{aligned}
B_1(t) &= \Big(q'(t)t^{3/2}\Big)\Big((1-s)t^{(s-1)/2-1} + st^{-s/2-1}\Big) \\
B_2(t) &= \Big(\frac{d}{dt}\,(t^{3/2}q'(t))\Big)\Big(-2t^{(s-1)/2} - 2t^{-s/2}\Big) \\
B_3(t) &= \Big(t^{3/2}q'(t)\Big)\Big(-2t^{(s-1)/2} - 2t^{-s/2})\Big)
\end{aligned}$$

It is clear that $B_3' = B_1 + B_2$, thus

$$B_1(t) = B_3'(t) - B_2(t) \ .$$

Therefore,

$$\begin{aligned}
\xi(s) &= \frac{1}{2} + q(1) + \int_1^\infty B_1(t)\,dt \\
&= \frac{1}{2} + q(1) + \int_1^\infty B_3'(t)\,dt - \int_1^\infty B_2(t)\,dt
\end{aligned}$$

Here

$$\int_1^\infty B_3'(t)\,dt = -B_3(1) = 4q'(1)\ .$$

Using (9.13) we obtain that

$$\xi(s) = -\int_1^\infty B_2(t)\,dt = 2\int_1^\infty \Phi(t)\Big(t^{(s-1)/2} + t^{-s/2}\Big)\,dt \tag{9.15}$$

with

$$\Phi(t) = \frac{d}{dt}\left(t^{3/2}q'(t)\right)\ .$$

Set

$$y = \frac{\ln t}{2}\left(s - \frac{1}{2}\right)\ .$$

We have

$$y = (\ln t)\left(s - \frac{1}{2}\right)/2 = \ln\left(t^{(s-\frac{1}{2})/2}\right)$$

thus

$$e^y = t^{(s-\frac{1}{2})/2} \quad \text{and} \quad e^{-y} = t^{-(s-\frac{1}{2})/2}\ .$$

Since

$$t^{(s-1)/2} = t^{-1/4}\,t^{(s-\frac{1}{2})/2} = t^{-1/4}e^y \quad \text{and} \quad t^{-s/2} = t^{-1/4}\,t^{-(s-\frac{1}{2})/2} = t^{-1/4}e^{-y}$$

we obtain that

$$\begin{aligned} t^{(s-1)/2} + t^{-s/2} &= t^{-1/4}\Big(e^y + e^{-y}\Big) \\ &= 2t^{-1/4}\sum_{n=0}^\infty \frac{y^{2n}}{(2n)!} \\ &= 2t^{-1/4}\sum_{n=0}^\infty \frac{1}{2^{2n}(2n)!}\,(\ln t)^{2n}\left(s - \frac{1}{2}\right)^{2n} \end{aligned}$$

Using this series in formula (9.15) yields that

$$\xi(s) = \sum_{n=0}^\infty g_{2n}\Big(s - \frac{1}{2}\Big)^{2n}$$

with

$$g_{2n} = \frac{4}{2^{2n}(2n)!}\int_1^\infty \Phi(t)t^{-1/4}(\ln t)^{2n}\,dt$$

where

$$\Phi(t) = \frac{d}{dt}\left(t^{3/2}q'(t)\right) > 0 \quad \text{for} \quad 1 \le t < \infty\ .$$

The positivity of $\Phi(t)$ for $t \ge 1$ has been proved in Lemma 9.6.

Since $g_{2n} > 0$ for all $n = 1, 2, \dots$ the proof of Theorem 9.4 is complete. $\diamond$

10 Analytic Continuation

10.1 Analytic Continuation Using the Cauchy Riemann Equations

Discussion of the 2π–periodic Cauchy problem for the Cauchy–Riemann equations. It is ill–posed.

10.2 Exponential Decay of Fourier Coefficients and the Strip of Analyticity

10.3 The Schwarz Reflection Principle

10.4 Examples for Analytic Continuation

10.5 Riemann Surfaces: Intuitive Approach

10.6 Riemann Surfaces: Germs, Sheafs, and Fibers

11 Fourier Series

11.1 History

Marc–Antoine Parseval, French, 1755–1836
Jean–Baptiste Joseph Fourier, French, 1768–1830
Peter Gustav Lejeune Dirichlet, German, 1805–1859
Karl Theodor Wilhelm Weierstrass, German, 1815–1897
Paul Du Bois–Reymond, German, 1831–1889
Ernesto Cesàro, Italian, 1859–1906
René–Louis Baire, French, 1874–1932
Lepot Fejér, Hungarian, 1880–1959
Henri Léon Lebesgue, French, 1875–1941
Hugo Steinhaus, Polish, 1887–1972
Stefan Banach, Polish, 1892–1945

11.2 Convergence Results: Overview

Let X denote the linear space of all continuous functions $f : [0,1] \to \mathbb{C}$ with $f(0) = f(1)$. (Some of the results that we discuss hold for more general functions than functions in X.)

For $f, g \in X$ we use the L_2–inner product,

$$(f,g)_{L_2} = \int_0^1 \bar{f}(x) g(x)\, dx \ ,$$

the L_2 norm,

$$\|f\|_{L_2} = (f,f)_{L_2}^{1/2} \ ,$$

and the maximum norm,

$$|f|_\infty = \max_x |f(x)| \ .$$

The sequence of functions

$$\phi_k(x) = e^{2\pi i k x}, \quad k \in \mathbb{Z} \ ,$$

is orthonormal in X, i.e.,

$$(\phi_j, \phi_k)_{L_2} = \delta_{jk} \ .$$

Let $f \in X$ be given by a series,

$$f(x) = \sum_k a_k \phi_k(x) \ .$$

If the series converges uniformly for $0 \le x \le 1$, then one obtains that

$$a_k = (\phi_k, f)_{L_2} \ .$$

This motivates the definition of the Fourier coefficients: If $f \in L_1(0,1)$ then set

$$\hat{f}(k) = (\phi_k, f)_{L_2}, \quad k \in \mathbb{Z} .$$

The series

$$\sum_k \hat{f}(k)\phi_k(x)$$

is called the Fourier series of f. Basic questions are: Under what assumptions on f does the Fourier series converge? In which sense does it converge (e.g., pointwise, or uniformly, or in L_2–norm)? If the series converges, does it converge to f?

With

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)\phi_k(x) \quad \text{for} \quad 0 \le x \le 1 \quad \text{and} \quad n = 0, 1, 2, \ldots$$

we denote the n–th partial sum of the Fourier series of f.

A reasonable question is: Given $f \in X$, is it true that

$$S_n f(x) \to f(x) \quad \text{as} \quad n \to \infty$$

for all $x \in [0, 1]$? In 1873, Du Bois–Raymond proved that the answer is *No*, in general. It is difficult to construct an explicit example. However, using the *Principle of Uniform Boundedness* of functional analysis, one can show rather easily that a function $f \in X$ *exists* for which the sequence of numbers $S_n f(0)$ is unbounded.

Let $f \in X$. The sequence of arithmetic means,

$$\sigma_n f(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k f(x) ,$$

can be shown to converge uniformly to f. These means are the so–called **Cesaro means** of the partial sums $S_n f(x)$.

Theorem 11.1 *Let $f \in X$. Then we have*

$$|f - \sigma_n f|_\infty \to 0 \quad as \quad n \to \infty .$$

We will prove this important result below.

Let $\mathcal{T}_n = span\{\phi_k : \; -n \le k \le n\}$ and let

$$\mathcal{T} = \cup_n \mathcal{T}_n .$$

The functions in $\mathcal{T}$ are called trigonometric polynomials.

Using Theorem 11.1 it is easy to prove:

Theorem 11.2 *(Weierstrass) The space $\mathcal{T}$ of trigonometric polynomials is dense in X with respect to $|\cdot|_\infty$.*

From Weierstrass' theorem it follows rather easily that the Fourier series $S_n f$ converges to f w.r.t. $\|\cdot\|_{L_2}$:

Theorem 11.3 *Let $f \in X$ (or, more generally, let $f \in L_2(0, 1)$). Then we have*

$$\|f - S_n f\|_{L_2} \to 0 \quad as \quad n \to \infty .$$

11.3 The Dirichlet Kernel

Let $f \in X$. We have

$$\hat{f}(k) = \int_0^1 e^{-2\pi iky} f(y)\,dy$$

and

$$\begin{aligned} S_n f(x) &= \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi ikx} \\ &= \int_0^1 \Big(\sum_{k=-n}^{n} e^{2\pi ik(x-y)} \Big) f(y)\,dy \\ &= \int_0^1 D_n(x-y) f(y)\,dy \end{aligned}$$

where

$$D_n(t) = \sum_{k=-n}^{n} e^{2\pi ikt}, \quad t \in \mathbb{R} \;, \tag{11.1}$$

is called the Dirichlet kernel.

Lemma 11.1 *The Dirichlet kernel (11.1) has the following properties:*

1. $$D_n \in C^\infty(\mathbb{R}) \;;$$
2. $$D_n(t) \equiv D_n(t+1) \;;$$
3. $$\int_0^1 D_n(t)\,dt = 1 \;;$$
4. $$D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)} \;;$$
5. $$L_n := \int_0^1 |D_n(t)|\,dt = \frac{4}{\pi^2} \ln n + \mathcal{O}(1) \quad \textit{as} \quad n \to \infty \;.$$

Proof: We prove 4. using the geometric sum formula. Let $q = e^{2\pi it} \neq 1$, thus $t \in \mathbb{R} \setminus \mathbb{Z}$. We have

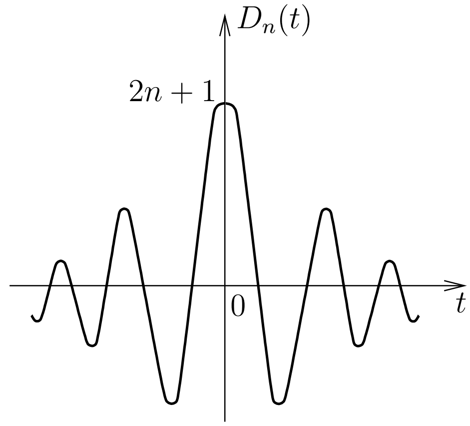

Figure 11.1: Dirichlet Kernel

$$\begin{aligned}
D_n(t) &= q^{-n} \sum_{j=0}^{2n} q^j \\
&= q^{-n} \frac{q^{2n+1}-1}{q-1} \\
&= \frac{q^{n+1/2}-q^{-n-1/2}}{q^{1/2}-q^{-1/2}} \\
&= \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}
\end{aligned}$$

Property 5 will be shown below in Lemma 11.4. $\diamond$

Note that, because of Property 4,

$$D_n(t) \sim 2n+1 \quad \text{for} \quad t \sim 0 \; .$$

A plot of $D_n(t)$ shows that $D_n(t), -\frac{1}{2} \le t \le \frac{1}{2}$, is concentrated near $t = 0$ for large n. Together with Property 3, this makes it plausible that

$$S_n f(x) = \int_0^1 D_n(x-y) f(y)\, dy \sim f(x)$$

for large n. In fact, convergence $S_n f(x) \to f(x)$ holds under suitable assumptions on f. However, the oscillatory nature of $D_n(t)$ makes a convergence analysis as $n \to \infty$ difficult. In fact, in Section 11.9 we will use Property 5 to show existence of a function $f \in X$ for which $S_n f(0)$ diverges.

11.4 The Fejér Kernel

Let $f \in X$. We have

$$\sigma_n f(x) = \int_0^1 F_n(x-y) f(y)\, dy$$

where $F_n(t)$ is the Fejér kernel:

$$F_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) \ .$$

The following result shows that $F_n(t) \geq 0$. This crucial inequality will allow us to prove Theorem 11.1.

Lemma 11.2 *We have*

$$F_n(t) = \frac{1}{n+1} \frac{\sin^2(\pi(n+1)t)}{\sin^2(\pi t)} \ .$$

Proof: Using the formula

$$D_k(t) = \frac{\sin(\pi(2k+1)t)}{\sin(\pi t)}$$

we must show that

$$\sum_{k=0}^{n} \sin(\pi(2k+1)t) = \frac{\sin^2(\pi(n+1)t)}{\sin(\pi t)} \ . \tag{11.2}$$

We will prove this equation using again the geometric sum formula. Let

$$r = e^{\pi it} \quad \text{and} \quad q = r^2 = e^{2\pi it} \ .$$

We have

$$\begin{aligned} 2i\sin(\pi(2k+1)t) &= e^{i\pi(2k+1)t} - e^{-i\pi(2k+1)t} \\ &= rq^k - \frac{1}{r}\,q^{-k} \end{aligned}$$

Therefore,

$$\begin{aligned} 2i\sum_{k=0}^{n} \sin(\pi(2k+1)t) &= r\,\frac{q^{n+1}-1}{q-1} - \frac{1}{r}\,\frac{q^{-n-1}-1}{q^{-1}-1} \quad (\text{ recall that } q = r^2) \\ &= \frac{q^{n+1}-1}{r-r^{-1}} - \frac{q^{-n-1}-1}{r^{-1}-r} \\ &= \frac{r^{2(n+1)} - 2 + r^{-2(n+1)}}{r-r^{-1}} \end{aligned}$$

Here the numerator is (recall that $r = e^{\pi it}$):

$$N = \Big(r^{n+1} - r^{-n-1}\Big)^2 = (2i)^2 \sin^2(\pi(n+1)t)$$

and we obtain

$$\sum_{k=0}^{n} \sin(\pi(2k+1)t) = 2i \cdot \frac{\sin^2(\pi(n+1)t)}{r - r^{-1}} \ .$$

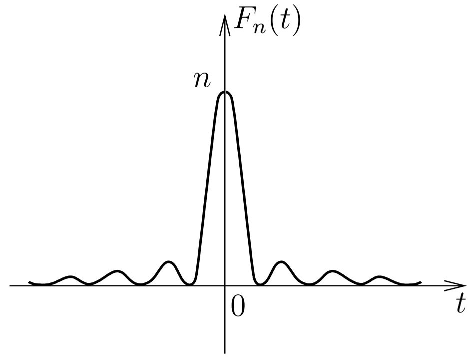

Figure 11.2: Fejér Kernel

Since

$$r - r^{-1} = 2i\sin(\pi t)$$

the equation (11.2) follows and the lemma is proved. $\diamond$

11.5 Convergence of $\sigma_n f$ in Maximum Norm

Let $f \in X$. We extend f as a 1–periodic continuous function defined for all $x \in \mathbb{R}$.

Below we will use that

$$\int_0^1 F_n(t)\,dt = 1 \ .$$

Furthermore, if $0 < \delta \le \frac{1}{2}$, then we have

$$\sin(\pi t) \ge \delta > 0 \quad \text{for} \quad \delta \le t \le \frac{1}{2} \ .$$

Therefore,

$$F_n(t) \le \frac{1}{n+1}\,\frac{1}{\delta^2} \quad \text{for} \quad \delta \le t \le \frac{1}{2} \ . \tag{11.3}$$

We have

$$\begin{aligned} \sigma_n f(x) &= \int_0^1 F_n(x-y)f(y)\,dy \\ &= \int_0^1 F_n(t)f(x-t)\,dt \\ &= \int_{-1/2}^{1/2} F_n(t)f(x-t)\,dt \end{aligned}$$

and

$$f(x) - \sigma_n f(x) = \int_{-1/2}^{1/2} F_n(t)(f(x) - f(x-t))\,dt \ .$$

First fix an arbitrary (small) $\delta > 0$. We have

$$\begin{aligned} |f(x) - \sigma_n f(x)| &\le \int_{|t|\le\delta} F_n(t)|f(x) - f(x-t)|\,dt + \int_{\delta\le|t|\le 1/2} F_n(t)|f(x) - f(x-t)|\,dt \\ &\le M_\delta + 4|f|_\infty \int_\delta^{1/2} F_n(t)\,dt \end{aligned} \tag{11.4}$$

with

$$M_\delta = \max\{|f(x) - f(x-t)| \ : \ x \in \mathbb{R},\ |t| \le \delta\}\ .$$

Note that $M_\delta \to 0$ as $\delta \to 0$ since f is uniformly continuous. The integral in (11.4) is bounded by $1/(\delta^2(n+1))$. If $\varepsilon > 0$ is given, we can find $\delta > 0$ with $M_\delta \le \varepsilon/2$ and then have

$$4|f|_\infty \int_\delta^{1/2} F_n(t)\,dt \le \frac{\varepsilon}{2} \quad \text{for} \quad n \ge N(\varepsilon)\ .$$

This proves Theorem 11.1. $\diamond$

11.6 Weierstrass' Approximation Theorem for Trigonometric Polynomials

Since $\sigma_n f \in \mathcal{T}_n$ it is clear that Theorem 11.2 follows from Theorem 11.1.

11.7 Convergence of $S_n f$ in L_2

Let $f \in X$. Recall that

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}$$

denotes the n–th partial sum of the Fourier series of f. It is clear that $S_n f \in \mathcal{T}_n$. The following result says that $S_n f$ is the unique best approximation to f in $\mathcal{T}_n$ w.r.t. the L_2–norm.

Theorem 11.4 *Let $f \in X$. If $g \in \mathcal{T}_n$ is arbitrary and $g \ne S_n f$, then we have*

$$\|f - S_n f\|_{L_2} < \|f - g\|_{L_2}\ .$$

Proof: We have

$$0 = (e^{2\pi i j x}, f - S_n f)_{L_2} \quad \text{for} \quad |j| \le n\ .$$

Therefore,

$$0 = (h, f - S_n f)_{L_2} \quad \text{for all} \quad h \in \mathcal{T}_n\ .$$

If $h \in \mathcal{T}_n$ is arbitrary, $h \ne 0$, then

$$\begin{aligned} \|f - S_n f - h\|^2_{L_2} &= \|f - S_n f\|^2_{L_2} + \|h\|^2_{L_2} \\ &> \|f - S_n f\|^2 \end{aligned}$$

which proves the theorem. $\diamond$

To prove Theorem 11.3 for $f \in L_2(0,1)$ we will need the following result:

Lemma 11.3 *a) The space X is dense in $L_2(0,1)$ with respect to the L_2–norm.*
b) If $g \in L_2(0,1)$ is arbitrary, then $\|S_n g\|_{L_2} \le \|g\|_{L_2}$.

Proof: a) Step functions are dense in L_2 (integration theory). Then, every step function can be approximated in L_2–norm by an element in X by 'rounding the corners' and enforcing periodicity.

b) Let $g \in L_2(0,1)$ be arbitrary. By construction of $S_n g$ we have

$$(\phi_j, g - S_n g)_{L_2} = 0 \quad \text{for} \quad |j| \le n \ .$$

This says that the approximation error, $\eta_n := g - S_n g$, is orthogonal to the space of trigonometric polynomials of degree $\le n$, i.e., to $\mathcal{T}_n$. In particular, $\eta_n = g - S_n g$ is orthogonal to $S_n g$. Therefore,

$$\begin{aligned} \|g\|_{L_2}^2 &= (g - S_n g + S_n g, g - S_n g + S_n g)_{L_2} \\ &= (\eta_n + S_n g, \eta_n + S_n g)_{L_2} \\ &= \|\eta_n\|_{L_2}^2 + \|S_n g\|_{L_2}^2 \ . \end{aligned}$$

$\diamond$

Proof of Theorem 11.3: First let $f \in X$. By Theorem 11.1, given any $\varepsilon > 0$, there is $g \in \mathcal{T}$ with

$$|f - g|_\infty < \varepsilon \ .$$

If $g \in \mathcal{T}_N$ then we have for all $n \ge N$:

$$\|f - S_n f\|_{L_2} \le \|f - g\|_{L_2} \le |f - g|_\infty < \varepsilon \ .$$

This proves that $\|f - S_n f\|_{L_2} \to 0$ as $n \to \infty$ if $f \in X$.

Second, let $f \in L_2(0,1)$. Given $\varepsilon > 0$ there is $f^\varepsilon \in X$ with

$$\|f - f^\varepsilon\|_{L_2} < \varepsilon \ .$$

We have

$$\|f - S_n f\|_{L_2} \le \|f - f^\varepsilon\|_{L_2} + \|f^\varepsilon - S_n f^\varepsilon\|_{L_2} + \|S_n f^\varepsilon - S_n f\|_{L_2} \ .$$

The last term is

$$\|S_n(f^\varepsilon - f)\|_{L_2} \le \|f^\varepsilon - f\|_{L_2} < \varepsilon \ .$$

It follows that

$$\|f - S_n f\|_{L_2} \le 2\varepsilon + \|f^\varepsilon - S_n f^\varepsilon\|_{L_2} \ .$$

Therefore, if $n \ge N(\varepsilon)$,

$$\|f - S_n f\|_{L_2} \le 3\varepsilon \ .$$

11.8 The Lebesgue Constant of the Dirichlet Kernel

Recall that

$$D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}$$

denotes the Dirichlet kernel.

Lemma 11.4 *There is a constant $c > 0$ with*

$$L_n := \int_{-1/2}^{1/2} |D_n(t)|\, dt \geq c \ln n \ .$$

More precisely,

$$L_n = \frac{4}{\pi^2} \ln n + \mathcal{O}(1) \ .$$

The number L_n is called the Lebesgue constant of the Dirichlet kernel $D_n(t)$.

Proof: 1) For $0 < t \leq \varepsilon$ we have

$$0 < \sin(\pi t) = \pi t(1 + \mathcal{O}(t^2)) \ ,$$

thus

$$0 < \frac{1}{\sin \pi t} = \frac{1}{\pi t}(1 + \mathcal{O}(t^2)) \ .$$

This yields that

$$\frac{1}{\sin \pi t} = \frac{1}{\pi t} + \mathcal{O}(1) \quad \text{for} \quad 0 < t \leq \frac{1}{2} \ .$$

2) We have

$$\begin{aligned}
L_n &= 2 \int_0^{1/2} |D_n(t)|\, dt \\
&= 2 \int_0^{1/2} \frac{|\sin(\pi t(2n+1))|}{\pi t}\, dt + \mathcal{O}(1) \quad (\text{set } t(2n+1) = s) \\
&= \frac{2}{\pi} \int_0^{n+\frac{1}{2}} \frac{1}{s} |\sin(\pi s)|\, ds + \mathcal{O}(1) \\
&= \frac{2}{\pi} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{1}{s} |\sin(\pi s)|\, ds + \mathcal{O}(1) \\
&= \frac{2}{\pi} \int_0^1 |\sin(\pi s)| \sum_{k=1}^{n-1} \frac{1}{s+k}\, ds + \mathcal{O}(1)
\end{aligned}$$

For $0 \leq s \leq 1$ and $k \geq 1$ we have

$$\frac{1}{1+k} \leq \frac{1}{s+k} \leq \frac{1}{k} \ ,$$

thus

$$
\begin{aligned}
\ln n + \Big(\sum_{k=1}^{n-1} \frac{1}{1+k} - \ln n \Big) &= \sum_{k=1}^{n-1} \frac{1}{1+k} \\
&\le \sum_{k=1}^{n-1} \frac{1}{s+k} \\
&\le \sum_{k=1}^{n-1} \frac{1}{k} \\
&= \ln n + \Big(\sum_{k=1}^{n-1} \frac{1}{k} - \ln n \Big) \ .
\end{aligned}
$$

This yields that there exists a constant $C > 0$ so that [5]

$$\ln n - C \le \sum_{k=1}^{n-1} \frac{1}{s+k} \le \ln n + C \ .$$

Since

$$\int_0^1 \sin(\pi s)\, ds = \frac{2}{\pi}$$

the claim,

$$L_n = \frac{4}{\pi^2} \ln n + \mathcal{O}(1) \ ,$$

follows. $\diamond$

11.8.1 The Lebesgue Constant as Norm of a Functional

As above, let X denote the linear space of all continuous functions $f : [0,1] \to \mathbb{C}$ with $f(0) = f(1)$.

On X we use the maximum norm, $|\cdot|_\infty$. Then $(X, |\cdot|_\infty)$ is a Banach space. Define the linear functional $A_n : X \to \mathbb{C}$ by

$$A_n f = S_n f(0) = \int_0^1 D_n(y) f(y)\, dy \ .$$

One defines the norm of A_n by

$$\|A_n\| = \sup\{ |A_n f| \ : \ f \in X, \ |f|_\infty = 1 \} \ .$$

It is not difficult to show that $A_n : X \to \mathbb{C}$ is a bounded linear functional with norm $\|A_n\| = L_n$. In fact, the estimate $\|A_n\| \le L_n$ is easy to show. To see that it is sharp, set [6]

$$h(y) = sgn(D_n(y))$$

[5] Recall that $\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \to \gamma = 0.57721\ldots$

[6] Here sgn denotes the sign–function, $sgn(x) = 1$ for $x > 0$, $sgn(x) = -1$ for $x < 0$, and $sgn(0) = 0$.

and let $h_\varepsilon \in X$ denote a smooth approximation [7] of h with $|h_\varepsilon|_\infty = 1$. We then have for $\varepsilon \to 0$:

$$A_n h_\varepsilon = \int_0^1 D_n(y) h_\varepsilon(y)\, dy \to \int_0^1 |D_n(y)|\, dy = L_n \ .$$

The main point is that, by the previous lemma, we obtain that

$$\|A_n\| \to \infty \quad \text{as} \quad n \to \infty \ .$$

11.9 Divergence of the Fourier Series of a Continuous Function at a Point

In 1873, du Bois–Raymond constructed a continuous function whose Fourier series diverges at a point.

We use the *Uniform–Boundedness Principle* of functional analysis. The corresponding theorem is also called the Banach–Steinhaus Theorem or the *Resonance Principle*. Its proof uses the Baire category theorem of topology. The proof of Baire's category theorem uses the axiom of choice. This indicates that the proof is not constructive.

First recall some simple concepts from functional analysis. If X, Y are normed spaces, then a linear operator

$$T : X \to Y$$

is called bounded if there exists a constant $C \geq 0$ with

$$\|Tf\|_Y \leq C \|f\|_X \quad \text{for all} \quad f \in X \ . \tag{11.5}$$

If $T : X \to Y$ is a bounded linear operator, then its operator norm is defined by

$$\begin{aligned} \|T\| &= \min\{C \geq 0 \ : \ (11.5) \text{ holds}\} \\ &= \sup\{\|Tx\|_Y \ : \ x \in X, \ \|x\|_X = 1\} \end{aligned}$$

The linear space $L(X, Y)$ of all bounded linear operators from X to Y, together with the operator norm, is a normed space.

The *Uniform–Boundedness Principle* is the following remarkable result:

Theorem 11.5 *Let X be a Banach space and Y be a normed space. Let $\mathcal{T} \subset L(X, Y)$, i.e., $\mathcal{T}$ is a set of bounded linear operators $T : X \to Y$. Assume that for every $f \in X$ there is a constant C_f with*

$$\|Tf\|_Y \leq C_f \quad \textit{for all} \quad T \in \mathcal{T} \ .$$

Then there exists a constant C with

$$\|T\| \leq C \quad \textit{for all} \quad T \in \mathcal{T} \ .$$

The following is a reformulation:

[7] The functions $h_\varepsilon(y)$ must satisfy $\int_0^1 |h(y) - h_\varepsilon(y)| dy \to 0$ as $\varepsilon \to 0$. One can choose $h_\varepsilon(y)$ as a continuous, piecewise linear function.

Theorem 11.6 *Let X be a Banach space and Y be a normed space. Let $\mathcal{T} \subset L(X,Y)$. Assume that*

$$\sup\{\|T\| \ : \ T \in \mathcal{T}\} = \infty \ . \tag{11.6}$$

Then there exists $f \in X$ so that

$$\sup\{\|Tf\|_Y \ : \ T \in \mathcal{T}\} = \infty \ . \tag{11.7}$$

This reformulation is called *Resonance Principle*. Roughly, if (11.6) holds, then there exists $f \in X$ which resonates with the family $\mathcal{T}$ of operators, and (11.7) holds.

Application: Let $(X, |\cdot|_\infty)$ denote the Banach space of all continuous functions

$$f : [0,1] \to \mathbb{C}$$

with $f(0) = f(1)$ equipped with the maximum norm. Let $Y = \mathbb{C}$ with $\|z\|_Y = |z|$. For $n = 1, 2, \ldots$ let

$$A_n : X \to \mathbb{C}$$

be defined by

$$A_n f = S_n f(0) = \int_0^1 D_n(y) f(y)\, dy, \quad f \in X \ .$$

As stated above, the operator norm of A_n is L_n and $L_n \to \infty$ by Lemma 11.4.

We apply the resonance principle to the family

$$\mathcal{T} = \{A_n \ : \ n = 1, 2, \ldots\} \ .$$

Since $\|A_n\| \to \infty$ as $n \to \infty$ the resonance principle implies existence of a function $f \in X$ for which

$$A_n f = S_n f(0)$$

is unbounded.

Clearly, this implies that the sequence

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}$$

does not converge for $x = 0$. The Fourier series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}$$

diverges for $x = 0$.

11.10 Isomorphy of $L_2(0,1)$ and l_2

So far, we have shown that

$$|f - \sigma_n f|_\infty \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad f \in X$$

and

$$\|f - S_n f\|_{L_2} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad f \in L_2(0,1) \ . \tag{11.8}$$

Also, if $f \in L_2(0,1)$, then the construction of $S_n f$ shows that the approximation error

$$f - S_n f$$

is orthogonal to $\mathcal{T}_n$. Therefore,

$$\|f\|_{L_2}^2 = \|f - S_n f\|_{L_2}^2 + \|S_n f\|^2 \quad \text{for} \quad f \in L_2(0,1) \ .$$

Since

$$\|S_n f\|_{L_2}^2 = \sum_{k=-n}^{n} |\hat{f}(k)|^2$$

one obtains Parseval's relation:

$$\|f\|_{L_2}^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \ . \tag{11.9}$$

Definition: *Let l_2 denote the linear space of all sequences*

$$a = (a_k)_{k \in \mathbb{Z}}, \quad a_k \in \mathbb{C} \ ,$$

with

$$\sum_k |a_k|^2 < \infty \ .$$

For $a, b \in l_2$ define the l_2 inner product by

$$(a, b)_{l_2} = \sum_k \bar{a}_k b_k \ .$$

As usual, the corresponding norm is defined by

$$\|a\|_{l_2}^2 = (a, a)_{l_2} \ .$$

It is not difficult to show:

Theorem 11.7 *The sequence space l_2 with the above inner product is complete, i.e., it is a Hilbert space.*

Theorem 11.8 *The mapping $F : L_2(0,1) \to l_2$ defined by*

$$F(f) = (\hat{f}(k))_{k\in\mathbb{Z}}, \quad f \in L_2(0,1) \ ,$$

is a Hilbert space isomorphism.

Proof: Parseval's relation (11.9) shows that F maps $L_2(0,1)$ into l_2 and is norm preserving. The linearity of F is clear. Since $\|Ff\|_{l_2} = \|f\|_{L_2}$ it is also clear that F is one–to–one. To prove that F is onto, let $a \in l_2$ be given and define

$$f_n(x) = \sum_{k=-n}^{n} a_k \phi_k(x) \ .$$

Then f_n is a Cauchy sequence in $L_2(0,1)$, thus there exists $f \in L_2(0,1)$ with

$$\|f - f_n\|_{L_2} \to 0 \quad \text{as} \quad n \to \infty \ .$$

We have

$$(\phi_k, f)_{L_2} = \lim_{n\to\infty} (\phi_k, f_n)_{L_2} = a_k \ .$$

Therefore, $Ff = a$, showing that F is onto.

It remains to prove that F preserves the inner product:

$$(Ff, Fg)_{l_2} = (f,g)_{L_2} \ . \tag{11.10}$$

The polarization equality,

$$4(f,g) = \|f+g\|^2 - \|f-g\|^2 + i\|if+g\|^2 - i\|if-g\|^2 \ ,$$

which is valid in any inner product space over $\mathbb{C}$, shows that one can express the inner product in terms of the norm. Therefore, (11.10) follows from (11.9). $\diamond$

11.11 Convergence of $S_n f$ in Maximum Norm

Lemma 11.5 *Let $f : \mathbb{R} \to \mathbb{C}, f(x+1) \equiv f(x)$. If $f \in C^r$ then*

$$|\hat{f}(k)| \le C_r |k|^{-r}, \quad k \in \mathbb{Z}, \quad k \neq 0 \ ,$$

with

$$C_r = (2\pi)^{-r} \int_0^1 |f^{(r)}(x)|\, dx \ .$$

Consequently,

$$|f - S_n f|_\infty \to 0 \quad \textit{as} \quad n \to \infty$$

if $r \ge 2$.

Proof: Through integration by parts one finds that

$$\hat{f}(k) = \frac{1}{2\pi ik}(f')\hat{}(k) \quad \text{for} \quad k \in \mathbb{Z}, \quad k \neq 0 \ .$$

Applying this result r times and noting that

$$(f^{(r)})\hat{}(k) = \int_0^1 e^{-2\pi ikx} f^{(r)}(x)\, dx$$

the estimate of $|\hat{f}(k)|$ follows. Therefore, if $r \geq 2$, we have for $n > m \geq N(\varepsilon)$:

$$|S_n f - S_m f|_\infty \leq \sum_{m<|k|\leq n} |\hat{f}(k)| < \varepsilon \ .$$

It follows that $S_n f$ is a Cauchy sequence in $(X, |\cdot|_\infty)$, and there exists $g \in X$ with $|g - S_n f|_\infty \to 0$. This implies $\|g - S_n f\|_{L_2} \to 0$. We have shown that $\|f - S_n f\|_{L_2} \to 0$, and conclude that $g = f$. $\diamond$

One can relax the assumption $f \in C^2$ slightly.

Theorem 11.9 *Let $f : \mathbb{R} \to \mathbb{C}, f(x+1) \equiv f(x), f \in C^1$. Then the Fourier sums*

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi ikx}$$

converge uniformly to $f(x)$,

$$|f - S_n f|_\infty \to 0 \quad \textit{as} \quad n \to \infty \ .$$

Proof: From

$$\hat{f}(k) = \frac{1}{2\pi ik}(f')\hat{}(k)$$

and

$$\sum_{k=-\infty}^{\infty} |(f')\hat{}(k)|^2 = \|f'\|^2_{L_2}$$

we conclude that

$$\sum_{k=-\infty}^{\infty} |k|^2 |\hat{f}(k)|^2 =: Q < \infty \ .$$

Using the Cauchy–Schwarz inequality,

$$\begin{aligned}
|S_n f - S_m f|_\infty &\leq \sum_{m<|k|\leq n} |\hat{f}(k)| \\
&= \sum_{m<|k|\leq n} \frac{1}{|k|} |k||\hat{f}(k)| \\
&\leq \Big(\sum_{m<|k|\leq n} \frac{1}{|k|^2} \Big)^{1/2} Q^{1/2} \ .
\end{aligned}$$

As in the proof of the previous lemma, it follows that $S_n f$ is a Cauchy sequence in $(X, |\cdot|_\infty)$, and the claim follows. $\diamond$

11.12 Smoothness of $f(x)$ and Decay of $\hat{f}(k)$

Let $f : \mathbb{R} \to \mathbb{C}, f(x) \equiv f(x+1), f \in C$. Since

$$\|f\|_{L_2}^2 = \sum_k |\hat{f}(k)|^2$$

we know that

$$\hat{f}(k) \to 0 \quad \text{as} \quad |k| \to \infty \ .$$

We want to show that the smoothness of f is related to the decay rate of $\hat{f}(k)$.

Smoothness implies decay:

Lemma 11.6 *Let $r \in \{1, 2, \ldots\}$. Let $f \in C^{r-1}(\mathbb{R})$ have period one, and assume that $D^r f = f^{(r)} \in L_1$. Then we have*

$$|\hat{f}(k)| \le C_r |k|^{-r} \quad \text{for} \quad k \neq 0$$

with

$$C_r = (2\pi)^{-r} \int_0^1 |D^r f(x)| \, dx \ .$$

Proof: This follows through integration by parts,

$$\begin{aligned} \hat{f}(k) &= \int_0^1 e^{-2\pi ikx} f(x) \, dx \\ &= \frac{1}{2\pi ik} \int_0^1 e^{-2\pi ikx} Df(x) \, dx \\ &= \frac{1}{(2\pi ik)^r} \int_0^1 e^{-2\pi ikx} D^r f(x) \, dx \end{aligned}$$

◇

Decay implies smoothness:

Lemma 11.7 *Let $r \in \{0, 1, 2, \ldots\}$. Let $f : \mathbb{R} \to \mathbb{C}$ have period one and let $f \in L_1(0, 1)$. Assume that there exists $A > 0, \delta > 0$ with*

$$|\hat{f}(k)| \le A|k|^{-(r+1+\delta)} \quad \text{for all} \quad k \neq 0 \ .$$

Then $f \in C^r$.

Proof: We assume that the above estimate holds for $r = 1$ and must show that $f \in C^1$. We have

$$\begin{aligned} S_n f(x) &= \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi ikx} \\ (S_n f)'(x) &= \sum_{k=-n}^{n} 2\pi ik \hat{f}(k) e^{2\pi ikx} \end{aligned}$$

For $n > m$:

$$\begin{aligned} |(S_n f)' - (S_m f)'|_\infty &\leq 2\pi \sum_{m<|k|\leq n} |k||\hat{f}(k)| \\ &\leq 2\pi A \sum_{m<|k|\leq n} |k|^{-(1+\delta)} \\ &\leq \varepsilon \end{aligned}$$

for $n > m \geq N(\varepsilon)$. Therefore, there exists $g \in X$ with

$$|g - (S_n f)'|_\infty \to 0 \quad \text{as} \quad n \to \infty \ .$$

Also,

$$|f - S_n f|_\infty \to 0 \quad \text{as} \quad n \to \infty \ .$$

Let $n \to \infty$ in the equation

$$(S_n f)(x) - (S_n f)(0) = \int_0^x (S_n f)'(y)\, dy$$

to obtain that

$$f(x) - f(0) = \int_0^x g(y)\, dy \ .$$

It follows that $f \in C^1$ and $f' = g$. The proof for other values of r is similar. $\diamond$

11.13 Exponential Decay of $\hat{f}(k)$ and Analyticity of f

For $\alpha > 0$ let S_α denote the horizontal strip of width 2α along the real axis:

$$S_\alpha = \{z = x + iy \ : \ x \in \mathbb{R},\ |y| < \alpha\} \ .$$

Lemma 11.8 *Let $f \in H(S_\alpha), f(z) \equiv f(z+1)$. Then, for any $0 \leq \beta < \alpha$, there is a constant $C_\beta = C_\beta(f)$ with*

$$|\hat{f}(k)| \leq C_\beta e^{-2\pi|k|\beta} \quad \textit{for all} \quad k \in \mathbb{Z} \ .$$

Proof: For definiteness, let $k < 0$. Consider the rectangle

$$\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$$

where

$$\begin{aligned} \Gamma_1 &: \ z(x) = x, \quad 0 \leq x \leq 1 \ , \\ \Gamma_2 &: \ z(y) = 1 + iy, \quad 0 \leq y \leq \beta \ , \\ -\Gamma_3 &: \ z(x) = x + i\beta, \quad 0 \leq x \leq 1 \ , \\ -\Gamma_4 &: \ z(y) = iy, \quad 0 \leq y \leq \beta \ . \end{aligned}$$

Let

$$g(z) = f(z)e^{-2\pi ikz} \; .$$

We then have

$$\hat{f}(k) = \int_{\Gamma_1} g(z)\, dz$$

and, by Cauchy's theorem,

$$\sum_{j=1}^{4} \int_{\Gamma_j} g(z)\, dz = 0 \; .$$

The periodicity of f also implies that

$$\int_{\Gamma_2} g(z)\, dz + \int_{\Gamma_4} g(z)\, dz = 0 \; .$$

Consequently,

$$\hat{f}(k) = \int_{-\Gamma_3} g(z)\, dz \; .$$

We have justified to move the integration path Γ_1 upwards by β. If $z \in \Gamma_3$ then $z = x + i\beta$ and $-2\pi ikz = -2\pi ikx + 2\pi k\beta$, thus

$$|e^{-2\pi ikz}| = e^{2\pi k\beta} = e^{-2\pi|k|\beta} \quad \text{for} \quad k < 0 \; .$$

It follows that

$$|\hat{f}(k)| \le C_\beta e^{-2\pi|k|\beta}$$

with

$$C_\beta = \max_{0 \le x \le 1} |f(x + i\beta)| \; .$$

If $k > 0$ we can argue similarly, moving the path of integration down by β. $\diamond$

The previous lemma says that the Fourier coefficients $\hat{f}(k)$ of a periodic function $f(x)$ decay exponentially as $|k| \to \infty$ if $f(x)$ can be continued analytically into a strip along the real axis. The following lemma shows a converse.

Lemma 11.9 *Let $\gamma > 0$ and let $a = (a_k) \in l_2$ denote a sequence with*

$$|a_k| \le Ce^{-2\pi|k|\gamma} \quad \textit{for all} \quad k \in \mathbb{Z} \; .$$

Then there exists a unique function $f \in H(S_\gamma)$ with $f(z) \equiv f(z+1)$ and

$$\hat{f}(k) = a_k, \quad k \in \mathbb{Z} \; .$$

Proof: Set

$$f_n(z) = \sum_{k=-n}^{n} a_k e^{2\pi ikz} \ .$$

Fix any β with $0 < \beta < \gamma$. Let $z = x + iy$ where $|y| \le \beta < \gamma$ and obtain that

$$|e^{2\pi ikz}| \le e^{2\pi|k|\beta} \ .$$

Therefore,

$$|a_k||e^{2\pi ikz}| \le Ce^{-2\pi|k|(\gamma-\beta)} \quad \text{where} \quad \gamma - \beta > 0 \ .$$

If $K \subset S_\gamma$ is any compact set, then $K \subset \bar{S}_\beta$ for some β with $0 < \beta < \gamma$, and the above estimate implies that the series

$$\sum_k a_k e^{2\pi ikz}$$

converges normally in S_γ to an analytic function $f \in H(S_\gamma)$. It is clear that $\hat{f}(k) = a_k$ and that f is unique. $\diamond$

11.14 Divergence of $S_n f(0)$: Explicit Construction of $f \in X$

The following is close to [Koerner].

We show existence of a function $f \in X$ for which

$$S_n f(0)$$

is unbounded. Recall that

$$S_n f(y) = \int_{-1/2}^{1/2} D_n(y-x) f(x)\, dx \ ,$$

thus

$$S_n f(0) = \int_{-1/2}^{1/2} D_n(-x) f(x)\, dx = \int_{-1/2}^{1/2} D_n(x) f(x)\, dx \ .$$

Here

$$D_n(x) = \frac{\sin(\pi(2n+1)x)}{\sin(\pi x)}$$

is the Dirichlet kernel. We have shown that the Lebesgue constants diverge,

$$L_n = \int_{-1/2}^{1/2} |D_n(x)|\, dx \to \infty \quad \text{as} \quad n \to \infty \ .$$

Let

$$h_n(x) = sgn\, D_n(x) \ .$$

We then have

$$S_n h_n(0) = L_n \to \infty \quad \text{as} \quad n \to \infty \ .$$

The piecewise constant function h_n is not continuous, but we can approximate h_n by a piecewise linear function $g_n \in X$ so that

$$\int_{-1/2}^{1/2} |h_n(x) - g_n(x)| \, dx \le \frac{1}{L_n} \quad \text{and} \quad |g_n|_\infty = 1 \ .$$

One finds that

$$S_n g_n(0) \ge L_n - 1, \quad \text{for} \quad n \ge N \ .$$

Since the trigonometric polynomials are dense in $(X, |\cdot|_\infty)$, there exists $f_n \in \mathcal{T}$ with $|g_n - f_n|_\infty \le \frac{1}{L_n}$. One then finds that

$$S_n f_n(0) \ge L_n - 2, \quad |f_n|_\infty \le 2 \ .$$

Dividing by 2, we obtain a trigonometric polynomial $\frac{1}{2} f_n$ with

$$S_n(\frac{1}{2} f_n)(0) \ge \frac{1}{2}(L_n - 2), \quad |\frac{1}{2} f_n|_\infty \le 1 \ .$$

Since $L_n \to \infty$ as $n \to \infty$, the following lemma is proved:

Lemma 11.10 *Given any constant $A > 0$ there exists $H \in \mathcal{T}$ and $N \in \mathbb{N}$ with*

$$S_N H(0) \ge A, \quad |H|_\infty \le 1 \ .$$

The remaining part of the proof may be called *piling up of bad functions.* Take $A = A_k = 2^{2k}$ for $k = 1, 2, \ldots$ By the previous lemma we obtain sequences $H_k \in \mathcal{T}$ and $n(k) \in \mathbb{N}$ with

$$S_{n(k)} H_k(0) \ge 2^{2k}, \quad |H_k|_\infty \le 1 \ .$$

Since $H_k(x)$ is a trigonometric polynomial, there exists $q(k) \in \mathbb{N}$ so that

$$H_k(x) = \sum_{|j| \le q(k)} \hat{H}_k(j) e^{2\pi i j x} \tag{11.11}$$

and we may assume that

$$q(k+1) \ge q(k) \ge n(k) \ .$$

Set

$$p(k) = \sum_{j=1}^{k} (2q(j) + 1)$$

and define the trigonometric polynomials

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} H_k(x) e^{2\pi i p(k) x} \ , \quad n = 1, 2, \ldots \tag{11.12}$$

The factor 2^{-k} makes the sequence $f_n(x)$ a Cauchy sequence with the respect to $|\cdot|_\infty$; the factor $e^{2\pi i p(k)x}$ makes every frequency of $H_{k+1}(x)e^{2\pi i p(k+1)x}$ larger than every frequency of $H_k(x)e^{2\pi i p(k)x}$. See below.

If $n > m$ then

$$|f_n - f_m|_\infty \le \sum_{k=m+1}^{n} 2^{-k} \le 2^{-m} \ .$$

This shows that f_n is a Cauchy sequence in X. Since $(X, |\cdot|_\infty)$ is complete there exists $f \in X$ with

$$|f - f_n|_\infty \to 0 \ .$$

We will show that the sequence $S_n f(0)$ is unbounded.

Convergence $|f - f_n|_\infty \to 0$ implies that

$$\hat{f}_n(r) \to \hat{f}(r) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad r \in \mathbb{Z} \ .$$

The main point of the construction is the following: The integers $p(k)$ are so large that the terms in the sum (11.12) have no common frequencies. This makes the following analysis possible.

Substituting the right–hand side of (11.11) for $H_k(x)$ into (11.12) yields that

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} \sum_{|j| \le q(k)} \hat{H}_k(j)\, e^{2\pi i (p(k)+j)x} \ .$$

Set

$$Q_k(x) = \sum_{|j| \le q(k)} \hat{H}_k(j)\, e^{2\pi i (p(k)+j)x} \ .$$

Note that the largest frequency of $Q_k(x)$ is

$$2\pi \Big(p(k) + q(k) \Big)$$

and the smallest frequency of $Q_{k+1}(x)$ is

$$2\pi \Big(p(k+1) - q(k+1) \Big) \ .$$

Since $p(k+1) = p(k) + 2q(k+1) + 1$ and $q(k+1) \ge q(k)$ we have

$$\begin{aligned} p(k+1) - q(k+1) &= p(k) + q(k+1) + 1 \\ &> p(k) + q(k) \end{aligned}$$

This shows that any two of the functions $Q_l(x), l = 1, 2, \ldots$ have no common frequency.

We obtain

$$\hat{f}_n(p(k)+j) = 2^{-k} \hat{H}_k(j) \quad \text{if} \quad n \ge k \quad \text{and} \quad |j| \le q(k) \quad \text{and} \quad 1 \le k \le n \ .$$

Letting $n \to \infty$ we obtain that

$$\hat{f}(p(k)+j) = 2^{-k} \hat{H}_k(j) \quad \text{if} \quad |j| \le q(k) \ .$$

If $l \in \mathbb{Z}$ is not of the form $l = p(k) + j$ for some j with $|j| \le q(k)$ and some $k \in \mathbb{N}$, then $\hat{f}_n(l) = 0$ for all n and $\hat{f}(l) = 0$.

Now consider

$$\begin{aligned} S_{p(k)+n(k)}f(0) - S_{p(k)-n(k)-1}f(0) &= \sum_{|j|\le n(k)} \hat{f}(p(k)+j) \\ &= 2^{-k} \sum_{|j|\le n(k)} \hat{H}_k(j) \\ &= 2^{-k} S_{n(k)} H_k(0) \end{aligned}$$

(In the second equation we have used that $n(k) \le q(k)$.) We now recall the estimate

$$S_{n(k)} H_k(0) \ge 2^{2k}$$

to obtain that

$$|S_{p(k)+n(k)}f(0) - S_{p(k)-n(k)-1}f(0)| \ge 2^k, \quad k = 1, 2, \ldots$$

This implies that the sequence $S_n f(0)$ is unbounded.

11.15 Fourier Series and the Dirichlet Problem for Laplace's Equation on the Unit Disk

The Poisson kernel $P_r(\alpha)$ is the Abel sum of

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik\alpha} .$$

In other words,

$$P_r(\alpha) = \frac{1}{2\pi} \sum_{k=0}^{\infty} (re^{i\alpha})^k + \sum_{k=1}^{\infty} (re^{-i\alpha})^k \quad \text{for} \quad 0 \le r < 1 .$$

12 Fourier Transformation

12.1 Motivation: Application to PDEs

Example 1: Consider the PDE

$$u_t + a u_x = 0$$

for a function $u(x,t)$ where $a \in \mathbb{R}$. Give an initial condition

$$u(x,0) = e^{ikx}, \quad x \in \mathbb{R} .$$

The initial function

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

is a wave with wavenumber k (assumed to be real). Roughly, the wave e^{ikx} has $|k|$ waves in the interval $0 \le x \le 2\pi$ and the wavelength is $2\pi/|k|$. To obtain a solution of the PDE $u_t + au_x = 0$ with initial condition $u(x,0) = e^{ikx}$ try the ansatz

$$u(x,t) = \alpha(t) e^{ikx} .$$

One obtains that

$$\alpha'(t) + iak\,\alpha(t) = 0, \quad \alpha(0) = 1 ,$$

thus

$$\alpha(t) = e^{-iakt}, \quad u(x,t) = e^{ik(x-at)} .$$

The solution $u(x,t)$ describes that the initial wave e^{ikx} moves undistorted at speed a. If $a > 0$ the wave moves to the right, if $a < 0$ it moves to the left.

Example 2: Consider the heat equation

$$u_t = u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t \ge 0 ,$$

with initial condition

$$u(x,0) = e^{ikx}, \quad x \in \mathbb{R} .$$

The ansatz

$$u(x,t) = \alpha(t) e^{ikx}$$

leads to

$$\alpha'(t) = -k^2 \alpha(t), \quad \alpha(0) = 1 ,$$

with solution

$$\alpha(t) = e^{-k^2 t} .$$

The solution of the PDE $u_t = u_{xx}$ with initial condition $u(x,0) = e^{ikx}$ is

$$u(x,t) = e^{-k^2 t} e^{ikx} \ .$$

We note that the amplitude decays exponentially as t increases if $k \neq 0$. The smaller the wavelength of the initial function $u(x,0) = e^{ikx}$, the larger is $|k|$ and the more rapid is the decay in time of the solution $u(x,t)$. We expect, therefore, that the solution of the heat equation will become smooth if we start with rough initial data $u(x,0) = f(x)$.

Example 3: Consider the linearized Korteweg–de Vries equation

$$u_t = u_{xxx} \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with initial condition

$$u(x,0) = e^{ikx}, \quad x \in \mathbb{R} \ .$$

The ansatz

$$u(x,t) = \alpha(t) e^{ikx}$$

leads to

$$\alpha'(t) = -ik^3 \alpha(t), \quad \alpha(0) = 1 \ ,$$

with solution

$$\alpha(t) = e^{-ik^3 t} \ .$$

The solution of the PDE $u_t = u_{xxx}$ with initial condition $u(x,0) = e^{ikx}$ is

$$u(x,t) = e^{ik(x - k^2 t)} \ .$$

We note that the amplitude $\alpha(t)$ satisfies $|\alpha(t)| = 1$, thus there is neither decay nor growth. Here the wave speed is k^2, i.e., the wave speed depends on the wave length of the initial function e^{ikx}. We have the effect of dispersion.

Example 4: Consider the free–space Schrödinger equation

$$u_t = i u_{xx} \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

with initial condition

$$u(x,0) = e^{ikx}, \quad x \in \mathbb{R} \ .$$

The ansatz

$$u(x,t) = \alpha(t) e^{ikx}$$

leads to

$$\alpha'(t) = -ik^2 \alpha(t), \quad \alpha(0) = 1 \ ,$$

with solution

$$\alpha(t) = e^{-ik^2 t} \ .$$

The solution of the PDE $u_t = iu_{xx}$ with initial condition $u(x,0) = e^{ikx}$ is

$$u(x,t) = e^{ik(x-kt)} \ .$$

We note that the amplitude $\alpha(t)$ satisfies $|\alpha(t)| = 1$, thus there is neither decay nor growth. Here the wave speed is k, i.e., the wave speed depends on the wave length of the initial function e^{ikx}. We have the effect of dispersion.

More General Equation: Consider the constant–coefficient equation

$$u_t = Lu \equiv \sum_{j=0}^{m} a_j D^j u \quad \text{for} \quad x \in \mathbb{R}, \quad t \geq 0 \ ,$$

where

$$a_j \in \mathbb{C}, \quad D = \frac{\partial}{\partial x} \ .$$

Given an initial condition

$$u(x,0) = e^{ikx}, \quad x \in \mathbb{R} \ ,$$

the ansatz

$$u(x,t) = \alpha(t) e^{ikx}$$

leads to

$$\alpha'(t) = \hat{L}(ik)\alpha(t), \quad \alpha(0) = 1 \ ,$$

where

$$\hat{L}(ik) = \sum_{j=0}^{m} a_j (ik)^j$$

is the so–called symbol of L. The solution of the amplitude equation is

$$\alpha(t) = e^{\hat{L}(ik)t}$$

and the solution of the PDE $u_t = Lu$ with initial condition $u(x,0) = e^{ikx}$ is

$$u(x,t) = e^{\hat{L}(ik)t} e^{ikx} \ .$$

So far we have only considered the initial condition $u(x,0) = e^{ikx}$. If one wants to solve the initial value problem

$$u_t = Lu, \quad u(x,0) = f(x) \ ,$$

with more general functions $f(x)$ then one may try to write $f(x)$ as a superposition of the simple waves e^{ikx}. The tool is the Fourier transform.

12.2 The Fourier Transform of an L_1–Function

For $f \in L_1 = L_1(\mathbb{R}, \mathbb{C})$ one defines its Fourier transform by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx, \quad \xi \in \mathbb{R} \; . \tag{12.1}$$

(We follow the conventions in [Stein, Shakarchi]. Other forms, like

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i x \xi}\, dx, \quad \xi \in \mathbb{R} \; ,$$

are also used.)

Remark: The condition $f \in L_1$ is necessary if one wants to use the standard meaning of the integral in (12.1). However, using the notions of distribution theory, one can define a Fourier transform for functions and distributions that are not in L_1.

Lemma 12.1 *If $f \in L_1$ then $\hat{f}$ is uniformly continuous.*

Proof: Let $\varepsilon > 0$ be given. We must show existence of $\delta > 0$ so that $|\xi_1 - \xi_2| < \delta$ implies $|\hat{f}(\xi_1) - \hat{f}(\xi_2)| < \varepsilon$.

Since $f \in L_1$ there exists $R > 0$ with

$$\int_{|x|>R} |f(x)|\, dx \le \frac{\varepsilon}{4} \; .$$

This can be shown by a cut–off argument and Lebesgue's dominated convergence theorem. We have

$$\begin{aligned} |\hat{f}(\xi_1) - \hat{f}(\xi_2)| &\le \int_{|x|>R} |f(x)||e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}|\, dx \\ &\quad + \int_{|x|\le R} |f(x)||e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}|\, dx \\ &\le \frac{\varepsilon}{4} \cdot 2 + \|f\|_{L_1} \max_{|x|\le R} |e^{-2\pi i x \xi_1} - e^{-2\pi i x \xi_2}| \; . \end{aligned}$$

Here

$$\|f\|_{L_1} = \int_{-\infty}^{\infty} |f(x)|\, dx \; .$$

The maximum in the above formula equals

$$M = \max_{|x|\le R} |1 - e^{-2\pi i x \xi}| \quad \text{where} \quad \xi = \xi_2 - \xi_1 \; .$$

Since the function

$$g(x,\xi) = |1 - e^{-2\pi i x \xi}|$$

is uniformly continuous on the set

$$|x| \le R, \quad |\xi| \le 1$$

and since

$$g(x,0) = 0 \ ,$$

there exists $\delta > 0$ so that

$$g(x,\xi) < \varepsilon' \quad \text{for} \quad |\xi| < \delta, \quad |x| \le R \ .$$

Choosing

$$\varepsilon' = \frac{\varepsilon}{2} \, \frac{1}{\|f\|_{L_1}}$$

we obtain $|\hat{f}(\xi_1) - \hat{f}(\xi_2)| < \varepsilon$ for $|\xi_1 - \xi_2| < \delta$. $\diamond$

What can be said about the decay of $|\hat{f}(\xi)|$ as $|\xi| \to \infty$? Through integration by parts one obtains:

Lemma 12.2 *Let $f \in C_0^1$, i.e., f is a C^1 function with compact support. Then*

$$|\hat{f}(\xi)| \le \frac{1}{2\pi|\xi|} \, \|f'\|_{L_1}, \quad \xi \neq 0 \ .$$

Proof: We have

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx \\ &= f(x) \frac{1}{-2\pi i \xi} \, e^{-2\pi i x \xi} \Big|_{x=-\infty}^{x=\infty} + \frac{1}{2\pi i \xi} \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi} \, dx \end{aligned}$$

The boundary term equals zero since $f(x)$ has compact support. The estimate follows. $\diamond$

12.3 The Riemann–Lebesgue Lemma

The following result is known as the Riemann–Lebesgue Lemma:

Lemma 12.3 *If $f \in L_1$ then $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.*

To prove the Riemann–Lebesgue Lemma, we need the following auxiliary result:

Lemma 12.4 *The set C_0^∞ of all C^∞ functions with compact support is dense in $(L_1, \| \cdot \|_{L_1})$.*

Proof: Let $f \in L_1$ and let $\varepsilon > 0$ be given. Choose $R > 0$ so large that

$$\int_{|x|>R} |f(x)| \, dx < \frac{\varepsilon}{2} \ .$$

We apply a cut–off to the function f and set

$$g(x) = \begin{cases} f(x) & |x| \le R \\ 0 & |x| > R \end{cases} \ .$$

We then have $\|f - g\|_{L_1} \le \frac{\varepsilon}{2}$ and g has compact support.

We now mollify g. This process is as follows: Choose a C^∞ function $\Phi : \mathbb{R} \to \mathbb{R}$ with

$$\Phi(y) > 0 \quad \text{for} \quad -1 < y < 1, \quad \Phi(y) = 0 \quad \text{for} \quad |y| \geq 1$$

and

$$\int_{-\infty}^{\infty} \Phi(y)\, dy = 1 \ .$$

Note that

$$\int_{-\infty}^{\infty} \frac{1}{\varepsilon}\Phi(y/\varepsilon)\, dy = 1 \quad \text{for} \quad \varepsilon > 0$$

and the function $y \to \Phi(y/\varepsilon)$ is supported in $|y| \leq \varepsilon$. Define

$$g_\varepsilon(x) = \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\Phi((x-y)/\varepsilon) g(y)\, dy \ .$$

It is not difficult to prove that $g_\varepsilon \in C_0^\infty$ and $\|g - g_\varepsilon\|_{L_1} \to 0$ as $\varepsilon \to 0$. $\diamond$

Proof of Riemann–Lebesgue Lemma: Let $f \in L_1$ and let $\varepsilon > 0$ be given. By Lemma 12.4 there exists $f_\varepsilon \in C_0^\infty$ so that

$$\|f - f_\varepsilon\|_{L_1} < \frac{\varepsilon}{2} \ .$$

This implies that

$$|\hat{f}(\xi) - \hat{f}_\varepsilon(\xi)| < \frac{\varepsilon}{2} \quad \text{for} \quad \xi \in \mathbb{R} \ .$$

Since (by Lemma 12.2)

$$|\hat{f}_\varepsilon(\xi)| \leq \frac{C_\varepsilon}{|\xi|} \quad \text{for} \quad \xi \neq 0$$

we obtain the estimate

$$|\hat{f}(\xi)| \leq \frac{\varepsilon}{2} + \frac{C_\varepsilon}{|\xi|} < \varepsilon \quad \text{for} \quad |\xi| \geq R_\varepsilon \ .$$

$\diamond$

Example 1: Let

$$f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \ .$$

We have

$$\begin{aligned} \hat{f}(\xi) &= \int_{-1}^{1} e^{-2\pi i x\xi}\, dx \\ &= \frac{1}{-2\pi i\xi}\, e^{-2\pi i x\xi}\Big|_{x=-1}^{x=1} \\ &= \frac{\sin(2\pi\xi)}{\pi\xi} \ . \end{aligned}$$

Since

$$\int_0^R |\hat{f}(\xi)|\,d\xi \to \infty \quad \text{as} \quad R \to \infty$$

we conclude that $\hat{f} \notin L_1$. The example shows that, in general, the assumption $f \in L_1$ does not imply $\hat{f} \in L_1$.

12.4 The Fourier Transform on the Schwartz Space $\mathcal{S}$

A function $f : \mathbb{R} \to \mathbb{C}$ is called rapidly decreasing if

$$|x^n f(x)|, \quad x \in \mathbb{R} \ ,$$

is bounded for all $n \in \mathbb{N}$. The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}, \mathbb{C})$ consists of all $f \in C^\infty(\mathbb{R}, \mathbb{C})$ for which all derivatives $f^{(k)}(x), k = 0, 1, 2, \ldots$ are rapidly decreasing.

For $f \in \mathcal{S}$ the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\,dx, \quad \xi \in \mathbb{R} \ ,$$

is well–defined.

Theorem 12.1 *a) If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$.*

b) If $f \in \mathcal{S}$ then

$$(f')\hat{}(\xi) = 2\pi i \xi\, \hat{f}(\xi) \ .$$

c) If $f \in \mathcal{S}$ then $-2\pi i x f(x) \in \mathcal{S}$ and

$$(\hat{f})'(\xi) = \Big(-2\pi i x f(x) \Big)\hat{}(\xi) \ .$$

Proof: We know that $\hat{f} \in C$ and $\hat{f}$ is bounded. Further, $f' \in \mathcal{S}$ and, through integration by parts,

$$\begin{aligned} (f')\hat{}(\xi) &= \int_{-\infty}^{\infty} f'(x) e^{-2\pi i x \xi}\,dx \\ &= \int_{-\infty}^{\infty} f(x)(2\pi i \xi) e^{-2\pi i x \xi}\,dx \\ &= 2\pi i \xi \hat{f}(\xi) \end{aligned}$$

This shows that $\xi \hat{f}(\xi)$ is bounded and b) holds. By further differentiations, we obtain that $|\xi^n \hat{f}(\xi)|$ is bounded for every n.

We now show that $\hat{f} \in C^1$ and

$$(\hat{f})'(\xi) = \hat{g}(\xi) \quad \text{with} \quad g(x) = -2\pi i x f(x) \ .$$

To this end, let $\xi \in \mathbb{R}$ be fixed and let h be a real number, $h \neq 0$. Consider

$$\begin{aligned} Q_h &:= \frac{1}{h}(\hat{f}(\xi + h) - \hat{f}(\xi)) - \hat{g}(\xi) \\ &= \int f(x) e^{-2\pi i x \xi} \Big(\frac{1}{h}(e^{-2\pi i x h} - 1) + 2\pi i x \Big)\,dx \ . \end{aligned}$$

We must show that for any $\varepsilon > 0$ there is $h_0 > 0$ so that $|Q_h| < \varepsilon$ for $0 < |h| \le h_0$.

Set

$$B(h,x) = \frac{1}{h}(e^{-2\pi ixh} - 1) + 2\pi ix \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad h \neq 0 \ .$$

We have

$$e^{-2\pi xh} = 1 - 2\pi ixh + \mathcal{O}(x^2h^2) \quad \text{for} \quad |xh| \le 1 \ .$$

Therefore,

$$|B(h,x)| \le C|hx^2| \quad \text{for} \quad |hx| \le 1 \ .$$

Also, for all $h \neq 0$ and all $x \in \mathbb{R}$

$$|B(h,x)| \le \frac{2}{|h|} + 2\pi|x| \le C|x| \quad \text{if} \quad |xh| \ge 1 \ .$$

One obtains that

$$|B(h,x)| \le C|x| \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad h \neq 0 \ .$$

Therefore,

$$|Q_h| \le C \int_{|x|\ge R} |f(x)||x| \, dx + \int_{|x|\le R} |f(x)||B(h,x)| \, dx \ .$$

Choose $R > 0$ so that

$$C \int_{|x|\ge R} |f(x)||x| \, dx \le \frac{\varepsilon}{2} \ .$$

Then choose $h_0 > 0$ so that $h_0 R \le 1$ and obtain that $|xh| \le 1$ for $0 < |h| \le h_0$ and $|x| \le R$, thus $|B(h,x)| \le C|hx^2|$. Therefore,

$$\int_{|x|\le R} |f(x)||B(h,x)| \, dx \le C|h| \le \frac{\varepsilon}{2}$$

if $|h|$ is small enough. This proves that $|Q_h| < \varepsilon$ if $|h|$ is small enough. This proof of c) is complete.

With the same arguments as above, it follows that $|\xi^n (\hat{f})'(\xi)|$ is bounded for every n. The argument can be repeated for every derivative of $\hat{f}(\xi)$, and the theorem is proved. $\diamond$

12.5 The Fourier Inversion Formula on the Schwartz Space: Preparations

The following is an important definite integral:

Lemma 12.5

$$J := \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = 1 \ .$$

Proof: Using Fubini's theorem and transformation to polar coordinates, we have

$$
\begin{aligned}
J^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,dxdy \\
&= 2\pi \int_0^\infty re^{-\pi r^2}\,dr \\
&= \int_0^\infty e^{-\rho}\,d\rho \\
&= 1\ .
\end{aligned}
$$

$\diamond$

We next show that the Fourier transform of a Gaussian is a Gaussian:

Lemma 12.6 *The Fourier transform of*

$$f(x) = e^{-\pi x^2}$$

is

$$\hat{f}(\xi) = e^{-\pi \xi^2}\ .$$

Proof: Set

$$F(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\,dx\ .$$

We then have $F(0) = 1$ and, using integration by parts,

$$
\begin{aligned}
F'(\xi) &= \int e^{-\pi x^2}(-2\pi i x)e^{-2\pi i x\xi}\,dx \\
&= i\int \frac{d}{dx}(e^{-\pi x^2})\,e^{-2\pi i x\xi}\,dx \\
&= -2\pi\xi F(\xi)
\end{aligned}
$$

Thus we have shown that the function $\hat{f}(\xi) = F(\xi)$ satisfies

$$F'(\xi) = -2\pi\xi F(\xi), \quad F(0) = 1\ .$$

Unique solubility of this initial value problem proves that

$$\hat{f}(\xi) = F(\xi) = e^{-\pi\xi^2}\ .$$

$\diamond$

By linear scalings $(x/\sqrt{\delta} = y)$ one obtains:

Lemma 12.7 *For* $\delta > 0$ *let*

$$K_\delta(x) = \frac{1}{\sqrt{\delta}}\,e^{-\pi x^2/\delta}, \quad G_\delta(\xi) = e^{-\pi\xi^2\delta}\ .$$

Then we have

$$\hat{K}_\delta = G_\delta \quad \textit{and} \quad \hat{G}_\delta = K_\delta\ .$$

Further properties of the family of functions $K_\delta(x)$ are:

Lemma 12.8 *a)*

$$\int_{-\infty}^{\infty} K_\delta(x)\,dx = 1, \quad \delta > 0 \ .$$

b) If $\eta > 0$ *is fixed, then*

$$\int_{|x|\ge\eta} K_\delta(x)\,dx \to 0 \quad \textit{as} \quad \delta \to 0 \ .$$

c)

$$\int_{-\infty}^{\infty} |x| K_\delta(x)\,dx \to 0 \quad \textit{as} \quad \delta \to 0 \ .$$

d) If $f \in \mathcal{S}$ *then*

$$\int_{-\infty}^{\infty} f(x) K_\delta(x)\,dx \to f(0) \quad \textit{as} \quad \delta \to 0 \ .$$

Proof: a) follows from

$$\int_{-\infty}^{\infty} K_\delta(x)\,dx = G_\delta(0) = 1 \ .$$

Using the substitution $x/\sqrt{\delta} = y$ we have

$$\begin{aligned} \int_{\eta}^{\infty} K_\delta(x)\,dx &= \frac{1}{\sqrt{\delta}} \int_{\eta}^{\infty} e^{-\pi(x/\sqrt{\delta})^2}\,dx \\ &= \int_{\eta/\sqrt{\delta}}^{\infty} e^{-\pi y^2}\,dy \end{aligned}$$

and b) follows. To show c) we compute

$$\begin{aligned} \int_{0}^{\infty} x K_\delta(x)\,dx &= \frac{1}{\sqrt{\delta}} \int_{0}^{\infty} x e^{-\pi(x/\sqrt{\delta})^2}\,dx \\ &= \sqrt{\delta} \int_{0}^{\infty} e^{-\pi y^2}\,dy \ . \end{aligned}$$

Finally,

$$\int_{-\infty}^{\infty} f(x) K_\delta(x)\,dx = f(0) + \int_{-\infty}^{\infty} (f(x) - f(0)) K_\delta(x)\,dx \ .$$

Here $|f(x) - f(0)| \le C|x|$, and the claim follows from c). $\diamond$

We next prove the following multiplication rule:

Theorem 12.2 *a) If* $f, g \in \mathcal{S}$ *then* $fg \in \mathcal{S}$. *Therefore,* $f\hat{g} \in \mathcal{S}$.
b) If $f, g \in \mathcal{S}$ *then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(y) g(y)\,dy \ .$$

Proof: If $f, g \in \mathcal{S}$ then Leibniz' rule yields $fg \in \mathcal{S}$. Furthermore, using Fubini's theorem,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi ixy}\,dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi ixy}\,dxdy \\
&= \int_{-\infty}^{\infty} \hat{f}(y)g(y)\,dy
\end{aligned}$$

This proves the multiplication rule. $\diamond$

12.6 The Fourier Inversion Formula on the Schwartz Space

Any function $f \in \mathcal{S}$ can be written as a Fourier integral. The formula (12.2) is called the Fourier representation of $f(x)$ or the Fourier inversion formula.

Theorem 12.3 *If $f \in \mathcal{S}$ then we have*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}\,d\xi, \quad x \in \mathbb{R}\ . \tag{12.2}$$

Proof: First, since the Fourier transform of $G_\delta(\xi)$ is $K_\delta(x)$, we have for any $\delta > 0$:

$$\int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{\delta}}\,e^{-\pi x^2/\delta}\,dx = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\pi\xi^2\delta}\,d\xi\ . \tag{12.3}$$

In this equation we take the limit $\delta \to 0$ and obtain

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\,d\xi\ . \tag{12.4}$$

This proves the Fourier inversion formula for $x = 0$. Next, fix x and set

$$F(y) = f(x+y), \quad y \in \mathbb{R}\ .$$

We have

$$\begin{aligned}
\hat{F}(\xi) &= \int_{-\infty}^{\infty} f(x+y)e^{-2\pi iy\xi}\,dy \\
&= \int_{-\infty}^{\infty} f(z)e^{-2\pi iz\xi}e^{2\pi ix\xi}\,dz \\
&= \hat{f}(\xi)\,e^{2\pi ix\xi}\ .
\end{aligned}$$

Using (12.4) with f replaced by F yields

$$\begin{aligned}
f(x) &= F(0) \\
&= \int_{-\infty}^{\infty} \hat{F}(\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi ix\xi}\,d\xi\ .
\end{aligned}$$

This proves the inversion formula. $\diamond$

12.7 Operators

We summarize the results by defining the following operators from $\mathcal{S}$ into itself:

$$\begin{aligned}
(\mathcal{F}f)(\xi) &= \int f(x)e^{-2\pi ix\xi}\,dx \\
(\mathcal{G}g)(x) &= \int g(\xi)e^{2\pi ix\xi}\,d\xi \\
(Sf)(x) &= f(-x) \\
(Bf)(x) &= \bar{f}(x)
\end{aligned}$$

The Fourier Inversion Theorem says that

$$\mathcal{G}\mathcal{F} = id_{\mathcal{S}}\ .$$

Since

$$\mathcal{G} = \mathcal{F}S = S\mathcal{F}$$

one obtains that

$$\mathcal{F}\mathcal{G} = \mathcal{F}(S\mathcal{F}) = (\mathcal{F}S)\mathcal{F} = \mathcal{G}\mathcal{F} = id_{\mathcal{S}}\ .$$

The operators

$$\mathcal{F}:\mathcal{S}\to\mathcal{S} \quad \text{and} \quad \mathcal{G}:\mathcal{S}\to\mathcal{S}$$

are linear, one–to–one and onto. We also note that

$$\mathcal{G} = B\mathcal{F}B\ .$$

Since $SS = BB = id_{\mathcal{S}}$ one obtains that

$$\mathcal{F}\mathcal{F} = S, \quad \mathcal{F}^4 = id_{\mathcal{S}}\ .$$

The following result is Plancherel's formula. It is a;so called Parseval's formula.

Theorem 12.4 *For $f, g \in \mathcal{S}$ define the L_2 inner product by*

$$(f,g)_{L_2} = \int_{-\infty}^{\infty} \bar{f}(x)g(x)\,dx\ .$$

Then we have

$$(f,g)_{L_2} = (\hat{f},\hat{g})_{L_2}\ .$$

Proof: Define

$$\phi = \mathcal{G}Bf = B\mathcal{F}f\ .$$

Then, using the multiplication formula,

$$
\begin{aligned}
(f,g)_{L_2} &= \int \bar{f} g \, dx \\
&= \int \hat{\phi} g \, dx \\
&= \int \phi \hat{g} \, d\xi \\
&= \int (B\hat{f})\hat{g} d\xi \\
&= (\hat{f}, \hat{g})_{L_2}
\end{aligned}
$$

Corollary: For all $f \in \mathcal{S}$ we have

$$\|f\| = \|\hat{f}\| \; . \tag{12.5}$$

12.8 Elementary Theory of Tempered Distributions

On the Schwartz space $\mathcal{S}$ one defines a convergence concept as follows: If $g_n \in \mathcal{S}$ then $g_n \to 0$ in $\mathcal{S}$ means that for all $j, k \in \mathbb{N}$ we have

$$\sup_x |x^j g_n^{(k)}(x)| \to 0 \quad \text{as} \quad n \to \infty \; .$$

If $g_n, g \in \mathcal{S}$ then $g_n \to g$ in $\mathcal{S}$ means that $g - g_n \to 0$ in $\mathcal{S}$. This convergence concept is very restrictive.

A linear functional $\phi : \mathcal{S} \to \mathbb{C}$ is called continuous if $g_n \to 0$ in $\mathcal{S}$ implies that $\phi(g_n) \to 0$ in $\mathbb{C}$. Since the convergence concept in $\mathcal{S}$ is very restrictive, the continuity requirement of a linear functional is very mild.

The linear space of all continuous linear functionals on $\mathcal{S}$ is called the dual space of $\mathcal{S}$, denote by $\mathcal{S}'$. Every element of $\mathcal{S}'$ is called a tempered distribution.

Notation: If $\phi \in \mathcal{S}'$ and $g \in \mathcal{S}$ we often write

$$\phi(g) = (\phi, g)_{\mathcal{S}'\mathcal{S}}$$

to denote the value of ϕ at g. This notation emphasizes the linearity of the expression $\phi(g)$ in ϕ and in g.

12.8.1 Ordinary Functions as Tempered Distributions

If $f \in L_{1,loc}$ satisfies an estimate

$$|f(x)| \le K|x|^k \quad \text{for} \quad |x| \ge K_1 \; , \tag{12.6}$$

for some constants K, K_1 and k, then f determines the tempered distribution ϕ_f defined by

$$\phi_f(g) = \int_{-\infty}^{\infty} f(x) g(x) \, dx, \quad g \in \mathcal{S} \; .$$

We also denote the expression by

$$(f,g)_{\mathcal{S}'\mathcal{S}}$$
and identify ϕ_f with f. In particular, since every $f \in \mathcal{S}$ satisfies an estimate (12.6), we obtain the inclusion $\mathcal{S} \subset \mathcal{S}'$.

12.9 The Fourier Transform on $\mathcal{S}'$: An Example Using Complex Variables

For $f, g \in \mathcal{S}$ we have
$$\int_{-\infty}^{\infty} (\mathcal{F}f)(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x)(\mathcal{F}g)(x)\, dx \ .$$
Therefore, if $\phi \in \mathcal{S}'$ we define $\mathcal{F}\phi \in \mathcal{S}'$ by
$$(\mathcal{F}\phi, g)_{\mathcal{S}'\mathcal{S}} = (\phi, \mathcal{F}g)_{\mathcal{S}'\mathcal{S}} \quad \text{for all} \quad g \in \mathcal{S} \ .$$

We want to show by an example how the theory of complex variables is used in determining the Fourier transform of the distribution (determined by)
$$f(x) = e^{-\pi x^2 is}, \quad x \in \mathbb{R} \ ,$$
where $s \in \mathbb{R}, s \neq 0$, is fixed. Note that $f \in L_\infty$, so $f \in \mathcal{S}'$. However, $f \notin L_1$, so the Fourier transform of f cannot be obtained directly by evaluating the Fourier integral formula.

Recall the definitions
$$G_\varepsilon(x) = e^{-\pi x^2 \varepsilon}, \quad K_\varepsilon(\xi) = \frac{1}{\sqrt{\varepsilon}}\, e^{-\pi \xi^2/\varepsilon}$$
where x and ξ are real and where $\varepsilon > 0$. The functions G_ε and K_ε belong to $\mathcal{S}$ and we have
$$\hat{G}_\varepsilon = K_\varepsilon \quad \text{in} \quad \mathcal{S}$$
for every $\varepsilon > 0$. Formally setting $\varepsilon = is$ suggests that the Fourier transform of $f(x)$ is
$$F(\xi) = \frac{1}{\sqrt{is}}\, e^{-\pi \xi^2/is}$$
in the sense of distributions. Here $\sqrt{z}$ is obtained by analytically continuing the function $\sqrt{\varepsilon}, \varepsilon > 0$, to $\mathbb{C} \setminus (-\infty, 0]$.

Note that f and F both belong to L_∞, thus f and F both belong to $\mathcal{S}'$. We claim that $\mathcal{F}(f) = F$ in $\mathcal{S}'$. This means that we have for all $g \in \mathcal{S}$:
$$\int_{-\infty}^{\infty} e^{-\pi x^2 is} \hat{g}(x)\, dx = \frac{1}{\sqrt{is}} \int_{-\infty}^{\infty} e^{-\pi \xi^2/is} g(\xi)\, d\xi \ .$$
To show this, fix $g \in \mathcal{S}$ and introduce the functions
$$\begin{aligned} L(z) &= \int_{-\infty}^{\infty} e^{-\pi x^2 z} \hat{g}(x)\, dx \\ R(z) &= \frac{1}{\sqrt{z}} \int_{-\infty}^{\infty} e^{-\pi \xi^2/z} g(\xi)\, d\xi \end{aligned}$$
for $\operatorname{Re} z \geq 0, z \neq 0$. Then we know that

$$L(\varepsilon) = R(\varepsilon) \quad \text{for} \quad \varepsilon > 0 \ .$$

Furthermore, $L(z)$ and $R(z)$ are analytic functions in the open right half–plane

$$H_r = \{z \ : \ \mathrm{Re}\, z > 0\} \ .$$

Therefore, $L(z) = R(z)$ in H_r. Now fix $s \in \mathbb{R}, s \neq 0$. The point $z_0 = is$ is a boundary point of the half–plane H_r. We claim that

$$\lim_{z \to is, z \in H_r} L(z) = L(is) \ ,$$

and a similar relation hold for $R(z)$. To see this, let $z \in H_r$ and let $\varepsilon > 0$ be given. For sufficiently large x_0 we have

$$\begin{aligned} |L(is) - L(z)| &\leq \int_{-\infty}^{\infty} |e^{-\pi x^2 is} - e^{-\pi x^2 z}||\hat{g}(x)| \, dx \\ &\leq 2 \int_{|x| \geq x_0} |\hat{g}(x)| \, dx + \int_{|x| \leq x_0} |e^{-\pi x^2 is} - e^{-\pi x^2 z}||\hat{g}(x)| \, dx \\ &\leq \varepsilon + \|\hat{g}\|_{L^1} \max_{|x| \leq x_0} |e^{-\pi x^2 is} - e^{-\pi x^2 z}| \ . \end{aligned}$$

This estimate shows that $|L(is) - L(z)| \leq 2\varepsilon$ if $z \in H_r$ is sufficiently close to is. Since a similar result holds for $R(z)$ and since $L(z) = R(z)$ for $z \in H_r$, the claim $\hat{f} = F$ follows.

12.10 Decay of the Fourier Transform of f and Analyticity of f

12.11 The Paley–Wiener Theorem

Application to the finite speed of propagation for the wave equation and the Klein–Gordon equation.

12.12 The Laplace Transform and Its Inversion

Relation between the Laplace and the Fourier transform:

Let $f : [0, \infty) \to \mathbb{C}$ denote a continuous function satisfying the estimate

$$|f(t)| \leq Ce^{\alpha t}, \quad t \geq 0 \ ,$$

for some $C > 0$ and real α. For complex s with $\mathrm{Re}\, s > \alpha$ the Laplace transform of f is

$$(\mathcal{L}f)(s) = \int_0^{\infty} e^{-st} f(t) \, dt \ .$$

Using certain conventions, the Fourier transform of a function $g : \mathbb{R} \to \mathbb{C}$ is

$$(\mathcal{F}g)(y) = \int_{-\infty}^{\infty} e^{-iyt} g(t) \, dt, \quad y \in \mathbb{R} \ .$$

Denote the Heaviside function by $H(t)$ and let $s = x + iy$. We have

$$\begin{aligned}(\mathcal{L}f(t))(x+iy) &= \int_{-\infty}^{\infty} H(t)f(t)e^{-xt}e^{-iyt}\,dt \\ &= \Big(\mathcal{F}(H(t)f(t)e^{-xt})\Big)(y)\end{aligned}$$

Assuming the Fourier inversion formula to be valid for the function $t \to H(t)f(t)e^{-xt}$ one obtains that

$$H(t)f(t)e^{-xt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iyt}\mathcal{L}(f(t))(x+iy)\,dy \ .$$

This yields

$$H(t)f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(x+iy)t}\mathcal{L}(f(t))(x+iy)\,dy \ .$$

In terms of a line integral one obtains that

$$H(t)f(t) = \frac{1}{2\pi i}\int_{\Gamma_x} e^{st}(\mathcal{L}f)(s)\,ds \ .$$

Here Γ_x is the straight line parameterized by

$$s = x+iy, \quad -\infty < y < \infty \ .$$

For applications, it is important that the function $(\mathcal{L}f)(s)$ is often analytic in a region larger than the half–plane $\text{Re}\, s > \alpha$ and that the curve Γ_x may be deformed in the region of analyticity.

13 Growth and Zeros of Entire Functions

If $p(z)$ is a polynomial of degree n then $p(z)$ has n zeros and $|p(z)|$ grows like $|z|^n$ as $|z| \to \infty$. Thus, the number of zeros of $p(z)$ is related to the growth of $|p(z)|$ as $|z| \to \infty$.

This chapter deals with generalizations to entire functions $f(z)$. It relates the zeros a_j of $f(z)$ to growth estimates of $|f(z)|$ as $|z| \to \infty$: If $|f(z)|$ satisfies some growth estimate as $|z| \to \infty$, then the number of zeros,

$$n(r) := \text{number of } \Big\{ a_j \in \mathbb{C} \ : \ f(a_j) = 0,\, |a_j| \le r \Big\}$$

cannot grow too fast as $r \to \infty$. See Theorem 13.2.

13.1 Jensen's Formula

Theorem 13.1 *Let $D = D(0, R+\varepsilon)$ and let $f \in H(D)$. Assume that*

$$f(0) \neq 0 \quad and \quad f(z) \neq 0 \quad for \quad |z| = R \ .$$

Let $a_1, \ldots, a_N$ denote the zeros of f with $0 < |a_j| < R$, repeated by their multiplicity. Then the following formula holds:

$$\ln|f(0)| = \sum_{j=1}^N \ln\Big(\frac{|a_j|}{R}\Big) + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{it})|\,dt \ . \qquad (13.1)$$

Formula (13.1) is called Jensen's formula. Note that $\ln(|a_j|/R)$ is negative since $0 < |a_j| < R$. If the function $f(z)$ has many zeros a_j with $0 < |a_j| < R$ then the sum term in Jensen's formula is negative with large absolute value. Equation (13.1) then implies that $|f(z)|$ will be large for some z on the circle $z = Re^{it}$. Roughly, the existence of many zeros a_j of $f(z)$ implies some growth of $|f(z)|$ as $|z| = R$ gets large. Conversely, growth estimates of $|f(z)|$ as $|z| \to \infty$ imply growth estimates of $n(r)$ as $r \to \infty$.

Before proving the formula, we recall Cauchy's integral formula: If $f \in D(z_0, R+\varepsilon)$ then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz \ .$$

Here Γ has the parameterization

$$z(t) = z_0 + Re^{it}, \quad 0 \le t \le 2\pi \ .$$

Using that $dz = Rie^{it}\,dt$ we can also write the formula for $f(z_0)$ in mean–value form:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt \ .$$

Here we have used that

$$dz = Rie^{it}\,dt = (z - z_0)i\,dt \ ,$$

thus

$$\frac{dz}{i(z - z_0)} = dt \ .$$

If $u : D \to \mathbb{R}$ is harmonic, then u is the real–part of a function $f \in H(D)$. Therefore,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{it})\, dt \ . \tag{13.2}$$

Lemma 13.1 *Let $a \in \mathbb{C}, |a| < 1$. Then we have*

$$\int_0^{2\pi} \ln|e^{it} - a|\, dt = 0 \ .$$

Proof: Set $f(z) = 1 - za$. Then f has no zero in $D(0, 1+\varepsilon) =: D$ if $\varepsilon > 0$ is small. We have $f(z) = e^{\phi(z)}$ for some $\phi \in H(D)$. Therefore, $|f(z)| = e^{\mathrm{Re}\,\phi(z)}$ and

$$u(z) := \ln|f(z)| = \mathrm{Re}\,\phi(z)$$

is harmonic in D. Also, $u(0) = \ln|f(0)| = \ln 1 = 0$. Formula (13.2) with $R = 1$ and $u_0 = 0$ yields that

$$0 = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} \ln|1 - ae^{it}|\, dt \ ,$$

thus

$$0 = \int_0^{2\pi} \ln|1 - ae^{it}|\, dt \ . \tag{13.3}$$

(Note that (13.3) is the special case of (13.1) obtained for $R = 1, f(z) = 1 - az$.)

In (13.3) substitute $t = -s$ to obtain that

$$\begin{aligned} 0 &= -\int_0^{-2\pi} \ln|1 - ae^{-is}|\, ds \\ &= \int_{-2\pi}^{0} \ln|1 - ae^{-is}|\, ds \\ &= \int_0^{2\pi} \ln|1 - ae^{-is}|\, ds \ . \end{aligned}$$

In the last equation we have used 2π–periodicity of the function which is integrated. Note that

$$|1 - ae^{-is}| = |e^{-is}(e^{is} - a)| = |e^{is} - a| \ .$$

This proves the lemma. $\diamond$

Proof of Theorem 13.1: a) Assume first that f has no zero a_j with $|a_j| < R$. Then we can write

$$f(z) = e^{\phi(z)}, \quad \phi \in H(D) \ .$$

The function

$$u(z) := \ln|f(z)| = \mathrm{Re}\,\phi(z)$$

is harmonic in D. Formula (13.2) with $z_0 = 0$ yields that

$$\ln|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{it})|\,dt\ .$$

This proves formula (13.1) if $f(z)$ has no zero a_j with $|a_j| < R$.

b) Fix $w \in \mathbb{C}$ with $0 < |w| < R$ and consider the function $f(z) = z - w$. Then formula (13.1) claims that

$$\ln|w| = \ln\Big(\frac{|w|}{R}\Big) + \frac{1}{2\pi}\int_0^{2\pi} \ln|Re^{it} - w|\,dt\ . \tag{13.4}$$

Set $a = w/R$. Then we have

$$\ln|Re^{it} - w| = \ln R + \ln|e^{it} - a|\ .$$

Lemma 13.1 yields that

$$\int_0^{2\pi} \ln|e^{it} - a|\,dt = 0\ .$$

Formula (13.4) follows. This shows that (13.1) holds for a function of the form $f(z) = z - w$ with $0 < |w| < R$.

c) Assume that two functions $f_1(z)$ and $f_2(z)$ satisfy the assumptions of the theorem and assume that the formula (13.1) holds for f_1 and f_2. Then the product $f(z) = f_1(z)f_2(z)$ also satisfies the assumptions of the theorem and formula (13.1) holds for f.

d) Let $f \in H(D)$ denote any function satisfying the assumptions of the theorem. We can write

$$f(z) = g(z)(z - a_1)(z - a_2)\ldots(z - a_N)$$

where $g \in H(D)$ has no zeros in D. Using a), b), and c), the formula (13.1) follows for the function $f(z)$. $\diamond$

13.2 The Order of Growth of an Entire Function

Definition: Let f denote an entire function. One says that f has finite growth order if there exist positive constants A, B, ρ so that

$$|f(z)| \le Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C}\ . \tag{13.5}$$

If f is an entire function of finite growth order, then one defines its order of growth, denoted by $\rho_0 = \rho_0(f)$, as the infimum of all positive numbers ρ for which the estimate (13.5) holds for some constants $A = A_\rho$ and $B = B_\rho$.

Examples:
1. Every polynomial has order of growth $\rho_0 = 0$.
2. The function $f(z) = e^z$ has order of growth $\rho_0 = 1$.
3. The function $f(z) = \cos z$ has order of growth $\rho_0 = 1$.
4. Recall that

$$\cos w = \sum_{j=0}^{\infty} (-1)^j \frac{w^{2^j}}{(2j)!}$$

and define

$$f(z) = \cos\sqrt{z} = \sum_{j=0}^{\infty}(-1)^j \frac{z^j}{(2j)!} \ .$$

It is then clear that

$$f(z^2) = \cos z \ .$$

The order of growth of f is $\rho_0 = \frac{1}{2}$.

5. One can show that

$$\frac{1}{|\Gamma(z)|} \leq Ae^{B|z|\ln|z|} \quad \text{for} \quad |z| \geq 1 \ .$$

The function $1/\Gamma(z)$ has order of growth $\rho_0 = 1$.

[Stein, Sharkarchi], p. 165

(The estimate should also follow from the infinite product representation.)

6. It can be shown that the function $(s-1)\zeta(s)$ has order of growth $\rho_0 = 1$.

[Stein, Sharkarchi], p. 202

13.3 Zeros and Growth Estimates of Entire Functions

Let f denote an entire functions which has a sequence of zeros a_j. We order the zeros a_j so that

$$|a_1| \leq |a_2| \leq \ldots$$

Each zero is listed according to its multiplicity. For $0 \leq r < \infty$ let $n(r)$ denote the number of zeros a_j with $|a_j| \leq r$. Clearly, $n(r)$ is a piecewise constant function which increases. We will show that a growth estimate for $|f(z)|$ implies a growth estimate for $n(r)$ as $r \to \infty$.

Theorem 13.2 *Assume that the entire function $f(z)$ satisfies the growth estimate*

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \textit{for all} \quad z \in \mathbb{C} \tag{13.6}$$

with positive constants A, B, ρ. Then there exist positive constants C and r_0 so that

$$n(r) \leq Cr^\rho \quad \textit{for} \quad r \geq r_0 \ . \tag{13.7}$$

Proof: a) Using the Heaviside function

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

we have

$$H(r - |a_j|) = \begin{cases} 1, & |a_j| \leq r \\ 0, & |a_j| > r \end{cases}$$

and can write

$$n(r) = \sum_{j=1}^{\infty} H(r - |a_j|) \ .$$

For each fixed r the sum is finite.

b) Assume first that $f(0) \neq 0$ and $f(z) \neq 0$ for $|z| = R$. Let $a_1, \ldots, a_N$ denote the zeros of $f(z)$ with $0 < |a_j| < R$. We can write

$$\begin{aligned}
\int_0^R n(r)\,\frac{dr}{r} &= \sum_{j=1}^N \int_0^R H(r - |a_j|)\,\frac{dr}{r} \\
&= \sum_{j=1}^N \int_{|a_j|}^R \frac{dr}{r} \\
&= \sum_{j=1}^N \ln(R/|a_j|) \\
&= -\sum_{j=1}^N \ln(|a_j|/R)\ .
\end{aligned}$$

Here the zeros $a_1, \ldots, a_N$ are the zeros of f with $0 < |a_j| < R$. Using Theorem 13.1 we obtain the following result:

Lemma 13.2 *As above, let $f(z)$ denote an entire function and assume that*

$$f(0) \neq 0 \quad and \quad f(z) \neq 0 \quad for \quad |z| = R\ . \tag{13.8}$$

Then we have

$$\int_0^R n(r)\,\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{it})|\,dt - \ln|f(0)|\ .$$

c) We continue the proof of Theorem 13.2 and let $f(z)$ denote an entire function with (13.8) satisfying the growth estimate (13.6).

We let $R = 2r$ and have

$$\begin{aligned}
\ln|f(Re^{it})| &\leq \ln A + BR^\rho \\
&\leq C_1 r^\rho \quad \text{for} \quad r \geq r_0\ .
\end{aligned}$$

The equation of the previous lemma yields that (with $R = 2r$):

$$\int_0^R n(s)\frac{ds}{s} \leq C_2 r^\rho \quad \text{for} \quad r \geq r_0\ .$$

Furthermore,

$$\begin{aligned}
n(r)\ln 2 &= n(r)\int_r^{2r} \frac{ds}{s} \\
&\leq \int_r^{2r} n(s)\,\frac{ds}{s} \\
&\leq C_2 r^\rho
\end{aligned}$$

for $r \geq r_0$.

This proves the estimate(13.7) as long as f satisfies (13.8) for $R = 2r$. If $f(z)$ has a zero a_j with $|a_j| = R = 2r$, we simply choose $0 < \varepsilon \leq r_0$ so that $f(z)$ has no zero on the circle with radius $2r + 2\varepsilon$. We then have

$$n(r) \leq n(r+\varepsilon) \leq C_2(r+\varepsilon)^\rho \leq C_3 r^\rho \quad \text{for} \quad r \geq r_0 \ .$$

d) Assume that f has a zero of order m at $z = 0$. We set $F(z) = f(z)/z^m$. The entire function $F(z)$ satisfies the same growth estimate as f does. Since $n_f(r) = n_F(r) + m$ the claim $n_f(r) \leq Cr^\rho$ for $r \geq r_0$ follows. $\diamond$

Theorem 13.3 *Let $f \in H(\mathbb{C})$ satisfy the growth estimate of the previous theorem and let a_j denote the non–zero zeros of f. If $s > \rho$ then we have*

$$\sum_j |a_j|^{-s} < \infty \ .$$

Proof: For any positive integer l the number of zeros a_j of f with

$$2^l < |a_j| \leq 2^{l+1}$$

is less than or equal to $n(2^{l+1})$. Therefore, for any large l,

$$\begin{aligned}
\sum_{2^l < |a_j| \leq 2^{l+1}} |a_j|^{-s} &\leq 2^{-ls}\, n(2^{l+1}) \\
&\leq C\, 2^{-ls}\, 2^{(l+1)\rho} \\
&\leq C_1\, 2^{-l(s-\rho)}
\end{aligned}$$

(In the first estimate we have used that $2^l < |a_j|$ implies $|a_j|^{-s} \leq 2^{-ls}$.)

Since $2^{-(s-\rho)} < 1$ the geometric series

$$\sum_l 2^{-l(s-\rho)}, \quad s > \rho \ ,$$

converges, and the claim follows. $\diamond$

13.4 Hadamard's Factorization Theorem

Let f denote an entire function with growth order ρ_0 and let $k := [\rho_0]$ denote the integer part of ρ_0. We choose s and ρ so that

$$k = [\rho_0] \leq \rho_0 < \rho < s < k+1 \ .$$

Then f satisfies a growth estimate

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all} \quad z \in \mathbb{C} \ .$$

We assume that the zeros a_j of $f(z)$ form a sequence; we order the zeros so that

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \leq |a_{m+2}| \leq \ldots$$

As above, $n(r)$ denotes the number of zeros a_j with $|a_j| \le r$. We know from Theorems 13.2 and 13.3 that $n(r) \le Cr^\rho$ for large r and

$$\sum_{j=m+1}^{\infty} |a_j|^{-s} < \infty \ . \tag{13.9}$$

Recall the Weierstrass' canonical factors,

$$E_k(w) = (1-w)\exp\Big(w + \frac{w^2}{2} + \ldots + \frac{w^k}{k}\Big) \ .$$

Convergence of the series

$$\sum_{j=m+1}^{\infty} |a_j|^{-k-1} < \infty \tag{13.10}$$

implies that the product

$$F(z) = z^m \, \Pi_{j=m+1}^{\infty} E_k\Big(\frac{z}{a_j}\Big) \tag{13.11}$$

defines an entire function, which has precisely the same zeros as $f(z)$. Consequently, one can write the given function $f(z)$ in the form

$$f(z) = e^{g(z)} F(z)$$

where $g(z)$ is an entire function.

Hadamard's Factorization Theorem (for an entire function with infinitely many zeros) is the following remarkable result. It gives information about the function $g(z)$.

Theorem 13.4 *Let $f(z)$ denote an entire function with growth order ρ_0 and let $k = [\rho_0]$ denote the integer part of ρ_0. Assume that $f(z)$ has a sequence of zeros a_j with*

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \le |a_{m+2}| \le \ldots$$

Then $f(z)$ has the form

$$f(z) = e^{g(z)} \, z^m \, \Pi_{j=m+1}^{\infty} E_k\Big(\frac{z}{a_j}\Big) \tag{13.12}$$

where $g(z)$ is a polynomial of degree $\le k$. The function $g(z)$ is unique, except that one can add a constant $2\pi i\, l, l \in \mathbb{Z}$, to $g(z)$.

An idea of the proof is to turn the growth estimate of $|f(z)|$ into an estimate of $\mathrm{Re}\, g(z)$. This requires to show lower bounds for the product $|F(z)|$ (see (13.11)) and is rather technical. Once these lower bounds are derived, the following lemma will complete the proof of Hadamard's Factorization Theorem.

Lemma 13.3 *Let $g(z)$ denote an entire function. Let $k \in \{0, 1, \ldots\}$ and let $k \le s < k+1$. Assume that there exists a constant $C > 0$ and a sequence of positive numbers r_ν with $r_\nu \to \infty$ so that $\mathrm{Re}\, g(z)$ satisfies the following upper bound:*

$$\mathrm{Re}\, g(z) \le C r_\nu^s \quad \text{if} \quad |z| = r_\nu, \quad \nu = 1, 2, \ldots \tag{13.13}$$

Under these assumptions, the function $g(z)$ is a polynomial of degree $\le k$.

Proof: First fix any $r > 0$ and any integer $n \geq 0$. With Γ_r we denote the circle with parameterization $z = re^{it}, 0 \leq t \leq 2\pi$. Let

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \ .$$

We obtain that

$$\frac{g(z)}{z^{n+1}} = \ldots + \frac{g_n}{z} + \ldots$$

thus

$$2\pi i g_n = \int_{\Gamma_r} \frac{g(z)}{z^{n+1}}\, dz = i \int_0^{2\pi} g(re^{it}) r^{-n} e^{-int}\, dt \ .$$

(We have used the substitution $z = re^{it}, dz/z = idt$.)

This shows that

$$2\pi g_n r^n = \int_0^{2\pi} g(re^{it}) e^{-int}\, dt \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{13.14}$$

Also, by Cauchy's theorem,

$$\int_{\Gamma_r} g(z) z^n \frac{dz}{z} = 0 \quad \text{for} \quad n \geq 1 \ ,$$

thus

$$\int_0^{2\pi} g(re^{it}) e^{int}\, dt = 0 \quad \text{for} \quad n \geq 1 \ ,$$

thus

$$\int_0^{2\pi} \bar{g}(re^{it}) e^{-int}\, dt = 0 \quad \text{for} \quad n \geq 1 \ .$$

Therefore, if we set $u(z) = \mathrm{Re}\, g(z) = \frac{1}{2}(g(z) + \bar{g}(z))$, then we obtain that

$$\pi g_n r^n = \int_0^{2\pi} u(re^{it}) e^{-int}\, dt \quad \text{for} \quad n \geq 1 \ .$$

Furthermore, setting $n = 0$ in (13.14) and taking real–parts yields that

$$2\pi\, \mathrm{Re}\, g_0 = \int_0^{2\pi} u(re^{it})\, dt \ .$$

Using that

$$\int_0^{2\pi} e^{-int}\, dt = 0 \quad \text{for} \quad n \geq 1$$

we have, for $n \geq 1$,

$$\begin{aligned}
\pi g_n &= \frac{1}{r^n}\int_0^{2\pi} u(re^{it})e^{-int}\,dt \\
&= \frac{1}{r^n}\int_0^{2\pi} \Big(u(re^{it}) - Cr^s\Big)e^{-int}\,dt
\end{aligned}$$

Here the term in $\Big(\ldots\Big)$ is ≤ 0 for $r = r_\nu$, by assumption. Taking absolute values and using (13.13) we obtain that

$$\begin{aligned}
\pi|g_n| &\le \frac{1}{r^n}\int_0^{2\pi} \Big(Cr^s - u(re^{it})\Big)\,dt \\
&= 2\pi\, C\, r^{s-n} - 2\pi(\mathrm{Re}\, g_0)r^{-n}
\end{aligned}$$

Now recall the assumption $s < k+1$ and assume that $n \ge k+1$, thus $s-n<0$. Letting $r = r_\nu \to \infty$ in the above estimate yields that $g_n = 0$ for $n > k$. This proves the lemma. $\diamond$

13.5 Entire Functions of Non–Integer Order of Growth

The previous lemma has the following important implication.

Theorem 13.5 *Let $f(z)$ denote an entire function with order of growth ρ_0. If ρ_0 is not an integer, then f has infinitely many zeros.*

Proof: Suppose that $f(z)$ has only finitely many zeros $a_1, \ldots, a_N$. We may assume that $N \ge 1$ and let

$$p(z) = (z - a_1)\ldots(z - a_N)$$

and write

$$f(z) = e^{g(z)}p(z)$$

where $g(z)$ is an entire function. (If $f(z)$ has no zero, then take $p(z) \equiv 1$.) If $k < \rho_0 < s < k+1$ with integer k then $f(z)$ satisfies the estimate

$$|f(z)| \le Ae^{B|z|^s} \quad \text{for all} \quad z \in \mathbb{C}\ .$$

Since $|p(z)| \ge 1$ for all large $|z|$ we obtain that

$$\begin{aligned}
e^{\mathrm{Re}\, g(z)} &= |e^{g(z)}| \\
&\le Ae^{B|z|^s}
\end{aligned}$$

if $|z| = r$ is large. This yields that

$$\mathrm{Re}\, g(z) \le Cr^s \quad \text{for} \quad |z| = r$$

if r is large. The previous lemma then yields that $g(z)$ is a polynomial, and the representation $f(z) = e^{g(z)}p(z)$ implies the order of growth of $f(z)$ to be an integer. This contradiction proves the theorem. $\diamond$

13.6 Proof of Hadamard's Factorization Theorem

Recall the Weierstrass' canonical factors

$$E_k(z) = (1-z)e^{h_k(z)} \quad \text{where} \quad h_k(z) = z + \frac{z}{2} + \ldots + \frac{z^k}{k} \ .$$

The main technical lemma for proving Hadamard's Factorization Theorem is the following lower bound for the product

$$E(z) = \Pi_{j=1}^{\infty} E_k\Big(\frac{z}{a_j}\Big) \ . \tag{13.15}$$

Lemma 13.4 *Let $0 \le k \le \rho < s < k+1$ where k is an integer. Let a_j denote a sequence of complex numbers with*

$$0 < |a_1| \le |a_2| \ldots$$

and assume that $n(r) \le Cr^\rho$ for $r \ge r_0$, where $n(r)$ is the number of a_j with $|a_j| \le r$. Consider the entire function $E(z)$ defined in (13.15). There exists a constant $c > 0$, independent of z, and there exists a sequence r_ν of positive radii with $r_\nu \to \infty$ so that

$$|E(z)| \ge e^{-c|z|^s} \quad \text{if} \quad |z| = r_\nu, \quad \nu = 1, 2, \ldots$$

For later reference, we recall that the assumption $n(r) \le Cr^\rho$ implies that

$$\sum_{j=1}^{\infty} |a_j|^{-s} < \infty \quad \text{for} \quad s > \rho \ .$$

Before we prove the main lemma, Lemma 13.4, we show two simple auxiliary results for the Weierstrass function $E_k(z)$.

Lemma 13.5 *For small $|z|$ we have*

$$|E_k(z)| \ge e^{-2|z|^{k+1}} \quad \text{if} \quad |z| \le \frac{1}{2} \ .$$

Proof: We have $E_k(z) = e^w$ with

$$\begin{aligned} w &= \log(1-z) + h_k(z) \\ &= -\Big(\frac{z^{k+1}}{k+1} + \frac{z^{k+2}}{k+2} + \ldots\Big) \end{aligned}$$

This yields the bound $|w| \le 2|z|^{k+1}$ if $|z| \le \frac{1}{2}$. Therefore,

$$|E_k(z)| = |e^w| = e^{\mathrm{Re}\, w} \ge e^{-|w|} \ge e^{-2|z|^{k+1}} \ .$$

$\diamond$

If $|z|$ is bounded away from zero, we have the following lower bound for $|E_k(z)|$:

Lemma 13.6 *There exists a constant $c = c_k > 0$ with*

$$|E_k(z)| \ge |1-z|\, e^{-c|z|^k} \quad \text{if} \quad |z| \ge \frac{1}{2} \ .$$

Proof: The definition of $E_k(z)$ yields that

$$\begin{aligned} |E_k(z)| &= |1-z||e^{h_k(z)}| \\ &\geq |1-z|e^{-|h_k(z)|} \end{aligned}$$

Here

$$|h_k(z)| \leq \Big| z + \frac{z^2}{2} + \ldots + \frac{z^k}{k} \Big| \leq c_k |z|^k \quad \text{if} \quad |z| \geq \frac{1}{2} \ .$$

$\diamond$

The estimate of the next lemma is central for the proof of the main lemma, Lemma 13.4.

Lemma 13.7 *Under the assumptions of Lemma 13.4, define the following union of open discs:*

$$\Omega = \cup_{j=1}^{\infty} D\Big(a_j, |a_j|^{-k-1}\Big) \ .$$

There exist positive constants c and R so that

$$|E(z)| \geq e^{-c|z|^s} \quad \textit{if} \quad |z| \geq R \quad \textit{and} \quad z \in \mathbb{C} \setminus \Omega \ .$$

Proof: a) Write $|E(z)| = P_1 P_2$ with

$$P_1 = \Pi_{|a_j| \leq 2|z|} \Big| E_k(z/a_j) \Big| \quad \text{and} \quad P_2 = \Pi_{|a_j| > 2|z|} \Big| E_k(z/a_j) \Big| \ .$$

We first estimate P_2 from below: If $|a_j| > 2|z|$ then, by Lemma 13.5,

$$|E_k(z/a_j)| \geq e^{-2|z/a_j|^{k+1}} \ .$$

Therefore,

$$P_2 \geq e^{-2|z|^{k+1} Q} \quad \text{with} \quad Q = \sum_{|a_j| > 2|z|} |a_j|^{-k-1} \ .$$

Since $|z|^{-1} > 2|a_j|^{-1}$ we have

$$|a_j|^{-k-1} = |a_j|^{-s} \, |a_j|^{s-k-1} \leq |a_j|^{-s} |z|^{s-k-1} \ .$$

Here we have used that $s-k-1 < 0$. This yields the upper bound

$$\begin{aligned} Q &\leq |z|^{s-k-1} \sum_j |a_j|^{-s} \\ &\leq C_1 |z|^{s-k-1} \ . \end{aligned}$$

The lower bound

$$P_2 \geq e^{-c|z|^s}$$

follows.

b) Next consider the finite product

$$P_1 = \Pi_{|a_j| \leq 2|z|} \Big| E_k(z/a_j) \Big| \ .$$

Since $|z/a_j| \geq \frac{1}{2}$ we can apply Lemma 13.6 and obtain that

$$|E_k(z/a_j)| \geq \left|1 - \frac{z}{a_j}\right| e^{-c|z/a_j|^k} \ .$$

Therefore,

$$P_1 \geq P_3 P_4$$

with

$$P_3 = \Pi_{|a_j| \leq 2|z|} \left|1 - \frac{z}{a_j}\right|$$

and

$$P_4 = \Pi_{|a_j| \leq 2|z|} \, e^{-c|z/a_j|^k} \ .$$

c) To estimate P_4 from below we write

$$P_4 = e^{-c|z|^k Q_1}$$

with

$$Q_1 = \sum_{|a_j| \leq 2|z|} |a_j|^{-k} \ .$$

Since

$$|a_j|^{-k} = |a_j|^{-s} \, |a_j|^{s-k} \leq C|a_j|^{-s} \, |z|^{s-k}$$

we have

$$Q_1 \leq C|z|^{s-k} \sum_j |a_j|^{-s} = C_1 |z|^{s-k} \ .$$

This shows that

$$P_4 \geq e^{-c_1|z|^s}, \quad c_1 = cC_1 \ .$$

d) It remains to estimate

$$P_3 = \Pi_{|a_j| \leq 2|z|} \left|1 - \frac{z}{a_j}\right|$$

from below. Here it is important that z does not lie in any of the discs $D(a_j, |a_j|^{-k-1})$, i.e.,

$$|a_j - z| \geq |a_j|^{-k-1} \ .$$

One obtains that

$$\left|1 - \frac{z}{a_j}\right| = \frac{|a_j - z|}{|a_j|} \geq |a_j|^{-k-2}$$

and

$$P_3 \geq \Pi_{|a_j| \leq 2|z|} \, |a_j|^{-k-2} \ .$$

Therefore,

$$\ln P_3 \geq -(k+2) \sum_{|a_j| \leq 2|z|} \ln |a_j| \ .$$

Finally,

$$\begin{aligned}
\sum_{|a_j| \leq 2|z|} \ln |a_j| &\leq n(2|z|) \, \ln(2|z|) \\
&\leq C|z|^\rho \ln(2|z|) \\
&\leq C_1 |z|^s
\end{aligned}$$

for all sufficiently large $|z|$. The lower bound

$$\ln P_3 \geq -c|z|^s, \quad c > 0 \ ,$$

follows and the inequality

$$P_3 \geq e^{-c|z|^s}$$

is shown.

e) To summarize, we have for $|z| \geq R, z \in \mathbb{C} \setminus \Omega$:

$$\begin{array}{rll}
|E(z)| & = P_1 P_2 & \text{(part a)} \\
P_2 & \geq e^{-c|z|^s} & \text{(part a)} \\
P_1 & \geq P_3 P_4 & \text{(part b)} \\
P_3 & \geq e^{-c|z|^s} & \text{(part d)} \\
P_4 & \geq e^{-c|z|^s} & \text{(part c)}
\end{array}$$

where $c > 0$ is a constant. This implies that

$$|E(z)| \geq e^{-3c|z|^s} \quad \text{for} \quad |z| \geq R, \quad z \in \mathbb{C} \setminus \Omega \ .$$

The lemma is proved. $\diamond$

Recall that

$$0 < |a_1 \leq |a_2| \leq \ldots$$

and $k \leq \rho < k+1$. Consequently,

$$\sum_{j=1}^\infty |a_j|^{-k-1} < \infty \ .$$

Also, recall the definition

$$\Omega = \cup_{j=1}^\infty D\Big(a_j, |a_j|^{-k-1}\Big) \ .$$

Lemma 13.8 *Under the above assumptions, there exists a sequence of positive radii* $r_1, r_2, \ldots$ *with* $r_\nu \to \infty$ *so that none of the circles*

$$|z| = r_\nu, \quad \nu = 1, 2, \ldots$$

intersects the union of discs Ω.

Proof: Choose J so large that

$$\sum_{j=J}^{\infty} |a_j|^{-k-1} \leq \frac{1}{10} \ .$$

The set

$$\Omega_J = \cup_{j=1}^{J-1} D\Big(a_j, |a_j|^{-k-1}\Big)$$

is bounded and we choose an integer n so large that $\Omega_J \subset D(0, n)$.

We claim that the interval

$$n \leq r \leq n+1$$

contains a number r so that the circle $|z| = r$ does not intersect the set Ω. In fact, for any $r \geq n$, the circle $|z| = r$ does not intersect Ω_J since $\Omega_J \subset D(0, r)$ and it remains to show that we can choose r in the interval $n \leq r \leq n+1$ so that the circle $|z| = r$ does not intersect the union

$$\cup_{j=J}^{\infty} D\Big(a_j, |a_j|^{-k-1}\Big) \ .$$

To see this, consider the closed interval

$$I_j = \Big[|a_j| - |a_j|^{-k-1}, |a_j| + |a_j|^{-k-1}\Big], \quad j \geq J \ ,$$

of length

$$length(I_j) = 2|a_j|^{-k-1} \ .$$

We have

$$\sum_{j \geq J} length(I_j) \leq \frac{1}{5} \ .$$

Therefore, there exists a number r in the interval $n \leq r \leq n+1$ which does not lie in the union $\cup_{j \geq J} I_j$.

It is then clear that the circle $|z| = r$ does not intersect Ω.

This proves Lemma 13.8. The main technical lemma, Lemma 13.4, follows as a consequence of Lemma 13.7 and Lemma 13.8.

We summarize:

Theorem 13.6 *(Hadamard) Let* $f(z)$ *denote an entire function of finite order of growth* ρ_0 *and let* $k = [\rho_0]$ *denote the integer part of* ρ_0. *Assume that* f *is not identically zero.*

Case 1: $f(z)$ *has only finitely many zeros. Then* $f(z)$ *has the form*

$$f(z) = e^{g(z)}\, p(z)$$

where $p(z)$ and $g(z)$ are polynomials. The polynomial $g(z)$ has degree k. The order of growth of $f(z)$ equals $k = \rho_0$.

Case 2: *$f(z)$ has infinitely many zeros a_j with*

$$a_1 = \ldots = a_m = 0 < |a_{m+1}| \le |a_{m+2}| \le \ldots$$

Then $f(z)$ has the form

$$f(z) = e^{g(z)}\, z^m\, \Pi_{j=m+1}^{\infty} E_k(z/a_j)$$

where $g(z)$ is a polynomial of degree $\le k$.

In particular, Case 1 cannot occur unless the order of growth of $f(z)$ is an integer. If the order of growth of $f(z)$ is not an integer, then $f(z)$ has infinitely many zeros.

The next result follows from Hadamard's Theorem:

Theorem 13.7 *Let $f(z)$ denote an entire function of finite order of growth ρ_0. If $f(z)$ is not constant then either $f(\mathbb{C}) = \mathbb{C}$ or there exists a unique $\alpha \in \mathbb{C}$ so that*

$$f(\mathbb{C}) = \mathbb{C} \setminus \{\alpha\}\ .$$

Proof: Suppose that there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ with $\alpha_1 \ne \alpha_2$ so that the equations

$$f(z) = \alpha_1 \quad \text{and} \quad f(z) = \alpha_2$$

both have no solution. Then the equation

$$f(z) - \alpha_1 = \alpha_2 - \alpha_1$$

has no solution and

$$f(z) - \alpha_1 \ne 0 \quad \text{for all} \quad z \in \mathbb{C}\ .$$

There exists a polynomial $g(z)$ (of degree k with $k = [\rho_0]$) so that

$$f(z) - \alpha_1 = e^{g(z)}\ .$$

Set $\alpha_2 - \alpha_1 =: q$. We have

$$q = |q| e^{it} = e^{\beta} \quad \text{with} \quad \beta = \ln |q| + it\ .$$

Since the equation

$$f(z) - \alpha_1 = \alpha_2 - \alpha_1 = q$$

has no solution $z \in \mathbb{C}$, the equation

$$e^{g(z)} = e^{\beta}$$

has no solution $z \in \mathbb{C}$, thus $g(z) - \beta \ne 0$ for all $z \in \mathbb{C}$. Since $g(z)$ is a polynomial, one obtains that $g(z)$ is constant. This contradicts the assumption that $f(z)$ is not constant. $\diamond$

Remark: Picard's little theorem says that

$$f(\mathbb{C}) = \mathbb{C} \quad \text{or} \quad f(\mathbb{C}) = \mathbb{C} \setminus \{\alpha\}$$

for some $\alpha \in \mathbb{C}$ holds for every entire function $f(z)$, unless $f(z)$ is constant. Thus, the assumption that $f(z)$ has finite growth order is not needed to obtain the result of the Theorem 13.7.

14 The Prime Number Theorem

For $1 \le x < \infty$ let $\pi(x)$ denote the number of primes p less than or equal to x. The Prime Number Theorem states that

$$\lim_{x\to\infty} \frac{\pi(x)\ln(x)}{x} = 1 \ .$$

If one defines a remainder $R(x)$ by

$$\pi(x) = \frac{x}{\ln x}\,(1 + R(x))$$

then $R(x) \to 0$ as $x \to \infty$. Sharp estimates of the remainder are related to the zeros of the Riemann ζ–function in the critical strip $0 < \operatorname{Re} s < 1$.

14.1 Functions

The following functions will be used:

If $n \le x < n+1$, where n is an integer, then let

$$x = n + \{x\} = [x] + \{x\}$$

where $[x]$ is the integer part and $\{x\}$ is the fractional part of the real number x.

Von Mangoldt function $\Lambda : \mathbb{N} \to [0,\infty)$:

$$\Lambda(n) = \begin{cases} \ln p & \text{if} \quad n = p^m,\ p \ \text{ prime}, \quad m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

n	$\Lambda(n)$
1	0
2	$\ln 2$
3	$\ln 3$
4	$\ln 2$
5	$\ln 5$
6	0
7	$\ln 7$
8	$\ln 2$
9	$\ln 3$
10	0

$$\begin{aligned}
\pi(x) &= \text{number of primes} \quad \le x \\
\psi(y) &= \sum_{1\le n\le y} \Lambda(n) \\
&= \sum_{p^m\le y} \ln p \quad (\text{Chebyshev's } \psi-\text{function}) \\
&= \sum_{p\le y} \left[\frac{\ln y}{\ln p}\right] \ln p \\
\psi(y) &= \sum_{1\le n\le y} \Lambda(n)H(y-n) \quad (\text{with Heaviside function}) \\
&= \sum_{n\ge 1} \Lambda(n)H(y-n)
\end{aligned}$$

Here we use the Heaviside function

$$H(\xi) = \begin{cases} 1 & \text{for} \quad \xi \ge 0 \\ 0 & \text{for} \quad \xi < 0 \end{cases}$$

An auxiliary integral:

Lemma 14.1 *Let Γ_c denote the straight line with parameterization*

$$\Gamma_c \ : \ s(t) = c + it, \quad -\infty < t < \infty \ .$$

For $c > 0$ we have

$$\frac{1}{2\pi i} \int_{\Gamma_c} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 0 & \textit{for} \quad 0 < a \le 1 \\ 1 - \frac{1}{a} & \textit{for} \quad a > 1 \end{cases}$$

Proof: See [Stein, Sharkarchi], Chapter 7, Lemma 2.4. Homework.

We will show below that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad \text{for} \quad \operatorname{Re} s > 1 \ .$$

(This follows from the product formula $\zeta(s) = \Pi_p (1-p^{-s})^{-1}$.)

Multiply the above equation by

$$\frac{1}{2\pi i} \frac{x^{s+1}}{s(s+1)} \quad \text{where} \quad x \ge 1$$

to obtain:

$$\frac{1}{2\pi i} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) = \frac{x}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \frac{(x/n)^s}{s(s+1)} \ . \tag{14.1}$$

Now let $c > 1$ and integrate the equation along Γ_c. We will justify below that we may interchange summation and integration. We have, by the previous lemma with $a = x/n$:

$$\frac{1}{2\pi i}\int_{\Gamma_c}\frac{(x/n)^s}{s(s+1)}\,ds=\begin{cases} 0 & \text{for} \quad 1\le x<n \\ 1-\frac{n}{x} & \text{for} \quad 1\le n\le x \end{cases}$$

Therefore, equation (14.1) yields that

$$\frac{1}{2\pi i}\int_{\Gamma_c}\frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds=x\sum_{1\le n\le x}\Lambda(n)\Big(1-\frac{n}{x}\Big)\,.$$

In the following we will use that

$$\int_1^x H(y-n)\,dy=\begin{cases} x-n & \text{for} \quad x\ge n \\ 0 & \text{for} \quad x\le n \end{cases}$$

The integrated Chebyshev function and its integral representation:

$$\begin{aligned}
\psi_1(x) &= \int_1^x \psi(y)\,dy \\
&= \sum_{n\ge 1}\Lambda(n)\int_1^x H(y-n)\,dy \\
&= \sum_{1\le n\le x}\Lambda(n)(x-n) \\
&= x\sum_{1\le n\le x}\Lambda(n)\Big(1-\frac{n}{x}\Big) \\
&= \frac{x}{2\pi i}\int_{\Gamma_c}\frac{x^s}{s(s+1)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\,ds
\end{aligned}$$

Riemann's ζ–function:

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\
&= \Pi_{p\in P}\Big(1-p^{-s}\Big)^{-1}, \quad \operatorname{Re} s>1 \\
-\frac{d}{ds}\log\zeta(s) &= \sum_{n=1}^{\infty}\Lambda(n)n^{-s} \\
\zeta(s) &= \frac{1}{s-1}+1-s\int_1^{\infty}\frac{\{x\}}{x^{s-1}}\,dx, \quad \operatorname{Re} s>0 \\
h(s) &= \pi^{-s/2}\Gamma(s/2)\zeta(s)
\end{aligned}$$

14.2 Reduction to Asymptotics of $\psi_1(x)$

Let $g:[1,\infty)\to\mathbb{R}$ denote a function. We will use the following notation:

$$\liminf_{x\to\infty} g(x)=\lim_{x\to\infty}\Big(\inf_{y\ge x} g(y)\Big)\,.$$

Similarly,

$$\limsup_{x\to\infty} g(x) = \lim_{x\to\infty}\left(\sup_{y\geq x} g(y)\right) .$$

The following is easy to show: If

$$\liminf_{x\to\infty} g(x) \geq A \quad \text{and} \quad \limsup_{x\to\infty} g(x) \leq A$$

then

$$\lim_{x\to\infty} g(x) = A .$$

Write

$$\begin{aligned}
\pi(x) &= \frac{x}{\ln x}\,(1+R(x)) \\
\psi(x) &= x(1+r(x)) \\
\psi_1(x) &= \frac{x^2}{2}(1+r_1(x))
\end{aligned}$$

Theorem 14.1 *a) If $r(x)\to 0$ as $x\to\infty$ then $R(x)\to 0$ as $x\to\infty$.*
b) If $r_1(x)\to 0$ as $x\to\infty$ then $r(x)\to 0$ as $x\to\infty$.

Proof: Part a1) lower bound for

$$\frac{\pi(x)\ln x}{x}$$

For $x\geq 1$ we have

$$\begin{aligned}
\psi(x) &= \sum_{p\leq x}\left[\frac{\ln x}{\ln p}\right]\ln p \\
&\leq \sum_{p\leq x}\frac{\ln x}{\ln p}\ln p \\
&= \sum_{p\leq x}\ln x \\
&= \pi(x)\ln x
\end{aligned}$$

Therefore,

$$\frac{\psi(x)}{x} \leq \frac{\pi(x)\ln x}{x} \quad \text{for all} \quad x\geq 1 .$$

Therefore, if

$$\frac{\psi(x)}{x}\to 1 \quad \text{as} \quad x\to\infty$$

then

$$\liminf_{x\to\infty}\frac{\pi(x)\ln x}{x}\geq 1 .$$

Part a2) upper bound for

$$\frac{\pi(x)\ln x}{x}$$

Fix $0<\alpha<1$. Note that if $x\geq 1$ and $x^\alpha<p$ then $\ln x^\alpha<\ln p$. We have

$$\begin{aligned}
\psi(x) &= \sum_{p\leq x}\left[\frac{\ln x}{\ln p}\right]\ln p \\
&\geq \sum_{p\leq x}\ln p \\
&\geq \sum_{x^\alpha<p\leq x}\ln p \\
&\geq \sum_{x^\alpha<p\leq x}\ln(x^\alpha) \\
&= \Big(\pi(x)-\pi(x^\alpha)\Big)\alpha\ln x
\end{aligned}$$

Therefore,

$$\psi(x)+\alpha\pi(x^\alpha)\ln x\geq \alpha\pi(x)\ln x$$

and

$$\frac{\psi(x)}{x}+\alpha\frac{\pi(x^\alpha)\ln x}{x}\geq\alpha\frac{\pi(x)\ln x}{x}\ .$$

Since $0<\alpha<1$ it is clear that

$$\pi(x^\alpha)\leq x^\alpha\ ,$$

thus

$$\frac{\pi(x^\alpha)\ln x}{x}\to 0\quad\text{as}\quad x\to\infty\ .$$

Therefore, if

$$\frac{\psi(x)}{x}\to 1\quad\text{as}\quad x\to\infty$$

then

$$1\geq\alpha\limsup_{x\to\infty}\frac{\pi(x)\ln x}{x}\ .$$

Since $0<\alpha<1$ was arbitrary, we obtain that

$$1\geq\limsup_{x\to\infty}\frac{\pi(x)\ln x}{x}\ .$$

The upper and lower bounds imply that

$$\lim_{x\to\infty}\frac{\pi(x)\ln x}{x}=1\ .$$

Part b) We claim that if

$$\frac{\psi_1(x)}{x^2/2} \to 1 \quad \text{as} \quad x \to \infty$$

then

$$\frac{\psi(x)}{x} \to 1 \quad \text{as} \quad x \to \infty \ .$$

The proof essentially only uses that the function $\psi(x)$ increases monotonically.

Let $\beta > 1$. We have

$$\begin{aligned}
\psi(x) &\leq \frac{1}{(\beta-1)x} \int_x^{\beta x} \psi(y)\, dy \\
&= \frac{1}{(\beta-1)x}\Big(\psi_1(\beta x) - \psi_1(x)\Big)
\end{aligned}$$

Therefore,

$$\frac{\psi(x)}{x} \leq \frac{1}{\beta - 1}\Big(\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\Big) \ .$$

As $x \to \infty$ we obtain that

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{1}{\beta-1}\Big(\frac{\beta^2}{2} - \frac{1}{2}\Big) = \frac{\beta+1}{2} \ .$$

Since $\beta > 1$ was arbitrary we obtain that

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq 1 \ .$$

Similarly, fix $0 < \alpha < 1$ and obtain that

$$\begin{aligned}
\psi(x) &\geq \frac{1}{(1-\alpha)x} \int_{\alpha x}^{x} \psi(y)\, dy \\
&= \frac{1}{(1-\alpha)x}\Big(\psi_1(x) - \psi_1(\alpha x)\Big)
\end{aligned}$$

Therefore,

$$\frac{\psi(x)}{x} \geq \frac{1}{1-\alpha}\Big(\frac{\psi_1(x)}{x^2} - \alpha^2 \frac{\psi_1(\alpha x)}{(\alpha x)^2}\Big) \ .$$

As $x \to \infty$ we obtain that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{1}{1-\alpha}\,\frac{1-\alpha^2}{2} = \frac{1+\alpha}{2} \ .$$

Since $\alpha < 1$ was arbitrary we obtain that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1 \ .$$

◇

14.3 Integral Representation of $\psi_1(x)$

For $c \in \mathbb{R}$ let Γ_c denote the vertical line with parameterization

$$\Gamma_c : \quad s(t) = c + it, \quad -\infty < t < \infty \ .$$

Theorem 14.2 *For $c > 1$ and $x \geq 1$ the function $\psi_1(x)$ has the integral representation*

$$\psi_1(x) = \frac{1}{2\pi i} \int_{\Gamma_c} \frac{x^{s+1}}{s(s+1)} \Big(- \frac{\zeta'(s)}{\zeta(s)} \Big) \, ds \ .$$

The proof is based on auxiliary results given below.

Remark: Set

$$F(x,s) = \frac{x^{s+1}}{s(s+1)} \Big(- \frac{\zeta'(s)}{\zeta(s)} \Big) \ ,$$

i.e., $F(x,s)$ is the integrand in the above representation of $\psi_1(x)$. We know that

$$\zeta(s) = \frac{1}{s-1} + h(s), \quad \zeta'(s) = -(s-1)^{-2} + h'(s)$$

where $h(s)$ is holomorphic near $s = 1$. Therefore,

$$Res\Big(\zeta(s), s = 1\Big) = 1 \ .$$

It follows that

$$Res\Big(F(x,s), s = 1\Big) = \frac{x^2}{2} \ .$$

The claim is that this term gives the asymptotic behavior of $\psi_1(x)$ to leading order. If $\mathcal{C}_\varepsilon$ denotes the circle of radius ε centered at $s = 1$, then we have

$$\frac{1}{2\pi i} \int_{\mathcal{C}_\varepsilon} F(x,s) \, ds = \frac{x^2}{2} \ .$$

14.4 Auxiliary Results

Lemma 14.2 *For $c > 0$ and $a > 0$ we have*

$$\frac{1}{2\pi i} \int_{\Gamma^c} \frac{a^s}{s(s+1)} \, ds = \begin{cases} 0 & \text{if} \quad 0 < a \leq 1 \\ 1 - \frac{1}{a} & \text{if} \quad a \geq 1 \end{cases}$$

Lemma 14.3 *For $|\varepsilon| < 1$ we have*

$$-\log(1 - \varepsilon) = \sum_{j=1}^{\infty} \frac{1}{j} \, \varepsilon^j \ .$$

Proof: Set

$$f(\varepsilon) = -\log(1-\varepsilon) \quad \text{for} \quad |\varepsilon| < 1 \ .$$

Then $f(0) = 0$ and

$$f'(\varepsilon) = \frac{1}{1-\varepsilon} = \sum_{j=0}^{\infty} \varepsilon^j \ .$$

The claim follows by integration. ⋄

Lemma 14.4 *For* $\mathrm{Re}\, s > 1$ *we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \ .$$

Proof: From the previous lemma,

$$-\log(1-\varepsilon) = \sum_{j=1}^{\infty} \frac{1}{j}\, \varepsilon^j \quad \text{for} \quad |\varepsilon| < 1 \ .$$

Also, for $\mathrm{Re}\, s > 1$:

$$\zeta(s) = \Pi_p (1 - p^{-s})^{-1} \ ,$$

$$\begin{aligned} \log \zeta(s) &= -\sum_p \log(1-p^{-s}) \\ &= \sum_p \sum_{j=1}^{\infty} \frac{1}{j}\, p^{-js} \\ &= \sum_{n=1}^{\infty} c_n n^{-s} \end{aligned}$$

where

$$c_n = \left\{ \begin{array}{lll} \frac{1}{j} & \text{if} & n = p^j \\ 0 & \text{if} & p \ \text{ is not a prime power} \end{array} \right. \ .$$

Therefore,

$$\log \zeta(s) = \sum_{n=1}^{\infty} c_n e^{-s \ln n}$$

and differentiation yields that

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds} \log \zeta(s) \\ &= -\sum_{n=1}^{\infty} c_n n^{-s} \ln n \ . \end{aligned}$$

If $n = p^j$ then

$$c_n \ln n = \frac{1}{j} \ln(p^j) = \ln p = \Lambda(n) \ .$$

If n is not a prime power, then

$$c_n \ln n = 0 = \Lambda(n) \ .$$

This proves the formula

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \ .$$

$\diamond$

14.5 The ζ–Function has no Zero on the Line $s = 1 + it$

Let

$$s = \sigma + it, \quad \sigma > 1, \quad t \in \mathbb{R} \ .$$

We have

$$\begin{aligned} \zeta(s) &= \Pi_p (1 - p^{-s})^{-1} \\ \log \zeta(s) &= -\sum_p \log(1 - p^{-s}) \\ &= \sum_p \sum_{j=1}^{\infty} \frac{1}{j} (p^j)^{-s} \\ &= \sum_{n=1}^{\infty} c_n n^{-s} \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{j} & \text{if} \quad n = p^j \\ 0 & \text{if} \quad p \ \text{ is not a prime power} \end{cases} \ .$$

We have

$$\begin{aligned} n^{-s} &= n^{-\sigma} n^{-it} \\ &= n^{-\sigma} e^{-it \ln n} \\ &= n^{-\sigma} \Big(\cos(t \ln n) - i \sin(t \ln n) \Big) \end{aligned}$$

thus

$$\mathrm{Re}\, n^{-s} = n^{-\sigma} \cos(t \ln n) \ .$$

Therefore,

$$\text{Re } \log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-\sigma} \cos(t \ln n) \ .$$

From

$$\zeta(s) = e^{\log \zeta(s)}$$

we obtain that

$$|\zeta(s)| = e^{\text{Re } \log \zeta(s)}$$

and

$$\ln |\zeta(s)| = \text{Re } \log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-\sigma} \cos(t \ln n) \ .$$

Now consider

$$\begin{aligned} \ln \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| &= 3 \ln |\zeta(\sigma)| + 4 \ln |\zeta(\sigma + it)| + \ln |\zeta(\sigma + 2it)| \\ &= \sum_{n=1}^{\infty} c_n n^{-\sigma} \Big(3 + 4 \cos(t \ln n) + \cos(2t \ln n) \Big) \end{aligned}$$

The term in brackets turns out to be non–negative:

We know that

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = 2\cos^2(\alpha) - 1$$

thus

$$\begin{aligned} 3 + 4\cos(\alpha) + \cos(2\alpha) &= 2(1 + 2\cos(\alpha) + \cos^2(\alpha)) \\ &= 2(1 + \cos(\alpha))^2 \end{aligned}$$

and obtain that

$$\ln \left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| \geq 0 \ .$$

This yields the lower bound

$$\left| \zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it) \right| \geq 1 \ . \tag{14.2}$$

for all

$$\sigma > 1 \quad \text{and} \quad t \in \mathbb{R} \ .$$

Now suppose that

$$\zeta(1 + it) = 0$$

for some real $t \neq 0$. Then consider the function

$$f(\sigma) = \Big|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\Big| \quad \text{for} \quad 1 < \sigma \le 2 \ .$$

We obtain for some constant $C > 0$ the estimates

$$\begin{aligned}
|\zeta(\sigma+it)| &\le C(\sigma-1) \quad (\text{since } \zeta(1+it)=0) \\
|\zeta(\sigma+2it)| &\le C \\
|\zeta(\sigma)| &\le 2(\sigma-1)^{-1} \quad (\text{since } \zeta(s) \text{ has a first order pole at } s=1)
\end{aligned}$$

for $1 < \sigma \le 2$. The constant $C > 0$ may depend on t, but is independent of $1 < \sigma \le 2$.

Therefore,

$$f(\sigma) \le K(\sigma-1)^{-3}(\sigma-1)^4 = K(\sigma-1) \quad \text{for} \quad 1 < \sigma \le 2 \ .$$

This estimate contradicts the lower bound which was established in (14.2). The contradiction shows that the ζ–function has no zero on the line

$$s = 1 + it, \quad t \in \mathbb{R} \ .$$

14.6 Growth Estimates for $\Big|\zeta'(s)/\zeta(s)\Big|$

Let $s = \sigma + it$ where σ and t are real.

To prove the PNT one needs a growth estimate for the function

$$\Big|\zeta'(\sigma+it)/\zeta(\sigma+it)\Big|$$

for $1 \le \sigma \le 2$ as $|t| \to \infty$.

We will prove:

Theorem 14.3 *For all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ with*

$$\Big|\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\Big| \le C(\varepsilon)\,|t|^{\varepsilon} \quad \textit{for} \quad 1 \le \sigma \le 2 \quad \textit{and} \quad |t| \ge 2 \; .$$

The theorem follows from three Lemmas. We first prove a growth estimate for $|\zeta(s)|$. Using Cauchy's inequality it implies a growth estimate for $|\zeta'(s)|$. Using the inequality (14.2) and the estimates for $|\zeta(s)|$ and $|\zeta'(s)|$ we then prove a lower bound for $|\zeta(s)|$ as $|t| \to \infty$. The upper bound for $|\zeta'(s)|$ and the lower bound for $|\zeta(s)|$ then imply the upper bound for $|\zeta'(s)/\zeta(s)|$ stated in the theorem.

14.6.1 Growth Estimates for $|\zeta(s)|$

Lemma 14.5 *Let $0 < \sigma_0 \le 1$. For all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ so that*

$$|\zeta(\sigma+it)| \le C(\varepsilon)\,|t|^{1-\sigma_0+\varepsilon} \quad \textit{for} \quad 0 < \sigma_0 \le \sigma \le 3 \quad \textit{and} \quad |t| \ge 1 \; . \tag{14.3}$$

Proof: We may assume that $0 < \varepsilon \le \sigma_0$, thus

$$0 < \eta := 1 - \sigma_0 + \varepsilon \le 1 \; .$$

a) We use the formula

$$\zeta(s) = \frac{1}{s-1} + H(s) \quad \text{for} \quad \operatorname{Re} s > 0, \quad s \ne 1 \; , \tag{14.4}$$

where

$$H(s) = \sum_{n=1}^{\infty} \delta_n(s) \quad \text{for} \quad \operatorname{Re} s > 0$$

with

$$\delta_n(s) = \int_n^{n+1} (n^{-s} - x^{-s})\, dx \; .$$

b) We have $|s-1| \ge |t| \ge 1$, thus

$$\frac{1}{|s-1|} \le \frac{1}{|t|} \le 1 \quad \text{for} \quad |t| \ge 1 \; .$$

c) First estimate of $|\delta_n(s)|$: We have $s = \sigma + it$ with $\sigma > 0$ and claim that

$$|\delta_n(s)| \le \frac{2}{n^\sigma} \ . \tag{14.5}$$

Since $n = e^{\ln n}$ we have $n^{-s} = e^{-(\sigma+it)\ln n}$ and

$$|n^{-s}| = e^{-\sigma \ln n} = n^{-\sigma} \ .$$

Also, for $x \ge n$ we have

$$|x^{-s}| = x^{-\sigma} \le n^{-\sigma} \ .$$

Therefore,

$$|\delta_n(s)| \le \int_n^{n+1} \Big(|n^{-s}| + |x^{-s}| \Big) \, dx \le \frac{2}{n^\sigma} \ . \tag{14.6}$$

d) Second estimate of $|\delta_n(s)|$: We have $s = \sigma + it$ with $\sigma > 0$ and claim that

$$|\delta_n(s)| \le \frac{|s|}{n^{1+\sigma}} \ . \tag{14.7}$$

For the function $f(x) = x^{-s}$ we have $f'(x) = -sx^{-s-1}$ for $x > 0$. Therefore,

$$\begin{aligned}
|\delta_n(s)| &= \Big| \int_n^{n+1} \Big(n^{-s} - x^{-s} \Big) \, dx \Big| \\
&= \Big| \int_n^{n+1} \Big(f(n) - f(x) \Big) \, dx \Big| \\
&\le \max \Big\{ |f'(q)| \ : \ n \le q \le n+1 \Big\} \\
&= \frac{|s|}{n^{1+\sigma}}
\end{aligned}$$

e) Let $0 < \varepsilon \le \sigma_0 \le 1$ and let $\sigma \ge \sigma_0$. We have

$$|\delta_n(s)| \le \frac{2}{n^\sigma} \le \frac{2}{n^{\sigma_0}} \tag{14.8}$$

and

$$|\delta_n(s)| \le \frac{|s|}{n^{1+\sigma}} \le \frac{|s|}{n^{1+\sigma_0}} \ . \tag{14.9}$$

Since

$$0 < \eta = 1 - \sigma_0 + \varepsilon \le 1$$

we obtain that

$$\begin{aligned}
|\delta_n(s)| &= \Big| \delta_n(s) \Big|^\eta \, \Big| \delta_n(s) \Big|^{1-\eta} \\
&\le \Big(\frac{|s|}{n^{1+\sigma_0}} \Big)^\eta \Big(\frac{2}{n^{\sigma_0}} \Big)^{1-\eta} \\
&\le \frac{2|s|^\eta}{n^{1+\varepsilon}} \ .
\end{aligned}$$

Here we have used that

$$(1+\sigma_0)\eta + \sigma_0(1-\eta) = \eta + \sigma_0 = 1+\varepsilon \ .$$

Therefore,

$$|\delta_n(s)| \le \frac{2}{n^{1+\varepsilon}}\, |s|^{1-\sigma_0+\varepsilon} \ .$$

Using the equation

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty \delta_n(s)$$

one obtains that

$$|\zeta(s)| \le 1 + 2\zeta(1+\varepsilon)\, |s|^{1-\sigma_0+\varepsilon}$$

for $0 < \sigma_0 \le \sigma \le 3$ and $|t| \ge 1$. If $|\sigma| \le 3$ and $|t| \ge 1$ then $|s|^2 \le 9 + |t|^2 \le 10|t|^2$, thus $|s| \le \sqrt{10}\,|t|$. The estimate (14.3) follows. $\diamond$

14.6.2 Growth Estimates for $|\zeta'(s)|$

Lemma 14.6 *For all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ so that*

$$|\zeta'(\sigma + it)| \le C(\varepsilon)\, |t|^\varepsilon \quad \textit{for} \quad 1 \le \sigma \le 2 \quad \textit{and} \quad |t| \ge 2 \ . \tag{14.10}$$

Proof: We will use the previous lemma and a Cauchy estimate for $\zeta'(s)$. Let $s = \sigma + it$ with $1 \le \sigma \le 2$ and $|t| \ge 2$. Let Γ denote the circle of radius ε centered at s. We have

$$\begin{aligned} \zeta(s) &= \frac{1}{2\pi i} \int_\Gamma \frac{\zeta(z)}{z-s}\, dz \\ \zeta'(s) &= \frac{1}{2\pi i} \int_\Gamma \frac{\zeta(z)}{(z-s)^2}\, dz \end{aligned}$$

Therefore,

$$|\zeta'(s)| \le \frac{1}{\varepsilon} \max\{|\zeta(z)| \ : \ z \in \Gamma\} \ .$$

In the previous lemma, choose $\sigma_0 = 1 - \varepsilon$, thus

$$1 - \sigma_0 + \varepsilon = 2\varepsilon \ .$$

Obtain that

$$|\zeta(z)| \le C(\varepsilon)\, (|t| + \varepsilon)^{2\varepsilon} \quad \text{for} \quad z \in \Gamma$$

if $\varepsilon > 0$ is small. The estimate

$$|\zeta'(s)| \le \frac{1}{\varepsilon}\, C(\varepsilon) \Big(|t| + \varepsilon\Big)^{2\varepsilon} \le \frac{1}{\varepsilon}\, C(\varepsilon) 2^{2\varepsilon} |t|^{2\varepsilon}$$

for $1 \le \sigma \le 2$ and $|t| \ge 2$ follows. $\diamond$

14.6.3 Lower Bounds for $|\zeta(s)|$

Lemma 14.7 *For all $\varepsilon > 0$ there exists a constant $K(\varepsilon) > 0$ so that*

$$|\zeta(\sigma + it)| \geq K(\varepsilon)|t|^{-\varepsilon} \quad \textit{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 2 \; . \tag{14.11}$$

Proof: Let's first recall estimates that we have shown above:

$$\Big|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)\Big| \geq 1 \quad \text{for} \quad \sigma > 1, \quad t \in \mathbb{R} \; . \tag{14.12}$$

$$|\zeta(\sigma + it)| \leq C(\varepsilon)|t|^{\varepsilon} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 1 \; . \tag{14.13}$$

$$|\zeta'(\sigma + it)| \leq C(\varepsilon)|t|^{\varepsilon} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 2 \; . \tag{14.14}$$

$$0 < \frac{1}{\sigma - 1} \leq \zeta(\sigma) \leq \frac{2}{\sigma - 1} \quad \text{for} \quad 1 < \sigma \leq 2 \; . \tag{14.15}$$

From (14.15) obtain that

$$\frac{1}{8}\,(\sigma - 1)^3 \leq \Big(\zeta(\sigma)\Big)^{-3} \quad \text{for} \quad 1 < \sigma \leq 2 \; . \tag{14.16}$$

Then (14.12) yields that

$$\Big|\zeta(\sigma + it)\Big|^4 \geq \frac{1}{8}\,(\sigma - 1)^3\,\frac{1}{|\zeta(\sigma + 2it)|} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 1 \; . \tag{14.17}$$

The estimate

$$|\zeta(\sigma + 2it)| \leq C(\varepsilon)|t|^{\varepsilon} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 1 \tag{14.18}$$

yields that

$$\frac{1}{|\zeta(\sigma + 2it)|} \geq \frac{1}{C(\varepsilon)}\,|t|^{-\varepsilon} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 1 \; . \tag{14.19}$$

Therefore, from (14.17) one obtains that

$$\Big|\zeta(\sigma + it)\Big| \geq K_1(\varepsilon)\,(\sigma - 1)^{3/4}\,|t|^{-\varepsilon/4} \quad \text{for} \quad 1 \leq \sigma \leq 2, \quad |t| \geq 1 \; . \tag{14.20}$$

Let $1 \leq \sigma < \tilde{\sigma} \leq 2$ and let $|t| \geq 2$. Use the bound (14.14) for $|\zeta'(s)|$ to obtain that

$$\Big|\zeta(\tilde{\sigma} + it) - \zeta(\sigma + it)\Big| \leq K_2(\varepsilon)(\tilde{\sigma} - \sigma)\,|t|^{\varepsilon} \leq K_2(\varepsilon)(\tilde{\sigma} - 1)\,|t|^{\varepsilon} \; . \tag{14.21}$$

Set

$$A_\varepsilon = \Big(\frac{K_1(\varepsilon)}{2K_2(\varepsilon)}\Big)^4 \; . \tag{14.22}$$

This choice of A_ε) will be explained below. Since $K_1(\varepsilon)$ is a (small) constant in a lower bound and $K_2(\varepsilon)$ is a (large) constant in an upper bound, one may assume that

$$0 < A_\varepsilon \leq 1 \; .$$

In the following, let

$$s = \sigma + it \quad \text{where} \quad 1 \le \sigma \le 2 \quad \text{and} \quad |t| \ge 2 \ .$$

Case 1:

$$\sigma - 1 \ge A_\varepsilon |t|^{-5\varepsilon}$$

(In Case 1 $s = \sigma + it$ is not too close to Γ_1.)

Case 2:

$$0 \le \sigma - 1 < A_\varepsilon |t|^{-5\varepsilon}$$

(In Case 2 $s = \sigma + it$ gets very close to Γ_1 for large $|t|$.)

First consider Case 1. Obtain that

$$(\sigma - 1)^{3/4} \ge A_\varepsilon^{3/4} \, |t|^{-15\varepsilon/4} \ .$$

The estimate (14.20) yields that

$$\Big| \zeta(\sigma + it) \Big| \ge K_1(\varepsilon) \, A_\varepsilon^{3/4} \, |t|^{-4\varepsilon} \tag{14.23}$$

in Case 1.

Second, consider Case 2. Since

$$0 \le \sigma - 1 < A_\varepsilon |t|^{-5\varepsilon} \le 1$$

there exists $\tilde\sigma$ with

$$1 \le \sigma < \tilde\sigma \le 2 \quad \text{and} \quad \tilde\sigma - 1 = A_\varepsilon |t|^{-5\varepsilon} \ .$$

For $\tilde s = \tilde\sigma + it$ we have Case 1 and obtain from (14.23) that

$$\Big| \zeta(\tilde\sigma + it) \Big| \ge K_1(\varepsilon) \, A_\varepsilon^{3/4} \, |t|^{-4\varepsilon} \tag{14.24}$$

Using this lower bound and using (14.21) one obtains that

$$\begin{aligned} |\zeta(\sigma + it)| &\ge |\zeta(\tilde\sigma + it)| - K_2(\varepsilon)(\tilde\sigma - 1)|t|^\varepsilon \\ &\ge |t|^{-4\varepsilon} \Big(K_1(\varepsilon) A_\varepsilon^{3/4} - K_2(\varepsilon) A_\varepsilon \Big) \end{aligned}$$

If

$$K_1(\varepsilon) A_\varepsilon^{3/4} = 2 K_2(\varepsilon) A_\varepsilon$$

then

$$A_\varepsilon = \Big(\frac{K_1(\varepsilon)}{2 K_2(\varepsilon)} \Big)^4 \ .$$

This is the choice for A_ε in (14.22). With this definition of A_ε one obtains that

$$K_1(\varepsilon) A_\varepsilon^{3/4} - K_2(\varepsilon) A_\varepsilon = K_2(\varepsilon) A_\varepsilon$$

and the above lower bound for $|\zeta(\sigma + it)|$ yields that

$$|\zeta(\sigma + it)| \geq K_2(\varepsilon) A_\varepsilon |t|^{-4\varepsilon}$$

in Case 2. Together with the Case 1 estimate (14.23) one obtains that

$$\left|\zeta(\sigma + it)\right| \geq K_3(\varepsilon)\,|t|^{-4\varepsilon} \tag{14.25}$$

where $K_3(\varepsilon) > 0$. This completes the proof of Lemma 14.7. $\diamond$

14.7 Proof of the Prime Number Theorem

Recall:

$$\begin{aligned}\psi(y) &= \sum_{1\le n\le y} \Lambda(n) \\ &= \sum_{n\ge 1} \Lambda(n) H(y-n)\end{aligned}$$

and

$$\begin{aligned}\psi_1(x) &= \int_1^x \psi(y)\, dy \\ &= \sum_{1\le n\le x} \Lambda(n)(x-n)\end{aligned}$$

We have shown that

$$\psi_1(x) \sim \frac{x^2}{2} \quad \text{as} \quad x \to \infty$$

implies the PNT.

The equation

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad \text{for} \quad \operatorname{Re} s > 1 \tag{14.26}$$

follows from the product formula

$$\zeta(s) = \Pi_p \frac{1}{1-p^{-s}} \quad \text{for} \quad \operatorname{Re} s > 1\ .$$

Multiply the equation (14.26) by

$$\frac{1}{2\pi i} \frac{x^{s+1}}{s(s+1)} \quad \text{for} \quad x \ge 1, \quad \operatorname{Re} s > 1$$

to obtain that

$$\frac{1}{2\pi i} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) = \frac{x}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \frac{(x/n)^2}{s(s+1)}\ . \tag{14.27}$$

Integrate along the straight line Γ_2 and use Lemma 14.1 to obtain that

$$\begin{aligned}\frac{1}{2\pi i} \int_{\Gamma_2} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big)\, ds &= \sum_{n=1}^{\infty} \Lambda(n)\, (x-n) \\ &= \psi_1(x)\end{aligned}$$

Set

$$F(x,s) = \frac{x^{s+1}}{s(s+1)} \Big(- \frac{\zeta'(s)}{\zeta(s)} \Big) \ .$$

We know that the function $\zeta(s)$ has a simple pole at $s=1$. Therefore,

$$Res\Big(F(x,s), s=1\Big) = \frac{x^2}{2} \ .$$

This is a good reason for

$$\psi_1(x) \sim \frac{x^2}{2} \quad \mbox{as} \quad x \to \infty \ .$$

We will use the following growth estimate: There exists a constant $C>0$ so that

$$\Big|\frac{\zeta'(s)}{\zeta(s)}\Big| \le C|t|^{1/2} \quad \mbox{for} \quad s = \sigma + it, \quad 1 \le \sigma \le 2, \quad |t| \ge 2 \ . \tag{14.28}$$

For $1 \le c \le 2$ let $\tilde\Gamma_c$ denote the half–line with parameterization

$$\tilde\Gamma_c : \quad s = c + it \quad \mbox{where} \quad 2 \le t < \infty \ .$$

Using the estimate (14.28) one obtains that the integral

$$\int_{\tilde\Gamma_c} F(x,s)\, ds$$

is well defined. This follows from the bound

$$|F(x,s)| \le C \frac{x^{\sigma+1}}{t^2}\, t^{1/2} \quad \mbox{for} \quad s = \sigma + it, \quad t \ge 2$$

and

$$\int_2^\infty t^{-3/2}\, dt = (-2 t^{-1/2})\Big|_2^\infty = \frac{2}{\sqrt{2}} < \infty \ .$$

Using (14.28) one also obtains the following:

Lemma 14.8 *Let $\tau \ge 2$ and let γ_τ denote the horizontal straight line with parameterization*

$$\gamma_\tau : \quad s = \sigma + i\tau \quad \mbox{where} \quad 1 \le \sigma \le 2 \ .$$

Then we have

$$\Big| \int_{\gamma_\tau} F(x,s)\, ds \Big| \to 0 \quad \mbox{as} \quad \tau \to \infty \ .$$

Proof: We have

$$|F(x,s)| \le C \frac{x^{\sigma+1}}{t^2}\, t^{1/2} = K t^{-3/2} \quad \mbox{for} \quad s = \sigma + it, t \ge \tau \ge 2$$

and

$$\Big| \int_{\gamma_\tau} F(x,s)\, ds \Big| \le K \tau^{-3/2} \to 0 \quad \mbox{as} \quad \tau \to \infty \ .$$

$\diamond$

For $T \geq 2$ let

$$\gamma_T = \gamma_{1T} + \gamma_{2T} + \gamma_{3T} + \gamma_{4T} + \gamma_{5T}$$

where γ_{jT} is the straight line with parameterization:

$$\begin{aligned}
\gamma_{1T} &: \quad s = 1 + it, \quad -\infty < t \leq -T \\
\gamma_{5T} &: \quad s = 1 + it, \quad T \leq t < \infty \\
\gamma_{3T} &: \quad s = 2 + it, \quad -T \leq t \leq T \\
\gamma_{2T} &: \quad s = \sigma - iT, \quad 1 \leq \sigma \leq 2 \\
\gamma_{4T} &: \quad s = 2 - \sigma + iT, \quad 1 \leq \sigma \leq 2
\end{aligned}$$

Using the previous lemma, it is not difficult to show that

$$\int_{\Gamma_2} F(x,s)\, ds = \int_{\gamma_T} F(x,s)\, ds \quad \text{for} \quad T \geq 2 \; .$$

For $T \geq 2$ and small $\delta > 0$ consider the rectangle $R(T,\delta)$ with corners

$$1 - \delta - iT, \quad 2 - iT, \quad 2 + iT, \quad 1 - \delta + iT \; .$$

Since

$$\zeta(1 + it) \neq 0 \quad \text{for} \quad 0 < |t| \leq T$$

and since $\zeta(s)$ has a pole at $s = 1$ one obtains: For every $T \geq 2$ there exists $\delta = \delta(T) > 0$ so that

$$\zeta(s) \neq 0 \quad \text{for} \quad s \in R(T,\delta), \quad s \neq 1 \; . \tag{14.29}$$

In the following, we fix $\delta = \delta(T) > 0$ so that (14.29) holds. Let $\partial R(T,\delta)$ denote the boundary curve of $R(T,\delta)$, positively oriented.

We obtain that

$$\frac{1}{2\pi i} \int_{\partial R(T,\delta)} F(x,s) = Res\Big(F(x,s), s = 1\Big) = \frac{x^2}{2} \; .$$

Let $\mathcal{C} = \mathcal{C}(T,\delta)$ denote the curve consisting of five straight lines,

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 + \mathcal{C}_5$$

where

$$\mathcal{C}_1 = \gamma_{1T} \quad \text{and} \quad \mathcal{C}_5 = \gamma_{5T}$$

and

$$\begin{aligned}
\mathcal{C}_3 &: \quad s = 1 - \delta + it, \quad -T \leq t \leq T \\
\mathcal{C}_2 &: \quad s = 1 - \sigma - iT, \quad 0 \leq \sigma \leq \delta \\
\mathcal{C}_4 &: \quad s = 1 - \delta + \sigma + iT, \quad 0 \leq \sigma \leq \delta
\end{aligned}$$

Obtain that

$$\int_{\Gamma_T} F(x,s)\,ds - \int_{\mathcal{C}} F(x,s)\,ds = \int_{\partial R(T,\delta)} F(x,s)\,ds\ .$$

Here the last integral equals $2\pi i \cdot \frac{x^2}{2}$. Therefore,

$$\psi_1(x) = \frac{x^2}{2} + \frac{1}{2\pi i} \int_{\mathcal{C}} F(x,s)\,ds \tag{14.30}$$

and we have to estimate

$$\frac{1}{x^2} \int_{\mathcal{C}} F(x,s)\,ds \quad \text{as} \quad x \to \infty\ .$$

Here $\mathcal{C} = \mathcal{C}(T,\delta)$.

We must estimate essentially three terms:

$$\begin{aligned} term_5(x,T) &= \frac{1}{x^2}\Big|\int_{\mathcal{C}_5} F(x,s)\,ds\Big| \\ term_3(x,T,\delta) &= \frac{1}{x^2}\Big|\int_{\mathcal{C}_3} F(x,s)\,ds\Big| \\ term_4(x,T,\delta) &= \frac{1}{x^2}\Big|\int_{\mathcal{C}_4} F(x,s)\,ds\Big| \end{aligned}$$

Estimate of $term_5(x,T)$:

We have

$$\begin{aligned} \Big|\int_{\mathcal{C}_5} F(x,s)\,ds\Big| &\le \int_T^\infty |F(x,s)|\,ds \\ &\le Cx^2 \int_T^\infty t^{-3/2}\,dt \\ &= 2Cx^2/\sqrt{T} \end{aligned}$$

thus

$$term_5(x,T) = \frac{1}{x^2}\Big|\int_{\mathcal{C}_5} F(x,s)\,ds\Big| \le K/\sqrt{T}\ .$$

Estimate of $term_3(x,T,\delta)$:

Recall that

$$|F(x,s)| = x^{1+\sigma}\Big|\frac{1}{s(1+s)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\Big| \quad \text{for} \quad s = \sigma + it\ .$$

There exists a constant $C(T,\delta)$ so that

$$\Big|\frac{1}{s(1+s)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\Big| \le C(T,\delta) \quad \text{for} \quad s \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4\ .$$

For $s \in \mathcal{C}_3$ we have $s = 1 - \delta + it$, thus

$$|x^{1+s}| = x^{2-\delta} \quad \text{for} \quad s \in \mathcal{C}_3 \ .$$

Therefore,

$$\frac{1}{x^2} \Big| \int_{\mathcal{C}_3} F(x,s)\, ds \Big| \le x^{-\delta} 2TC(T,\delta) \ .$$

Obtain:

$$term_3(x,T,\delta) \le \frac{K(T,\delta)}{x^\delta} \ .$$

Estimate of $term_4(x,T,\delta)$:
For $s \in \mathcal{C}_4$ we have $s = 1-\delta+\sigma+iT, 0 \le \sigma \le \delta$, thus

$$|x^{1+s}| = x^{2-\delta} x^\sigma = x^{2-\delta} e^{\sigma \ln x}$$

and

$$\Big| \int_{\mathcal{C}_4} F(x,s)\, ds \Big| \le C(T,\delta) x^{2-\delta} \int_0^\delta e^{\sigma \ln x}\, d\sigma \ .$$

Here

$$\begin{aligned} \int_0^\delta e^{\sigma \ln x}\, d\sigma &= \frac{1}{\ln x} e^{\sigma \ln x} \Big|_{\sigma=0}^{\sigma=\delta} \\ &\le \frac{x^\delta}{\ln x} \end{aligned}$$

Therefore,

$$\frac{1}{x^2} \Big| \int_{\mathcal{C}_4} F(x,s)\, ds \Big| \le \frac{C(T,\delta)}{\ln x} \ ,$$

i.e.,

$$term_4(x,T,\delta) \le \frac{C(T,\delta)}{\ln x} \ .$$

Obtain that

$$\Big| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \Big| \le \frac{K}{\sqrt{T}} + \frac{K_1(T,\delta)}{x^\delta} + \frac{C(T,\delta)}{\ln x} \ .$$

If $\varepsilon > 0$ is given, choose $T > 0$ so that

$$\frac{K}{\sqrt{T}} \le \varepsilon \ .$$

Then choose $\delta = \delta(T)$ so that (14.29) holds. Obtain that

$$\Big| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \Big| \le 2\varepsilon \quad \text{for} \quad x \ge x_\varepsilon$$

if x_ε is sufficiently large.

This proves that

$$\psi_1(x) \sim \frac{x^2}{2} \quad \text{as} \quad x \to \infty \ .$$

15 The Mertens Function and the Riemann Hypothesis

History
August Ferdinand Möbius, 1790–1868, German
Franz Mertens, 1840–1927, Polish

The Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } \; n = 1 \\ 1 & \text{if } n > 1 \text{ is a product of an even number of distinct primes} \\ -1 & \text{if } n > 1 \text{ is a product of an odd number of distinct primes} \\ 0 & \text{if } n \text{ contains a quadratic prime factor} \end{cases}$$

The Mertens function $M : \mathbb{N} \to \mathbb{Z}$ is

$$M(j) = \sum_{n=1}^{j} \mu(n) \; .$$

The Mertens conjecture was that $|M(j)| < \sqrt{j}$ for all $j > 1$.

A property of the Möbius function:

Lemma 15.1 *The following holds:*

$$\sum_{j|n} \mu(j) = \begin{cases} 1 & \text{for} \quad n = 1 \\ 0 & \text{for} \quad n > 1 \end{cases}$$

Proof: The formula is obvious for $n = 1$. Let

$$n = p_1^{a_1} \ldots p_k^{a_k}, \quad k \geq 1 \; ,$$

with distinct primes p_j and exponents $a_j \geq 1$.

Let $j|n$. If $j = 1$ then

$$\mu(j) = \mu(1) = 1 \; .$$

One obtains that

$$\begin{aligned} \sum_{j|n} \mu(j) &= \mu(1) + \mu(p_1) + \ldots + \mu(p_k) + \mu(p_1 p_2) + \ldots + \mu(p_{k-1} p_k) + \ldots + \mu(p_1 \cdots p_k) \\ &= 1 + \binom{k}{1} (-1)^1 + \binom{k}{2} (-1)^2 + \binom{k}{3} (-1)^3 + \ldots + \binom{k}{k} (-1)^k \\ &= (1-1)^k \\ &= 0 \end{aligned}$$

This proves the lemma. $\diamond$

Let a_j and b_k denote two bounded sequences of complex numbers. Set

$$f(s) = \sum_{j=1}^{\infty} \frac{a_j}{j^s} \quad \text{and} \quad g(s) = \sum_{k=1}^{\infty} \frac{b_k}{k^s} \quad \text{for} \quad \mathrm{Re}\, s > 1 \; .$$

The functions $f(s)$ and $g(s)$ are holomorphic for $\mathrm{Re}\, s > 1$.

We have

$$f(s)g(s) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_j b_k}{(jk)^s} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

with

$$c_n = \sum_{jk=n} a_j b_k = \sum_{j|n} a_j b_{n/j} \ .$$

Apply this to the functions

$$f(s) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^s} \quad \text{and} \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

Since

$$\sum_{j|n} \mu(j) = \begin{cases} 1 & \text{for} \quad n = 1 \\ 0 & \text{for} \quad n > 1 \end{cases}$$

one obtains that

$$f(s)\zeta(s) = 1 \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

This proves that

$$\frac{1}{\zeta(s)} = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^s} \quad \text{for} \quad \mathrm{Re}\, s > 1 \ .$$

Assume that there exists a constant $C > 0$ so that

$$|M(j)| \le C\sqrt{j} \quad \text{for all} \quad j \in \mathbb{N} \ .$$

We claim that the formula

$$f(s) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^s}$$

defines a holomorphic function for $\mathrm{Re}\, s > \frac{1}{2}$.

We set $M(0) = 0$ and have

$$\mu(j) = M(j) - M(j-1) \quad \text{for} \quad j = 1, 2, \ldots$$

We also set

$$M(x) = M(n) \quad \text{for} \quad n \le x < n+1 \ .$$

We have, formally,

$$
\begin{aligned}
f(s) &= \sum_{j=1}^{\infty} \frac{M(j) - M(j-1)}{j^s} \\
&= \sum_{j=1}^{\infty} \frac{M(j)}{j^s} - \sum_{j=1}^{\infty} \frac{M(j-1)}{j^s} \\
&= \sum_{j=1}^{\infty} \frac{M(j)}{j^s} - \sum_{j=1}^{\infty} \frac{M(j)}{(j+1)^s} \\
&= \sum_{j=1}^{\infty} M(j) \Big(\frac{1}{j^s} - \frac{1}{(j+1)^s} \Big)
\end{aligned}
$$

Note that

$$\int_j^{j+1} s x^{-s-1}\, dx = -x^{-s}\Big|_j^{j+1} = -\frac{1}{(j+1)^s} + \frac{1}{j^s} \ ,$$

thus

$$f(s) = s \sum_{j=1}^{\infty} M(j) \int_j^{j+1} x^{-s-1}\, dx = s \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx \ .$$

If $|M(x)| \le C\sqrt{x}$ for $x \ge 1$ then

$$\Big| \int_1^{\infty} \frac{M(x)}{x^{s+1}}\, dx \Big| \le C \int_1^{\infty} \frac{dx}{x^{\sigma + \frac{1}{2}}} < \infty \quad \text{for} \quad \sigma = \operatorname{Re} s > \frac{1}{2} \ .$$

If the estimate $|M(x)| \le C\sqrt{x}$ holds, then the formal calculations can be justified and the formula

$$f(s) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^s}$$

defines a holomorphic functions for $\operatorname{Re} s > \frac{1}{2}$. By analytic continuation, the formula

$$f(s)\zeta(s) = 1$$

holds for $\operatorname{Re} s > \frac{1}{2}, s \neq 1$. One obtains that $\zeta(s) \neq 0$ for $\operatorname{Re} s > \frac{1}{2}, s \neq 1$.

Instead of assuming the bound $|M(x)| \le C\sqrt{x}$ the following is sufficient: For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that

$$|M(x)| \le C_\varepsilon x^{\frac{1}{2} + \varepsilon} \quad \text{for} \quad x \ge 1 \ .$$

This condition is equivalent to the Riemann hypothesis.

Set

$$m(n) = \frac{M(n)}{\sqrt{n}} \quad \text{for} \quad n = 1, 2, \ldots$$

Mertens conjecture was that $|m(n)| < 1$ for $n > 1$.

According to Wikipedia, the estimate

$$|m(n)| \le 0.6 \quad \text{for} \quad 1 < n \le 10^{16}$$

has been checked numerically. No value of $n > 1$ is known with $|m(n)| \ge 1$. However, in 2016 it has been proved that

$$\limsup_{n\to\infty} |m(n)| > 1.8 \ .$$

16 Complex Variables and PDEs

16.1 2D Irrotational Euler Flows

The Euler equations for 2D incompressible flows read

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

Here the vector

$$\Big(u(x,y,t), v(x,y,t)\Big) \in \mathbb{R}^2$$

is the velocity and $p(x,y,t)$ is the pressure. The equation $u_x + v_y = 0$ is the incompressibility condition. The corresponding 3D velocity field

$$(u, v, 0)$$

has the vorticity

$$\begin{aligned} \omega &= curl(u,v,0) \\ &= \nabla \times (u,v,0) \\ &= (0, 0, v_x - u_y) \end{aligned}$$

A velocity field is called irrotational if its vorticity is zero. For 2D incompressible, irrotational flows one obtains the conditions

$$u_x + v_y = 0, \quad v_x - u_y = 0$$

which can also be written as

$$u_x = -v_y, \quad u_y = -(-v_x) \ .$$

These are the Cauchy–Riemann equations for the pair

$$(u, -v) \ .$$

Lemma 16.1 *Let $U \subset \mathbb{C}$ denote an open set and let $f \in H(U)$. Define $u, v : U \to \mathbb{R}$ by*

$$f(x+iy) = u(x,y) - iv(x,y)$$

and set

$$p(x,y) = -\frac{1}{2}\Big(u^2(x,y) + v^2(x,y)\Big) \ .$$

Then

$$(u, v, p)$$

is an irrotational solution of the stationary incompressible Euler equations.

Proof: We know that $(u, -v)$ satisfy the Cauchy–Riemann equations, thus

$$u_x + v_y = 0, \quad v_x - u_y = 0 \ .$$

Also,

$$\begin{aligned}
\Big(\frac{1}{2}(u^2+v^2)\Big)_x &= uu_x + vv_x \\
&= uu_x + vu_y \\
&= -p_x
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\Big(\frac{1}{2}(u^2+v^2)\Big)_y &= uu_y + vv_y \\
&= uv_x + vv_y \\
&= -p_y
\end{aligned}$$

$\diamond$

Remark: For the constructed solution (u, v, p) we have

$$\frac{1}{2}\Big(u^2+v^2\Big) + p = 0 \ .$$

Roughly, this implies that the pressure is low (negative, large in absolute value) where flow speed is high and, conversely, the pressure is high (negative, small in absolute value) where the flow speed is low. This is a simple form of Bernoulli's law relating the pressure to the flow speed.

The Velocity Field has a Potential Let $F \in H(U)$ and let $F' = f$. As above, define u and v by

$$f(x+iy) = u(x,y) - iv(x,y) \ .$$

Define ϕ and ψ by

$$F(x+iy) = \phi(x,y) + i\psi(x,y) \ .$$

We have

$$\begin{aligned}
u - iv &= f \\
&= F' \\
&= \phi_x + i\psi_x \\
&= \phi_x - i\phi_y
\end{aligned}$$

thus

$$u = \phi_x \quad \text{and} \quad v = \phi_y \; .$$

This says that the function $\phi(x,y)$ is a potential of the velocity field

$$\Big(u(x,y), v(x,y)\Big) = \nabla \phi(x,y) \; .$$

Flow Lines From the Cauchy–Riemann equations

$$\phi_x = \psi_y, \quad \phi_y = -\psi_x$$

one obtains that

$$\phi_x \psi_x + \phi_y \psi_y = 0 \; .$$

If Γ is a line described by

$$\psi(x,y) = const$$

and if $P = (x,y) \in \Gamma$, then the vector

$$\Big(\psi_x(P), \psi_y(P)\Big)$$

is orthogonal to the tangent to Γ at P. At $P = (x,y)$ we have

$$\begin{aligned} 0 &= \phi_x \psi_x + \phi_y \psi_y \\ &= u \psi_x + v \psi_y \end{aligned}$$

which implies that the velocity vector

$$\Big(u(P), v(P)\Big)$$

is tangent to Γ at P. In other words, the lines

$$\psi(x,y) = const$$

are flow lines.

Example 1: $F(z) = z = x + iy$
We have

$$\phi = x, \quad \psi = y$$

and

$$u = 1, \quad v = 0 \; .$$

The velocity field is

$$\Big(u(x,y), v(x,y)\Big) = (1,0)$$

is a uniform flow in the direction of the x–axis. The pressure is constant. The value of the constant is irrelevant since only p_x and p_y occur in the Euler equations.

Example 2: $F(z) = az = (a_1 + ia_2)(x + iy)$
We have

$$f(z) = a = a_1 + ia_2 = u - iv \ .$$

The velocity vector

$$\Big(u(x,y), v(x,y)\Big) = (a_1, -a_2)$$

is constant in space. The velocity field describes a uniform flow.

Example 3: $F(z) = \frac{z^2}{2}$ (flow around a corner in the first quadrant)
Here

$$f(z) = z = x + iy \ ,$$

thus

$$u = x, \quad v = -y \ .$$

We have

$$F(z) = \frac{(x+iy)^2}{2} = \frac{1}{2}(x^2 - y^2) + ixy \ .$$

Therefore,

$$\psi(x,y) = xy \ .$$

The flow lines are hyperbolas

$$y = \frac{c}{x} \ .$$

On the y–axis the velocity is

$$(u(0,y), v(0,y)) = -(0,y) \ ,$$

i.e., the flow is parallel to the y–axis.
On the x–axis the velocity is

$$(u(x,0), v(x,0)) = (x,0)$$

i.e., the flow is parallel to the x–axis.
The first quadrant is

$$U = Q_1 = \{(x,y) \ : \ 0 < x, y < \infty\} \ .$$

On the boundary of U the velocity is tangent to the boundary. This is the usual boundary condition for Euler flow.

Example 4: $F(z) = z + \frac{1}{z}$ (flow around a cylinder)
We have

$$F(z) = x + iy + \frac{x - iy}{x^2 + y^2}$$

thus

$$\psi(x,y) = y - \frac{y}{x^2+y^2} \ .$$

The stream lines are given by the equation

$$y - \frac{y}{x^2+y^2} = c = const \ .$$

If we take $c = 0$ we see that the circle

$$x^2 + y^2 = 1$$

and the y–axis are flow lines. We can interpret the flow as a flow around the unit–circle. We have

$$u = \phi_x = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

and

$$v = \phi_y = -\frac{2xy}{(x^2+y^2)^2}$$

This yields that

$$u(x,y) \to 1 \quad \text{and} \quad v(x,y) \to 0$$

for $|x| + |y| \to \infty$.

Remarks: The Navier–Stokes equations for 2D incompressible flows read

$$\begin{aligned} u_t + uu_x + vu_y + p_x &= \nu\Delta u \\ v_t + uv_x + vv_y + p_y &= \nu\Delta v \\ u_x + v_y &= 0 \end{aligned}$$

where $\nu > 0$ is the kinematic viscosity of the fluid. If u and v are constructed as above, then

$$\Delta u = \Delta v = 0 \ ,$$

thus

$$(u, v, p)$$

is a solution of the stationary Navier–Stokes equations. However, for the Navier–Stokes equations one typically requires the boundary condition

$$u = v = 0$$

on every wall of the domain. For the Euler equations one only requires that (u, v) is tangential to the wall. The solutions of the Euler equations which we have constructed will not satisfy the boundary conditions $u = v = 0$ at a wall unless the solution is trivial.

In regions away from walls the Euler equations may still be useful since often one has that $0 < \nu << 1$. The viscosity terms are important, however, in boundary layers near walls.

16.2 Laplace Equation

We will to derive the Poisson kernel for the Dirichlet problem for Laplace equation in the unit disc.

Dirichlet Problem: Let $D = D(0,1)$ denote the unit disc with boundary curve $\Gamma = \partial D$. Let $g : \Gamma \to \mathbb{R}$ denote a continuous function. Find

$$u : \bar{D} \to \mathbb{R} \quad \text{with} \quad u \in C^2(D) \cap C(\bar{D})$$

so that

$$\Delta u = 0 \quad \text{in} \quad D, \quad u = g \quad \text{on} \quad \Gamma \, .$$

16.2.1 Derivation of the Poisson Kernel via Complex Variables

Let $f \in H(D(0,1+\varepsilon))$. By Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for} \quad z \in D \, .$$

For $z \in D, z \neq 0$, define the reflected (w.r.t. Γ) point

$$z_1 = \frac{z}{|z|^2} = \frac{1}{\bar{z}} \, .$$

Since $|z_1| > 1$ we have that

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z_1} \, d\zeta \quad \text{for} \quad z \in D \, .$$

Subtracting this equation from the equation for $f(z)$ we obtain

$$f(z) = \frac{1}{2\pi} \int_\Gamma \Big(\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - z_1} \Big) f(\zeta) \, \frac{d\zeta}{i\zeta} \tag{16.1}$$

for $0 < |z| < 1$.

We have for $0 < |z| < 1$ and $\zeta \in \Gamma$:

$$\begin{aligned}
\frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - z_1} &= \frac{\zeta}{\zeta - z} - \frac{\zeta}{\zeta - \frac{\zeta\bar{\zeta}}{\bar{z}}} \\
&= \frac{\zeta}{\zeta - z} - \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \\
&= \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2} \\
&= \frac{1 - |z|^2}{|\zeta - z|^2}
\end{aligned}$$

The kernel

$$K(z,\zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\zeta - z|^2} \quad \text{where} \quad \zeta \in \Gamma \quad \text{and} \quad z \in D \tag{16.2}$$

is called the Poisson kernel for the unit disc D. If $f \in H(D(0,1+\varepsilon))$ then we have

$$f(z) = \int_\Gamma K(z,\zeta) f(\zeta) \, \frac{d\zeta}{i\zeta} \quad \text{for} \quad z \in D \; . \tag{16.3}$$

(In (16.1) we had to exclude $z = 0$ since z_1 is undefined for $z = 0$. However, both sides of (16.1) extend continuously to $z = 0$ and one obtains the validity of (16.3) also for $z = 0$.)

Properties of $K(z,\zeta)$:

1. $K(z,\zeta) > 0$ for $z \in D, \zeta \in \Gamma$.

2.

$$\int_\Gamma K(z,\zeta) \frac{d\zeta}{i\zeta} = 1 \quad \text{for} \quad z \in D$$

(Take $f \equiv 1$ in (16.3).)

3. The function $z \to K(z,\zeta)$ is harmonic in D for every $\zeta \in \Gamma$.

This follows since the function

$$z \to \frac{1}{\zeta - z}, \quad z \in D \; ,$$

is holomorphic and the function

$$z \to \frac{1}{\zeta - z_1}, \quad z \in D \setminus \{0\} \; ,$$

is the complex conjugate of a holomorphic function. Therefore,

$$z \to \text{Re} \Big(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_1} \Big)$$

is harmonic in $D \setminus \{0\}$, and this function extends smoothly to $z = 0$.

Other notation: In formula (16.2) let

$$z = re^{i\theta} \quad \text{and} \quad \zeta = e^{i\phi} \; .$$

One obtains that

$$\begin{aligned} K(z,\zeta) &= K(re^{i\theta}, e^{i\phi}) \\ &= \frac{1}{2\pi} \frac{1 - r^2}{|e^{i\phi} - re^{i\theta}|^2} \end{aligned}$$

Here

$$\begin{aligned} |e^{i\phi} - re^{i\theta}|^2 &= (\cos\phi - r\cos\theta)^2 + (\sin\phi - r\sin\theta)^2 \\ &= 1 - 2r\cos(\phi - \theta) + r^2 \end{aligned}$$

One defines

$$P(r,\alpha) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\alpha + r^2} \; .$$

The kernel $P(r,\alpha)$ is also called the Poisson kernel for the unit disc.

We have

$$K(re^{i\theta}, e^{i\phi}) = P(r, \phi - \theta) \ .$$

Formula (16.3) becomes

$$f(re^{i\theta}) = \int_0^{2\pi} P(r, \phi - \theta) f(e^{i\phi}) \, d\phi \quad \text{for} \quad 0 \le r < 1 \ .$$

This result suggests the following:

Theorem 16.1 *The solution of the Dirichlet problem is*

$$u(re^{i\theta}) = \int_0^{2\pi} P(r, \phi - \theta) g(e^{i\phi}) \, d\phi \quad \textit{for} \quad 0 \le r < 1 \tag{16.4}$$

and

$$u(re^{i\theta}) = g(e^{i\theta}) \quad \textit{for} \quad r = 1 \ .$$

Proof: Since the Poisson kernel $K(z, \zeta)$ is a harmonic function of $z \in D$ for every fixed $\zeta \in \Gamma$ it follows that $u(z)$ is harmonic in D. The difficulty is to prove that u attains its boundary values continuously.

Fix $\zeta_0 = e^{i\theta_0} \in \Gamma$. We must prove that

$$\lim_{z \to \zeta_0, z \in D} u(z) = g(\zeta_0) \ .$$

Using the notation

$$\zeta = e^{i\phi}, \quad \zeta_0 = e^{i\theta_0}$$

we have for all $z \in D$:

$$\begin{aligned} u(z) - g(\zeta_0) &= \int_\Gamma K(z, \zeta) \Big(g(\zeta) - g(\zeta_0) \Big) \frac{d\zeta}{i\zeta} \\ &= \int_0^{2\pi} K(z.e^{i\phi}) \Big(g(e^{i\phi}) - g(e^{i\theta_0}) \Big) \, d\phi \end{aligned}$$

Let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ so that

$$|g(e^{i\phi}) - g(e^{i\theta_0})| \le \frac{\varepsilon}{2} \quad \text{for} \quad |\phi - \theta_0| \le 2\delta_1 \ .$$

Therefore,

$$|u(z) - g(\zeta_0)| \le \frac{\varepsilon}{2} + 2|g|_\infty \int_{|\phi - \theta_0| \ge 2\delta_1} K(z, e^{i\phi}) \, d\phi \ .$$

Let

$$z = re^{i\theta} \quad \text{and} \quad |\theta - \theta_0| \le \delta_1 \ .$$

We must bound

$$K(z,e^{i\phi}) = K(re^{i\theta},e^{i\phi})$$

for

$$|\phi-\theta_0| \geq 2\delta_1 \ .$$

Note that

$$|\phi-\theta_0| \geq 2\delta_1 \quad \text{and} \quad |\theta-\theta_0| \leq \delta_1$$

implies that

$$|\phi-\theta| \geq \delta_1 \ .$$

We have

$$\begin{aligned} K(z,e^{i\phi}) &= K(re^{i\theta},e^{i\phi}) \\ &= \frac{1}{2\pi}\,\frac{1-r^2}{|e^{i\phi}-re^{i\theta}|^2} \end{aligned}$$

Here

$$|e^{i\phi}-re^{i\theta}|^2 = |e^{i\alpha}-r|^2 \quad \text{with} \quad |\alpha| = |\phi-\theta| \geq \delta_1 \ .$$

Set

$$c_1 := \sin\delta_1 > 0 \ .$$

Then we have that

$$|e^{i\alpha}-r|^2 = (\cos\alpha - r)^2 + \sin^2\alpha \geq c_1^2 \ .$$

This estimate yields the bound

$$K(z,e^{i\phi}) = K(re^{i\theta},e^{i\phi}) \leq \frac{1}{2\pi}\,\frac{1-r^2}{c_1^2}$$

for

$$|\phi-\theta_0| \geq 2\delta_1, \quad |\theta-\theta_0| \leq \delta_1, \quad 0 \leq r < 1 \ .$$

Therefore, there exists $\delta_2 > 0$ so that

$$2|g|_\infty \int_{|\phi-\theta_0|} K(z,e^{i\phi})\,d\phi \leq \frac{\varepsilon}{2}$$

if

$$|\theta-\theta_0| \leq \delta_1 \quad \text{and} \quad 1-\delta_2 \leq r < 1 \ .$$

We have proved the limit relation

$$\lim_{z\to\zeta_0, z\in D} u(z) = g(\zeta_0) \ ,$$

which, together with the continuity of g, implies that $u \in C(\bar{D})$. $\diamond$

Remark: In the circle $D(0,R)$ of radius $R > 0$ the solution formula (16.4) is replaced by

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\phi - \theta)}\, g(e^{i\phi})\, d\phi \quad \text{for} \quad 0 \le r < R\ . \tag{16.5}$$

16.2.2 Derivation of the Poisson Kernel via Separation of Variables

rearrangement

17 Rearrangement of Series

17.1 Rearrangement of Absolutely Convergent Series

Theorem 17.1 *Assume*

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

and let

$$\sum_{j=1}^{\infty} a_j =: A \ .$$

If $\beta : \mathbb{N} \to \mathbb{N}$ *is bijective then*

$$\sum_{j=1}^{\infty} a_{\beta(j)} = A \ .$$

Proof: For $\varepsilon > 0$ let $\nu_\varepsilon \in \mathbb{N}$ be chosen so that

$$\sum_{j>\nu_\varepsilon} |a_j| \le \varepsilon \ . \qquad (17.1)$$

We then have

$$\Big| \sum_{j=1}^{\nu_\varepsilon} a_j - A \Big| \le \varepsilon \ . \qquad (17.2)$$

Set

$$F_\varepsilon := \beta^{-1}\Big(\{1, 2, \ldots, \nu_\varepsilon\}\Big) \ ;$$

clearly,

$$\beta(F_\varepsilon) = \{1, 2, \ldots, \nu_\varepsilon\} \ .$$

Set

$$n_\varepsilon := \max F_\varepsilon \ .$$

We then have for all $n \ge n_\varepsilon$:

$$F_\varepsilon \subset \{1, 2, \ldots, n_\varepsilon\} \subset \{1, 2, \ldots, n\} \ ,$$

thus

$$\beta(F_\varepsilon) = \{1, 2, \ldots, \nu_\varepsilon\} \subset \beta\Big(\{1, 2, \ldots, n\}\Big) \ .$$

For all $n \ge n_\varepsilon$ obtain the following:

$$
\begin{aligned}
\Big|\sum_{j=1}^{n} a_{\beta(j)} - A\Big| &= \Big|\sum_{\nu\in F_\varepsilon} a_{\beta(\nu)} - A + \sum_{\nu\in\{1,2,\dots,n\}\setminus F_\varepsilon} a_{\beta(\nu)}\Big| \\
&\le \Big|\sum_{\nu\in F_\varepsilon} a_{\beta(\nu)} - A\Big| + \sum_{\nu\in\mathbb{N}\setminus F_\varepsilon} |a_{\beta(\nu)}| \\
&= \Big|\sum_{j=1}^{\nu_\varepsilon} a_j - A\Big| + \sum_{j>\nu_\varepsilon} |a_j| \\
&\le 2\varepsilon
\end{aligned}
$$

by (17.1) and (17.2). $\diamond$

17.2 Interchanging Double Sums

Let $a_{kl} \in \mathbb{C}$ for $kl = (k,l) \in \mathbb{N}\times\mathbb{N}$. Assume

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_{kl}| =: S < \infty \ .$$

Then, for all $K, L \in \mathbb{N}$:

$$\sum_{l=1}^{L}\sum_{k=1}^{K} |a_{kl}| = \sum_{k=1}^{K}\sum_{l=1}^{L} |a_{kl}| \le S < \infty \ .$$

This implies that for all $L \in \mathbb{N}$:

$$\sum_{l=1}^{L}\sum_{k=1}^{\infty} |a_{kl}| \le S < \infty \ ,$$

and

$$\tilde{S} := \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} |a_{kl}| \le S < \infty \ .$$

With a similar argument one obtains that $S \le \tilde{S}$, thus

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_{kl}| = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} |a_{kl}| \ .$$

The following result about interchanging double sums is more difficult to prove:

Theorem 17.2 *Assume that*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_{kl}| < \infty \ .$$

Then

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl} = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_{kl} \ .$$

Proof: For $K, L \in \mathbb{N}$ set

$$A(K,L) = \sum_{k=1}^{K} \sum_{l=1}^{L} a_{kl}$$
$$B(K,L) = \sum_{l=1}^{L} \sum_{k=1}^{K} a_{kl}$$

These are finite sums, and it is clear that

$$A(K,L) = B(K,L) \quad \text{for all} \quad K, L \in \mathbb{N} \ .$$

The proof of the theorem will be based on the following lemma.

Lemma 17.1 *Assume that*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |a_{kl}| < \infty$$

and set

$$A := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} \ .$$

Then, for all $\varepsilon > 0$, there exists $K_\varepsilon, L_\varepsilon \in \mathbb{N}$ with

$$|A - A(K,L)| \le \varepsilon \quad \textit{for} \quad K \ge K_\varepsilon, \quad L \ge L_\varepsilon \ .$$

Proof: Set

$$S_k = \sum_{l=1}^{\infty} |a_{kl}|, \quad \sum_{k=1}^{\infty} S_k = S \ .$$

There exists K_ε with

$$\sum_{k > K_\varepsilon} S_k \le \varepsilon \ , \tag{17.3}$$

thus

$$\sum_{k > K_\varepsilon} \sum_{l=1}^{\infty} |a_{kl}| = \sum_{k > K_\varepsilon} S_k \le \varepsilon \ . \tag{17.4}$$

Since

$$\sum_{l=1}^{\infty} |a_{kl}| < \infty$$

for all $k \in \mathbb{N}$ there exists $L(k, \varepsilon) \in \mathbb{N}$ with

$$\sum_{l>L(k,\varepsilon)} |a_{kl}| \le \frac{\varepsilon}{k^2} \; . \qquad (17.5)$$

Set

$$L_\varepsilon := \max_{1\le k\le K_\varepsilon} L(k,\varepsilon)$$

where K_ε is determined so that (17.3) holds.

In the following, let $K \ge K_\varepsilon$ and $L \ge L_\varepsilon$. We have:

$$\begin{aligned} A &= \sum_{k=1}^\infty \sum_{l=1}^\infty a_{kl} \\ &= \sum_{k=1}^K \sum_{l=1}^\infty a_{kl} + \sum_{k>K} \sum_{l=1}^\infty a_{kl} \\ &= \sum_{k=1}^K \sum_{l=1}^L a_{kl} + \sum_{k=1}^K \sum_{l>L} a_{kl} + \sum_{k>K} \sum_{l=1}^\infty a_{kl} \end{aligned}$$

Thus,

$$|A - A(K,L)| \le \sum_{k=1}^K \sum_{l>L} |a_{kl}| + \sum_{k>K} \sum_{l=1}^\infty |a_{kl}| \; . \qquad (17.6)$$

The second term on the right–hand side is $\le \varepsilon$ by (17.4). For the first term on the right–hand side of (17.6) we have, using (17.5):

$$\begin{aligned} \sum_{k=1}^K \sum_{l>L} |a_{kl}| &= \sum_{k=1}^{K_\varepsilon} \sum_{l>L} |a_{kl}| + \sum_{K_\varepsilon<k\le K} \sum_{l>L} |a_{kl}| \\ &\le \sum_{k=1}^{K_\varepsilon} \frac{\varepsilon}{k^2} + \sum_{k>K_\varepsilon} S_k \\ &\le \varepsilon \frac{\pi^2}{6} + \varepsilon \end{aligned}$$

This proves the lemma. $\diamond$

We can complete the proof of Theorem 17.2. Recall that for all finite K, L we have

$$B(K,L) = \sum_{l=1}^L \sum_{k=1}^K a_{kl} = \sum_{k=1}^K \sum_{l=1}^L a_{kl} = A(K,L) \; .$$

We set

$$B := \sum_{l=1}^\infty \sum_{k=1}^\infty a_{kl} \; .$$

As in Lemma 17.1, it follows that

$$|B - B(K,L)| \le \varepsilon$$

if K and L are sufficiently large. Therefore, since $B(K,L) = A(K,L)$, the equation $B = A$ follows. This completes the proof of Theorem 17.2. $\diamond$

17.3 Rearrangement of a Double Series

Theorem 17.3 *Let $a_{kl} \in \mathbb{C}$ for $kl = (k,l) \in \mathbb{N} \times \mathbb{N}$ and assume that*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |a_{kl}| < \infty \ . \tag{17.7}$$

Let

$$A := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}$$

and let $\beta : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be bijective. Then

$$\sum_{j=1}^{\infty} a_{\beta(j)} = A \ .$$

Proof: For $K, L \in N$ let

$$R(K,L) = \{kl = (k,l) \in \mathbb{N} \times \mathbb{N} \ : \ 1 \le k \le K \quad \text{and} \quad 1 \le l \le L\} \ ,$$

thus $R(K,L)$ denotes the sets of all points in $\mathbb{N} \times \mathbb{N}$ which lie in the rectangle $[1,K] \times [1,L]$.

Let $\varepsilon > 0$ be given. Because of (17.7) there exist $K_\varepsilon, L_\varepsilon \in \mathbb{N}$ so that

$$\sum_{kl \in \mathbb{N} \times \mathbb{N} \setminus R_\varepsilon} |a_{kl}| \le \varepsilon \tag{17.8}$$

where

$$R_\varepsilon = R(K_\varepsilon, L_\varepsilon) \ .$$

We then have

$$\Big| \sum_{kl \in R_\varepsilon} a_{kl} - A \Big| \le \varepsilon \ . \tag{17.9}$$

Set

$$F_\varepsilon := \beta^{-1}(R_\varepsilon)$$

and let

$$n_\varepsilon := \max F_\varepsilon \ .$$

For $n \ge n_\varepsilon$ we have

$$F_\varepsilon \subset \{1, 2, \ldots, n_\varepsilon\} \subset \{1, 2, \ldots, n\} \ ,$$

thus

$$R_\varepsilon = \beta(F_\varepsilon) \subset \beta\Big(\{1, 2, \ldots, n\}\Big) \; .$$

For all $n \geq n_\varepsilon$ obtain the following:

$$\begin{aligned}
\Big|\sum_{j=1}^{n} a_{\beta(j)} - A\Big| &= \Big|\sum_{\nu\in F_\varepsilon} a_{\beta(\nu)} - A + \sum_{\nu\in\{1,2,\ldots,n\}\setminus F_\varepsilon} a_{\beta(\nu)}\Big| \\
&\leq \Big|\sum_{\nu\in F_\varepsilon} a_{\beta(\nu)} - A\Big| + \sum_{\nu\in\mathbb{N}\setminus F_\varepsilon} |a_{\beta(\nu)}| \\
&= \Big|\sum_{kl\in R_\varepsilon} a_{kl} - A\Big| + \sum_{kl\in\mathbb{N}\times\mathbb{N}\setminus R_\varepsilon} |a_{kl}| \\
&\leq 2\varepsilon
\end{aligned}$$

by (17.8) and (17.9). $\diamond$

18 Supplements

18.1 The Function $|\sin z|^2$

Recall that

$$\sinh y = \frac{1}{2}(e^y - e^{-y}) \; .$$

Thus, if y is real and large in absolute value, then $\sinh y$ is exponentially large in absolute value.

Lemma 18.1 *For $z = x + iy$:*

$$\begin{aligned} |\sin z|^2 &= \sin^2 x + \sinh^2 y \\ |\cos z|^2 &= \cos^2 x + \sinh^2 y \end{aligned}$$

The lemma implies that $|sin(x_i y)|$ and $|\cos(x + iy)|$ are exponentially large as a function of y if $|y|$ is large.

Proof: Let

$$s = \sin x, \quad c = \cos x \; .$$

We have

$$\begin{aligned} \sin z &= \frac{1}{2i}\,(e^{iz} - e^{-iz}) \\ &= \frac{1}{2i}\,(e^{ix}e^{-y} - e^{-ix}e^{y}) \\ &= \frac{1}{2i}\,\Big((c + is)e^{-y} - (c - is)e^{y}\Big) \\ &= \frac{1}{2i}\,\Big(c(e^{-y} - e^{y}) + is(e^{-y} + e^{y})\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} |\sin z|^2 &= \frac{1}{4}\Big(c^2(e^y - e^{-y})^2 + s^2(e^y + e^{-y})^2\Big) \\ &= \frac{1}{4}\Big((1 - s^2)(e^y - e^{-y})^2 + s^2(e^y + e^{-y})^2\Big) \\ &= \sinh^2 y + \frac{s^2}{4}\Big((e^{2y} + 2 + e^{-2y}) - (e^{2y} - 2 + e^{-2y})\Big) \\ &= \sinh^2 y + \sin^2 x \end{aligned}$$

The proof for $|\cos z|^2$ is similar. $\diamond$

18.2 Find the Error

Let n denote any integer. Where is the error in the following:

$$\begin{aligned} e^{2\pi in} &= 1 \\ e^{1+2\pi in} &= e \\ e &= e^{1+2\pi in} \\ &= \left(e^{1+2\pi in}\right)^{1+2\pi in} \\ &= e^{1+4\pi in-4\pi n^2} \\ &= ee^{-4\pi n^2} \end{aligned}$$

Therefore,

$$1 = e^{-4\pi n^2} \ .$$

18.3 The Functions sin and arcsin

Let

$$\mathcal{H} = \{w = u + iv \ : \ u \in \mathbb{R}, v > 0\}$$

denote the open upper half–plane.

Let S denote the strip

$$S = \{z = x + iy \ : \ -\frac{\pi}{2} < x < \frac{\pi}{2}, \ y > 0\}$$

in the open upper half–plane $\mathcal{H}$. We claim that the map

$$z \to \sin z$$

from S to $\mathcal{H}$ is 1–1 and onto.

We have

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\frac{1}{2}\left(ie^{iz} + \frac{1}{ie^{iz}}\right) \ .$$

Claim 1: Let R denote the open semi–circle

$$R = \{w = u + iv \ : \ |w| < 1, \ v > 0\} \ .$$

Then the map

$$z \to ie^{iz}$$

from S to R is 1–1 and onto.

Proof: If $z = x + iy \in S$ then

$$e^{iz} = e^{ix}e^{-y}$$

with

$$-\infty < e^{-y} < 0, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \ .$$

One obtains that the map

$$z \to e^{iz}$$

maps S 1–1 and onto a semi–circle of radius 1 in the right–half plane. Multiplying by i, one obtains the semi–circle R as the image of S under the map $z \to ie^{iz}$.

Claim 2: Changing notation, we let R denote the open semi–circle

$$R = \{z = x + iy \ : \ |z| < 1, \ y > 0\}$$

and consider the map

$$z \to z + \frac{1}{z} =: g(z)$$

on R. We claim that

$$g : R \to -\mathbb{H}$$

is 1–1 and onto.

Proof: For $z = re^{i\phi} \in R$ we have

$$z = r(c + is), \quad \frac{1}{z} = \frac{1}{r}(c - is)$$

with

$$c = \cos\phi, \quad s = \sin\phi > 0 \ .$$

One obtains that

$$w := z + \frac{1}{z} = (r + \frac{1}{r})c + i(r - \frac{1}{r})s \ .$$

It follows that

$$\mathrm{Im}\, w < 0 \ .$$

thus $w \in -\mathcal{H}$.

We must show: For all $w \in -\mathcal{H}$ there exists a unique $z \in R$ with $g(z) = w$.

The equation

$$z + \frac{1}{z} = w$$

is equivalent to

$$z^2 - wz + 1 = 0 \ .$$

The solutions

$$z_{1,2} = \frac{w}{2} \pm \sqrt{\frac{w^2}{4} - 1}$$

are distinct unless

$$w^2 = 4, \quad w = \pm 2 \ .$$

Thus, for all $w \in -\mathcal{H}$ the two solutions $z_{1,2}$ of the quadratic are distinct. We have

$$z_1 z_2 = 1, \quad z_1 + z_2 = w$$

and may assume that

$$|z_1| \leq 1 \leq |z_2| \ .$$

Write

$$z_1 = re^{i\phi} \quad \text{with} \quad 0 < r \leq 1, \quad 0 \leq \phi < 2\pi \ .$$

Suppose that $r = 1$. Then $z_2 = e^{-i\phi}$ and

$$z_1 + z_2 = 2\cos\phi \in \mathbb{R} \ .$$

This contradicts that

$$z_1 + z_2 = w \in -\mathcal{H} \ .$$

Therefore,

$$z_1 = re^{i\phi} \quad \text{with} \quad 0 < r < 1, \quad 0 \leq \phi < 2\pi \ .$$

From

$$w = z_1 + z_2 = (r + \frac{1}{r})\cos\phi + i(r - \frac{1}{r})\sin\phi$$

and

$$\mathrm{Im}\, w < 0$$

it follows that

$$\sin\phi > 0 \ ,$$

thus

$$0 < \phi < \pi \ .$$

We have shown that

$$z_1 = re^{i\phi} \in R$$

which proves Claim 2.

Recall that

$$\sin z = -\frac{1}{2}\Big(ie^{iz} + \frac{1}{ie^{iz}}\Big) \ .$$

The map

$$z \to ie^{iz}$$

from S to R is 1–1 and onto. The map

$$w \to w + \frac{1}{w}$$

from R to $-\mathcal{H}$ is 1–1 and onto. Multiplying by $-\frac{1}{2}$ we obtain that the map

$$z \to \sin z$$

from R to $\mathcal{H}$ is 1–1 and onto. The inverse map from $\mathcal{H}$ onto R is arcsin.

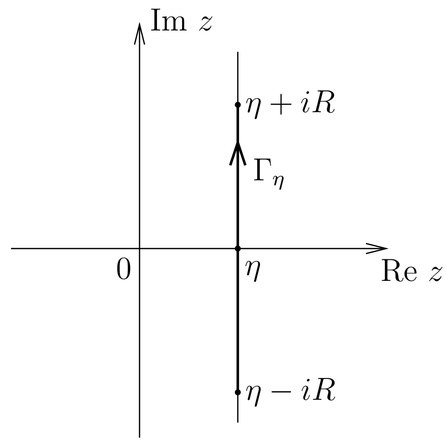

Figure 19.1: Contour

19 Graphs

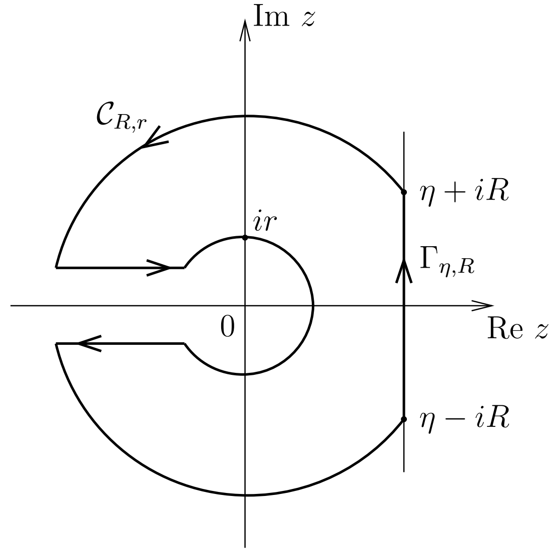

Figure 19.2: Contour

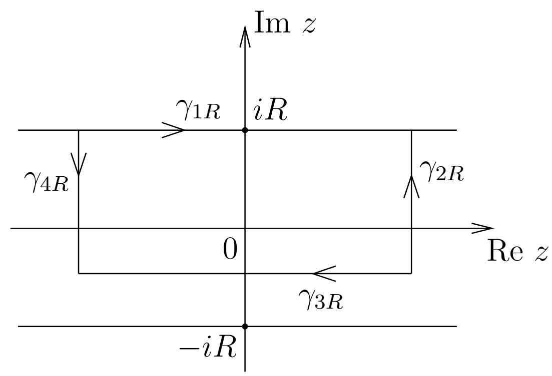

Figure 19.3: Contour

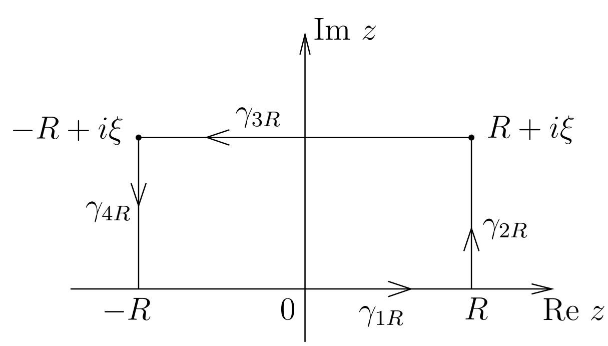

Figure 19.4: Contour

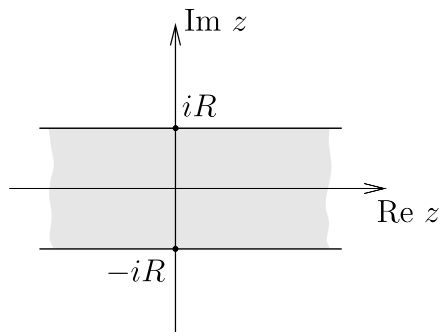

Figure 19.5: Strip $\{z \ : \ z = x + iy, -R < y < R\}$