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THE MAGNETO–HYDRODYNAMIC EQUATIONS: LOCAL THEORY AND BLOW-UP OF SOLUTIONS

Jens Lorenz*

Department of Mathematics and Statistics, The University of New Mexico
Albuquerque NM 87131, United States of America

Wilberclay G. Melo

Departamento de Matematica, Universidade Federal de Sergipe
Sao Cristovao SE 49100-000, Brasil

Natã Firmino Rocha

Departamento de Matematica, Universidade Federal de Minas Gerais
Belo Horizonte MG 31270-901, Brasil

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Abstract. This work establishes local existence and uniqueness as well as blow-up criteria for solutions $(u,b)(x,t)$ of the Magneto–Hydrodynamic equations in Sobolev–Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. More precisely, we prove that there is a time $T>0$ such that $(u,b)\in C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for $a>0,\sigma\geq 1$ and $\frac{1}{2}<s<\frac{3}{2}$. If the maximal time interval of existence is finite, $0\leq t<T^*$, then the blow–up inequality

$$\frac{C_1\exp\{C_2(T^*-t)^{-\frac{1}{3\sigma}}\}}{(T^*-t)^q}\leq \|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \quad \text{with } q=\frac{2(s\sigma+\sigma_0)+1}{6\sigma}$$

holds for $0\leq t<T^*$, $\frac{1}{2}<s<\frac{3}{2}$, $a>0$, $\sigma>1$ ($2\sigma_0$ is the integer part of 2σ).

1. Introduction.

Consider the unforced Magneto–Hydrodynamic (MHD) equations for incompressible flows on all space $\mathbb{R}^3$:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla(p+\frac{1}{2}|\,b\,|^2) = \mu\Delta u + b\cdot\nabla b, & x\in\mathbb{R}^3, \quad t\geq 0,\\ b_t + u\cdot\nabla b = \nu\,\Delta b + b\cdot\nabla u, & x\in\mathbb{R}^3, \quad t\geq 0,\\ \operatorname{div} u = \operatorname{div} b = 0, & x\in\mathbb{R}^3, \quad t\geq 0,\\ u(x,0) = u_0(x),\ b(x,0) = b_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1}$$

Here $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))\in\mathbb{R}^3$ denotes the incompressible velocity field, $b(x,t)=(b_1(x,t),b_2(x,t),b_3(x,t))\in\mathbb{R}^3$ the magnetic field and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure. The positive constants μ and ν are associated with specific properties of the fluid: The constant μ is the kinematic viscosity and ν^{-1} is the magnetic Reynolds number. The initial data for the velocity and magnetic fields, given by u_0 and b_0 in (1), are assumed to be divergence free, i.e., $\operatorname{div} u_0=\operatorname{div} b_0=0$.

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* Corresponding author: Jens Lorenz.

Note that the MHD system reduces to the classical incompressible Navier–Stokes system if $b = 0$.

We shall study the above system using the Sobolev–Gevrey spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. (See the next section for notations.) More precisely, we shall obtain solutions with $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ where $\frac{1}{2} < s < \frac{3}{2}$, $a > 0$ and $\sigma \geq 1$. Here $[0, T^*)$ denotes the maximal interval of existence of a classical solution. Even in the Navier–Stokes case it is not known if $T^* = \infty$ always holds. In this paper we shall derive blow–up rates for the solution if T^* is finite.

In a recent paper, J. Benameur and L. Jlali [4] proved blow–up criteria for the Navier–Stokes equations in Sobolev–Gevrey spaces. Our current paper extends the results of [4] from the Navier–Stokes to the MHD system. Also, we prove the blow–up inequality for $\frac{1}{2} < s < \frac{3}{2}$ whereas only the value $s = 1$ was considerded in [4]. For further blow–up results for the Navier–Stokes and MHD systems we refer to [1, 2, 4, 6, 7, 8, 11, 12, 13, 14, 15] and references therein.

Our main results are stated in following two theorems. The first one guarantees the existence of a finite time $T > 0$ and a unique solution $(u, b) \in C([0, T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2}, \frac{3}{2})$, $a > 0$ and $\sigma \geq 1$, for the MHD equations (1).

Theorem 1.1. *Assume that $a > 0$, $\sigma \geq 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Then, there exist an instant $T = T_{s,\mu,\nu,u_0,b_0} > 0$ and a unique solution $(u, b) \in C([0, T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for the MHD equations (1).*

Remark 1. It is important to point out that the existence result obtained for the space $\dot{H}^1_{a,\sigma}(\mathbb{R}^3)$ by J. Benameur and L. Jlali [4] is a particular case of Theorem 1.1. In fact, it is enough to take $s = 1$ and $b = 0$ in this last statement. Furthermore, Theorem 1.1 generalizes [4] from the Navier-Stokes equations to MHD system (1).

By assuming that $[0, T^*)$ is the maximal interval of existence for the solution $(u, b)(x, t)$ obtained in Theorem 1.1 with T^* finite, let us present our blow-up criteria for the solution $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2}, \frac{3}{2})$ of the MHD equations (1).

Theorem 1.2. *Assume that $a > 0$, $\sigma > 1$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $(u_0, b_0) \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ such that $div\, u_0 = div\, b_0 = 0$. Assume that $(u, b) \in C([0, T^*); \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ is the solution for the MHD equations (1) in the maximal time interval $0 \leq t < T^*$. If $T^* < \infty$, then the following holds:*

i): $\limsup_{t \nearrow T^*} \|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)} = \infty$;

ii): $\displaystyle\int_t^{T^*} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\, d\tau = \infty$;

iii): $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \geq \dfrac{2\pi^3\sqrt{\theta}}{\sqrt{T^* - t}}$;

iv): $\|(u, b)(t)\|_{\dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{n}},\sigma}(\mathbb{R}^3)} \geq \dfrac{2\pi^3\sqrt{\theta}}{C_1\sqrt{T^* - t}}$;

v): $\dfrac{a^{\sigma_0 + \frac{1}{2}} C_2 \exp\{aC_3(T^* - t)^{-\frac{1}{3\sigma}}\}}{(T^* - t)^{\frac{2(s\sigma + \sigma_0) + 1}{6\sigma}}} \leq \|(u, b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$, *if* $(u_0, b_0) \in L^2(\mathbb{R}^3)$,

for all $t \in [0, T^*), n \in \mathbb{N}$, where $\theta = \min\{\mu, \nu\}$,

$$C_1 = C_{a,\sigma,s,n} := \left\{ 4\pi\sigma \left[2\frac{a}{(\sqrt{\sigma})^{(n-1)}} \left(\frac{1}{\sqrt{\sigma}} - \frac{1}{\sigma} \right) \right]^{-\sigma(3-2s)} \Gamma(\sigma(3 - 2s)) \right\}^{\frac{1}{2}},$$

$C_2 = C_{\mu,\nu,s,\sigma,u_0,b_0}, C_3 = C_{\mu,\nu,\sigma,s,u_0,b_0}$ *and* $2\sigma_0$ *is the integer part of* 2σ.

Remark 2. Under the assumptions of Theorem 1.2, let us list implications of the results.

1. First of all, let us emphasize that the blow–up criteria obtained by J. Benameur and L. Jlali [4] for the space $\dot{H}^1_{a,\sigma}(\mathbb{R}^3)$ are particular cases of Theorem 1.2. In fact, it is enough to assume $s = 1$ and $b = 0$ in this last result. Moreover, we have extended all the information stated in [4] from the classical Navier-Stokes equations to MHD system (1).
2. Notice that the item **iii)** of Theorem 1.2 shows a non trivial inequality; since, $\|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}$ is finite for all $t \in [0,T^*), n \in \mathbb{N}, a > 0, \sigma > 1$. It can be concluded due to the estimate (7) below and the continuous embedding $\dot{H}^s_{a,\sigma}(\mathbb{R}^3) \hookrightarrow \dot{H}^s_{\frac{a}{(\sqrt{\sigma})^{(n-1)}},\sigma}(\mathbb{R}^3)$ $(s \geq 0)$.
3. By applying Dominated Convergence Theorem in Theorem 1.2 **iii)**, one obtains:
$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} \leq \lim_{n\to\infty} \|e^{\frac{a}{\sigma(\sqrt{\sigma})^{(n-1)}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} = \|(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}, \tag{2}$$
for all $t \in [0,T^*)$. Moreover, if $(u_0,b_0) \in L^2(\mathbb{R}^3)$, then $\|(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}$ is finite for all $t \in [0,T^*)$. This follows from Lemmas 2.3 and 2.4, and (46) below.
4. Observe also that Theorem 1.2 **v)**, by assuming $s = 1$ and $b = 0$, presents the same lower bound as the one determined in [4].
5. It is easy to check that Theorem 1.2 **v)** implies
$$\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \geq \frac{a^{\sigma_0+\frac{1}{2}}C_2}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{6\sigma}}}, \quad \forall t \in [0,T^*),$$
where $s \in (\frac{1}{2},\frac{3}{2})$.

Section 2 describes notations and definitions and presents some important lemmas. Section 3 contains the proof of Theorem 1.1; Section 4 the proof of Theorem 1.2.

2. **Preludes.** This section presents notations and definitions as well as lemmas that will be needed for the proofs of the main theorem.

2.1. **Notations and definitions.**

1. The vector fields are denoted by
$$f = f(t) = f(x,t) = (f_1(x,t), f_2(x,t), ..., f_n(x,t)),$$
where $x \in \mathbb{R}^3$, $t \in [0,T^*)$ and $n \in \mathbb{N}$.
2. The gradient field is defined by $\nabla f = (\nabla f_1, \nabla f_2, ..., \nabla f_n)$ $(f = (f_1, f_2, .., f_n))$, $\nabla f_j = (D_1 f_j, D_2 f_j, D_3 f_j)$ $(j = 1,2,...,n)$, with $D_i = \partial/\partial x_i$ $(i = 1,2,3)$.
3. The Laplacian $f = (f_1, f_2, .., f_n)$ is established by $\Delta f = (\Delta f_1, \Delta f_2, ..., \Delta f_n)$, where $\Delta f_j = \sum_{i=1}^3 D_i^2 f_j$.
4. The standard divergence is given by $\operatorname{div} f = \sum_{i=1}^3 D_i f_i$ for $f = (f_1, f_2, f_3)$.
5. In the MHD equations (1), the notation $f \cdot \nabla g$ means $\sum_{i=1}^3 f_i D_i g$ where $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$.

6. Define the Fourier transform of f by
$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-i\xi\cdot x} f(x)\,dx, \quad \forall\,\xi \in \mathbb{R}^3,$$
where $\xi \cdot x := \sum_{j=1}^3 \xi_j x_j$, with $\xi = (\xi_1, \xi_2, \xi_3), x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and its inverse by
$$\mathcal{F}^{-1}(g)(x) := (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i\xi\cdot x} g(\xi)\,d\xi, \quad \forall\,x \in \mathbb{R}^3.$$
7. $L^p(X)$ denotes the Lebesgue space $(1 \leq p \leq \infty)$. Here the L^p-norm of f is given by
$$\|f\|_{L^p(X)} := \left(\int_X |f(x)|^p\,dx\right)^{\frac{1}{p}}, \forall\,1 \leq p < \infty,\ \|f\|_{L^\infty(X)} := \text{esssup}_{x\in X}\{|f(x)|\}.$$
8. Assuming that $(X, \|\cdot\|)$ is a Banach space and $T > 0$, the space $L^\infty([0,T];X)$ contains all measurable functions $f : [0,T] \to X$ for which the following norm is finite:
$$\|f\|_{L^\infty([0,T];X)} := \text{esssup}_{t\in[0,T]}\{\|f(t)\|\}.$$
9. $\dot{H}^s(\mathbb{R}^3)$ denotes the homogeneous Sobolev space
$$\Big\{f \in S'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{f}(\xi)|^2\,d\xi < \infty\Big\},$$
where $S'(\mathbb{R}^3)$ is the space of tempered distributions. The $\dot{H}^s(\mathbb{R}^3)$-norm is given by
$$\|f\|^2_{\dot{H}^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{f}(\xi)|^2\,d\xi,$$
where $|x|^2 := |x_1|^2 + |x_2|^2 + ... + |x_n|^2$, with $x = (x_1, x_2, ..., x_n) \in \mathbb{C}^n$ $(n \in \mathbb{N})$.
10. The non–homogeneous Sobolev space $H^s(\mathbb{R}^3)$ is
$$\Big\{f \in S'(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi < \infty\Big\}.$$
The corresponding $H^s(\mathbb{R}^3)$-norm is
$$\|f\|^2_{H^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} (1 + |\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi.$$
11. Let $a > 0, \sigma \geq 1$ and $s \in \mathbb{R}$. The Sobolev-Gevrey space
$$\dot{H}^s_{a,\sigma}(\mathbb{R}^3) := \Big\{f \in S'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi)|^2\,d\xi < \infty\Big\},$$
is endowed with the $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$-norm
$$\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi)|^2\,d\xi.$$
Moreover, the $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$-inner product is given by
$$\langle f, g\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}}\,\widehat{f}(\xi)\cdot\widehat{g}(\xi)\,d\xi,$$
where $x \cdot y := x_1\overline{y}_1 + x_2\overline{y}_2 + ... + x_n\overline{y}_n$, with $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{C}^n$ $(n \in \mathbb{N})$.

12. The tensor product and the usual convolution, respectively, are given by
$$f \otimes g := (g_1 f, g_2 f, g_3 f),$$
where $f, g : \mathbb{R}^3 \to \mathbb{R}^3$,
$$\varphi * \psi(x) = \int_{\mathbb{R}^3} \varphi(x-y)\psi(y)\, dy,$$
where $\varphi, \psi : \mathbb{R}^3 \to \mathbb{R}$.
13. In Section 4.4, $T^*_\omega < \infty$ denotes the first blow-up time for the solution $(u, b) \in C([0, T^*_\omega); \dot{H}^s_{\omega,\sigma}(\mathbb{R}^3))$, where $\omega > 0$.
14. As usual, constants that appear in this paper may change in value from line to line without change of notation. With $C_{q,r,s}$ we denote constants that depend on q, r and s, for example.

2.2. **Auxiliary lemmas.** We establish results that will play key roles in the proofs of our main theorems. We start with two lemmas used for the proof of Theorem 1.1.

Lemma 2.1 (see [9]). *Let $(X, \|\cdot\|)$ be a Banach space and let $B : X \times X \to X$ denote a continuous bilinear operator, i.e, there exists a positive constant C_1 such that*
$$\|B(x, y)\| \le C_1 \|x\|\|y\|, \quad \forall\, x, y \in X.$$
Then, for each $x_0 \in X$ that satisfies $4C_1\|x_0\| < 1$, the equation $a = x_0 + B(a, a)$ with unknown $a \in X$ admits a solution $a = x \in X$. Moreover, the solution $a = x$ obeys the inequality $\|x\| \le 2\|x_0\|$ and is the only solution with $\|x\| \le \frac{1}{2C_1}$.

Proof. For details see [9]. □

The next result is due to J.-Y. Chemin [10].

Lemma 2.2 (see [10]). *Let $(s_1, s_2) \in \mathbb{R}^2$ and assume $s_1 < \frac{3}{2}$ and $s_1 + s_2 > 0$. Then there exists a positive constant C_{s_1,s_2} such that, for all $f, g \in \dot{H}^{s_1}(\mathbb{R}^3) \cap \dot{H}^{s_2}(\mathbb{R}^3)$, we have*
$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}(\mathbb{R}^3)} \le C_{s_1,s_2}\Big(\|f\|_{\dot{H}^{s_1}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}(\mathbb{R}^3)} + \|f\|_{\dot{H}^{s_2}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_1}(\mathbb{R}^3)}\Big).$$
If $s_1 < \frac{3}{2}$, $s_2 < \frac{3}{2}$ and $s_1 + s_2 > 0$, then there is a positive constant C_{s_1,s_2} such that
$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}(\mathbb{R}^3)} \le C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}(\mathbb{R}^3)}.$$

Proof. For details see [10]. □

The next lemma is a result of interpolation theory that will be used in the proof of Theorem 1.2 **v)**. It has been proved by J. Benameur [3].

Lemma 2.3 (see [3]). *Let $\delta > 3/2$ and $f \in \dot{H}^\delta(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Then, the following inequality is valid:*
$$\|\widehat{f}\|_{L^1(\mathbb{R}^3)} \le C_\delta \|f\|^{1-\frac{3}{2\delta}}_{L^2(\mathbb{R}^3)}\|f\|^{\frac{3}{2\delta}}_{\dot{H}^\delta(\mathbb{R}^3)},$$
where C_δ is a positive constant that depends on δ only. Moreover, for each $\delta_0 > 3/2$ there exists a positive constant C_{δ_0}, that depends on δ_0 only, such that $C_\delta \le C_{\delta_0}$ for all $\delta \ge \delta_0$.

Proof. For details see [3]. □

The next lemma is important to prove the estimate (2).

Lemma 2.4. *Let $a > 0$, $\sigma \geq 1$, $s \in [0, \frac{3}{2})$ and $\delta \geq \frac{3}{2}$. For every $f \in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$, we have that $f \in \dot{H}^\delta(\mathbb{R}^3)$. More precisely, one concludes that there is a positive constant $C_{a,s,\delta,\sigma}$ such that*

$$\|f\|_{\dot{H}^\delta(\mathbb{R}^3)} \leq C_{a,s,\delta,\sigma} \|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

Proof. It is well known that $\mathbb{R}_+ \subseteq \cup_{n\in\mathbb{N}\cup\{0\}}[n, n+1)$. Notice that $2\sigma(\delta - s) \in \mathbb{R}_+$. As a result, there is $n_0 \in \mathbb{N} \cup \{0\}$ that depends on σ, δ and s such that $n_0 \leq 2\sigma(\delta - s) < n_0 + 1$. Consequently, one obtains $t \in [0, 1]$ such that, by Young's inequality, we infer

$$|\xi|^{2\delta-2s} = |\xi|^{t\cdot\frac{n_0}{\sigma}+(1-t)\cdot\frac{n_0+1}{\sigma}} = |\xi|^{t\cdot\frac{n_0}{\sigma}}|\xi|^{(1-t)\cdot\frac{n_0+1}{\sigma}} \leq |\xi|^{\frac{n_0}{\sigma}} + |\xi|^{\frac{n_0+1}{\sigma}}.$$

Therefore, one has

$$\begin{aligned}
\|f\|^2_{\dot{H}^\delta(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2\delta}|\hat{f}(\xi)|^2\,d\xi \leq \int_{\mathbb{R}^3} [|\xi|^{\frac{n_0}{\sigma}} + |\xi|^{\frac{n_0+1}{\sigma}}]|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi \\
&\leq \int_{\mathbb{R}^3} \Big[\frac{(2a+1)(2a)^{n_0}(n_0+1)!}{(2a)^{n_0+1}n_0!}|\xi|^{\frac{n_0}{\sigma}} \\
&\quad + \frac{(2a+1)(2a)^{n_0+1}(n_0+1)!}{(2a)^{n_0+1}(n_0+1)!}|\xi|^{\frac{n_0+1}{\sigma}}\Big]|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi \\
&= \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}} \int_{\mathbb{R}^3} \Big[\frac{(2a|\xi|^{\frac{1}{\sigma}})^{n_0}}{n_0!} + \frac{(2a|\xi|^{\frac{1}{\sigma}})^{n_0+1}}{(n_0+1)!}\Big]|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi.
\end{aligned}$$

Hence, we deduce

$$\begin{aligned}
\|f\|^2_{\dot{H}^\delta(\mathbb{R}^3)} &\leq \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}} \int_{\mathbb{R}^3} |\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi \\
&= \frac{(n_0+1)!(2a+1)}{(2a)^{n_0+1}} \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.
\end{aligned}$$

This completes the proof of Lemma 2.4. □

The following result has been proved in [3].

Lemma 2.5 (see [3])**.** *The following inequality holds:*

$$(a+b)^r \leq ra^r + b^r, \quad \forall\, 0 \leq a \leq b, r \in (0, 1].$$

Proof. For details see [3]. □

Let us present two consequences of Lemma 2.5.

Lemma 2.6. *The following inequality holds:*

$$e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a\max\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}} e^{\frac{a}{\sigma}\min\{|\xi-\eta|,|\eta|\}^{\frac{1}{\sigma}}}, \quad \forall\, \xi, \eta \in \mathbb{R}^3, a > 0, \sigma \geq 1.$$

Proof. Apply Lemma 2.5 to obtain

$$\begin{aligned}
a|\xi|^{\frac{1}{\sigma}} &= a|\xi - \eta + \eta|^{\frac{1}{\sigma}} \leq a(|\xi - \eta| + |\eta|)^{\frac{1}{\sigma}} \\
&\leq a(\max\{|\xi - \eta|, |\eta|\} + \min\{|\xi - \eta|, |\eta|\})^{\frac{1}{\sigma}} \\
&\leq a\max\{|\xi - \eta|, |\eta|\}^{\frac{1}{\sigma}} + \frac{a}{\sigma}\min\{|\xi - \eta|, |\eta|\}^{\frac{1}{\sigma}}.
\end{aligned}$$

This proves Lemma 2.6. □

Lemma 2.7. *Let $\xi,\eta\in\mathbb{R}^3, a>0,$ and $\sigma\geq 1$. Then, it holds*

$$e^{a|\xi|^{\frac{1}{\sigma}}}\leq e^{a|\xi-\eta|^{\frac{1}{\sigma}}}e^{a|\eta|^{\frac{1}{\sigma}}}. \tag{3}$$

Proof. Apply Lemma 2.6. □

J. Benameur and L. Jlali [4] proved a version of Chemin's Lemma (see [10]) by considering the spaces $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. Let us introduce this result exactly as it was enunciated in [4].

Lemma 2.8 (see [4])**.** *Let $a>0$, $\sigma\geq 1$ and $(s_1,s_2)\in\mathbb{R}^2$ such that $s_1<\frac{3}{2}$ and $s_1+s_2>0$. Then, there exists a positive constant C_{s_1,s_2} such that, for all $f,g\in\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)\cap\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)$, we have*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}\leq C_{s_1,s_2}\Big(\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}+\|f\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\Big).$$

If $s_1<\frac{3}{2}$, $s_2<\frac{3}{2}$ and $s_1+s_2>0$, then there is a positive constant C_{s_1,s_2} such that

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)}\leq C_{s_1,s_2}\|f\|_{\dot{H}^{s_1}_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^{s_2}_{a,\sigma}(\mathbb{R}^3)}.$$

Proof. For details see Lemma 2.2 in [4]. □

The next result presents our extension for Lemma 2.5.

Lemma 2.9. *Let $a>0$, $\sigma>1$, and $s\in[0,\frac{3}{2})$. For every $f,g\in\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$, we have $fg\in\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$. More precisely, one obtains*

- **i):** $\|fg\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\leq C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|_{L^1(\mathbb{R}^3)}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^3)}\|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]$;
- **ii):** $\|fg\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\leq C_{a,\sigma,s}\|f\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$,

where $C_s=2^{\frac{2s-5}{2}}\pi^{-3}$ and $C_{a,\sigma,s}:=2^{s-2}\pi^{-3}\sqrt{\frac{4\pi\sigma\Gamma(\sigma(3-2s))}{[2(a-\frac{a}{\sigma})]^{\sigma(3-2s)}}}<\infty$. Here Γ is the standard gamma function.

Proof. First note that

$$\begin{aligned}\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}&=\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{fg}(\xi)|^2\,d\xi\\&=(2\pi)^{-6}\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}*\widehat{g}(\xi)|^2\,d\xi\\&\leq(2\pi)^{-6}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|\xi|^se^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta\Big)^2d\xi\\&\leq(2\pi)^{-6}\int_{\mathbb{R}^3}\Big(\int_{|\eta|\leq|\xi-\eta|}|\xi|^se^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta\\&\quad+\int_{|\eta|>|\xi-\eta|}|\xi|^se^{a|\xi|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\,d\eta\Big)^2d\xi.\end{aligned}$$

It is easy to check that

$$|\xi|^s\leq[|\xi-\eta|+|\eta|]^s\leq[2\max\{|\xi-\eta|,|\eta|\}]^s=2^s[\max\{|\xi-\eta|,|\eta|\}]^s. \tag{4}$$

Apply Lemma 2.6 to obtain

$$\begin{aligned}\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}&\leq\frac{2^{2s-6}}{\pi^6}\int_{\mathbb{R}^3}\Big(\int_{|\eta|\leq|\xi-\eta|}|\xi-\eta|^se^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\\&\quad+\int_{|\eta|>|\xi-\eta|}e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\eta|^se^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\Big)^2d\xi\end{aligned}$$

$$\le 2^{2s-5}\pi^{-6}\Big[\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}|\xi-\eta|^s e^{a|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)|e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\Big)^2 d\xi$$
$$+\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}}|\widehat{f}(\xi-\eta)||\eta|^s e^{a|\eta|^{\frac{1}{\sigma}}}|\widehat{g}(\eta)|\,d\eta\Big)^2 d\xi\Big].$$

In other notation,

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le 2^{2s-5}\pi^{-6}\|[|\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|]*[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^3)}$$
$$+(2\pi)^{-6}2^{2s+1}\|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}|\widehat{f}|]*[|\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}|\widehat{g}|]\|^2_{L^2(\mathbb{R}^3)}.$$

Consequently, it follows from Young's inequality that

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le 2^{2s-5}\pi^{-6}[\||\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^2(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^1(\mathbb{R}^3)}$$
$$+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^1(\mathbb{R}^3)}\||\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|^2_{L^2(\mathbb{R}^3)}]. \quad (5)$$

Notice that

$$\||\cdot|^s e^{a|\cdot|^{\frac{1}{\sigma}}}\widehat{f}\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\hat{f}(\xi)|^2\,d\xi = \|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \quad (6)$$

Using this result in (5), the proof of **i)** is given.

Let us prove **ii)**. Applying the Cauchy-Schwarz's inequality one obtains

$$\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{g}\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3}e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|\,d\xi$$
$$\le\Big(\int_{\mathbb{R}^3}|\xi|^{-2s}e^{2(\frac{a}{\sigma}-a)|\xi|^{\frac{1}{\sigma}}}\,d\xi\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{g}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}$$
$$=: C_{a,\sigma,s}\|g\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}, \quad (7)$$

where

$$C^2_{a,\sigma,s} = \frac{4\pi\sigma\Gamma(\sigma(3-2s))}{[2(a-\frac{a}{\sigma})]^{\sigma(3-2s)}},$$

since $\sigma > 1$ and $s < 3/2$. Hence, by combining (5), (6) and (7), we have

$$\|fg\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le 2^{2s-4}\pi^{-6}C^2_{a,\sigma,s}\|f\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|g\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

This completes the proof of Lemma 2.9. □

We state an elementary result:

Lemma 2.10 (see [5])**.** *Let $a, b > 0$. Then $\lambda^a e^{-b\lambda} \le a^a(eb)^{-a}$ for all $\lambda > 0$.*

3. **Proof of Theorem 1.1.** In this section, we prove the existence of a time $T = T_{s,\mu,\nu,u_0,b_0} > 0$ and a unique solution $(u,b) \in C([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ with $s \in (\frac{1}{2},\frac{3}{2})$ for the MHD system (1). As noted above, Theorem 1.1 is an improvement of previous results even for the Navier-Stokes equations. It extends Theorem 3.1 in [4]. Our main point is, however, the extension from the Navier-Stokes to the MHD equations (1).

We first proceed formally and apply the heat semigroup $e^{\mu\Delta(t-\tau)}$, with $\tau \in [0,t]$, to the velocity equation in (1). Integration in time yields

$$\int_0^t e^{\mu\Delta(t-\tau)}u_\tau\,d\tau + \int_0^t e^{\mu\Delta(t-\tau)}\Big(u\cdot\nabla u - b\cdot\nabla b + \nabla(p+\frac{1}{2}|b|^2)\Big)\,d\tau =$$

$$\mu \int_0^t e^{\mu\Delta(t-\tau)} \Delta u \, d\tau.$$

Using integration by parts one deduces

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} \Big(u \cdot \nabla u - b \cdot \nabla b + \nabla(p + \frac{1}{2}|b|^2) \Big) \, d\tau.$$

Let us recall that the Helmholtz's projector P_H (see Section 7.2 in [13] and references therein) is well defined, yielding

$$P_H(u \cdot \nabla u - b \cdot \nabla b) = u \cdot \nabla u - b \cdot \nabla b + \nabla(p + \frac{1}{2}|b|^2),$$

and also

$$\mathcal{F}[P_H(f)](\xi) = \widehat{f}(\xi) - \frac{\widehat{f}(\xi) \cdot \xi}{|\xi|^2} \xi. \tag{8}$$

As a result, it follows that

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H(u \cdot \nabla u - b \cdot \nabla b) \, d\tau.$$

Therefore,

$$\begin{aligned} u(t) &= e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H(u \cdot \nabla u - b \cdot \nabla b) \, d\tau \\ &= e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^3 (u_j D_j u - b_j D_j b)] \, d\tau \\ &= e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^3 D_j(u_j u - b_j b)] \, d\tau, \end{aligned}$$

provided that $\operatorname{div} u = \operatorname{div} b = 0$. Hence,

$$u(t) = e^{\mu\Delta t} u_0 - \int_0^t e^{\mu\Delta(t-\tau)} P_H[\sum_{j=1}^3 D_j(u_j u - b_j b)] \, d\tau. \tag{9}$$

Next, our goal is to present an equality for the field b analogous to (9). By applying the heat semigroup $e^{\nu\Delta(t-\tau)}$, with $\tau \in [0, t]$, to the second equation in (1) and integrating in time, we obtain

$$\int_0^t e^{\nu\Delta(t-\tau)} b_\tau \, d\tau + \int_0^t e^{\nu\Delta(t-\tau)} [u \cdot \nabla b - b \cdot \nabla u] \, d\tau = \nu \int_0^t e^{\nu\Delta(t-\tau)} \Delta b \, d\tau.$$

Using integrating by parts again, we have

$$b(t) = e^{\nu\Delta t} b_0 - \int_0^t e^{\nu\Delta(t-\tau)} [u \cdot \nabla b - b \cdot \nabla u] \, d\tau.$$

As u and b are divergence free (see (1)), it follows that

$$\begin{aligned} b(t) &= e^{\nu\Delta t} b_0 - \int_0^t e^{\nu\Delta(t-\tau)} [\sum_{j=1}^3 (u_j D_j b - b_j D_j u)] \, d\tau \\ &= e^{\nu\Delta t} b_0 - \int_0^t e^{\nu\Delta(t-\tau)} [\sum_{j=1}^3 D_j(u_j b - b_j u)] \, d\tau, \end{aligned}$$

that is

$$b(t) = e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3} D_j(u_j b - b_j u)]\,d\tau. \tag{10}$$

By (9) and (10), one obtains

$$(u,b)(t) = (e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0) + B((u,b),(u,b))(t), \tag{11}$$

where

$$B((w,v),(\gamma,\phi))(t) = \int_0^t (-e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)], \\ - e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3} D_j(w_j\phi - v_j\gamma)])\,d\tau. \tag{12}$$

Here w, v, γ, and ϕ belong to a suitable function space that we now discuss.

Let us estimate $B((w,v),(\gamma,\phi))(t)$ in $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with $1/2 < s < 3/2$, $a > 0$ and $\sigma \geq 1$. It follows from the definition of the space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ that

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \\ \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\,d\xi.$$

As a consequence, we have

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = \\ \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\}(\xi)|^2\,d\xi.$$

By applying (8), we can write

$$\begin{aligned} &\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \\ &\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\sum_{j=1}^{3}\mathcal{F}[D_j(\gamma_j w - v_j\phi)](\xi)|^2\,d\xi \\ &\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)\cdot\xi|^2\,d\xi \\ &\leq \int_{\mathbb{R}^3} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{2s+2} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)|^2\,d\xi. \end{aligned}$$

Rewriting the last integral, we have

$$\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \\ \int_{\mathbb{R}^3} |\xi|^{5-2s} e^{-2\mu(t-\tau)|\xi|^2}|\xi|^{4s-3} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}(w\otimes\gamma - \phi\otimes v)(\xi)|^2\,d\xi.$$

As a result, by using Lemma 2.10, it follows that

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\
&\le \frac{\left(\frac{5-2s}{4e\mu}\right)^{\frac{5-2s}{2}}}{(t-\tau)^{\frac{5-2s}{2}}}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(w\otimes\gamma-\phi\otimes v)(\xi)|^2\,d\xi\\
&=: \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{2}}}\|w\otimes\gamma-\phi\otimes v\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)},
\end{aligned}$$

since $s<3/2$.

On the other hand, by using Lemma 2.8, one infers

$$\begin{aligned}
\|w\otimes\gamma\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{w\otimes\gamma}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{4s-3}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{\gamma_j w_k}(\xi)|^2\,d\xi\\
&= \sum_{j,k=1}^{3}\|\gamma_j w_k\|^2_{\dot{H}^{2s-\frac{3}{2}}_{a,\sigma}(\mathbb{R}^3)} \le C_s\|w\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|\gamma\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)},
\end{aligned}\tag{13}$$

provided that $0<s<3/2$. Therefore, one deduces

$$\begin{aligned}
&\|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le\\
&\qquad \frac{C_{s,\mu}}{(t-\tau)^{\frac{5-2s}{4}}}\|(w,v)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|(\gamma,\phi)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.
\end{aligned}$$

By integrating the above estimate over time from 0 to t, we conclude

$$\begin{aligned}
&\int_0^t \|e^{\mu\Delta(t-\tau)}P_H[\sum_{j=1}^{3} D_j(\gamma_j w - v_j\phi)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\\
&\le C_{s,\mu}\int_0^t \frac{\|(w,v)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|(\gamma,\phi)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}}{(t-\tau)^{\frac{5-2s}{4}}}\,d\tau\\
&\le C_{s,\mu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned}\tag{14}$$

for all $t\in[0,T]$ (recall that $s>1/2$).

Analogously, we can write

$$\begin{aligned}
&\int_0^t \|e^{\nu\Delta(t-\tau)}[\sum_{j=1}^{3} D_j(w_j\phi - v_j\gamma)]\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau \le\\
&\qquad C_{s,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned}\tag{15}$$

for all $t\in[0,T]$.

By (12), we can assure that (14) and (15) imply the bound

$$\begin{aligned}
&\|B((w,v),(\gamma,\phi))(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \le\\
&\qquad C_{s,\mu,\nu}T^{\frac{2s-1}{4}}\|(w,v)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))}\|(\gamma,\phi)\|_{L^\infty([0,T];\dot{H}^s_{a,\sigma}(\mathbb{R}^3))},
\end{aligned}\tag{16}$$

for all $t\in[0,T]$.

To summarize, it has been shown that

$$\begin{aligned}\|e^{\nu\Delta t}b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\mathcal{F}\{e^{\nu\Delta t}b_0\}(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^3} e^{-2\nu t|\xi|^2} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2\, d\xi \\ &\leq \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{b_0}(\xi)|^2\, d\xi = \|b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \end{aligned} \tag{17}$$

Therefore, we have established the following estimate:

$$\|(e^{\mu\Delta t}u_0, e^{\nu\Delta t}b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq \|(u_0, b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.$$

Notice that $B : C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \times C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3)) \to C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ (with $s \in (\frac{1}{2}, \frac{3}{2})$, $a > 0$ and $\sigma \geq 1$) is a bilinear operator, which is continuous (see (12) and (16)). Choosing a time $T > 0$ with

$$T < \frac{1}{[4C_{s,\mu,\nu}\|(u_0,b_0)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}]^{\frac{4}{2s-1}}},$$

where $C_{s,\mu,\nu}$ is given in (16), we can apply Lemma 2.1 to obtain a unique solution $(u,b) \in C([0,T]; \dot{H}^s_{a,\sigma}(\mathbb{R}^3))$ for the equation (11).

This completes the proof of Theorem 1.1.

4. **Proof of Theorem 1.2.** We prove the blow-up criteria for the solution of the MHD equations (1), asuming that the solution exists only in a finite time interval $0 \leq t < T^*$. As mentioned above, Theorems 3.3 and 4.1 obtained in [4] are particular cases of our Theorem 1.2. The structure of our proof follows [1, 2, 3, 4, 6, 7, 14].

4.1. **Proof of Theorem 1.2 i) (case $n = 1$).** We first generalize the arguments given in the Appendix of [4].

We prove Theorem 1.2 **i)** with $n = 1$ by contradiction. Suppose the solution $(u,b)(t)$ exists only in the finite time interval $0 \leq t < T^*$ and

$$\limsup_{t \nearrow T^*} \|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} < \infty. \tag{18}$$

We shall prove that the solution can be extended beyond $t = T^*$.

By (18) and Theorem 1.1 (since $s \in (\frac{1}{2}, \frac{3}{2})$), there exists an absolute constant C with

$$\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq C, \quad \forall\, t \in [0, T^*). \tag{19}$$

Integrating the inequality (35) below int time and applying (19) and (7), one concludes

$$\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + \theta \int_0^t \|\nabla(u,b)(\tau)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq \|(u_0,b_0)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\theta} C^4 T^*,$$

for all $t \in [0, T^*)$. Consequently,

$$\begin{aligned}\int_0^t \|\nabla(u,b)(\tau)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau &\leq \frac{1}{\theta}\|(u_0,b_0)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} + C_{s,a,\sigma,\theta} C^4 T^* \\ &=: C_{s,a,\sigma,\theta,u_0,b_0,T^*}, \end{aligned} \tag{20}$$

for all $t \in [0, T^*)$.

Let $(\kappa_n)_{n\in\mathbb{N}}$ denote a sequence if times with $0<\kappa_n<T^*$ and $\kappa_n\nearrow T^*$. We shall prove that

$$\lim_{n,m\to\infty}\|(u,b)(\kappa_n)-(u,b)(\kappa_m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0. \tag{21}$$

The following equality holds:

$$(u,b)(\kappa_n)-(u,b)(\kappa_m)=I_1(n,m)+I_2(n,m)+I_3(n,m), \tag{22}$$

where

$$I_1(n,m)=([e^{\mu\Delta\kappa_n}-e^{\mu\Delta\kappa_m}]u_0,[e^{\nu\Delta\kappa_n}-e^{\nu\Delta\kappa_m}]b_0), \tag{23}$$

$$\begin{aligned}I_2(n,m)=\Big(&\int_0^{\kappa_m}[e^{\mu\Delta(\kappa_m-\tau)}-e^{\mu\Delta(\kappa_n-\tau)}]P_H\,[u\cdot\nabla u-b\cdot\nabla b]\,d\tau,\\&\int_0^{\kappa_m}[e^{\nu\Delta(\kappa_m-\tau)}-e^{\nu\Delta(\kappa_n-\tau)}](u\cdot\nabla b-b\cdot\nabla u)\,d\tau\Big),\end{aligned} \tag{24}$$

and also

$$\begin{aligned}&I_3(n,m)=\\&-\Big(\int_{\kappa_m}^{\kappa_n}e^{\mu\Delta(\kappa_n-\tau)}P_H\,[u\cdot\nabla u-b\cdot\nabla b]\,d\tau,\int_{\kappa_m}^{\kappa_n}e^{\nu\Delta(\kappa_n-\tau)}(u\cdot\nabla b-b\cdot\nabla u)\,d\tau\Big).\end{aligned} \tag{25}$$

(See (11) and (12)). On the other hand, it is easy to check that

$$\begin{aligned}\|[e^{\nu\Delta\kappa_n}-e^{\nu\Delta\kappa_m}]b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}&=\int_{\mathbb{R}^3}[e^{-\nu\kappa_n|\xi|^2}-e^{-\nu\kappa_m|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{b}_0(\xi)|^2\,d\xi\\&\leq\int_{\mathbb{R}^3}[e^{-\nu\kappa_n|\xi|^2}-e^{-\nu T^*|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\widehat{b}_0(\xi)|^2\,d\xi.\end{aligned}$$

Since $b_0\in\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ and $e^{-\nu\kappa_n|\xi|^2}-e^{-\nu T^*|\xi|^2}\leq 1$ for all $n\in\mathbb{N}$ the Dominated Convergence Theorem yields that

$$\lim_{n,m\to\infty}\|[e^{\nu\Delta\kappa_n}-e^{\nu\Delta\kappa_m}]b_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

Similarly,

$$\lim_{n,m\to\infty}\|[e^{\mu\Delta\kappa_n}-e^{\mu\Delta\kappa_m}]u_0\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

Consequently, $\lim_{n,m\to\infty}\|I_1(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0$ (see (23)).

We also have:

$$\begin{aligned}&\int_0^{\kappa_m}\|[e^{\mu\Delta(\kappa_m-\tau)}-e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau=\\&\int_0^{\kappa_m}\Big(\int_{\mathbb{R}^3}[e^{-\mu(\kappa_m-\tau)|\xi|^2}-e^{-\mu(\kappa_n-\tau)|\xi|^2}]^2\\&\times|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[P_H(u\cdot\nabla u-b\cdot\nabla b)](\xi)|^2d\xi\Big)^{\frac{1}{2}}\,d\tau.\end{aligned}$$

By applying (8), we obtain that

$$\begin{aligned}&\int_0^{\kappa_m}\|[e^{\mu\Delta(\kappa_m-\tau)}-e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u-b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\,d\tau\leq\\&\int_0^{T^*}\Big(\int_{\mathbb{R}^3}[1-e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u-b\cdot\nabla b](\xi)|^2d\xi\Big)^{\frac{1}{2}}\,d\tau.\end{aligned}$$

The Cauchy-Schwarz's inequality yields that

$$\int_0^{\kappa_m} \|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau \leq$$
$$\sqrt{T^*}\Big(\int_0^{T^*}\int_{\mathbb{R}^3}[1-e^{-\mu(T^*-\kappa_m)|\xi|^2}]^2|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}[u\cdot\nabla u - b\cdot\nabla b](\xi)|^2 d\xi d\tau\Big)^{\frac{1}{2}}.$$

Observe that $1-e^{-\mu(T^*-\kappa_m)|\xi|^2} \leq 1$ for all $m\in\mathbb{N}$ and $\int_0^{T^*}\|u\cdot\nabla u - b\cdot\nabla b\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}d\tau < \infty$ since that

$$\begin{aligned}\|u\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq C_{a,\sigma,s}\sum_{j=1}^3\|u_j\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\|D_j u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\ &\leq C_{a,\sigma,s}C\|\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned} \tag{26}$$

(See Lemma 2.9 **ii)** ($0\leq s<3/2$ and $\sigma>1$), (19) and (20)). Application of the Dominated Convergence Theorem yields that

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{\mu\Delta(\kappa_m-\tau)} - e^{\mu\Delta(\kappa_n-\tau)}]P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

Analogously, we obtain

$$\lim_{n,m\to\infty}\int_0^{\kappa_m}\|[e^{\nu\Delta(\kappa_m-\tau)} - e^{\nu\Delta(\kappa_n-\tau)}](u\cdot\nabla b - b\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau = 0.$$

Therefore, $\lim_{n,m\to\infty}\|I_2(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$ (see (24)).

Finally, note that

$$\begin{aligned}\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq &\int_{\kappa_m}^{\kappa_n}\|e^{\mu\Delta(\kappa_n-\tau)}P_H(u\cdot\nabla u - b\cdot\nabla b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\ &+\int_{\kappa_m}^{\kappa_n}\|e^{\mu\Delta(\kappa_n-\tau)}(u\cdot\nabla b - b\cdot\nabla u)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.\end{aligned}$$

Following a similar process to the one proved in (17) and applying (8), one gets

$$\begin{aligned}\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq &\int_{\kappa_m}^{\kappa_n}\|u\cdot\nabla u - b\cdot\nabla b\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\\ &+\int_{\kappa_m}^{\kappa_n}\|u\cdot\nabla b - b\cdot\nabla u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.\end{aligned}$$

Use (26) to obtain

$$\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} \leq CC_{a,\sigma,s}\int_{\kappa_m}^{T^*}\|\nabla(u,b)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau.$$

Therefore, by the Cauchy-Schwarz's inequality and (20), one has

$$\begin{aligned}\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\leq C_{a,\sigma,s}\sqrt{T^*-\kappa_m}\left(\int_{\kappa_m}^{T^*}\|\nabla(u,b)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\, d\tau\right)^{\frac{1}{2}}\\ &\leq C_{s,a,\sigma,\theta,u_0,b_0,T^*}\sqrt{T^*-\kappa_m}.\end{aligned}$$

This implies that $\lim_{n,m\to\infty}\|I_3(n,m)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} = 0$. To summarize, we have derived the limit statement of (21) from equality (22). In other words, we have proved that

$((u,b)(\kappa_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ (recall that $s<3/2$). Therefore, there is $(u_1,b_1)\in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with

$$\lim_{n\to\infty}\|(u,b)(\kappa_n)-(u_1,b_1)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

The following simple argument shows that the limit (u_1,b_1) does not dependent on the sequence of times $(\kappa_n)_{n\in\mathbb{N}}$ approaching T^*. In fact, let $(\rho_n)_{n\in\mathbb{N}}\subseteq(0,T^*)$ with $\rho_n\nearrow T^*$ and let

$$\lim_{n\to\infty}\|(u,b)(\kappa_n)-(u_2,b_2)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0,$$

for some $(u_2,b_2)\in\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$.

We claim that $(u_2,b_2)=(u_1,b_1)$. To see this, define $(\varsigma_n)_{n\in\mathbb{N}}\subseteq(0,T^*)$ by $\varsigma_{2n}=\kappa_n$ and $\varsigma_{2n-1}=\rho_n$, for all $n\in\mathbb{N}$. It follows that $\varsigma_n\nearrow T^*$ and there exists $(u_3,b_3)\in \dot{H}^s_{a,\sigma}(\mathbb{R}^3)$ with

$$\lim_{n\to\infty}\|(u,b)(\varsigma_n)-(u_3,b_3)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0.$$

Therefore, $(u_1,b_1)=(u_3,b_3)=(u_2,b_2)$. Our arguments yield that $\lim_{t\nearrow T^*}\|(u,b)(t)-(u_1,b_1)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=0$.

Finally, consider the MHD equations (1) with the initial data (u_1,b_1) in instead of (u_0,b_0) and apply Theorem 1.1. As usual, we can piece the two solutions together to obtain a solution in an extended time interval, $0\le t\le T^*+T$ with $T>0$. This contradiction proves that

$$\limsup_{t\nearrow T^*}\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=\infty.$$

The proof of Theorem 1.2 **i)** $n=1$ is complete.

4.2. **Proof of Theorem 1.2 ii) (case $n=1$).** In this subsection we prove Theorem 1.2 **ii)** for $n=1$. Our result generalizes (4.1) of [4]. In fact, taking $s=1$ in Theorem 1.2 **ii)** (with $n=1$) yields (4.1) in [4].

Taking the $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)$-inner product of the velocity equation of (1) with $u(t)$ yields

$$\langle u,u_t\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=\langle u,-u\cdot\nabla u+b\cdot\nabla b-\nabla(p+\frac{1}{2}|b|^2)+\mu\Delta u\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}. \tag{27}$$

On the Fourier side, the second term on the right hand side of the above equation is

$$\begin{aligned}\mathcal{F}(u)\cdot\mathcal{F}[\nabla(p+\frac{1}{2}|b|^2)](\xi)&=-i\sum_{j=1}^3\mathcal{F}(u_j)(\xi)\xi_j\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)}\\&=-\sum_{j=1}^3\mathcal{F}(D_ju_j)(\xi)\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)}\\&=-\mathcal{F}(\operatorname{div}u)(\xi)\overline{\mathcal{F}[(p+\frac{1}{2}|b|^2)](\xi)}=0,\end{aligned} \tag{28}$$

because u is divergence free. As a consequence, we have

$$\langle u,\nabla(p+\frac{1}{2}|b|^2)\rangle_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(u)\cdot\mathcal{F}[\nabla(p+\frac{1}{2}|b|^2)](\xi)\,d\xi=0. \tag{29}$$

Furthermore, it holds that

$$\begin{aligned}\widehat{u}\cdot\widehat{\Delta u}(\xi) &= \sum_{j=1}^{3}\widehat{u}\cdot\widehat{D_j^2 u}(\xi) = -i\sum_{j=1}^{3}\widehat{u}\cdot[\xi_j\widehat{D_j u}(\xi)]\\ &= -\sum_{j=1}^{3}\widehat{D_j u}\cdot\widehat{D_j u}(\xi) = -|\widehat{\nabla u}(\xi)|^2. \qquad (30)\end{aligned}$$

Therefore,

$$\begin{aligned}\langle u,\Delta u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\,\widehat{u}\cdot\widehat{\Delta u}(\xi)\,d\xi = -\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\,|\widehat{\nabla u}(\xi)|^2\,d\xi\\ &= -\|\nabla u\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}. \qquad (31)\end{aligned}$$

Using (29) and (31) in (27), we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \mu\|\nabla u(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le |\langle u, u\cdot\nabla u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle u, b\cdot\nabla b\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}|. \qquad (32)$$

Next we consider the magnetic field equation of (1) and derive an estimate for $b(t)$ similar to the velocity estimate (32). Taking the $\dot H^s_{a,\sigma}(\mathbb{R}^3)$-inner product of the magnetic field equation with $b(t)$ yields that

$$\langle b, b_t\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = \langle u, -u\cdot\nabla b + b\cdot\nabla u + \nu\Delta b\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.$$

By applying (31), with b instead of u, it follows that

$$\frac{1}{2}\frac{d}{dt}\|b(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \nu\|\nabla b(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le |\langle b, u\cdot\nabla b\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle b, b\cdot\nabla u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}|. \qquad (33)$$

Combining (32) and (33), we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \theta\|\nabla(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\ &\le |\langle u, u\cdot\nabla u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle u, b\cdot\nabla b\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}| + |\langle b, u\cdot\nabla b\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}|\\ &+ |\langle b, b\cdot\nabla u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}|,\end{aligned}$$

where $\theta = \min\{\mu,\nu\}$. Furthermore, since $\operatorname{div} b = 0$, we have

$$\begin{aligned}\mathcal{F}(\nabla b)\cdot\mathcal{F}(b\otimes u)(\xi) &= \sum_{j=1}^{3}\mathcal{F}(\nabla b_j)\cdot\mathcal{F}(u_j b)(\xi) = \sum_{j,k=1}^{3}\mathcal{F}(D_k b_j)(\xi)\overline{\mathcal{F}(u_j b_k)(\xi)}\\ &= i\sum_{j,k=1}^{3}\xi_k\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(u_j b_k)(\xi)}\\ &= -\sum_{j,k=1}^{3}\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(D_k(u_j b_k))(\xi)},\end{aligned}$$

that is

$$\mathcal{F}(\nabla b)\cdot\mathcal{F}(b\otimes u)(\xi) = -\sum_{j,k=1}^{3}\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(b_k D_k u_j)(\xi)}$$

$$= -\sum_{j=1}^{3}\mathcal{F}(b_j)(\xi)\overline{\mathcal{F}(b\cdot\nabla u_j)(\xi)}$$
$$= -\mathcal{F}(b)\cdot\mathcal{F}(b\cdot\nabla u)(\xi).$$

It follows that

$$\begin{aligned}\langle b, b\cdot\nabla u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(b)\cdot\mathcal{F}(b\cdot\nabla u)(\xi)\,d\xi\\ &= -\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}\mathcal{F}(\nabla b)\cdot\mathcal{F}(b\otimes u)(\xi)\,d\xi\\ &= -\langle\nabla b, b\otimes u\rangle_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}$$

Using that u is divergence free and applying the Cauchy-Schwarz's inequality yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \theta\|\nabla(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le \|\nabla u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\ &+\|\nabla u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|\nabla b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\|u\otimes b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\ &+\|\nabla b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\|b\otimes u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}\tag{34}$$

We have to estimate the term $\|u\otimes b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}$ appearing above. Applying Lemma 2.9 **i)** $(0\le s<3/2)$ yields that

$$\begin{aligned}&\|u\otimes b\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} = \int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(u\otimes b)(\xi)|^2\,d\xi\\ &= \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}|\xi|^{2s}e^{2a|\xi|^{\frac{1}{\sigma}}}|\mathcal{F}(b_ju_k)(\xi)|^2\,d\xi\\ &= \sum_{j,k=1}^{3}\|b_ju_k\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}\\ &\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b_j}\|_{L^1(\mathbb{R}^3)}\|u_k\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|_{L^1(\mathbb{R}^3)}\|b_j\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}]^2\\ &\le C_s\sum_{j,k=1}^{3}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b_j}\|^2_{L^1(\mathbb{R}^3)}\|u_k\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u_k}\|^2_{L^1(\mathbb{R}^3)}\|b_j\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}]\\ &\le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|^2_{L^1(\mathbb{R}^3)}\|u\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|^2_{L^1(\mathbb{R}^3)}\|b\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}],\end{aligned}$$

or, equivalently,

$$\|u\otimes b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}\|u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}\|b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}].$$

Usinging this inequality in (34), we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \theta\|\nabla(u,b)(t)\|^2_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} \le\\ &C_s[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)} + \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}][\|u\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)} + \|b\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}]\|\nabla(u,b)(t)\|_{\dot H^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}$$

By Young's inequality:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\frac{\theta}{2}\|\nabla(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\\&\leq C_{s,\mu,\nu}[\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{u}\|_{L^1(\mathbb{R}^3)}+\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}\widehat{b}\|_{L^1(\mathbb{R}^3)}]^2[\|u\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}+\|b\|_{\dot{H}^1_{a,\sigma}(\mathbb{R}^3)}]^2\\&\leq C_{s,\mu,\nu}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^2_{L^1(\mathbb{R}^3)}\|(u,b)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}.\end{aligned}\tag{35}$$

Consider $0\leq t\leq T<T^*$ and apply the Gronwall's inequality to obtain:

$$\|(u,b)(T)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\leq\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}\exp\{C_{s,\mu,\nu}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau\}.$$

Passing to the limit superior, as $T\nearrow T^*$, Theorem 1.2 **i)** (with $n=1$) yields that

$$\int_t^{T^*}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau=\infty,\quad\forall\,t\in[0,T^*).$$

This completes the proof of Theorem 1.2 **ii)** for $n=1$.

4.3. **Proof of Theorem 1.2 iii) (case $n=1$).** In this subsection we prove Theorem 1.2 **iii)** for $n=1$. We point out that (4.2) in [4] is a particular case of Theorem 1.2 **iii)** obtained for $s=n=1$ and $b=0$ in (1).

Using Fourier transformation and taking the scalar product in $\mathbb{C}^3$ with $\widehat{u}(t)$, we obtain from the velocity equation of the MHD system:

$$\widehat{u}\cdot\widehat{u}_t=-\mu|\widehat{\nabla u}|^2-\widehat{u}\cdot\widehat{u\cdot\nabla u}+\widehat{u}\cdot\widehat{b\cdot\nabla b}.$$

We have used (28) and (30). Consequently,

$$\frac{1}{2}\partial_t|\widehat{u}(t)|^2+\mu|\widehat{\nabla u}|^2\leq|\widehat{u}\cdot\widehat{u\cdot\nabla u}|+|\widehat{u}\cdot\widehat{b\cdot\nabla b}|.\tag{36}$$

Similarly, by applying Fourier transformation and taking the scalar product in $\mathbb{C}^3$ with $\widehat{b}(t)$, we obtain from the magnetic field equation of the MHD system:

$$\widehat{b}\cdot\widehat{b}_t=-\nu|\widehat{\nabla b}|^2-\widehat{b}\cdot\widehat{u\cdot\nabla b}+\widehat{b}\cdot\widehat{b\cdot\nabla u}.$$

Therefore,

$$\frac{1}{2}\partial_t|\widehat{b}(t)|^2+\nu|\widehat{\nabla b}|^2\leq|\widehat{b}\cdot\widehat{u\cdot\nabla b}|+|\widehat{b}\cdot\widehat{b\cdot\nabla u}|.\tag{37}$$

Combining (36) and (37), it follows that

$$\frac{1}{2}\partial_t|(\widehat{u},\widehat{b})(t)|^2+\theta|(\widehat{\nabla u},\widehat{\nabla b})|^2\leq|\widehat{u}||\widehat{u\cdot\nabla u}|+|\widehat{u}||\widehat{b\cdot\nabla b}|+|\widehat{b}||\widehat{u\cdot\nabla b}|+|\widehat{b}||\widehat{b\cdot\nabla u}|,$$

where $\theta=\min\{\mu,\nu\}$. For $\delta>0$ arbitrary, it is easy to check that

$$\partial_t\sqrt{|(\widehat{u},\widehat{b})(t)|^2+\delta}+\theta\frac{|(\widehat{\nabla u},\widehat{\nabla b})|^2}{\sqrt{|(\widehat{u},\widehat{b})|^2+\delta}}\leq|\widehat{u\cdot\nabla u}|+|\widehat{b\cdot\nabla b}|+|\widehat{u\cdot\nabla b}|+|\widehat{b\cdot\nabla u}|.$$

Integrating from t to T (where $0\leq t\leq T<T^*<\infty$), one obtains that

$$\begin{aligned}&\sqrt{|(\widehat{u},\widehat{b})(T)|^2+\delta}+\theta|\xi|^2\int_t^T\frac{|(\widehat{u},\widehat{b})(\tau)|^2}{\sqrt{|(\widehat{u},\widehat{b})(\tau)|^2+\delta}}\,d\tau\leq\sqrt{|(\widehat{u},\widehat{b})(t)|^2+\delta}\\&+\int_t^T[|(\widehat{u\cdot\nabla u})(\tau)|+|(\widehat{b\cdot\nabla b})(\tau)|+|(\widehat{u\cdot\nabla b})(\tau)|+|(\widehat{b\cdot\nabla u})(\tau)|]\,d\tau,\end{aligned}$$

since $|(\widehat{\nabla u}, \widehat{\nabla b})| = |\xi||(\widehat{u}, \widehat{b})|$. Passing to the limit, as $\delta \to 0$, multiplying by $e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}}$ and integrating over $\xi \in \mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(T)\|_{L^1(\mathbb{R}^3)} \\
&+ \theta \int_t^T \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u}, \widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\, d\tau \le \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u}, \widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \\
&+ \int_t^T \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} [|(\widehat{u \cdot \nabla u})(\tau)| + |(\widehat{b \cdot \nabla b})(\tau)| + |(\widehat{u \cdot \nabla b})(\tau)| + |(\widehat{b \cdot \nabla u})(\tau)|]\, d\xi d\tau,
\end{aligned}
$$

because $|(\widehat{\Delta u}, \widehat{\Delta b})| = |\xi|^2|(\widehat{u}, \widehat{b})|$. Moreover, we have

$$
\begin{aligned}
|(\widehat{u \cdot \nabla b})(\xi)| &= \Big|\sum_{j=1}^{3} \widehat{u_j D_j b}(\xi)\Big| = (2\pi)^{-3}\Big|\sum_{j=1}^{3} \widehat{u_j} * \widehat{D_j b}(\xi)\Big| \\
&= (2\pi)^{-3}\Big|\sum_{j=1}^{3} \int_{\mathbb{R}^3} \widehat{u_j}(\eta)\widehat{D_j b}(\xi - \eta)\, d\eta\Big| \\
&\le (2\pi)^{-3}\Big|\int_{\mathbb{R}^3} \widehat{u}(\eta) \cdot \widehat{\nabla b}(\xi - \eta)\, d\eta\Big| \le (2\pi)^{-3} \int_{\mathbb{R}^3} |\widehat{u}(\eta)||\widehat{\nabla b}(\xi - \eta)|\, d\eta.
\end{aligned}
$$

Using the estimate (3), we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |(\widehat{u \cdot \nabla b})(\xi)|\, d\xi &\le (2\pi)^{-3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\eta)||\widehat{\nabla b}(\xi - \eta)|\, d\eta d\xi \\
&\le (2\pi)^{-3} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\eta|^{\frac{1}{\sigma}}} |\widehat{u}(\eta)| e^{\frac{a}{\sigma}|\xi-\eta|^{\frac{1}{\sigma}}} |\widehat{\nabla b}(\xi - \eta)|\, d\eta d\xi \\
&= (2\pi)^{-3} \int_{\mathbb{R}^3} [e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|] * [e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{\nabla b}(\xi)|]\, d\xi \\
&= (2\pi)^{-3} \|[e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} |\widehat{u}|] * [e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} |\widehat{\nabla b}|]\|_{L^1(\mathbb{R}^3)}.
\end{aligned}
$$

Applying Young's inequality it follows that

$$
\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |(\widehat{u \cdot \nabla b})(\xi)|\, d\xi \le (2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla b}\|_{L^1(\mathbb{R}^3)}. \tag{38}
$$

Furthermore, the Cauchy-Schwarz's inequality implies that

$$
\begin{aligned}
\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\nabla b}\|_{L^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{\nabla b}(\xi)|\, d\xi = \int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\xi||\widehat{b}(\xi)|\, d\xi \\
&\le \left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\xi|^2|\widehat{b}(\xi)|\, d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |\widehat{b}(\xi)|\, d\xi\right)^{\frac{1}{2}} \\
&= \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}},
\end{aligned} \tag{39}
$$

since $|\xi|^2|\widehat{b}| = |\widehat{\Delta b}|$ and $|\widehat{\nabla b}| = |\xi||\widehat{b}|$. Using the estimate (39) in (38) yields that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} e^{\frac{a}{\sigma}|\xi|^{\frac{1}{\sigma}}} |(\widehat{u \cdot \nabla b})(\xi)|\, d\xi \le \\
&(2\pi)^{-3} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{u}\|_{L^1(\mathbb{R}^3)} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}} \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}} \widehat{\Delta b}\|_{L^1(\mathbb{R}^3)}^{\frac{1}{2}}.
\end{aligned}
$$

Consequently,

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|_{L^1(\mathbb{R}^3)}\\&\quad+\theta\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau\le\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}\\&\quad+4(2\pi)^{-3}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^{\frac{3}{2}}_{L^1(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}\,d\tau.\end{aligned}$$

By using the Cauchy-Schwarz's inequality again, we conclude that

$$\begin{aligned}&4(2\pi)^{-3}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^{\frac{3}{2}}_{L^1(\mathbb{R}^3)}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)}\le\\&\frac{1}{8\pi^6\theta}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})\|^3_{L^1(\mathbb{R}^3)}+\frac{\theta}{2}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|_{L^1(\mathbb{R}^3)}+\frac{\theta}{2}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{\Delta u},\widehat{\Delta b})(\tau)\|_{L^1(\mathbb{R}^3)}\,d\tau\le\\&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)}+\frac{1}{8\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^3_{L^1(\mathbb{R}^3)}d\tau.\end{aligned}$$

By the Gronwall's inequality, it follows that

$$\begin{aligned}&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(T)\|^2_{L^1(\mathbb{R}^3)}\le\\&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}\exp\left\{\frac{1}{4\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\},\end{aligned}$$

for all $0\le t\le T<T^*$, or equivalently,

$$\begin{aligned}&\left(-4\pi^6\theta\right)\frac{d}{dT}\left[\exp\left\{-\frac{1}{4\pi^6\theta}\int_t^T\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\}\right]\le\\&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Integrate from t to t_0, with $0\le t\le t_0<T^*$, to obtain that

$$\begin{aligned}&\left(-4\pi^6\theta\right)\exp\left\{-\frac{1}{4\pi^6\theta}\int_t^{t_0}\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}d\tau\right\}+4\pi^6\theta\le\\&\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}(t_0-t).\end{aligned}$$

By passing to the limit, as $t_0\nearrow T^*$, and using Theorem 1.2 **ii)** with $n=1$, we have

$$4\pi^6\theta\le\|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|^2_{L^1(\mathbb{R}^3)}(T^*-t),\quad\forall\,t\in[0,T^*).$$

This completes the proof of Theorem 1.2 **iii)** for $n=1$.

4.4. **Proof of Theorem 1.2 iv) (case $n=1$).** One of the assumptions of Theorem 1.2 is that $\sigma>1$; consequently, $\frac{a}{\sqrt{\sigma}}\in(0,a)$. As a result, the embedding $\dot{H}^s_{a,\sigma}(\mathbb{R}^3)\hookrightarrow\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)$ holds. Therefore, Theorem 1.1 yields that $(u,b)\in C([0,T^*_a),\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3))$ since $(u,b)\in C([0,T^*_a),\dot{H}^s_{a,\sigma}(\mathbb{R}^3))$.

On the other hand, the inequality

$$\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)}\le\|(u,b)(t)\|_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)}$$

implies that

$$T^*_{\frac{a}{\sqrt{\sigma}}} \geq T^*_a. \tag{40}$$

Moreover, by applying Theorem 1.2 **iii)** with $n = 1$ and the Cauchy-Schwarz's inequality (analogously to (7)), it follows that

$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*_a - t}} \leq \|e^{\frac{a}{\sigma}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \leq C_{a,\sigma,s}\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)}, \tag{41}$$

for all $t \in [0, T^*_a)$, where

$$C^2_{a,\sigma,s} := \int_{\mathbb{R}^3} \frac{1}{|\xi|^{2s}} e^{-2a(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma})|\xi|^{\frac{1}{\sigma}}}\,d\xi = 4\pi\sigma\left[2a\left(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s)) < \infty.$$

(Recall that $s < 3/2$, $a > 0$ and $\sigma > 1$). This proves Theorem 1.2 **iv)** for $n = 1$.

4.5. **Proof of Theorem 1.2 i), ii), iii) and iv) (case $n > 1$).** First note that (41) implies

$$\limsup_{t\nearrow T^*_a} \|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sqrt{\sigma}},\sigma}(\mathbb{R}^3)} = \infty. \tag{42}$$

This yields Theorem 1.2 **i)** for $n = 2$. As above, we infer that

$$\int_t^{T^*} \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(\tau)\|^2_{L^1(\mathbb{R}^3)}\,d\tau = \infty, \quad \forall\, t \in [0, T^*).$$

This proves Theorem 1.2 **ii)** for $n = 2$ and Theorem 1.2 **iii)** for $n = 2$ follows (see Section 4.3). As an immediate consequence of (42), one obtains that

$$T^*_a \geq T^*_{\frac{a}{\sqrt{\sigma}}}. \tag{43}$$

Clearly, the inequalities (40) and (43) imply that

$$T^*_a = T^*_{\frac{a}{\sqrt{\sigma}}}. \tag{44}$$

Let us reexamine the above process with $\frac{a}{\sqrt{\sigma}}$ in the place of a. As in (41), we obtain that

$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*_{\frac{a}{\sqrt{\sigma}}} - t}} \leq \|e^{\frac{a}{\sigma\sqrt{\sigma}}|\cdot|^{\frac{1}{\sigma}}}(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \leq C_{\frac{a}{\sqrt{\sigma}},\sigma,s}\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)}, \tag{45}$$

for all $t \in [0, T^*_{\frac{a}{\sqrt{\sigma}}})$, where

$$C^2_{\frac{a}{\sqrt{\sigma}},\sigma,s} = \int_{\mathbb{R}^3} \frac{1}{|\xi|^{2s}} e^{-2a(\frac{1}{\sigma}-\frac{1}{\sigma\sqrt{\sigma}})|\xi|^{\frac{1}{\sigma}}}\,d\xi = 4\pi\sigma\left[2\frac{a}{\sqrt{\sigma}}\left(\frac{1}{\sqrt{\sigma}}-\frac{1}{\sigma}\right)\right]^{-\sigma(3-2s)}\Gamma(\sigma(3-2s))$$

is finite. By (44) and (45), one has

$$\|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} \geq \frac{2\pi^3\sqrt{\theta}}{C_{\frac{a}{\sqrt{\sigma}},\sigma,s}\sqrt{T^*_a - t}}, \quad \forall\, t \in [0, T^*_a).$$

This completes the proof of Theorem 1.2 **iv)** for $n = 2$. Passing to the limit as $t \nearrow T^*_a$, we deduce that

$$\limsup_{t\nearrow T^*_a} \|(u,b)(t)\|_{\dot{H}^s_{\frac{a}{\sigma},\sigma}(\mathbb{R}^3)} = \infty.$$

Consequently, Theorem 1.2 **i)** holds for $n=3$. Notice that, replacing a by $\frac{a}{\sqrt{\sigma}}$ in (44), one obtains that

$$T_a^* = T^*_{\frac{a}{\sqrt{\sigma}}} = T^*_{\frac{a}{\sigma}}.$$

Therefore, inductively, one concludes that $T_a^* = T^*_{\frac{a}{(\sqrt{\sigma})^n}}$ for all $n \in \mathbb{N}\cup\{0\}$. Theorem 1.2 **i)**, **ii)**, **iii)** and **iv)** holds for all $n \geq 1$.

4.6. **Proof of Theorem 1.2 v).** It remains to prove Theorem 1.2 **v)**. Note that Theorem 1.2 **v)** for $s=1$ and $b=0$ (in (1)) yields (1.3) in [4].

Choose $\delta = s + \frac{k}{2\sigma}$ with $k \in \mathbb{N}\cup\{0\}$ and $k \geq 2\sigma$ and set $\delta_0 = s+1$. By using Lemmas 2.3 and 2.4, and (2), we obtain

$$\frac{2\pi^3\sqrt{\theta}}{\sqrt{T^*-t}} \leq \|(\widehat{u},\widehat{b})(t)\|_{L^1(\mathbb{R}^3)} \leq C_s \|(u,b)(t)\|_{L^2(\mathbb{R}^3)}^{1-\frac{3}{2(s+\frac{k}{2\sigma})}} \|(u,b)(t)\|_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}^{\frac{3}{2(s+\frac{k}{2\sigma})}}.$$

Hence, using the inequality

$$\|(u,b)(t)\|_{L^2(\mathbb{R}^3)} \leq \|(u,b)(t_0)\|_{L^2(\mathbb{R}^3)}, \quad \forall\, 0 \leq t_0 \leq t < T^*, \tag{46}$$

(see (2) in [7]) we obtain that

$$\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}} \left(\frac{D_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k \leq \|(u,b)(t)\|^2_{\dot{H}^{s+\frac{k}{2\sigma}}(\mathbb{R}^3)}, \tag{47}$$

where $D_{\sigma,s,\theta,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta}\|(u_0,b_0)\|^{-1}_{L^2(\mathbb{R}^3)})^{\frac{2}{3\sigma}}$ and $C_{\theta,s,u_0,b_0} = (C_s^{-1}2\pi^3\sqrt{\theta})^{\frac{4s}{3}}$ $\times \|(u_0,b_0)\|^{\frac{6-4s}{3}}_{L^2(\mathbb{R}^3)}$. Multiplying (47) by $\frac{(2a)^k}{k!}$, one concludes that

$$\begin{aligned}\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}} \frac{\left(\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^k}{k!} &\leq \int_{\mathbb{R}^3} \frac{(2a)^k}{k!} |\xi|^{2(s+\frac{k}{2\sigma})} |(\widehat{u},\widehat{b})(t)|^2\, d\xi \\ &= \int_{\mathbb{R}^3} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!} |\xi|^{2s} |(\widehat{u},\widehat{b})(t)|^2\, d\xi.\end{aligned}$$

By summing over the set $\{k \in \mathbb{N}; k \geq 2\sigma\}$ and applying the Monotone Convergence Theorem, one obtains that

$$\begin{aligned}&\frac{C_{\theta,s,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}[\exp\left\{\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right\} - \sum_{0\leq k<2\sigma} \frac{(2aD_{\sigma,s,\theta,u_0,b_0})^k}{k!(T^*-t)^{\frac{k}{3\sigma}}}] \\ &\leq \int_{\mathbb{R}^3} [e^{2a|\xi|^{\frac{1}{\sigma}}} - \sum_{0\leq k<2\sigma} \frac{(2a|\xi|^{\frac{1}{\sigma}})^k}{k!}] |\xi|^{2s} |(\widehat{u},\widehat{b})(t)|^2\, d\xi \\ &\leq \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |(\widehat{u},\widehat{b})(t)|^2\, d\xi = \|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)},\end{aligned}$$

for all $t \in [0,T^*)$. Finally, if we define

$$f(x) = \Big[e^x - \sum_{k=0}^{2\sigma_0} \frac{x^k}{k!}\Big][x^{-(2\sigma_0+1)}e^{-\frac{x}{2}}], \quad \forall\, x \in (0,\infty),$$

where $2\sigma_0$ is the integer part of 2σ, then f is continuous on $(0,\infty)$, $f>0$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\nearrow 0} f(x) = \dfrac{1}{(2\sigma_0+1)!}$. Therefore, there is a positive constant

C_{σ_0} with $f(x) \geq C_{\sigma_0}$ for all $x > 0$. Therefore,

$$\begin{aligned}\|(u,b)(t)\|^2_{\dot{H}^s_{a,\sigma}(\mathbb{R}^3)} &\geq \frac{C_{\theta,s,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{2s}{3}}}\left(\frac{2aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right)^{2\sigma_0+1}\exp\left\{\frac{aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right\}\\ &= \frac{a^{2\sigma_0+1}C_{\theta,s,\sigma,\sigma_0,u_0,b_0}}{(T^*-t)^{\frac{2(s\sigma+\sigma_0)+1}{3\sigma}}}\exp\left\{\frac{aD_{\sigma,s,\theta,u_0,b_0}}{(T^*-t)^{\frac{1}{3\sigma}}}\right\},\end{aligned}$$

for all $t \in [0, T^*)$. The proof of Theorem 1.2 **v)** is completed.

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E-mail address: lorenz@math.unm.edu
E-mail address: wilberclay@gmail.com
E-mail address: nata@ufmg.br