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# A Note on Multistep Methods and Attracting Sets of Dynamical Systems*

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**Summary.** We consider a dynamical system described by an autonomous ODE with an asymptotically stable attractor, a compact set of orbitrary shape, for which the stability can be characterized by a Lyapunov function. Using recent results of Eirola and Nevanlinna [1], we establish a uniform estimate for the change in value of this Lyapunov function on discrete trajectories of a consistent, strictly stable multistep method approximating the dynamical system. This estimate can then be used to determine nearby attracting sets and attractors for the discretized system as done in Kloeden and Lorenz [3, 4] for 1-step methods.

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## 1 Introduction

If a dynamical system described by an autonomous ODE

$$\frac{dx}{dt} = F(x) \tag{1.1}$$

on $\mathbb{R}^n$ has a uniformly asymptotically stable attractor $\Lambda$, which is a compact set but can have arbitrary shape, then there exists a Lyapunov function $V$ which characterizes the stability of $\Lambda$. (See Yoshizawa [5] for definitions and details.) In particular, $V$ is defined on a neighborhood

$$S(\Lambda; R_0) := \{x \in \mathbb{R}^n: \operatorname{dist}(x, \Lambda) < R_0\}$$

of $\Lambda$ for some $R_0 > 0$, with

$$a(\operatorname{dist}(x, \Lambda)) \leqq V(x) \leqq b(\operatorname{dist}(x, \Lambda)), \tag{1.2}$$

$$|V(x) - V(y)| \leqq L|x - y| \tag{1.3}$$

and

$$V(\phi(t; x_0)) \leqq e^{-ct} V(x_0) \tag{1.4}$$

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for all $x, y, x_0 \in S(\Lambda; R_0)$. Here $a$ and $b$ are strictly increasing continuous functions with $a(0)=b(0)=0$ and $a(r)<b(r)$ for $r>0$; $L$ and $c$ are positive constants; and $\phi(t; x_0)$ is the solution of (1.1) with initial value $\phi(0; x_0)=x_0$. By (1.3) the Lyapunov function is Lipschitz, but without additional assumptions on the shape of $\Lambda$ it need not be differentiable. In [3, 4] Kloeden and Lorenz used this Lyapunov function to determine an attracting set

$$\tilde{\Lambda}_h := \{x \in \mathbb{R}^n : V(x) \leqq \eta(h)\}$$

for a 1-step $p$th-order numerical method approximating (1.1), with timestep $h$, considered as a discrete dynamical system on $\mathbb{R}^n$. Here the original attractor $\Lambda \subset \tilde{\Lambda}_h$ and

$$\operatorname{dist}(\tilde{\Lambda}_h, \Lambda) := \max\{\operatorname{dist}(x, \Lambda): x \in \tilde{\Lambda}_h\} \leqq a^{-1}(\eta(h)) \to 0,$$

for an appropriate choice of $\eta(h) \to 0$, as $h \to 0$. Moreover $\tilde{\Lambda}_h$ is penetrated after a finite number of steps by all trajectories of the 1-step method starting close to it. The attracting set $\tilde{\Lambda}_h$ is thus like a thin tube about the original attractor $\Lambda$, and contains an asymptotically stable attractor $\Lambda_h$ for the 1-step discrete dynamical system. (See Hale [2] and Kloeden and Lorenz [3, 4] for details.) The crucial step in establishing these results was an estimate

$$V(x_{j+1}^h) \leqq e^{-ch} V(x_j^h) + C h^{p+1} \tag{1.5}$$

for any trajectory $x_0^h, x_1^h, \ldots, x_j^h, \ldots$ of the 1-step scheme starting in some neighborhood $S(\Lambda; R_1)$ of $\Lambda$ where $0<R_1<R_0$. Here the constant $C$ depends on the neighborhood, but not on the particular index $j=0, 1, 2, \ldots$ of the trajectory. For a 1-step $p$-th order scheme estimate (1.5) follows from the error bound of the scheme and properties (1.3) and (1.4) of the Lyapunov function.

Similar results would also follow for a multistep discretization method once an estimate such as (1.5) has been established. This is, however, a nontrivial problem because, unlike 1-step methods, multistep methods do not directly approximate the flow of (1.1) but a mapping related to it. Nevertheless by using some recent results of Eirola and Nevanlinna [1] such an estimate can be established, and it is our purpose here to show how. We shall assume that $F \in C^p(\mathbb{R}^n)$ for some $p \geqq 1$ and, to simplify the exposition, that $F$ and its partial derivatives up to and including order $p$ are uniformly bounded on $\mathbb{R}^n$, and that $\Lambda$ is a global attractor for (1.1). This is not a serious loss of generality since when considering attractors one can restrict attention to some bounded region containing the attractor. In addition we shall closely follow the terminology of Eirola and Nevanlinna [1] and briefly state the result proven there.

As in [1] we consider a general $k$-step method involving $l-1$ derivatives of $F$ on the right side of (1.1), where $1 \leqq l \leqq p$. If $y_0, y_1 \ldots, y_{k-1} \in \mathbb{R}^n$ are given and $0<h \leqq h_0$, where $h_0$ is sufficiently small, then this method computes

$$y_k = \psi(h; y_0, y_1, \ldots, y_{k-1}),$$

where $\psi$ is Lipschitz in $(y_0, y_1, \ldots, y_{k-1})$ uniformly in $0<h \leqq h_0$. We assume that the method is of order $p$ and is strictly stable. In addition that a $p$-th

order starting routine is specified, so that given $y_0 \in \mathbb{R}^n$ and $0<h \leqq h_0$ it generates points $y_1, y_2, \ldots, y_{k-1}$ in $\mathbb{R}^n$ for which

$$|y_j - \phi(jh; y_0)| \leqq C_1 h^{p+1} \tag{1.6}$$

for $j=1, 2, \ldots, k-1$. Here and in the sequel $C, C_1, C_2, \ldots$ denote constants which are independent of $y_0$ and $0<h \leqq h_0$. Thus given $y_0$ and $0<h \leqq h_0$, the starting routine and the $k$-step method produce a sequence $y_0, y_1, y_2, \ldots$ where

$$y_j = \psi(h; y_{j-k}, y_{j-k+1}, \ldots, y_{j-1}) \tag{1.7}$$

for $j=k, k+1, \ldots$. Naturally this sequence depends on the timestep $h$, but we shall suppress this in our notation. The difficulty in proving our result stems from the fact that the sequences $y_j$ do *not* originate as trajectories of an autonomous discrete dynamical system on $\mathbb{R}^n$ because of the necessary presence of the starting routine. As is well known, the multistep method itself (without the starting routine) leads to a mapping $g_h$: $\mathbb{R}^{nk} \to \mathbb{R}^{nk}$ defined by

$$g_h(y_0, y_1, \ldots, y_{k-1}) = (y_1, y_2, \ldots, y_k) \tag{1.8}$$

for any $(y_0, y_1, \ldots, y_{k-1}) \in \mathbb{R}^{nk}$ where $y_k$ is given by (1.7) with $j=k$. Thus, there is an underlying autonomous discrete dynamical system on $\mathbb{R}^{nk}$. It is not obvious, however, how to apply the Lyapunov function $V$ defined on (a subset of) $\mathbb{R}^n$ to $g_h$.

## 2 Main Result

Our objective here is to verify that the estimate (1.5) for the decay of the Lyapunov function holds for trajectories generated by a strictly stable $k$-step method with any starting routine when both have the order $p$. This result holds uniformly for all initial $y_0 \in \mathbb{R}^n$ and timesteps $0<h \leqq h_0$ under the assumptions of global attraction and global uniform boundedness of $F$ and the partial derivatives up to and including order $p$. In this case the Lyapunov function $V$ is defined on all of $\mathbb{R}^n$. Generally it is defined only on some bounded neighborhood $S(\Lambda; R_0)$ of the original attractor $\Lambda$, in which case some caution is needed to ensure that the successive points remain within this domain of definition of $V$. This can be achieved as in [3]. While we shall not go into such details here, we shall nevertheless state our main result in Theorem 2.2 for this general case.

Let us first review some definitions and results of [1]. For $y \in \mathbb{R}^n$, let

$$D^j(y) := \frac{\partial^j}{\partial t^j} \phi(t; y)|_{t=0},$$

i.e., $D^0(y)=y$, $D^1(y)=F(y)$, $D^2(y)=F'(y)F(y)$ etc. A $k$-step multiderivative method can be written:

$$\sum_{j=0}^{l} h^j \sum_{i=0}^{k} \alpha_{ji} D^j(y_i)=0, \qquad \alpha_{0k}=-1.$$

Given $y_0, \ldots, y_{k-1}$, let $\psi(h; y_0, \ldots, y_{k-1}):=y_k$ denote the solution of the above equation.

The method is called strictly stable if the matrix

$$S=\begin{bmatrix} 0 & \\ \vdots & I \\ 0 & \\ \alpha_{00} & \alpha_{01}\ldots\alpha_{0k-1} \end{bmatrix}$$

has $k-1$ eigenvalues with absolute values $<1$. (By consistency, 1 is always an eigenvalue of $S$.)

The mapping $\psi$ is approximated by a mapping $\hat{\psi}$ which is defined next.

For $i=0, \ldots, k$; $v=0, \ldots, p$; $j=0, \ldots, p$ determine numbers $\delta^j_{iv}$ such that

$$\sum_{v=0}^{p} \delta^j_{iv} P(v)=\frac{d^j}{dt^j} P(t)\bigg|_{t=i}$$

for all polynomials $P(t)$ of degree $\leqq p$.

Then set

$$D^j_{h,i}(y):=h^{-j} \sum_{v=0}^{p} \delta^j_{iv} \phi((v-i)h; y)$$

and define $\hat{\psi}(h; y_0, \ldots, y_{k-1}):=y_k$ as the solution of

$$\sum_{j=0}^{l} h^j \sum_{i=0}^{k} \alpha_{ji} D^j_{h,i}(y_i)=0.$$

Thus, $\psi$ is determined in terms of $F$ and its derivative, whereas $\hat{\psi}$ is determined in terms of the values of the flow $\phi(t; y)$ of the dynamical system (1.1). Using $\psi$ we define a mapping $g_h$: $\mathbb{R}^{nk} \to \mathbb{R}^{nk}$ by

$$g_h(y_0, y_1, \ldots, y_{k-1})=(y_1, y_2, \ldots, y_{k-1}, \psi(h; y_0, y_1, \ldots, y_{k-1})),$$

and a similar mapping $\hat{g}_h$ using $\hat{\psi}$ instead of $\psi$. We shall need the following result which follows from [1].

**Theorem 2.1.** *Suppose $F$ has $p$ derivatives which are continuous and bounded on $\mathbb{R}^n$. We consider a multistep method using $l-1$ derivatives of $F$, $p \geqq l \geqq 1$, which is consistent of order $p$ and strictly stable. Then*

1) *There is* $h_0>0$ *and* $C>0$ *such that* $\psi(h;\cdot)$ *and* $\hat{\psi}(h;\cdot)$ *are defined on* $\mathbb{R}^{nk}$ *for* $0<h\leqq h_0$, *and*

$$|\psi(h;y)-\hat{\psi}(h;y)|\leqq Ch^{p+1}, \qquad y\in\mathbb{R}^{nk}.$$

2) *For any vector* $\bar{y}=(y_0, y_1, \ldots, y_{k-1})\in\mathbb{R}^{nk}$ *let*

$$v_h(\bar{y}):=\begin{bmatrix} y_{k-1}-\phi((k-1)h;y_0) \\ y_{k-1}-\phi((k-2)h;y_1) \\ \vdots \\ y_{k-1}-\quad\phi(h;y_{k-2}) \end{bmatrix}\in\mathbb{R}^{n(k-1)}.$$

*There is a norm* $|\cdot|_*$ *on* $\mathbb{R}^{n(k-1)}$ *and a number* $0<\rho<1$ *such that*

$$|v_h(\hat{g}_h(\bar{y}))|_*\leqq\rho\,|v_h(\bar{y})|_*$$

*for* $0<h\leqq h_0$, $\bar{y}\in\mathbb{R}^{nk}$.

*Remark.* The norm of $v_h(\bar{y})$ measures how well the points of $\bar{y}$ "belong to the same trajectory" of (1.1). The mapping $\hat{g}_h$ moves vectors in $\mathbb{R}^{nk}$ so that the image points lie closer to a trajectory of (1.1). The definition of $v_h(\bar{y})$ implies that the points of $\bar{y}$ lie on one trajectory of (1.1) if and only if $v_h(\bar{y})=0$.

Next we formulate our main result under the assumptions of Sect. 1.1 for a $p$-th order method. For simplicity we present the proof under the assumption that the Lyapunov function $V$ is defined on $\mathbb{R}^n$ and that $F$ has $p$ continuous bounded derivatives on $\mathbb{R}^n$.

**Theorem 2.2.** *Let* $y_0\in S(A;r_0)$ *and* $0<h\leqq h_0$, *where* $r_0>0$ *and* $h_0$ *are sufficiently small, and let* $y_0, y_1, y_2, \ldots$ *be the discrete trajectory generated by the starting routine and the strictly stable k-step method* (1.7). *Then*

$$V(y_{j+1})\leqq e^{-ch}V(y_j)+Ch^{p+1} \tag{2.1}$$

*for* $j=0, 1, 2\ldots$, *where the positive constant* $C$ *is independent of* $h, j$ *and* $y_0$.

*Proof. Step 1* (starting routine). For $j=0, 1, \ldots, k-2$ inequality (2.1) follows from the error bound (1.6) and from properties (1.3) and (1.4) of the Lyapunov function. Specifically

$$\begin{aligned} V(y_{j+1}) &\leqq V(\phi((j+1)h;y_0)+|V(y_{j+1})-V(\phi((j+1)h;y_0))| \\ &\leqq e^{-ch}V(\phi(jh;y_0))+L|y_{j+1}-\phi((j+1)h;y_0)| \\ &\leqq e^{-ch}V(\phi(jh;y_0))+LC_1h^{p+1} \\ &\leqq e^{-ch}V(y_j)+e^{-ch}|V(\phi(jh;y_0))-V(y_j)|+LC_1h^{p+1} \\ &\leqq e^{-ch}V(y_j)+(1+e^{-ch})LC_1h^{p+1} \end{aligned}$$

*Remark.* For a 1-step method Step 1 gives the desired result, but for a multistep method considerably more remains to be done.

*Step 2* (multistep routine). Let $j \geqq k-1$. Then by (1.3), (1.4) and (1.7):

$$V(y_{j+1}) \leqq V(\phi(h; y_j)) + |V(y_{j+1}) - V(\phi(h; y_j))|$$
$$\leqq e^{-ch} V(y_j) + LR_j$$

where

$$R_j = |y_{j+1} - \phi(h; y_j)|$$

Hence we need to show that $R_j$ is $O(h^{p+1})$ uniformly in $j$ and $y_0$.

We use the notation

$$\bar{y}_j = (y_{j-k+1}, \ldots, y_{j-1}, y_j), \quad j \geqq k-1,$$

i.e., $\bar{y}_j \in \mathbb{R}^{nk}$ contains $k$ consecutive vectors of the discrete trajectory, and $y_j$ is the last entry. Also, let

$$Y_j = (\phi((-k+1)h; y_j), \ldots, \phi(-h; y_j), y_j),$$

i.e., $Y_j \in \mathbb{R}^{nk}$ contains $k$ corresponding points of the continuous trajectory ending at $y_j$. We show

**Lemma 2.3.** *There exists a constant* $C_2$ *independent of* $0 < h \leqq h_0$, $y_0 \in \mathbb{R}^n$ *and* $j = k-1, k, \ldots$ *such that*

$$|v_h(\bar{y}_j)| \leqq C_2 h^{p+1}.$$

*Proof.* For $j = k-1$ the points of

$$\bar{y}_j = (y_0, \ldots, y_{k-1})$$

are generated by the $p$-th order starting routine, and the desired estimate follows. Furthermore, for $j \geqq k$,

$$|v_h(\bar{y}_j)|_* = |v_h(g_h(\bar{y}_{j-1}))|_*$$
$$\leqq |v_h(\hat{g}_h(\bar{y}_{j-1}))|_* + C_3 h^{p+1}$$
$$\leqq \rho\, |v_h(\bar{y}_{j-1})|_* + C_3 h^{p+1}$$

by Theorem 2.1. Recursively,

$$|v_h(\bar{y}_j)|_* \leqq (1 + \rho + \rho^2 + \ldots) C_3 h^{p+1} + \rho^{j-k+1} |v_h(\bar{y}_{k-1})|_*,$$

and the desired estimate follows.

Returning to the proof of Theorem 2.2, we note

$$R_j = |y_{j+1} - \phi(h; y_j)| = |\psi(h; \bar{y}_j) - \phi(h; y_j)|$$
$$\leqq |\psi(h; \bar{y}_j) - \psi(h; Y_j)| + |\psi(h; Y_j) - \phi(h; y_j)|.$$

From the definition of $Y_j$ and the fact that the method is consistent of order $p$, it follows that

$$|\psi(h; Y_j) - \phi(h; y_j)| \leqq C_4 h^{p+1}.$$

Furthermore, (by the lemma)

$$|v_h(\bar{y}_j) - v_h(Y_j)| = |v_h(\bar{y}_j)| \leqq C_2 h^{p+1}.$$

The points of $\bar{y}_j$ and of $Y_j$ agree in their last entry (which is $y_j$), and therefore we can conclude

$$|\bar{y}_j - Y_j| \leqq C_5 h^{p+1}$$

using the definition of $v_h$. From the above estimate the desired bound on $R_j$ follows because $\psi(h; \cdot)$ is Lipschitz uniformly in $h$. This completes the proof.

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