STABILITY OF VISCOUS PROFILES: PROOFS VIA DICHOTOMIES

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Abstract

In this paper we prove a nonlinear stability result, as $t \to \infty$, for a viscous profile corresponding to a strong shock of a system of conservation laws. The inital perturbation is assumed to be small and to have zero mass. As $t \to \infty$, the solution with perturbed initial data is shown to approach the viscous profile in maximum norm.

A complete proof of the stability result is given under slightly weaker assumptions than those in [5]; our assumptions, techniques, and results also differ from those in [11]. To derive resolvent estimates for a linearized problem, we use the theory of exponential dichotomies for ODEs extensively. A main tool provided by this theory is a quantitative L_1 perturbation theorem for dichotomies, which yields the delicate resolvent estimates for s near zero.

When showing that the resolvent estimates imply nonlinear stability, we essentially follow the arguments in [5, 6], but note some simplifications.

Key words: Viscous conservation laws, viscous profiles, stability, zero–mass condition
AMS subject classification: 34L05, 34A50, 65L10

[1]Supported by DFG-Research Group 'Spectral Analysis, Asymptotic Distributions and Stochastic Dynamics' Bielefeld University

Contents

1 Introduction

In this paper we consider systems of viscous conservation laws,

$$v_t + f(v)_x = v_{xx}, \quad x \in \mathbb{R} , \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, f \in C^\infty$, and assume the existence of a stationary profile, $U(x)$. We give conditions for asymptotic stability, as $t \to \infty$, of the profile $U(x)$ under small zero–mass perturbations. No assumption is needed about the shock strength, $|U_L - U_R|$, where $\lim_{x\to\infty} U(x) = U_R$, $\lim_{x\to-\infty} U(x) = U_L$, i.e., the theory applies to strong shocks. In fact, not to overburden the presentation, we restrict ourselves to the important case where the discontinuous function

$$U_0(x) = U_L \quad \text{for} \quad x < 0, \quad U_0(x) = U_R \quad \text{for} \quad x > 0$$

is a Lax shock (see [7]) for the system of conservation laws $v_t + f(v)_x$.

The proof of the main result, the stability result formulated in Theorem 1.1 below, consists of two parts:

1. resolvent estimates for the linearization about $U(x)$;
2. arguments which show that the resolvent estimates imply nonlinear stability under small zero–mass perturbations.

The main difficulty occurs in the derivation of the relevant resolvent estimates for s near zero, which are very delicate. The basic reason for the difficulty is well–known: The linearized operator about the profile $U(x)$ has $s = 0$ as an eigenvalue and there is no separation of this eigenvalue from the essential spectrum of the operator. For discussions of this difficulty when one uses the semigroup approach for stability analysis of shock waves we refer to [12]. The paper [12] also gives many references regarding the physical importance of shock waves and reviews known results on the stability of weak shocks.

The purpose of the present paper is to show the relevant resolvent estimates under slightly weaker assumptions than those in [5] and to use, in parts, different arguments from the ODE theory of exponential dichotomies. We will also derive nonlinear stability from the resolvent estimate (see Section 2). In this regard our arguments follow the proofs in [5, 6] closely; the only simplification that we observe is that L_1–estimates of *derivative* terms are not needed. (See, for example, condition (1.5) in [5]; this condition is only needed for $q_1 = q_2 = 0$.)

To describe our results in more detail, we proceed as in [5] and consider the system (1.1) with initial condition

$$v(x, 0) = U(x) + \varepsilon(v_0(x))_x \tag{1.2}$$

where $|\varepsilon|$ is small and v_0 is a smooth function that decays to zero as $|x| \to \infty$. In particular, the perturbation of $U(x)$ has zero mass. [1] Precisely, we will assume that v_0 satisfies the following: For some integer $k \geq 3$ let

$$v_0 \in H^{k+2} \quad \text{and} \quad D^j v_0 \in L_1 \quad \text{for} \quad j = 0, 1, 2 \tag{1.3}$$

and assume the normalization

$$\|v_0\|_{H^{k+2}} + \sum_{j=0}^{2} \|D^j v_0\|_{L_1} = 1 \ . \tag{1.4}$$

Here we use the following standard notations: By $L_p = L_p(\mathbb{R}), 1 \leq p \leq \infty$, we denote the usual Lebesgue space with norm $\|\cdot\|_{L_p}$. In particular, we use the norms

$$\begin{aligned}
\|u\|^2 &= \int_{-\infty}^{\infty} |u(x)|^2 \, dx \\
\|u\|_{L_1} &= \int_{-\infty}^{\infty} |u(x)| \, dx \\
\|u\|_{L_\infty} &= \sup_x |u(x)|
\end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm, for definiteness. With $W^{k,p} = W^{k,p}(\mathbb{R})$ we denote the usual Sobolev space of all vector functions that have weak derivatives up to order k in L_p. We write $H^k = W^{k,2}$ for the Hilbert space and $\|\cdot\|_{H^k}$ for the corresponding norm. Similar notations are used for matrix–valued functions.

Our main result is the following stability theorem.

Theorem 1.1 *Consider the system (1.1) with initial condition (1.2) under the assumptions (A0) to (A5) listed below. Assume that the function v_0 in (1.2) satisfies (1.3) and (1.4) for some $k \geq 3$. Then there are positive constants $\varepsilon_0 = \varepsilon_0(k)$ and C_k, which are independent of v_0, so that for $|\varepsilon| \leq \varepsilon_0$ the initial value problem (1.1), (1.2) has a unique classical solution $v(x,t)$, and we have the estimate*

$$\int_0^\infty \left(\|v(\cdot,t) - U(\cdot)\|^2_{H^{k+1}} + \|v_t(\cdot,t)\|^2_{H^{k-1}} \right) dt \leq C_k \varepsilon \ . \tag{1.5}$$

Consequently, if $|\varepsilon| \leq \varepsilon_0$, then

$$\sup_x |v(x,t) - U(x)| \to 0 \quad \text{as} \quad t \to \infty \ . \tag{1.6}$$

In the following we sketch the main arguments for step 2 above; see Section 2 for details. The change of variables

[1] If (1.2) is replaced by an initial condition $v(x,0) = U(x) + \varepsilon v_0(x)$ with $\int_{-\infty}^{\infty} v_0(x) dx \neq 0$, then the initial perturbation has non–zero mass. In this case it is reasonable to conjecture convergence to a perturbed profile close to $U(x)$.

$$v(x,t) = U(x) + \varepsilon w(x,t)$$

leads to a system of the form

$$\begin{aligned} w_t + (A(x)w)_x + \varepsilon(G(x,w))_x &= w_{xx} \\ w(x,0) &= (v_0(x))_x \end{aligned}$$

where

$$A(x) = f_u(U(x)) \; .$$

As in [5], we transform to homogeneous initial data: The function

$$u(x,t) = w(x,t) - e^{-t}(v_0(x))_x$$

satisfies a system of the form

$$\begin{aligned} u_t + (A(x)u)_x + \varepsilon(B(x,t)u)_x + \varepsilon(g(x,t,u))_x &= u_{xx} - F_x(x,t) && (1.7) \\ u(x,0) &= 0 && (1.8) \end{aligned}$$

Again, see Section 2 for details. If one first neglects the terms multiplied by ε in (1.7), then Laplace transformation in t leads to the resolvent equation

$$\hat{u}_{xx} - (A(x)\hat{u})_x - s\hat{u} = \hat{F}_x, \quad Re\, s \geq 0 \; . \tag{1.9}$$

Henceforth, we drop the hat notation.

Under the assumptions (A0) to (A5) listed below, we will prove the following resolvent estimate.

Theorem 1.2 *Let (A0) to (A5) hold and let $F \in L_1 \cap H^1$. If*

$$Re\, s \geq 0, \quad s \neq 0 \; , \tag{1.10}$$

then the equation

$$u_{xx} - (A(x)u)_x - su = F_x, \quad x \in \mathbb{R} \; , \tag{1.11}$$

has a unique solution u in H^1. Furthermore, for $\mathrm{Re}\, s \geq 0$*,*

$$\begin{aligned} \|u\|^2 + \|u_x\|^2 &\leq K_R(\|F\|^2 + \|F\|_{L^1}^2) \quad \text{if} \quad 0 < |s| \leq 1 \; , && (1.12) \\ \|u\|^2 + \|u_x\|^2 &\leq K_R\|F\|^2 \quad \text{if} \quad |s| \geq 1 \; , && (1.13) \\ & && (1.14) \end{aligned}$$

where the resolvent constant K_R is independent of F and s.

We use the following **notations and assumptions.**

(A0) We assume that $U \in C^\infty(\mathbb{R})$ is a stationary solution of (1.1),

$$U_x(x) = f(U(x)), \quad x \in \mathbb{R} \ ,$$

with

$$U(x) \to U_R \quad \text{as} \quad x \to \infty, \quad U(x) \to U_L \quad \text{as} \quad x \to -\infty, \quad U_L \neq U_R$$

and

$$U_x(x) \to 0 \quad \text{as} \quad |x| \to \infty \ .$$

(A1) We then set

$$A_L = f_u(U_L), \quad A_R = f_u(U_R)$$

and assume that the matrix function $A(x) = f_u(U(x))$ satisfies the following condition:

$$\int_l^\infty |A(x) - A_L| \, dx + \int_{-\infty}^{-l} |A(x) - A_R| \, dx \to 0 \quad \text{as} \quad l \to \infty \ . \tag{1.15}$$

Clearly, this assumption is equivalent to $A(\cdot) - A_L \in L_1[0,\infty)$ and $A(\cdot) - A_R \in L_1(-\infty, 0]$.

(A2) Both limit matrices, A_L and A_R, are assumed to be nonsingular, to have only real eigenvalues, and to be diagonalizable. [2]

Thus, there are real transformation matrices, S_L and S_R, with

$$S_R^{-1} A_R S_R = \Lambda_R = \begin{pmatrix} -\Lambda_R^I & 0 \\ 0 & \Lambda_R^{II} \end{pmatrix}$$
$$S_L^{-1} A_L S_L = \Lambda_L = \begin{pmatrix} -\Lambda_L^I & 0 \\ 0 & \Lambda_L^{II} \end{pmatrix}$$

where $\Lambda_{R,L}^{I,II}$ are diagonal matrices with positive diagonals. We assume the following dimensions:

$$\begin{array}{cll} \Lambda_R^I & \text{is} & k \times k \\ \Lambda_R^{II} & \text{is} & (n-k) \times (n-k) \\ \Lambda_L^I & \text{is} & (k-1) \times (k-1) \\ \Lambda_L^{II} & \text{is} & (n+1-k) \times (n+1-k) \end{array}$$

[2] In the language of dynamical systems, the fixed points U_L and U_R of the system $u_x = f(u)$ are hyperbolic. Therefore, the functions $|A_L - A(x)|$ and $|A_R - A(x)|$ decay exponentially as $x \to \infty$ and $x \to -\infty$, respectively, which implies (1.15). In our proof, we will only use decay in the form (1.15), which may be of use for generalizations to cases where U_L or U_R are non–hyperbolic.

This means that the system $v_t + (Av)_x = 0$ has

$$k + (n+1-k) = n+1$$

characteristics which enter the region around $x = 0$. In other words, we assume $U(x)$ to be a viscous profile for a stationary Lax shock.

(A3) We partition the transformation matrices S_R and S_L columnwise, in correspondence with the block structure of Λ_R and Λ_L,

$$S_R = (S_R^I, S_R^{II}), \quad S_L = (S_L^I, S_L^{II}) \ .$$

For example, S_R^{II} has $n-k$ and S_L^I has $k-1$ columns. We then assume that the $n \times n$ matrix

$$\mathcal{M} = (S_R^{II}, S_L^I, U_R - U_L) \tag{1.16}$$

is nonsingular.

(A4) We assume that the homogeneous system

$$u_{xx} - (Au)_x - su = 0, \quad x \in \mathbb{R} \ ,$$

has no nontrivial solution $u \in L_2$ if $Re\, s \geq 0, s \neq 0$. In other words, we assume that the operator $Lu = u_{xx} - (Au)_x$ has no eigenvalue $s \neq 0$ with $Re s \geq 0$ and L_2 eigenfunction.

(A5) Set $\varphi_0(x) = U_x(x)$. Note that the assumption

$$U_{xx} = f(U)_x = f_u(U)U_x$$

yields the relation

$$\varphi_{0x} = A\varphi_0 \ .$$

By assumption, $\varphi_0(x) \to 0$ as $|x| \to \infty$. Since $A(x)$ converges to the constant matrices $A_{L,R}$ with real, nonzero eigenvalues, it follows that there is $\beta > 0$ with

$$|\varphi_0(x)| \leq Ke^{-\beta|x|} \ .$$

In particular, $\varphi_0 \in L_2$. We assume that *all* L_2–solutions $y(x)$ of the system $y_x = Ay$ are multiples of the function $\varphi_0(x)$.

This completes the description of the assumptions of Theorem 1.2.

Remarks: 1. The assumptions are almost identical to those in [5]. Our assumption (1.15) is weaker than the corresponding assumption (1.3) of [5], but, as mentioned above, exponential decay of $|A_L - A(x)|$ etc. is implied by condition (A2). The assumption made in [5] that the matrices A_R and A_L each have distinct eigenvalues is not needed. We finally note that the eigenvalue assumption (A4) needs to be *required* only for

$$Re\, s \geq 0 \quad \text{and} \quad 0 < \delta \leq |s| \leq R$$

where δ is sufficiently small and R is sufficiently large. For the remaining s values,

$$Re\, s \geq 0 \quad \text{and} \quad 0 < |s| \leq \delta \quad \text{or} \quad |s| \geq R \ ,$$

the eigenvalue assumption can be deduced from the other assumptions. This is of some interest if one wants to check the eigenvalue assumption (A4) numerically.

2. Related and partially more general stability results for viscous shock waves are also stated in [11]. Assumptions and assertions are not directly comparable to ours, e.g. a certain algebraic decay (as $|x| \to \infty$) is required for initial perturbations and a corresponding decay rate (as $t \to \infty$) for the solutions is derived. An extensive linear theory centered around the Evans function and pointwise error estimates for time-dependent Green's functions is developed in [11],[12]. However, detailed arguments how to obtain stability for nonlinear problems using the linear theory are not given in [11], but a reference to the methods in [8] is made.

In the following we briefly outline the contents of the following sections. The proof of Theorem 1.2 is given in Sections 3 to 5. As mentioned above, the main difficulties occur for small $|s|$. Then problems in x–intervals

$$-\infty < x \leq -l + 1 \quad \text{or} \quad l - 1 \leq x < \infty \ ,$$

where l is large, so–called tail problems, are considered in Section 3. The idea is that a rather explicit discussion of constant–coefficient problems (with $A(x)$ replaced by A_L or A_R) is possible, and then a quantitative L_1 perturbation result using (1.15) can be applied. Details of the L_1 perturbation result for exponential dichotomies are given in Appendix A.

In Section 4 we supplement the results for tail problems by results for boundary value problems on finite but large intervals,

$$-l \leq x \leq l \ .$$

Note that there are two overlap intervals, $-l \leq x \leq -l + 1$ and $l - 1 \leq x \leq l$, when considering the two tail problems and the finite interval problem. The size of l is determined by properties of the tail problems. Solution estimates for the finite interval problem are again based on L_1 perturbation results for exponential dichotomies. More specifically, we will use sharp perturbation estimates for projectors of exponential dichotomies. The auxiliary results are proved in Appendices A and B.

Together, the results of Sections 3 and 4 allow the construction of an 'almost' solution of (1.11), i.e., the construction of a function that satisfies (1.11) with small defect and obeys suitable estimates. Abstractly speaking, one has constructed an approximate right–inverse of the operator $L(s)u = u_{xx} - (Au)_x - su$. A simple argument then proves Theorem 1.2 for small $|s|$; see Theorem 5.1. Together with standard estimates for $|s|$ large and for $0 < \delta \le |s| \le R$ the proof of Theorem 1.2 can be completed. The details are carried out in Section 5.

The derivation of the nonlinear stability result in Theorem 1.1, based on Theorem 1.2, is given in Section 2.

2 From Resolvent Estimates to Nonlinear Stability

In Section 2.1 we will describe a change of the dependent variable, leading to the problem (2.8). Estimates for the corresponding unperturbed problem, obtained for $\varepsilon = 0$, are then derived in Section 2.2. Essentially, these estimates are obtained from the resolvent estimate using nothing but Parseval's relation. The main estimate for the unperturbed problem is stated in Theorem 2.1. In Section 2.3 we then show how Theorem 2.1 can be used to obtain stability for the nonlinear problem.

Throughout, for simplicity of presentation, we assume that the flux function $f = f(u)$ and the viscous profile $U = U(x)$ are C^∞ smooth. Finite degrees of smoothness would suffice, however. For the function $v_0 = v_0(x)$, which describes the initial perturbation in condition (1.2), we will always assume (1.3) and (1.4) for some $k \ge 3$. The constants K, K_j etc. that appear in the following lemmas will be independent of v_0. The constants depend on the integer k, but we often suppress this dependence in our notation.

2.1 Pretransformations

Consider the system $v_t + f(v)_x = v_{xx}$ where $f : \mathbb{R}^n \to \mathbb{R}^n, f \in C^\infty$. Assume that $U : \mathbb{R} \to \mathbb{R}^n$ is a stationary C^∞ solution, $f(U) = U_x$, and set $A(x) = f_u(U(x))$. We seek a solution $v(x,t)$ with perturbed initial condition,

$$v(x,0) = U(x) + \varepsilon(v_0(x))_x \ .$$

Introduce a new unknown function $w(x,t)$ by setting

$$v(x,t) = U(x) + \varepsilon w(x,t) \ .$$

We can write

$$f(U(x) + \varepsilon w) = f(U(x)) + \varepsilon A(x) w + \varepsilon^2 q(x,w,\varepsilon), \quad A(x) = f_u(U(x)) \ ,$$

with

$$q(x,w,\varepsilon) = \int_0^1 (1-\tau) f_{uu}\Big(U(x) + \tau\varepsilon w\Big) ww \, d\tau \ .$$

The function $(x,w) \to q(x,w,\varepsilon)$ is C^∞ smooth, uniformly in ε, and vanishes quadratically at $w=0$. More precisely, we have the following:

Lemma 2.1 *a) For all $i=0,1,\ldots$ and all $j=0,1,\ldots$ and all $\gamma>0$ there is a constant $K_{ij}(\gamma)$ with*

$$|D_w^i D_x^j q(x,w,\varepsilon)| \le K_{ij}(\gamma) \tag{2.1}$$

for $x\in\mathbb{R}, |w| \le \gamma, |\varepsilon| \le 1$.

b) For all $\gamma>0$ there is a constant $K(\gamma)$ with

$$|D_w q(x,w,\varepsilon)| \le K(\gamma)|w| \tag{2.2}$$

for $x\in\mathbb{R}, |w| \le \gamma, |\varepsilon| \le 1$.

The system for $w(x,t)$ reads

$$w_t + (A(x)w)_x + \varepsilon(q(x,w,\varepsilon))_x = w_{xx}, \quad w(x,0) = v_{0x}(x) \ .$$

It will be convenient to transform to homogeneous initial conditions. To this end, define a new unknown function $u(x,t)$ by setting

$$w(x,t) = e^{-t}v_{0x}(x) + u(x,t) \ .$$

Obtain that

$$u_t + (A(x)u)_x + \varepsilon\Big(q(x,e^{-t}v_{0x}+u,\varepsilon)\Big)_x = u_{xx} + (G(x,t))_x$$

with

$$G(x,t) = e^{-t}\Big(v_0(x) - A(x)v_{0x}(x) + v_{0xx}\Big) \ .$$

Since the matrix function $x \to A(x)$ and its derivatives are bounded, our assumptions (1.3), (1.4), imply the following:

Lemma 2.2 *There is a constant K with*

$$\begin{aligned} \int_0^\infty \|G(\cdot,t)\|_{H^k}^2 \, dt \quad &\le \quad K \ , \\ \int_0^\infty \|G(\cdot,t)\|_{L_1} \, dt \quad &\le \quad K \ . \end{aligned}$$

We make a Taylor expansion of the term

$$q(x,e^{-t}v_{0x}+u,\varepsilon)$$

about $u=0$,

$$q(x,e^{-t}v_{0x}+u,\varepsilon)=q(x,e^{-t}v_{0x},\varepsilon)+B(x,t,\varepsilon)u+g(x,t,u,\varepsilon)\ .$$

Here

$$B(x,t,\varepsilon)=q_w(x,e^{-t}v_{0x}(x),\varepsilon)$$

and

$$g(x,t,u,\varepsilon)=\int_0^1(1-\tau)q_{ww}(x,e^{-t}v_{0x}(x)+\tau u,\varepsilon)uu\,d\tau\ .$$

In the next lemma we summarize properties of the matrix function $B(x,t,\varepsilon)$ that will be used below. The result follows easily from Lemma 2.1 and the assumptions (1.3) and (1.4).

Lemma 2.3 *There is a constant K, independent of $|\varepsilon|\le 1$, with*

$$\begin{aligned} |D_x^jB(x,t,\varepsilon)| &\le K \quad \text{for} \quad 0\le j\le k, \quad x\in\mathbb{R}, \quad t\ge 0\ ; && (2.3)\\ \int_0^\infty \|B(\cdot,t,\varepsilon)\|^2dt &\le K\ . && (2.4)\end{aligned}$$

The function $g(x,t,u,\varepsilon)$ vanishes quadratically at $u=0$ and, in bounded u–regions, its derivatives are uniformly bounded. In the following lemma we summarize estimates of g and its derivatives that we need below.

Lemma 2.4 *a) For all $\gamma>0$ there is a constant $K(\gamma)$ with*

$$|g(x,t,u,\varepsilon)|\le K(\gamma)|u|^2 \tag{2.5}$$

for $x\in\mathbb{R},t\ge 0,|\varepsilon|\le 1,|u|\le\gamma$.

b) For all $\gamma>0$ and all i with $0\le i\le k$ there is a constant $K_i(\gamma)$ with

$$|D_x^ig(x,t,u,\varepsilon)|\le K_i(\gamma)|u| \tag{2.6}$$

for $x\in\mathbb{R},t\ge 0,|\varepsilon|\le 1,|u|\le\gamma$.

c) For all $\gamma>0$ and all i with $0\le i\le k$ and all $j=0,1,\dots$ there is a constant $K_{ij}(\gamma)$ with

$$|D_x^iD_u^jg(x,t,u,\varepsilon)|\le K_{ij}(\gamma) \tag{2.7}$$

for $x\in\mathbb{R},t\ge 0,|\varepsilon|\le 1,|u|\le\gamma$.

The system for the new unknown function $u(x,t)$ has the form

$$u_t+(A(x)u)_x+\varepsilon\Big(B(x,t,\varepsilon)u\Big)_x+\varepsilon\Big(g(x,t,u,\varepsilon)\Big)_x=u_{xx}-F_x(x,t,\varepsilon),\quad u(x,0)=0\ , \tag{2.8}$$

with

$$F(x,t,\varepsilon) = -G(x,t) + \varepsilon q\Big(x, e^{-t} v_{0x}(x), \varepsilon\Big) \; .$$

The properties of $F(x,t,\varepsilon)$ are similar to those of $G(x,t)$ stated in Lemma 2.2 above.

Lemma 2.5 *There is a constant K, independent of ε with $|\varepsilon| \le 1$, so that*

$$\begin{aligned} \int_0^\infty \|F(\cdot,t,\varepsilon)\|_{H^k}^2 \, dt &\le K \; , \\ \int_0^\infty \|F(\cdot,t,\varepsilon)\|_{L_1} \, dt &\le K \; . \end{aligned}$$

2.2 Space–Time Estimates for the Linear Problem

For $\varepsilon = 0$ the problem (2.8) reads

$$u_t + (A(x)u)_x = u_{xx} - F_x(x,t), \quad u(x,0) = 0 \; . \tag{2.9}$$

In this section we consider the problem (2.9) and assume

$$\int_0^\infty \|F(\cdot,t)\|_{H^k}^2 \, dt < \infty, \quad \int_0^\infty \|F(\cdot,t)\|_{L_1} \, dt < \infty \; .$$

Here we only need to assume that $k \ge 1$. (The condition $k \ge 3$ will be needed in Section 2.3 below when we treat the nonlinear problem (2.8).)

We first proceed formally and denote by

$$\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t) \, dt, \quad \operatorname{Re} s \ge 0 \; ,$$

the Laplace transformation in t. The equation (2.9) becomes

$$s\hat{u} + (A(x)\hat{u})_x = \hat{u}_{xx} - \hat{F}_x, \quad \operatorname{Re} s \ge 0 \; .$$

The resolvent estimates of Theorem 1.2 yield:

a) If $s = i\xi, 0 < |\xi| \le 1$, then we have

$$\|\hat{u}(\cdot,s)\|^2 + \|\hat{u}_x(\cdot,s)\|^2 \le K_R \Big(\|\hat{F}(\cdot,s)\|^2 + \|\hat{F}(\cdot,s)\|_{L_1}^2 \Big) \; . \tag{2.10}$$

b) If $s = i\xi, |\xi| \ge 1$, then we have

$$\|\hat{u}(\cdot,s)\|^2 + \|\hat{u}_x(\cdot,s)\|^2 \le K_R \|\hat{F}(\cdot,s)\|^2 \; . \tag{2.11}$$

We will also need estimates for higher space derivatives of $\hat{u}$ in the L_2 norm.

Lemma 2.6 *There is a constant K_k with*

$$\|\hat{u}(\cdot,s)\|^2_{H^{k+1}} \le K_k\Big(\|\hat{F}(\cdot,s)\|^2_{H^k} + \|\hat{F}(\cdot,s)\|^2_{L_1}\Big) \quad \textit{if} \quad s = i\xi, \quad |s| \le 1 \;, \tag{2.12}$$

and

$$\|\hat{u}(\cdot,s)\|^2_{H^{k+1}} \le K_k\|\hat{F}(\cdot,s)\|^2_{H^k} \quad \textit{if} \quad s = i\xi, \quad |s| \ge 1 \;. \tag{2.13}$$

Proof. a) Let $|s| \le 1$. From

$$s\hat{u} + A_x\hat{u} + A\hat{u}_x = \hat{u}_{xx} - \hat{F}_x \tag{2.14}$$

we obtain that

$$\|\hat{u}_{xx}\|^2 \le C\Big(\|\hat{F}_x\|^2 + \|\hat{u}\|^2 + \|\hat{u}_x\|^2\Big) \;.$$

For $\|\hat{u}\|^2 + \|\hat{u}_x\|^2$ we use the bound (2.10). This proves (2.12) for $k = 1$. Estimates for $\|\hat{u}_{xxx}\|$ etc. follow in the same way by differentiating (2.14) repeatedly w.r.t. x.

b) Let $|s| \ge 1$. Differentiating (2.14) we obtain

$$s\hat{u}_x + (A\hat{u}_x)_x = \hat{u}_{xxx} - \hat{G}_x$$

with

$$\hat{G} = \hat{F} + A_x\hat{u} \;.$$

Thus the function $\hat{u}_x$ satisfies an equation of the same form as $\hat{u}$ does, with $\hat{F}$ replaced by $\hat{G}$. We obtain that

$$\|\hat{u}_{xx}\|^2 \le K_R\|\hat{G}\|^2 \le C\Big(\|\hat{F}\|^2 + \|\hat{F}_x\|^2\Big) \;.$$

Estimates for higher x–derivatives of $\hat{u}$ follow in the same way. ■

We now use Parseval's relation and the previous lemma to bound space–time integrals of u and its space derivatives in terms of space–time integrals of F and its space derivatives. We have

$$\begin{aligned}
\int_0^\infty \|u(\cdot,t)\|^2_{H^{k+1}}\,dt &= \frac{1}{2\pi}\int_{-\infty}^\infty \|\hat{u}(\cdot,i\xi)\|^2_{H^{k+1}}\,d\xi \\
&\le \frac{K_k}{2\pi}\int_{-\infty}^\infty \|\hat{F}(\cdot,i\xi)\|^2_{H^k}\,d\xi + \frac{K_k}{2\pi}\int_{-1}^1 \|\hat{F}(\cdot,i\xi)\|^2_{L_1}\,d\xi \\
&= K_k\int_0^\infty \|F(\cdot,t)\|^2_{H^k}\,dt + \frac{K_k}{2\pi}\int_{-1}^1 \|\hat{F}(\cdot,i\xi)\|^2_{L_1}\,d\xi \;.
\end{aligned} \tag{2.15}$$

The last integral can be estimated as follows: We have, for all $x \in \mathbb{R}$,

$$\begin{aligned}|\hat{F}(x,i\xi)| &= |\int_0^\infty e^{-i\xi t} F(x,t)\,dt| \\ &\le \int_0^\infty |F(x,t)|\,dt\end{aligned}$$

and integration in x yields,

$$\|\hat{F}(\cdot,i\xi)\|_{L_1} \le \int_0^\infty \|F(\cdot,t)\|_{L_1}\,dt\ .$$

If we square both sides of this estimate and then integrate over $-1 \le \xi \le 1$, we obtain from (2.15),

$$\int_0^\infty \|u(\cdot,t)\|^2_{H^{k+1}}\,dt \le K_k \int_0^\infty \|F(\cdot,t)\|^2_{H^k}\,dt + K_k\Big(\int_0^\infty \|F(\cdot,t)\|_{L_1}\,dt\Big)^2 \qquad (2.16)$$

where the constant K_k is independent of F.

Let $0 < T < \infty$ denote any fixed finite time. We can use a simple cut–off process for F and note that the solution $u(x,t)$ of (2.9) remains unchanged for $t < T$ if we alter $F(x,t)$ for $t > T$. Therefore, (2.16) yields the following result:

Lemma 2.7 *Let $u(x,t)$ denote the solution of (2.9). For every $k = 1,2,\dots$ there is a constant K_k, independent of T and F, so that*

$$\int_0^T \|u(\cdot,t)\|^2_{H^{k+1}}\,dt \le K_k \int_0^T \|F(\cdot,t)\|^2_{H^k}\,dt + K_k\Big(\int_0^T \|F(\cdot,t)\|_{L_1}\,dt\Big)^2 \qquad (2.17)$$

It is easy to extend this result and also include estimates of u_t. We have

$$u_t = -A_x u - A u_x + u_{xx} - F_x\ .$$

If we differentiate this equation $k-1$ times w.r.t. x, we note that we can bound $\|D^{k-1}u_t\|^2$ in terms of $\|u\|^2_{H^{k+1}}$ and $\|D^k F\|^2$. Therefore, we obtain the result formulated below in Theorem 2.1. To measure the solution u and the inhomogeneous term F, we use the following notations:

Notations:

$$\begin{aligned}U(u,k,T) &= \int_0^T \Big(\|u(\cdot,t)\|^2_{H^{k+1}} + \|u_t(\cdot,t)\|^2_{H^{k-1}}\Big)\,dt \\ R(F,k,T) &= \int_0^T \|F(\cdot,t)\|^2_{H^k}\,dt + \Big(\int_0^T \|F(\cdot,t)\|_{L_1}\,dt\Big)^2\end{aligned}$$

The functionals $U(\cdot,k,T)$ and $R(\cdot,k,T)$ are space–time measures, up to time T, for the solution u and the right–hand side F, respectively.

Theorem 2.1 *Let $u(x,t)$ denote the solution of (2.9). For every $k = 1, 2, \ldots$ there is a constant K_k, independent of T and F, so that*

$$U(u,k,T) \le K_k R(F,k,T) \ . \tag{2.18}$$

Remark: Global existence for the linear problem considered above is well–known, and its solution can grow at most exponentially in time. Therefore, the formal process of Laplace transformation in t is justified for $s = \eta + i\xi$ if η is sufficiently large. Then, when inverting the Laplace transform, our estimates show that no singularties are encountered for $\eta \ge 0$, and therefore the contour of integration can be deformed to $\eta = 0$. This justifies the formal use of the Laplace transform in t and the choice $\eta = 0$ in deriving solution estimates.

2.3 Nonlinear Stability

The stability proof is similar to the one given in [6] and [5]. For simplicity, we suppress the dependence of the functions $F(x,t), B(x,t)$, and $g(x,t,u)$ on ε in our notation since all bounds that we use for these functions hold uniformly in ε for $|\varepsilon| \le 1$. With $u(x,t)$ we always denote the solution of (2.8).

In this section, k denotes a fixed integer, $k \ge 3$. To prove the decay estimate (1.6), it would suffice to choose $k = 3$.

By Lemma 2.5 there is a constant M_k with

$$R(F,k,\infty) = \int_0^\infty \|F\|^2_{H^k}\, dt + \Big(\int_0^\infty \|F\|_{L_1}\, dt\Big)^2 \le M_k < \infty \ .$$

Set

$$\kappa_k = 1 + 2K_k M_k$$

where K_k is the constant in (2.18).

Theorem 2.2 *Let $k \ge 3$ be a fixed integer. There is a positive number $\varepsilon_0 = \varepsilon_0(k) > 0$ with the following property: If $|\varepsilon| \le \varepsilon_0$ then the solution $u(x,t)$ of (2.8) exists for all $t \ge 0$ and satisfies*

$$U(u,k,T) < \kappa_k \quad \textit{for all} \quad T \ge 0 \ . \tag{2.19}$$

The proof is given in several steps. Local existence (in time) of a solution $u(x,t)$ of the nonlinear problem (2.8) is well–known. We first fix ε with $|\varepsilon| \le 1$ and suppose that there exists a first time $T = T(\varepsilon,k)$ with

$$U(u,k,T) = \kappa_k \ .$$

We regard the terms $\varepsilon(Bu)_x$ and $\varepsilon(g)_x$ as part of the forcing, and Theorem 2.1 yields

$$U(u,k,T) \le K_k R(F + \varepsilon Bu + \varepsilon g, k, T) \ .$$

Here the term $R(F+\varepsilon Bu+\varepsilon g,k,T)$ can be estimated as follows:

$$\begin{aligned} R(F+\varepsilon Bu+\varepsilon g,k,T) &\leq 2R(F,k,T)+4\varepsilon^2\Big(\int_0^T \|Bu\|^2_{H^k}\,dt+\int_0^T \|g\|^2_{H^k}\,dt\Big) \\ &\quad + 4\varepsilon^2\Big(\int_0^T \|Bu\|_{L_1}\,dt\Big)^2+4\varepsilon^2\Big(\int_0^T \|g\|_{L_1}\,dt\Big)^2 . \end{aligned}$$

To summarize, if there is a time T with $U(u,k,T)=\kappa_k$, then we have

$$\begin{aligned} 1+2K_kM_k &= \kappa_k \\ &= U(u,k,T) \\ &\leq K_kR(F+\varepsilon Bu+\varepsilon g,k,T) \\ &\leq 2K_kM_k+4\varepsilon^2(X_1+X_2+X_3+X_4) \end{aligned} \tag{2.20}$$

where the four terms X_j read as follows:

$$\begin{aligned} X_1 &= \int_0^T \|Bu\|^2_{H^k}\,dt \\ X_2 &= \int_0^T \|g\|^2_{H^k}\,dt \\ X_3 &= \Big(\int_0^T \|Bu\|_{L_1}\,dt\Big)^2 \\ X_4 &= \Big(\int_0^T \|g\|_{L_1}\,dt\Big)^2 \end{aligned}$$

Assuming the equality $U(u,k,T)=\kappa_k$, we will prove below that each of the four terms X_j can be estimated in terms of the constant κ_k:

$$\sum_j X_j \leq \Phi_k(\kappa_k) .$$

Then the above inequality (2.20) yields that

$$1 \leq 4\varepsilon^2\Phi_k(\kappa_k) .$$

Consequently, if we make the smallness assumption

$$4\varepsilon^2\Phi_k(\kappa_k) < 1$$

then a time T with $U(u,k,T)=\kappa_k$ cannot exist and (2.19) holds.

Before we estimate the four terms X_j separately, we note the following maximum norm estimates:

Lemma 2.8 *a) We have*

$$\sup\{|u(x,t)|^2 \;:\; x\in\mathbb{R},\; 0\le t\le T\} \le \int_0^T \left(\|u(\cdot,t)\|_{H^1}^2 + \|u_t(\cdot,t)\|_{H^1}^2\right) dt \;. \tag{2.21}$$

b) For every $j=1,2,\dots$ we have

$$\sup\{|D_x^j u(x,t)|^2 \;:\; x\in\mathbb{R},\; 0\le t\le T\} \le U(u,j+2,T) \;. \tag{2.22}$$

Proof. By the Sobolev inequality with respect to t we have, for each x,

$$\max_{0\le t\le T} |u(x,t)|^2 \le \int_0^T \left(|u(x,t)|^2 + |u_t(x,t)|^2\right) dt \;.$$

Maximizing over x and using the Sobolev inequality

$$\sup_x |u(x,t)|^2 \le \|u(\cdot,t)\|_{H^1}^2$$

the claim (2.21) follows. The estimate (2.22) follows by applying (2.21) to $D_x^j u$ instead of u. ∎

Let us note the following implication of (2.21): For the solution $u(x,t)$ of (2.8) we have

$$|u(x,t)| \le \gamma := \sqrt{\kappa_k} \quad \text{for} \quad 0\le t\le T$$

and therefore the estimates for the nonlinear term g (see Lemma 2.4) can be used with $\gamma=\sqrt{\kappa_k}$.

We now consider the four terms X_j separately:

Estimate of X_1. Using (2.3) we have $\|Bu\|_{H^k} \le C_k\|u\|_{H^k}$, thus

$$\begin{aligned} X_1 &= \int_0^T \|Bu\|_{H^k}^2\, dt \\ &\le C_k U(u,k,T) \\ &= C_k\kappa_k \;. \end{aligned}$$

Estimate of X_2. For $0\le j\le k$ the derivative

$$\frac{\partial^j}{\partial x^j}\Big\{g(x,t,u(x,t))\Big\}$$

is a sum of terms of the form

$$(D_x^\alpha D_u^\beta g)\cdot D_x^{\sigma_1}u(x,t)\dots D_x^{\sigma_r}u(x,t) \tag{2.23}$$

where

$$\alpha+\sigma_1+\ldots+\sigma_r = j \quad \text{and} \quad \sigma_i\ge 1 \quad \text{for all} \quad i \;.$$

The argument of the term $D_x^\alpha D_u^\beta g$ is $(x,t,u(x,t))$.

a) If $r=0$ then also $\beta=0$ in (2.23), and we have

$$|D_x^\alpha g(x,t,u)| \le K_\alpha(\gamma)|u|$$

by (2.6). Therefore,

$$\int_0^T \int |D_x^\alpha g|^2 \, dxdt \le K_\alpha^2(\gamma) U(u,k,T) \ .$$

b) Consider a term (2.23) with $r=1$. Using (2.7) we have

$$\int_0^T \int |D_x^\alpha D_u^\beta g \, D_x^{\sigma_1} u|^2 \, dxdt \le K_{\alpha\beta}^2(\gamma) U(u,k,T) \ .$$

c) Consider a term (2.23) with $r \ge 2$. We again use (2.7) to bound the g–term in maximum norm. Also, we may assume that σ_1 and σ_2 are the two largest σ–values in (2.23). If

$$\sigma_1 \ge k-1 \quad \text{and} \quad \sigma_2 \ge k-1$$

then we would have

$$k \ge \sigma_1 + \sigma_2 \ge 2k-2$$

which contradicts our assumption $k \ge 3$. Therefore, every factor in (2.23), with at most one exception, can be estimated in maximum norm in terms of κ_k.

This implies that all terms (2.23) can be estimated in terms of κ_k.

Estimate of X_3. Using (2.4) we obtain,

$$\begin{aligned} X_3 &= \Big(\int_0^T \int |B(x,t)||u(x,t)| \, dxdt \Big)^2 \\ &\le \int_0^T \int |B(x,t)|^2 dxdt \int_0^T \|u(\cdot,t)\|^2 \, dt \\ &\le KU(u,k,T) \\ &= K\kappa_k \end{aligned}$$

Estimate of X_4. Using (2.5) with $\gamma = \sqrt{\kappa_k}$, we have

$$\begin{aligned} X_4 &= \Big(\int_0^T \int |g(x,t,u(x,t))| \, dxdt \Big)^2 \\ &\le K^2(\gamma) \Big(\int_0^T \int |u(x,t)|^2 dxdt \Big)^2 \\ &\le K^2(\gamma) U^2(u,k,T) \\ &\le K^2(\gamma)\kappa_k^2 \end{aligned}$$

These estimates of the four terms X_j prove that a bound of the form $\sum_j X_j \leq \Phi_k(\kappa_k)$ holds of one assumes that $U(u,k,T) = \kappa_k$. As noted above, this completes the proof of Theorem 2.2.

Proof of Theorem 1.1: Recall that we made the transfomation

$$v(x,t) = U(x) + \varepsilon\Big(e^{-t}v_{0x}(x) + u(x,t)\Big) \tag{2.24}$$

where $v(x,t)$ is the solution of the original perturbed problem (1.1), (1.2) and $u(x,t)$ is the solution of the transformed problem (2.8). If $|\varepsilon| \leq \varepsilon_0(k)$, where $k \geq 3$ is fixed and $v_0(x)$ satisfies (1.3), (1.4), then Theorem 2.2 yields

$$\int_0^\infty \Big(\|u(\cdot,t)\|^2_{H^{k+1}} + \|u_t(\cdot,t)\|^2_{H^{k-1}}\Big)\,dt \leq \kappa_k \ .$$

The estimate (1.5) then follows from the transformation formula (2.24). Finally, the decay estimate (1.6) follows from (1.5) by Sobolev's inequality: From (1.5) we have

$$I(T) := \int_T^\infty \Big(\|v(\cdot,t) - U(\cdot)\|^2_{H^{k+1}} + \|v_t(\cdot,t)\|^2_{H^{k-1}}\Big)\,dt \to 0 \quad \text{as} \quad T \to \infty \ .$$

As in the proof of Lemma 2.8 one can show that

$$\sup_x \sup_{t \geq T} |v(x,t) - U(x)|^2 \leq I(T) \ .$$

3 Reduction and Tail Problems

In this section we consider the resolvent equation

$$u_{xx} - (A(x)u)_x - su = F_x, \quad x \in \mathbb{R} \tag{3.1}$$

for small values of $|s|$ and $Re(s) \geq 0, s \neq 0$. Two major steps in proving the resolvent estimate (1.12) are the following:

1. It is sufficient to consider equation (3.1) with a small right hand side sw, where $w \in L_1$ can be estimated in terms of F. This reduction is carried out in Section 3.1.
2. Solutions of (3.1) with reduced right hand side sw can be estimated for tail problems, i.e., for $|x| \geq l-1$, l suffciently large, and with appropriate boundary conditions at $l-1$. Details are given in Section 3.2.

For both steps we use the technique of exponential dichotomies [3]. Consider a finite or infinite subinterval $J \subset \mathbb{R}$ and let

$$Lz = z_x - M(x)z, \quad x \in J \ , \tag{3.2}$$

denote a linear differential operator where $M(x) \in \mathbb{R}^{N,N}$ is a matrix function, continuous in $x \in J$. By $S(x,\xi), x, \xi \in J$ we denote the solution operator of L, i.e., the solution of

$$z_x - M(x)z = h(x), \quad z(x_0) = z_0$$

is given by

$$z(x) = S(x,x_0)z_0 + \int_{x_0}^{x} S(x,\xi)h(\xi)d\xi.$$

Definition 3.1 *The operator L has an* **exponential dichotomy on J with data** (β, K, π) *if $\beta > 0, K > 0$ are real numbers and $\pi(x), x \in J$, are projectors in $\mathbb{R}^N$ such that for all $x, \xi \in J$ the following holds:*

$$\pi(x)S(x,\xi) = S(x,\xi)\pi(\xi), \tag{3.3}$$

$$|S(x,\xi)\pi(\xi)| \leq Ke^{-\beta(x-\xi)}, \quad x \geq \xi, \tag{3.4}$$

$$|S(x,\xi)(I-\pi(\xi))| \leq Ke^{\beta(x-\xi)}, \quad x < \xi. \tag{3.5}$$

Relevant properties of exponential dichotomies, such as persistence under L_1-perturbations of the matrix function $M(x)$ and consequences for the solution of inhomogeneous systems $Lz = h$, will be summarized in Appendices A and B. There we will derive or cite corresponding results and also use more refined notions than that of an exponential dichotomy, called **generalized exponential dichotomy** and **exponential polychotomy**. Essentially, these notions replace the numbers $-\beta, \beta$ that appear in (3.4),(3.5) by general intervals $\alpha \leq \beta$ or by a collection of intervals. However, since these refinements are not needed for the proof of the resolvent estimate we do not use them in the main body of the text.

3.1 The Reduction Step

It will be convenient to introduce the quantity (see (1.15))

$$\varepsilon(l) = \int_{l}^{\infty} |A(x) - A_L|\,dx + \int_{-\infty}^{-l} |A(x) - A_R|\,dx\ .$$

Smallness requirements for $\varepsilon(l)$ are, effectively, size conditions for l in the following proofs. First consider the system

$$w_x - A(x)w = F, \tag{3.6}$$

which is formally obtained from (3.1) by neglecting the su-term and integrating.

Proposition 3.1 *The linear operator $Lz = z_x - A(x)z$ has an exponential dichotomy on both, $[0,\infty)$ and $(-\infty,0]$, with data (β, K_-, π_-) and (β, K_+, π_+), respectively. The projector π_- has rank $k-1$ and π_+ has rank k. There exists a constant C_0 such that, for any l sufficiently large,*

$$\sup_{x\leq -l} |\pi_-(x) - \pi^L| + \sup_{x\geq l} |\pi_+(x) - \pi^R| \leq C_0\varepsilon(l). \tag{3.7}$$

Here π^L and π^R are the projectors in $\mathbb{R}^n$ associated with the constant matrices A_L and A_R as follows (cf.Assumption **(A2)***)*

$$\pi^R = S_R \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} S_R^{-1}, \quad \pi^L = S_L \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix} S_L^{-1}. \tag{3.8}$$

If $F \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ then equation (3.6) has a solution $w \in W^{1,1}(\mathbb{R}) \cap W^{1,2}(\mathbb{R})$ that satisfies

$$||w||_{L_p} \leq C||F||_{L_p}, \quad for \quad p = 1, 2. \tag{3.9}$$

Proof. Due to assumption **(A1)** the constant coefficient operators $z \mapsto z_x - A_L z$ and $z \mapsto z_x - A_R z$ have exponential dichotomies on $\mathbb{R}$ with data (β, K, π^L) and (β, K, π^R), respectively. Choose l_1 so that $K^2\varepsilon(l_1) \leq \frac{1}{2}$. Then Theorem A.1 ensures that the operator L has exponential dichotomies on $(-\infty, l_1]$ and $[l_1, \infty)$, respectively, and (3.7) follows from A(1.11) with $C_0 = \frac{1}{2}K^2$. Moreover, according to Proposition A.1, the ranks of the projectors are $k_- = k - 1$ and $k_+ = k$, respectively. By Remark 3 preceding Theorem A.1 we can extend the dichotomies from $(-\infty, l_1]$ and $[l_1, \infty)$ to the intervals $(-\infty, 0]$ and $[0, \infty)$, which yields new constants $K_\pm$ depending on l_1. Theorem A.3 guarantees that the operator $L : W^{1,p}(\mathbb{R}) \mapsto L_p(\mathbb{R})$ is Fredholm of index 1. From Assumption **(A5)** we obtain that φ_0 is the only L_2-function in the kernel of L. Therefore,

$$range(\pi_+(0)) \cap ker(\pi_-(0)) = span\{\varphi_0(0)\} \ ,$$

and the second assertion of Theorem A.3 also applies. Hence all solutions of (3.6) are of the form $w + c\varphi_0, c \in \mathbb{R}$, and the estimate (3.9) follows from A(1.29) with $p' = p$. ■

Remark When estimating the solution of finite interval problems in section 4, we will use (3.7) to determine the size of l. It is important to note that the estimate (3.9) does not depend on this later choice of l; the estimate (3.9) can be obtained, as explained above, with a fixed value of l_1 satisfying $K^2\varepsilon(l_1) \leq \frac{1}{2}$.

Let w be a (special) solution of (3.6), satisfying (3.9), and let u be a solution of (3.1). Then $u_1 = u - w$ solves

$$u_{1xx} - (Au_1)_x - su_1 = sw$$

where the right hand side, sw, is in L_1 and is small since $|s|$ is small. In the next section we consider this inhomogeneous problem, with $h \in L_1$ replacing sw, and for tail problems $|x| \geq l - 1$.

3.2 Tail Problems

Let us rewrite the resolvent equation

$$u_{xx} - (Au)_x - su = h \tag{3.10}$$

as a first order system: With $v = u_x - Au, z = (u, v)$ we have

$$L(s)z = z_x - M(x, s)z = \begin{pmatrix} 0 \\ h \end{pmatrix}, \quad M(x, s) = \begin{pmatrix} A(x) & I \\ sI & 0 \end{pmatrix}. \tag{3.11}$$

Define the limit matrices obtained as $x \to \pm\infty$

$$M_R(s) = \begin{pmatrix} A_R & I \\ sI & 0 \end{pmatrix}, \quad M_L(s) = \begin{pmatrix} A_L & I \\ sI & 0 \end{pmatrix}. \tag{3.12}$$

In the following we investigate the dichotomy properties of the constant– coefficient, but $s-$ $-$dependent, differential operators

$$L_R(s)z = z_x - M_R(s)z, \quad L_L(s)z = z_x - M_L(s)z. \tag{3.13}$$

As in assumption **(A2)** we order the eigenvalues of A_R,

$$\lambda_1 \leq \ldots \leq \lambda_k < 0 < \lambda_{k+1} \leq \ldots \leq \lambda_n.$$

Lemma 3.1 *For any $s \in \mathbb{C}$ satisfying (1.10) each eigenvalue λ_j of A_R leads to two eigenvalues of $M_R(s)$ given by*

$$\kappa_{1,j}(s) = \frac{\lambda_j}{2} + \sqrt{\frac{\lambda_j^2}{4} + s}, \quad \kappa_{2,j}(s) = \frac{\lambda_j}{2} - \sqrt{\frac{\lambda_j^2}{4} + s}\ . \tag{3.14}$$

These are the roots of the quadratic

$$\kappa^2 - \lambda_j \kappa - s = 0$$

and they satisfy $\lambda_j \mathrm{Re}\, \kappa_{1j}(s) > 0, \lambda_j \mathrm{Re}\, \kappa_{2j}(s) < 0$.

The transformation $T_R(s)$ defined by

$$T_R(s) = \begin{pmatrix} S_R & S_R \\ sS_R\mathcal{K}_1^{-1} & sS_R\mathcal{K}_2^{-1} \end{pmatrix}, \quad \mathcal{K}_1 = diag(\kappa_{1j}),\ \mathcal{K}_2 = diag(\kappa_{2j}) \tag{3.15}$$

diagonalizes $M_R(s)$:

$$T_R^{-1}(s)M_R(s)T_R(s) = \begin{pmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{pmatrix}. \tag{3.16}$$

The operator $L_R(s)$ has an exponential dichotomy on $\mathbb{R}$ with suitable data $(\beta(s), K(s), \pi_{1,2}^R(s))$ where the projectors satisfy

$$\pi_{1,2}^R(s) = T_R(s) \begin{pmatrix} I_k & & & \\ & 0 & & \\ & & 0 & \\ & & & I_{n-k} \end{pmatrix} T_R(s)^{-1}. \tag{3.17}$$

There exist positive constants δ_R, β_R, K_R such that $K(s) \leq K_R$ for all $|s| \leq \delta_R$ and

$$\kappa_{1j}(s) = \lambda_j + \mathcal{O}(|s|), \quad \kappa_{2j}(s) = -\frac{s}{\lambda_j} + \mathcal{O}(|s|^2), \quad |\mathrm{Re}\, \kappa_{2j}(s)| \geq \beta_R |s|^2 = \beta(s). \tag{3.18}$$

The transformation $T_R(s)$ satisfies

$$T_R(s) = \begin{pmatrix} S_R & S_R \\ 0 & -S_R\Lambda_R \end{pmatrix} + \mathcal{O}(|s|), \quad T_R^{-1}(s) = \begin{pmatrix} S_R^{-1} & \Lambda_R^{-1}S_R^{-1} \\ 0 & -\Lambda_R^{-1}S_R^{-1} \end{pmatrix} + \mathcal{O}(|s|). \tag{3.19}$$

Proof. The first assertions hold for all $s \neq 0, \mathrm{Re}\, s \geq 0$. A computation shows that the real part of $\kappa_{1j}(s)$ and λ_j have the same sign; the real part of $\kappa_{2j}(s)$ and λ_j have opposite signs. With T_R defined by (3.15) one then verifies (3.16) by using the equality $S_R(\mathcal{K}_\nu^2 - \Lambda_R \mathcal{K}_\nu - sI) = 0, \nu = 1, 2$. The decaying modes belong to the first k eigenvalues $\kappa_{1j} (j = 1, \dots, k)$ and to the last $n - k$ eigenvalues $\kappa_{2j} (j = k + 1, \dots, n)$, which shows that (3.17) is the right projector.

The estimates (3.18) for small $|s|$ follow from a Taylor expansion of the eigenvalues with respect to s up to orders $1, 2$, and 3, respectively. This proves that the dichotomy constants are of form $\beta(s) = \beta_R |s|^2$. The s-dependence of $\mathcal{K}_2$ then leads to the formulas (3.19), and we obtain that the constant $K(s)$ can be chosen independently of s. ■

The calculation shows that for small s there are four types of solutions of the homogeneous system $L_R z = 0$:

strong decay	$\kappa_{1j}, j = 1, \dots, k,$
weak decay	$\kappa_{2j}, j = k + 1, \dots, n$
weak growth	$\kappa_{2j}, j = 1, \dots, k$
strong growth	$\kappa_{1j}, j = k + 1, \dots, n.$

For an illustration see Figures 1 and 2. In Appendix B we make this precise by showing that $L_R(s)$ has an exponential polychotomy with exponents

$$\alpha_1 = -C_1 < \beta_1 = -C_2|s| < \alpha_2 = -C_3|s|^2 < \beta_2 = C_3|s|^2 < \alpha_3 = C_4|s| < \beta_3 = C_5 \tag{3.20}$$

and associated projectors

$$\pi_1^R(s) = T_R(s) \begin{pmatrix} I_k & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} T_R(s)^{-1}, \quad \pi_2^R(s) = T_R(s) \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & I_{n-k} \end{pmatrix} T_R(s)^{-1}, \tag{3.21}$$

$$\pi_3^R(s) = T_R(s) \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & I_k & \\ & & & 0 \end{pmatrix} T_R(s)^{-1}, \quad \pi_4^R(s) = T_R(s) \begin{pmatrix} 0 & & & \\ & I_{n-k} & & \\ & & 0 & \\ & & & 0 \end{pmatrix} T_R(s)^{-1}. \tag{3.22}$$

Notice that $\pi_{1,2}(s) = \pi_1(s) + \pi_2(s)$ holds with $\pi_{1,2}(s)$ defined in (3.17). Similarly, we write $\pi_{3,4}(s) = \pi_3(s) + \pi_4(s)$. In the case of M_L one simply replaces k by $k - 1$ and defines $\pi_j^L(s), j = 1, \dots, 4$, in a completely analogous fashion.

It is useful to evaluate the limit projectors in $\mathbb{R}^{2n}$,

$$\pi_{1,2}^R(0) = \lim_{s \to 0} \pi_{1,2}(s) \ , \tag{3.23}$$

and to relate them to the projector π^R in $\mathbb{R}^n$ from (3.8). We insert the expansion (3.19) into the formula (3.17) and, after a short computation, obtain the following Lemma.

Lemma 3.2 *With the matrices from Assumption* **A2** *let*

$$B_R = S_R \begin{pmatrix} {\Lambda_R^I}^{-1} & 0 \\ 0 & {\Lambda_R^{II}}^{-1} \end{pmatrix} S_R^{-1}.$$

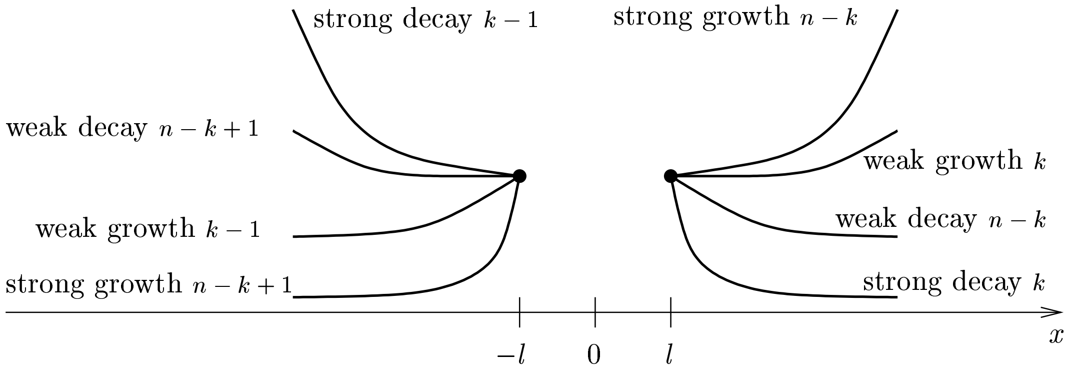

Figure 1: Illustration of the four types of exponential behavior for tail problems.

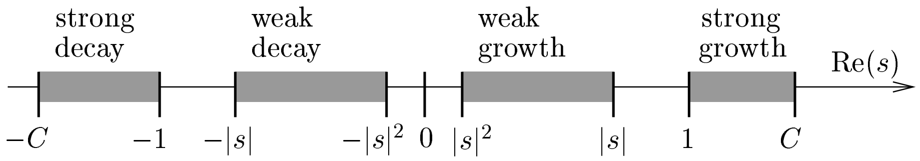

Figure 2: Separation of eigenvalues with respect to Re(s).

Then, for $\operatorname{Re}(s) \geq 0, s \neq 0$, $|s|$ *small, one has*

$$\pi_{1,2}^R(s) = \pi_{1,2}^R(0) + \mathcal{O}(|s|), \quad \pi_{1,2}^R(0) = \begin{pmatrix} \pi^R & -B_R \\ 0 & I_n - \pi^R \end{pmatrix} \tag{3.24}$$

and

$$\pi_{3,4}^R(s) = \pi_{3,4}^R(0) + \mathcal{O}(|s|), \quad \pi_{3,4}^R(0) = \begin{pmatrix} I_n - \pi^R & B_R \\ 0 & \pi^R \end{pmatrix}. \tag{3.25}$$

In the next theorem we study the dichotomy properties of the variable–coefficient and s–dependent operator $L(s)$ from (3.11) and compare with the projector from (3.17).

Theorem 3.1 *Let* K_R, δ_R, β_R *be the constants from Lemma 3.1 and choose* l *such that*

$$\varepsilon(l-1)K_R \leq \frac{1}{2}.$$

Then the operators $L(s), 0 < |s| \leq \delta_R, \operatorname{Re} s \geq 0$ *have exponential dichotomies on* $[l-1,\infty)$ *with data* $(\beta_R|s|^2, 2K_R, \pi_{1,2}(\cdot,s))$ *where*

$$\sup_{x \geq l-1} |\pi_{1,2}^R(s) - \pi_{1,2}(x,s)| \leq 2\varepsilon(l-1)K_R^2. \tag{3.26}$$

For any $h \in L_1[l-1,\infty)$ *the boundary value problem*

$$Lz = h \text{ in } [l-1,\infty), \quad \pi_{1,2}(l-1,s)z(l-1) = 0 \tag{3.27}$$

has a unique solution $z \in W^{1,1}[l-1,\infty)$. The solution satisfies $z \in L_p$ for $1 \le p \le \infty$; with some constant $C > 0$, independent of s and l,

$$||z||_{L_\infty} + |s|||z|| + |s|^2||z||_{L_1} \le C||h||_{L_1}. \tag{3.28}$$

Proof. For any $0 < |s| \le \delta_R, \operatorname{Re} s \ge 0$ we apply Theorem A.1 to $L = L_R$ with $\Delta(x) = \begin{pmatrix} A(x) - A_R & 0 \\ 0 & 0 \end{pmatrix}$. This yields the dichotomies as well as the estimate (3.26). In the second step we use Theorem A.2 with $\beta = \beta_R|s|^2$ and with the indices $p = 1$, $p' = 1, 2, \infty$. From A(1.25) we obtain the estimate (3.28). ∎

As a final consequence of Lemma 3.1 we notice that for general s with $\operatorname{Re} s \ge 0, s \ne 0$, the operator $L(s)$ has Fredholm properties.

Proposition 3.2 *For any s with $\operatorname{Re} s \ge 0, s \ne 0$, the operator $L(s)$ has exponential dichotomies on $[0,\infty)$ and on $(-\infty, 0]$, and is Fredholm of index 0 considered as an operator from $W^{1,p}(\mathbb{R})$ into $L_p(\mathbb{R}), 1 \le p \le \infty$.*

Proof. We do not quantify the dichotomy data but argue for any fixed value of s. Theorem A.1 shows that the exponential dichotomies hold on intervals $(-\infty, -l]$ and $[l, \infty)$ for l sufficiently large. By Remark 2. A.1 we can extend the dichotomies to $(-\infty, 0]$ and $[0, \infty)$ and by Proposition A.1 the projectors have rank n on both semi-intervals. An application of Theorem A.3 with $k_+ = k_- = n$, $N = 2n$ completes the proof. ∎

4 Estimates on Finite Intervals

We analyze the resolvent equation (3.10) for small values of $|s|$ and on intervals $(-l, l)$ such that Theorem 3.1 applies. By $\pi_{1,2}(x,s)$ and $\pi_{3,4}(x,s)$ we denote the projectors constructed in Lemma 3.1 for $x \ge l-1$, and also the corresponding projectors for the left tails, $x \le -l+1$. We assume $h \in L_1(-l,l)$ and consider the second order operator

$$P(s)u = (u_x - A(x)u)_x - su \tag{4.1}$$

on the domain $W^{2,1}(-l,l)$. Since $A(x) = f_u(U(x))$ is of class C^2 and bounded by Assumptions **(A1)**,**(A2)** we find that $u \in W^{2,1}(-l,l)$ is equivalent to $u \in W^{1,1}, u_x - Au \in W^{1,1}$. The operator $P(s)$ has the function φ_0 in its kernel at $s = 0$. Therefore, instead of (3.10), we consider the following regularized system:

$$P(s)u + \alpha\varphi_0 = h \text{ in } (-l,l), \quad \langle u, \varphi_0 \rangle = 0 \ . \tag{4.2}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(-l,l)$, and (4.2) will be solved for u and α.

Theorem 4.1 *There exist positive constants C, δ_0, l_1 such that, for all s, l with*

$$l|s| \le \delta_0, \quad l \ge l_1 \tag{4.3}$$

and for any $h \in L_1(-l,l)$, the boundary value problem (4.2) together with

$$\pi_{1,2}(-l,s)\begin{pmatrix} u(-l) \\ (u_x - Au)(-l)\end{pmatrix} = 0, \quad \pi_{3,4}(l,s)\begin{pmatrix} u(l) \\ (u_x - Au)(l)\end{pmatrix} = 0, \tag{4.4}$$

has a unique solution $u \in W^{2,1}(\mathbb{R}), \alpha \in \mathbb{R}$. The function u lies in $W^{1,\infty}(-l,l)$ and satisfies

$$||u||_{L_\infty} + ||u_x||_{L_\infty} + |\alpha| \le C||h||_{L_1}. \tag{4.5}$$

4.1 Estimates for the $s = 0$ Problem

For the proof of the theorem above we need two preparatory lemmata that deal with the $s = 0$ finite interval version of (4.2),(4.4). Consider first the reduced system

$$y_x - Ay = h, \; x \in [-l,l], \quad \langle y, \varphi_0 \rangle = 0 \tag{4.6}$$

$$\pi^L y(-l) = \eta_L, \quad (I - \pi^R)y(l) = \eta_R, \tag{4.7}$$

where the projectors π^R, π^L are defined in (3.8).

Lemma 4.1 *For any $h \in L_\infty(-l,l), l \ge l_1$, and any $\eta_R \in ker(\pi^R), \eta_L \in range(\pi^L)$, the boundary value problem (4.6),(4.7) has a unique solution $y \in W^{1,\infty}(-l,l)$, that satisfies*

$$||y||_{L_\infty} \le C(||h||_{L_\infty} + |\eta_R| + |\eta_L|). \tag{4.8}$$

Remark We note that $I - \pi^R$ is of rank $n-k$ while π^L is of rank $k-1$. Therefore, (4.7) contains only $n-1$ boundary conditions. This is compensated for by the orthogonality constraint in (4.6).

Proof. We apply Proposition 3.1 and obtain exponential dichotomies for L on $(-\infty, 0]$ and on $[0,\infty)$ with data (β, K_-, π_-) and (β, K_+, π_+), respectively, such that

$$|\pi_+(l) - \pi^R| + |\pi_-(-l) - \pi^L| \le C_0\varepsilon(l). \tag{4.9}$$

Moreover, as in the proof of Theorem A.3, we can modify the projectors π_- and π_+ (compare A(1.32)) such that

$$\mathbb{R}^n = V_0 \oplus V_+ \oplus V_-, \quad V_0 = range(\pi_+(0)) \cap ker(\pi_-(0)) = span(\varphi_0(0)), \tag{4.10}$$

$$range(\pi_+(0)) = V_0 \oplus V_+, \qquad ker(\pi_+(0)) = V_-, \tag{4.11}$$

$$ker(\pi_-(0)) = V_0 \oplus V_-, \qquad range(\pi_-(0)) = V_+. \tag{4.12}$$

With this choice of projectors we have $\dim V_+ = k-1$, $\dim V_- = n-k$, and for any $\gamma_- \in range(\pi_-(-l))$ and $\gamma_+ \in ker(\pi_+(l))$ the following estimate holds:

$$|S(x,-l)\gamma_-| \le e^{-\beta(x+l)}|\gamma_-|, \quad |S(x,l)\gamma_+| \le Ce^{-\beta(l-x)}|\gamma_+|, \quad x \in [-l,l]. \tag{4.13}$$

Because of the modification of the projectors, the estimate A(1.6) shows that (4.9) must be modified to read

$$|\pi_+(l) - \pi^R| + |\pi_-(-l) - \pi^L| \le C_0\varepsilon(l) + Ce^{-2\beta l}. \tag{4.14}$$

Now we proceed on the finite interval $(-l, l)$ as on the infinite interval $(-\infty, \infty)$ in A(1.35). To this end, let y_- and y_+ solve the boundary value problems

$$Ly_- = h \text{ in } [-l, 0], \quad \pi_-(-l)y_-(-l) = 0, \quad (I - \pi_-(0))y_-(0) = 0$$

$$Ly_+ = h \text{ in } [0, l], \quad (I - \pi_+(l))y_+(l) = 0, \quad \pi_+(0)y_+(0) = 0.$$

By Theorem A.3 (with $p' = \infty, p = 1$) we have the estimates $||y_\pm||_{L_\infty} \le C||h||_{L_1}$. By construction we have $y_+(0) \in V_-$ and $y_-(0) \in V_+$ and as in A(1.35) we define

$$y_{\rm sp}(x) = \begin{cases} y_-(x) + S(x, 0)y_+(0), & \text{for } -l \le x \le 0, \\ y_+(x) + S(x, 0)y_-(0), & \text{for } 0 \le x \le l. \end{cases} \tag{4.15}$$

Then $y_{\rm sp}$ is continuous at 0, solves the inhomogeneous equation $Ly = h$ on $(-l, l)$ and, by Theorem A.2, satisfies $||y_{\rm sp}||_{L_\infty} \le C||h||_{L_1}$ with a constant C independent of $l \ge l_1$.

In view of the decomposition (4.10), any solution y of $Ly = h$ can be written as follows (for suitable $\alpha \in \mathbb{R}, \gamma_- \in range(\pi_-(-l)), \gamma_+ \in ker(\pi_+(l))$):

$$y = y_{\rm sp} + y_{\rm hom}, \quad y_{\rm hom}(x) = \alpha\varphi_0(x) + S(x, l)\gamma_+ + S(x, -l)\gamma_- \tag{4.16}$$

We insert this expression into the boundary conditions and the orthogonality constraint and obtain

$$\begin{aligned} \eta_R = (I - \pi^R)y(l) &= (I - \pi^R)\gamma_+ + (I - \pi^R)S(l, -l)\pi_-(-l)\gamma_- + \alpha(I - \pi^R)\varphi_0(l) + (I - \pi^R)y_{\rm sp}(l), \\ \eta_L = \pi^L y(-l) &= \pi^L S(-l, l)\pi_+(l)\gamma_+ + \pi^L\gamma_- + \alpha\pi^L\varphi_0(-l) + \pi^L y_{\rm sp}(-l), \\ 0 = \langle y, \varphi_0\rangle &= \langle S(\cdot, l)\pi_+(l)\gamma_+, \varphi_0\rangle + \langle S(\cdot, -l)\pi_-(-l)\gamma_-, \varphi_0\rangle + \alpha\langle\varphi_0, \varphi_0\rangle + \langle y_{\rm sp}, \varphi_0\rangle. \end{aligned}$$

Using the previous estimates, in particular (4.13) and $|\varphi_0(x)| \le Ce^{-\beta|x|}$, we end up with a linear system for $(\gamma_+, \gamma_-, \alpha)$ of the form

$$\begin{pmatrix} I - \pi^R & \mathcal{O}(e^{-\beta l}) & \mathcal{O}(e^{-\beta l}) \\ \mathcal{O}(e^{-\beta l}) & \pi^L & \mathcal{O}(e^{-\beta l}) \\ \mathcal{O}(1) & \mathcal{O}(1) & \langle\varphi_0, \varphi_0\rangle \end{pmatrix} \begin{pmatrix} \gamma_+ \\ \gamma_- \\ \alpha \end{pmatrix} = \begin{pmatrix} \eta_R \\ \eta_L \\ 0 \end{pmatrix} - \begin{pmatrix} (I - \pi^R)y_{\rm sp}(l) \\ \pi^L y_{\rm sp}(-l) \\ \langle y_{\rm sp}, \varphi_0\rangle \end{pmatrix}. \tag{4.17}$$

Because of (4.14) the mappings $I - \pi^R : ker(\pi_+(l)) \mapsto ker(I - \pi^R)$ and $\pi^L : range(\pi_-(-l)) \mapsto range(\pi^L)$ have uniformly bounded inverses for l sufficiently large. Therefore, the matrix in (4.17) has a uniformly bounded inverse, and we obtain the estimate:

$$\begin{aligned} |\gamma_-| + |\gamma_+| + |\alpha| &\le C(|\eta_R| + |\eta_L| + |y_{\rm sp}(-l)| + |y_{\rm sp}(l)| + ||y_{\rm sp}||_{L_\infty}) \\ &\le C(|\eta_R| + |\eta_L| + ||h||_{L_\infty}). \end{aligned}$$

Combining this with (4.13) and (4.16) proves our assertion. ■

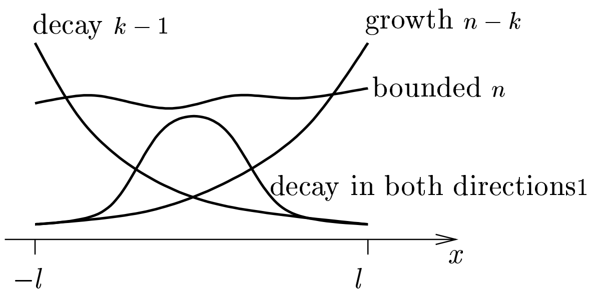

Figure 3: Illustration of homogeneous solutions on a finite interval at $s = 0$.

We interpret the result of this lemma and of Theorem A.3 for the homogeneous $s = 0$ system of dimension $2n$:

$$u_x - Au - v = 0, \quad v_x = 0. \tag{4.18}$$

This system has the followings sets of linearly independent solutions:

- $k-1$ solutions that decay on $\mathbb{R}$ and on any interval $(-l, l)$ with initial values in V_+
- $n-k$ growing solutions with initial value in V_-
- 1 solution that decays in both directions with initial value in V_0.

These solutions are obtained by setting $v = 0$ and using Lemma 4.1 as well as (4.10)-(4.12) for the u-part. In addition, there are n linearly independent bounded solutions $v \equiv const$ where u is a suitable bounded solution of $u_x - Au - v = 0$; see Theorem A.3. This behavior is illustrated in Figure 3.

Our task is to analyze the perturbation of these solutions when $s \neq 0$ and to properly match them with the solutions of the tail problems in Figure 1.

Since the homogeneous system has the nontrivial L_2-solution $u = \varphi_0, v = 0$ we consider a regularized finite interval boundary value problem of dimension $2n$ at $s = 0$:

$$\begin{aligned} u_x - Au - v &= 0, \quad \pi^R_{3,4}(0)\begin{pmatrix} u \\ v \end{pmatrix}(l) = \gamma_R & (4.19)\\ v_x + \alpha\varphi_0 &= h \quad \pi^L_{1,2}(0)\begin{pmatrix} u \\ v \end{pmatrix}(-l) = \gamma_L, & (4.20)\\ \langle u, \varphi_0\rangle &= 0. & (4.21) \end{aligned}$$

The projectors are given as the limits determined in (3.25) and in (3.24) (with the index R replaced by L). The assumption **(A3)** will be crucial for the following lemma.

Lemma 4.2 *There exist constants $C, l_1 > 0$ such that for any $l \geq l_1$ and any $h \in L_1[-l, l]$, $\gamma_R \in range(\pi^R_{3,4}(0)), \gamma_L \in range(\pi^L_{1,2}(0))$ the boundary value problem (4.19)-(4.21) has a unique solution $u \in W^{1,\infty}(-l, l), v \in W^{1,1}(-l, l), \alpha \in \mathbb{R}$, and this solution satisfies*

$$||u||_{L_\infty} + ||v||_{L_\infty} + |\alpha| \leq C(||h||_{L_1} + |\gamma_R| + |\gamma_L|). \tag{4.22}$$

Proof. Let us write $\gamma_R = \begin{pmatrix} \varrho_R \\ \sigma_R \end{pmatrix}$ and $\gamma_L = \begin{pmatrix} \varrho_L \\ \sigma_L \end{pmatrix}$. Due to the triangular block structure of the projectors $\pi^R_{3,4}(0)$ and $\pi^L_{1,2}(0)$ in (3.24),(3.25) the boundary value problem (4.19)-(4.21) decouples into two boundary value problems of dimension n:

$$v_x + \alpha\varphi_0 = h, \quad \pi^R v(l) = \sigma_R, \quad (I - \pi^L)v(-l) = \sigma_L, \tag{4.23}$$

$$u_x - Au = v, \qquad (I_n - \pi^R)u(l) = \varrho_R - B_R v(l) \tag{4.24}$$

$$\langle u, \varphi_0 \rangle = 0, \qquad \pi^L u(-l) = \varrho_L + B_L v(-l). \tag{4.25}$$

Let us first solve (4.23) by integration. Because of the boundary conditions we have, for some $\eta_R^{II} \in \mathbb{R}^k, \eta_L^I \in \mathbb{R}^{k-1}$:

$$v(-l) = \sigma_L + S_L^I \eta_L^I, \quad v(l) = \sigma_R + S_R^{II}\eta_R^{II}. \tag{4.26}$$

Integrating the differential equation in (4.23) leads to the condition

$$v(l) - v(-l) + \alpha(U(l) - U(-l)) = \int_{-l}^{l} h(x)dx,$$

which by (4.26) is equivalent to

$$S_R^{II}\eta_R^{II} - S_L^I\eta_L^I + \alpha(U(l) - U(-l)) = \int_{-l}^{l} h(x)dx - \sigma_R + \sigma_L. \tag{4.27}$$

Since $U(l) \to U_L$ and $U(-l) \to U_R$ as $l \to \infty$ condition **(A3)** shows that (4.27) has a unique solution $(\eta_R^{II}, \eta_L^I, \alpha) \in \mathbb{R}^n$ that satisfies an estimate

$$|\eta_R^{II}| + |\eta_L^I| + |\alpha| \leq C(||h||_{L_1} + |\sigma_R| + |\sigma_L|) \leq C(||h||_{L_1} + |\gamma_R| + |\gamma_L|). \tag{4.28}$$

It is easy to reverse the argument, i.e., with $(\eta_R^{II}, \eta_L^I, \alpha) \in \mathbb{R}^n$ determined from (4.27) define $v(-l)$ by (4.26) and then set

$$v(x) = v(-l) - \alpha(U(x) - U(-l)) + \int_{-l}^{x} h(\xi)d\xi.$$

Then the second equation in (4.26) also holds and v solves the boundary value problem (4.23). From (4.28) we obtain the estimate

$$||v||_{L_\infty} + |\alpha| \leq C(|v(-l)| + |\alpha| + ||h||_{L_1}) \leq C(||h||_{L_1} + |\gamma_R| + |\gamma_L|). \tag{4.29}$$

In the next step we apply Lemma 4.1 to the boundary value problem (4.24),(4.25). Notice that by assumption the right hand sides in the boundary conditions of (4.24) and (4.25) are in the ranges of the corresponding projectors. We find a unique solution $u \in W^{1,\infty}(-l, l)$ that, using (4.29), can be estimated as follows:

$$\begin{aligned} ||u||_{L_\infty} &\leq C(||v||_{L_\infty} + |\varrho_R - B_R v(l)| + |\varrho_L + B_L v(-l)|) \\ &\leq C(||h||_{L_1} + |\gamma_R| + |\gamma_L|). \end{aligned}$$

Together with (4.29) our proof is complete. ∎

So far we have only increased the size of l in order to solve $s = 0$ problems.

4.2 Proof of Theorem 4.1

. Let us rewrite (4.2), (4.4) as a first order boundary value problem.

$$\begin{aligned} u_x - Au - v &= 0, \quad \pi_{3,4}(l,s)\begin{pmatrix} u \\ v \end{pmatrix}(l) = 0 & (4.30) \\ v_x - su + \alpha\varphi_0 &= h, \quad \pi_{1,2}(-l,s)\begin{pmatrix} u \\ v \end{pmatrix}(-l) = 0, & (4.31) \\ \langle u, \varphi_0 \rangle &= 0. & (4.32) \end{aligned}$$

We consider this as a small perturbation of (4.19)-(4.21). We use Lemma B.1 to write the boundary conditions as inhomogeneous conditions with the unperturbed projectors $\pi^R_{3,4}(0), \pi^L_{1,2}(0)$. Notice that (3.25) and (3.26) imply

$$|\pi^R_{3,4}(0) - \pi(l,s)| + |\pi^L_{1,2}(0) - \pi(-l,s)| \le C(\varepsilon(l-1) + |s|). \tag{4.33}$$

Take l large and $|s|$ small so that $C(\varepsilon(l-1)+|s|) \le \frac{1}{2}$ and as in B (2.14) consider the matrices

$$\Gamma(-l,s) = \pi^L_{1,2}(0)\left(I_{2n} - (\pi^L_{1,2}(0) - \pi(-l,s))\right)^{-1}(\pi^L_{1,2}(0) - \pi(-l,s))(I_{2n} - \pi^L_{1,2}(0)) \tag{4.34}$$

$$\Gamma(l,s) = \pi^R_{3,4}(0)\left(I_{2n} - (\pi^R_{3,4}(0) - \pi(l,s))\right)^{-1}(\pi^R_{3,4}(0) - \pi(l,s))(I_{2n} - \pi^R_{3,4}(0)). \tag{4.35}$$

With these settings and $z(\pm l) = \begin{pmatrix} u(\pm l) \\ v(\pm l) \end{pmatrix}$ we rewrite the boundary value problem (4.30)-(4.32) as an operator equation,

$$\begin{pmatrix} u_x - Au - v \\ v_x + \alpha\varphi_0 \\ \langle u, \varphi_0 \rangle \\ \pi^L_{1,2}(0)z(-l) \\ \pi^R_{3,4}(0)z(l) \end{pmatrix} = B(s)\begin{pmatrix} u \\ v \\ \alpha \end{pmatrix} + \hat{h}, \quad B(s)\begin{pmatrix} u \\ v \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ su \\ 0 \\ \Gamma(-l,s)z(-l) \\ \Gamma(l,s)z(l) \end{pmatrix}, \quad \hat{h} = \begin{pmatrix} 0 \\ h \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.36}$$

Let us denote the solution operator of (4.19)-(4.21) by T, then (4.36) is equivalent to the following fixed point equation for $(u,v,\alpha) \in L_\infty(-l,l) \times L_\infty(-l,l) \times \mathbb{R}$

$$\begin{pmatrix} u \\ v \\ \alpha \end{pmatrix} = TB(s)\begin{pmatrix} u \\ v \\ \alpha \end{pmatrix} + T\hat{h}. \tag{4.37}$$

By Lemma 4.2 we can estimate $(\hat{u}, \hat{v}, \hat{\alpha}) := T\hat{h}$ by

$$||\hat{u}||_{L_\infty} + ||\hat{v}||_{L_\infty} + |\hat{\alpha}| \le C||h||_{L_1}. \tag{4.38}$$

We show that $TB(s)$ is a contraction. Let $(\tilde{u}, \tilde{v}, \tilde{\alpha}) = TB(s)(u,v,\alpha)$ and estimate by using Lemma 4.2 and (4.33)

$$\begin{aligned} ||\tilde{u}||_{L_\infty} + ||\tilde{v}||_{L_\infty} + |\tilde{\alpha}| &\le C\left(||su||_{L_1} + |\Gamma(-l,s)z(-l)| + |\Gamma(l,s)z(l)|\right) \\ &\le C\left(|s||l||u||_{L_\infty} + (\varepsilon(l-1)+|s|)(||u||_{L_\infty} + ||v||_{L_\infty})\right) \\ &\le C(|s|l + |s| + \varepsilon(l-1))(||u||_{L_\infty} + ||v||_{L_\infty}). \end{aligned}$$

Here the coupling of l and s arises since the L_1-norm is estimated by the L_∞-norm. Now we choose l so large and $\delta_0 > 0$ so small that we have $C(|s|l + |s| + \varepsilon(l-1)) \leq \frac{1}{2}$ if $|s|l \leq \delta_0$

Then (4.37) has a unique solution $(u, v, \alpha) \in L_\infty(-l, l) \times L_\infty(-l, l) \times \mathbb{R}$ which satisfies

$$||u||_{L_\infty} + ||v||_{L_\infty} + |\alpha| \leq 2(||\hat{u}||_{L_\infty} + ||\hat{v}||_{L_\infty} + |\hat{\alpha}|) \leq C||h||_{L_1}.$$

This completes the proof.

5 Resolvent Estimate

5.1 Estimates for $|s|$ Large

Consider the equation (1.11) for $|s| \geq R, \operatorname{Re} s \geq 0$, where R is large and $F_x \in L_2$. First assume that $u \in H^2$ satisfies (1.11). We will prove the estimate

$$\|u\|^2 + \|u_x\|^2 \leq K\|F\|^2 \tag{5.1}$$

with K independent of s and F. This implies uniqueness of a solution and existence follows by the Fredholm property in Proposition 3.2. From

$$(u, u_{xx}) - (u, (Au)_x) - s(u, u) = (u, F_x)$$

one obtains that

$$(u_x, Au) + (u_x, F) = \|u_x\|^2 + s\|u\|^2 \ . \tag{5.2}$$

Let $s = \eta + i\xi$. Taking real parts in (5.2) one finds that

$$\begin{aligned} \|u_x\|^2 + \eta\|u\|^2 &\leq \|A\|_\infty \|u\| \|u_x\| + \|F\| \|u_x\| \\ &\leq \frac{1}{2}\|u_x\|^2 + \|A\|_\infty^2 \|u\|^2 + \|F\|^2 \ . \end{aligned} \tag{5.3}$$

Since $\eta \geq 0$ this yields the bound

$$\|u_x\|^2 \leq C\|u\|^2 + 2\|F\|^2 \ . \tag{5.4}$$

Case 1: $\eta \geq |\xi|$

We have $2\eta^2 \geq \eta^2 + \xi^2 \geq R^2$, thus $\eta \geq R/\sqrt{2}$.

Estimate (5.3) yields

$$\frac{1}{2}\|u_x\|^2 + (\eta - \|A\|_\infty^2)\|u\|^2 \leq \|F\|^2 \ ,$$

and the desired bound (5.1) follows if R is large enough.

Case 2: $\eta \leq |\xi|$

We have $|\xi| \geq R/\sqrt{2}$. Take the absolute value of the imaginary part in (5.2) to obtain

$$
\begin{aligned}
|\xi|\|u\|^2 &\le \|A\|_\infty \|u\|\|u_x\| + \|F\|\|u_x\| \\
&\le \frac{1}{2}\|u_x\|^2 + \|A\|_\infty^2 \|u\|^2 + \|F\|^2 \ .
\end{aligned}
$$

Using (5.4),

$$
|\xi|\|u\|^2 \le C_1\|u\|^2 + C_1\|F\|^2 \ .
$$

If R is large enough, obtain that

$$
\|u\|^2 \le \|F\|^2 \ .
$$

Using (5.4) again, the desired bound (5.1) follows.

As usual, the bound (5.4) implies uniqueness of any H^2–solution, and existence follows. We summarize the result.

Lemma 5.1 *There are (large) constants $R > 0$ and $K > 0$ with the following property: If $|s| \ge R, \mathrm{Re}\, s \ge 0$, and $F_x \in L_2$, then the equation (1.11) has a unique H^2–solution u. This solution satisfies (5.1).*

5.2 Estimates for $|s|$ Near Zero

Consider the resolvent equation

$$
P(s)u = u_{xx} - (Au)_x - su = h \tag{5.5}
$$

for $0 < |s| < \delta, \mathrm{Re}\, s \ge 0$, where $h \in L_1$.

Lemma 5.2 *There exist positive constants δ, C_0 such that, for $0 < |s| \le \delta$, $\mathrm{Re}\, s \ge 0$, and for any $h \in L_1$, the equation (5.5) has a unique solution $u \in W^{2,1}$. This solution u lies in H^1 and the estimate*

$$
||u||_{H^1} \le \frac{C_0}{|s|}||h||_{L_1} \tag{5.6}
$$

holds.

Proof of Theorem 1.2 for $|s| \le \delta$. We use Lemma 5.2 in order to complete the proof of Theorem 1.2. Let $w \in W^{1,1} \cap H^1$ be a solution of (3.6) for which Proposition 3.1 yields the bound

$$
||w||_{L_1} \le C||F||_{L_1}, \quad ||w|| \le C||F||. \tag{5.7}
$$

From the assumption on F and the boundedness of A_x one has

$$
w_x = Aw + F \in H^1 \cap L_1, \quad ||w_x|| \le C(||w|| + ||F||).
$$

This shows that $w \in H^2$ and $P(s)w = F_x - sw$. Let $v \in W^{2,1} \cap H^1$ be the unique solution of $P(s)v = sw$ given by Lemma 5.2. For $u = v + w \in W^{2,1} \cap H^1$ we obtain $P(s)u = sw + F_x - sw = F_x$. Then, from (5.6) and (5.7), we obtain the final estimate

$$||u||_{H^1} \leq ||v||_{H^1} + ||w||_{H^1} \leq \frac{C_0}{|s|}||sw||_{L_1} + C||F|| \leq C(||F||_{L_1} + ||F||).$$

Proof of Lemma 5.2 We will use an abstract theorem for an operator equation $Pu = h$ where $P : U \to W$ is a linear operator (bounded or unbounded) from some normed linear space U into some Banach space W.

Theorem 5.1 *Let $(U, \|\cdot\|_U)$ be a normed space and let $(W, \|\cdot\|_W)$ be a Banach space. Consider a linear operator $P : U \to W$. Assume that there is a bounded linear operator $S : W \to U$ so that, for all $h \in W$,*

$$\begin{aligned} \|PSh - h\|_W &\leq q\|h\|_W && (5.8)\\ \|Sh\|_U &\leq K_0\|h\|_W, && (5.9) \end{aligned}$$

where K_0, q are positive constants and $q < 1$. Then, for any $h \in W$, the equation $Pu = h$ has a solution $u \in U$ with

$$\|u\|_U \leq \frac{K_0}{1-q}\,\|h\|_W \ .$$

Remark: The operator S is an approximate right inverse of P. We can interpret assumption (5.8) as a defect condition: For any $h \in W$ one can obtain an approximate solution $\tilde{u} = Sh$ of the equation $Pu = h$; the defect of $\tilde{u} = Sh$ satisfies the bound $\|P\tilde{u} - h\|_W \leq q\|h\|_W$.
Proof. Given $h \in W$, define the affine linear operator $\Phi : W \to W$ by

$$\Phi v = v - PSv + h, \quad v \in W \ .$$

Obtain

$$\begin{aligned} \|\Phi v_1 - \Phi v_2\|_W &= \|v_1 - v_2 - PS(v_1 - v_2)\|_W \\ &\leq q\|v_1 - v_2\|_W \ . \end{aligned}$$

Therefore, Φ has a unique fixed point, $\bar{w}$, say. We have $PS\bar{w} = w$. If we define $\bar{u} = S\bar{w}$, then we have $P\bar{u} = w$. Also,

$$\|\bar{w}\|_W \leq q\|\bar{w}\|_W + \|h\|_W \ ,$$

thus

$$\|\bar{w}\|_W \leq \frac{1}{1-q}\|h\|_W \ ,$$

and

$$\|\bar{u}\|_U \le K_0\|\bar{w}\|_W \le \frac{K_0}{1-q}\|h\|_W \ .$$

■

Let us apply this theorem to $P = P(s)$ with the settings $U = W^{2,1} \cap H^1$, $||\cdot||_U = ||\cdot||_{H^1}$ and $W = L_1$, $||\cdot||_W = ||\cdot||_{L_1}$. For $h \in L_1$ we construct $\tilde{u} = Sh \in U$ in four steps.

Step 1

First use Theorem 3.1 to solve the tail problems

$$P(s)u_R = h,\ x \ge l-1, \qquad \pi_{3,4}(l-1,s)\begin{pmatrix} u_R \\ u_{R,x} - Au_R\end{pmatrix}(l-1) = 0, \tag{5.10}$$

$$P(s)u_L = h,\ x \le -l+1, \qquad \pi_{1,2}(-l+1,s)\begin{pmatrix} u_L \\ u_{L,x} - Au_L\end{pmatrix}(-l+1) = 0. \tag{5.11}$$

By Theorem 3.1 we have $u_R \in W^{2,1} \cap W^{1,p}$ for $1 \le p \le \infty$ and the estimate

$$||u_R||_{W^{1,\infty}} + |s|||u_R||_{H^1} + |s|^2||u_R||_{W^{1,1}} \le C||h||_{L_1}. \tag{5.12}$$

For u_L a corresponding estimate holds.

Step 2

Choose a cutoff function $\chi \in C^\infty(\mathbb{R})$ satisfying

$$\chi(x)\begin{cases} = 0 & |x| \le l-1, \\ \in [0,1] & l-1 \le |x| \le l, \\ = 1 & |x| \ge l. \end{cases}$$

and join the tail solutions together to obtain the extended function

$$u_{\mathrm{ext}}(x) = \begin{cases} \chi(x)u_L(x) & x \le -l+1, \\ = 0 & |x| \le l-1, \\ \chi(x)u_R(x) & l-1 \le x. \end{cases} \tag{5.13}$$

We set $\hat{h} = h - P(s)u_{\mathrm{ext}}$; by construction,

$$\hat{h}(x) = \begin{cases} h(x) & |x| \le l-1, \\ 0 & |x| \ge l. \end{cases}$$

In the intermediate region $l-1 \le x \le l$ we obtain by (5.12)

$$\begin{aligned} |\hat{h}(x)| &= |(h - \chi P(s)u_R - \chi_{xx}u_R - 2\chi_x u_{R,x} - \chi_x Au_R)(x)| \le \\ &\le C(|h(x)| + |u_R(x)| + |u_{R,x}(x)|) \le C(|h(x)| + ||h||_{L_1}). \end{aligned}$$

Together with the corresponding estimate for $-l \le x \le -l+1$ an integration yields

$$||\hat{h}||_{L_1} \le C||h||_{L_1}. \tag{5.14}$$

Step 3

We invoke Theorem 4.1 and solve the finite–interval boundary value problem (4.2), (4.4) with $\hat{h}$ instead of h. For the unique solution $\hat{u} \in W^{2,1}(-l,l), \alpha \in \mathbb{R}$, we have $\hat{u} \in W^{1,\infty}(-l,l)$ and an estimate

$$\|\hat{u}\|_{L_\infty(-l,l)} + \|\hat{u}_x\|_{L_\infty(-l,l)} + |\alpha| \le C\|\hat{h}\|_{L_1(-l,l)} \le C\|h\|_{L_1(-l,l)}. \tag{5.15}$$

We turn to system variables $\hat{v} = \hat{u}_x - A\hat{u}, \hat{z} = (\hat{u}, \hat{v})$, and continue $\hat{z}$ outside $(-l,l)$ by solving the homogeneous equation

$$z_{\rm int}(x) = \begin{pmatrix} u_{\rm int}(x) \\ v_{\rm int}(x) \end{pmatrix} = \begin{cases} S(x,-l,s)\hat{z}(-l) & x < -l, \\ \hat{z}(x) & |x| \le l, \\ S(x,l,s)\hat{z}(l) & x > l. \end{cases} \tag{5.16}$$

Here $S(x,\xi,s)$ denotes the solution operator of $L(s)$ from (3.11). By definition (5.16) we have $z_{\rm int} \in W^{1,1}(\mathbb{R}), u_{\rm int} \in W^{2,1}(\mathbb{R})$ and

$$L(s)z_{\rm int}(x) = \begin{cases} 0 & |x| > l, \\ \begin{pmatrix} 0 \\ \hat{h}(x) \end{pmatrix} & |x| \le l. \end{cases} \tag{5.17}$$

Since $\hat{z}$ satisfies the homogeneous boundary conditons (4.4) at $\pm l$ the tails of $z_{\rm int}$ are weakly decaying. More precisely, by the first part of Theorem 3.1 and A (1.26) (with $p' = 2$ and $\beta = \beta_{R,L}|s|^2$) we find the following estimate

$$\begin{aligned} \|z_{\rm int}\|^2 &\le \|\hat{z}\|^2_{L_2(-l,l)} + \|S(\cdot,-l,s)\hat{z}(-l)\|^2_{L_2(-\infty,-l)} + \|S(\cdot,l,s)\hat{z}(l)\|^2_{L_2(l,\infty)} \\ &\le C\left(l\|\hat{z}\|^2_{L_\infty(-l,l)} + |s|^{-2}(|\hat{z}(-l)|^2 + |\hat{z}(l)|^2)\right) \\ &\le C|s|^{-2}(l|s|^2+1)\|h\|^2_{L_1}. \end{aligned}$$

In the last line we have used (5.15). So far, all our estimates hold for $l \ge l_1$ and $|s|l \le \delta_0$, with constants C that depend only on l_1 and δ_0 as determined by Theorems 3.1 and 4.1. Therefore, we can continue the estimate above and obtain

$$\|z_{\rm int}\|^2 \le C|s|^{-2}(\frac{\delta_0^2}{l}+1)\|h\|^2_{L_1} \le C|s|^{-2}\|h\|^2_{L_1}. \tag{5.18}$$

Step 4

The approximate solution is now defined as

$$\tilde{u} = u_{\rm ext} + u_{\rm int} + \frac{\alpha}{s}\varphi_0. \tag{5.19}$$

Note that $\tilde{u} \in W^{2,1} \cap H^1$ satisfies, by (5.12), (5.13), (5.15), (5.18),

$$\begin{aligned} \|\tilde{u}\|^2_{H^1} &\le C\left[\|u_R\|^2_{H^1} + \|u_L\|^2_{H^1} + \|u_{\rm int}\|^2_{H^1} + \frac{\alpha^2}{|s|^2}\|\varphi_0\|^2_{H^1}\right] \\ &\le \frac{C_0^2}{|s|^2}\|h\|^2_{L_1}. \end{aligned}$$

This proves (5.9) with $K_0 = \frac{C_0}{|s|}$.

Finally, we use the equation $P(s)\varphi_0 = s\varphi_0$, (5.17), (4.2) to obtain

$$\begin{aligned} P(s)\tilde{u} - h &= P(s)u_{\text{ext}} + P(s)u_{\text{int}} + \alpha\varphi_0 - h \\ &= \begin{cases} 0 & |x| < l, \\ \alpha\varphi_0(x) & |x| \geq l. \end{cases} \end{aligned}$$

We use (5.15) again and deduce the L_1-estimate

$$\|P(s)\tilde{u} - h\|_{L_1} \leq |\alpha| \int_{|x|\geq l} |\varphi_0(x)| dx \leq \frac{C}{\beta} e^{-\beta l} \|h\|_{L_1}. \tag{5.20}$$

This determines the final choice of l through the condition $\frac{C}{\beta}e^{-\beta l} \leq \frac{1}{2}$. Theorem 5.1 applies with $q = \frac{1}{2}$ and, for $0 < |s| \leq \frac{\delta_0}{l}$ and $\operatorname{Re} s \geq 0$, yields a solution $u \in W^{2,1} \cap H^1$ of $P(s)u = h$ that satisfies

$$\|u\|_{H^1} \leq \frac{2C_0}{|s|} \|h\|_{L_1}.$$

Uniqueness in $W^{2,1}$ follows from the Fredholm alternative in Proposition 3.2.

5.3 Estimates for Moderate $|s|$

These follow by a standard compactness argument. Note that in the previous sections we found constants $\delta > 0, R > 0$, such that Theorem 1.2 holds for $\operatorname{Re} s \geq 0$ and $|s| \in (0, \delta) \cup (R, \infty)$. Up to this point, Assumption (**A4**) has not been used, but it is essential in the domain $\delta \leq |s| \leq R$ and $\operatorname{Re} s \geq 0$. For any fixed s_0 in this domain the kernel of the operator $P(s_0) : H^2 \mapsto L^2$ is trivial by Assumption (**A4**). The Fredholm property in Proposition 3.2 guarantees that $P(s_0)u = h \in L_2$ has a unique solution $u \in H^2$ with an estimate

$$\|u\|_{H^2} \leq C(s_0)\|h\|.$$

A small perturbation argument shows that this holds, correspondingly, for all s with $|s - s_0| \leq \frac{1}{2C_0}$, with the constant $2C(s_0)$ in place of $C(s_0)$. By compactness, the proof of Theorem 1.2 is complete.

A Dichotomies and L_1-Perturbations

Consider a subinterval $J = [x_-, x_+] \subset \mathbb{R}$ where x_- or x_+ may be finite or infinite and with the understanding that $-\infty$ and ∞ are not contained in J. Let L be a linear differential operator on J,

$$Lz = z_x - M(x)z, \tag{1.1}$$

with $N \times N$ matrices $M(x)$ that are continuous in $x \in J$, and let $S(x, \xi)$ denote the solution operator of L.

Definition A.1 *The operator L has a* **generalized exponential dichotomy on J with data** $(\alpha, \beta, K, \pi(x))$ *if $\alpha \leq \beta$ are real numbers and $\pi(x)$ are projectors in $\mathbb{R}^N$ such that for all $x, \xi \in J$ the following holds:*

$$\pi(x)S(x,\xi) = S(x,\xi)\pi(\xi), \tag{1.2}$$

$$|S(x,\xi)\pi(\xi)| \leq Ke^{\alpha(x-\xi)}, \quad x \geq \xi, \tag{1.3}$$

$$|S(x,\xi)(I-\pi(\xi))| \leq Ke^{\beta(x-\xi)}, \quad x < \xi. \tag{1.4}$$

Remarks

1. In the definition above there is no sign restrictions on α and β. The number α limits the exponential growth of solutions in forward direction when started in $range(\pi(\xi))$; correspondingly, β limits the exponential growth in backward direction when started in $range(I - \pi(\xi))$. Note that in the case $-\alpha = \beta > 0$ we have a (standard) exponential dichotomy as in Definition 3.1; in the case $\alpha = \beta = 0$ we have an ordinary dichotomy in the sense of Coppel [3]. In the general case, when $\alpha < \beta$, our notion agrees with the shifted exponential dichotomy of Hale and Lin [4].

2. In general, the projectors $\pi(x)$ of a generalized exponential dichotomy are not unique. [3] However, if $\alpha < \beta$ and $J = [x_0, \infty)$, then the ranges are unique because they can be written as

$$range(\pi(\xi)) = \{z \in \mathbb{R}^N : e^{-\eta(x-\xi)}|S(x,\xi)z| \text{ is bounded for } x \geq \xi\}, \tag{1.5}$$

for any $\alpha < \eta < \beta$. While "$\subset$" is obvious the converse conclusion follows from

$$|(I-\pi(\xi))z| = |(I-\pi(\xi))S(\xi,x)S(x,\xi)z| \leq Ce^{(\eta-\beta)(x-\xi)} \to 0 \text{ as } x \to \infty.$$

In this case, the kernel of $\pi(x_0)$ is still not determined, however. Take, for example, another projector $\hat{\pi}(x_0)$ that satisfies $range(\pi(x_0)) = range(\hat{\pi}(x_0))$ and define $\hat{\pi}(x) = S(x,x_0)\hat{\pi}(x_0)S(x_0,x)$ for $x \in J$. Then L has a generalized exponential dichotomy on J with data $(\alpha, \beta, \hat{K}, \hat{\pi})$, where $\hat{K} = K(B+K), B = K^2|\pi(x_0) - \hat{\pi}(x_0)|$. In fact, the assumption on the ranges implies $\hat{\pi}(x)\pi(x) = \pi(x), \pi(x)\hat{\pi}(x) = \hat{\pi}(x)$ for $x \in J$ and then

$$|\pi(x) - \hat{\pi}(x)| = |S(x,x_0)\pi(x_0)(\pi(x_0) - \hat{\pi}(x_0))(\pi(x_0) - I)S(x_0,x)| \leq Be^{(\alpha-\beta)(x-x_0)}. \tag{1.6}$$

Using $|\pi(x)| \leq K$ yields the dichotomy estimates for $\hat{\pi}$

$$|S(x,\xi)\hat{\pi}(\xi)| = |S(x,\xi)\pi(\xi)\hat{\pi}(\xi)| \leq Ke^{\alpha(x-\xi)}(B+K) \quad \text{for} \quad x \geq \xi,$$

$$|S(x,\xi)(I-\hat{\pi}(\xi))| = |(I-\hat{\pi}(x))S(x,\xi)(I-\pi(\xi))| \leq (K+B)Ke^{\beta(x-\xi)} \quad \text{for} \quad x < \xi.$$

[3] Of course, by (1.2), if a projector $\pi(x_0)$ is determined at one point x_0, then all projectors $\pi(x)$ are determined uniquely.

3. Generalized exponential dichotomies can be extended over compact intervals. For example, assume that Definition A.1 is satisfied on $[x_0, \infty)$ with $x_0 \geq 0$ and suitable data $(\alpha, \beta, K, \pi(x))$. Then a simple calculation shows that the generalized exponential dichotomy also holds on $[0, \infty)$. Possible data are $(\alpha, \beta, \tilde{K}, \tilde{\pi}(x))$, where

$$\tilde{K} = K K_\alpha K_\beta, \quad \tilde{\pi}(x) = \begin{cases} \pi(x), & x_0 \leq x \\ S(x, x_0)\pi(x_0)S(x_0, x), & 0 \leq x < x_0, \end{cases}$$

where $K_\alpha = 1 + \sup_{0 \leq x \leq x_0} e^{\alpha(x - x_0)}|S(x_0, x)|$ and $K_\beta = 1 + \sup_{0 \leq x \leq x_0} e^{\beta(x_0 - x)}|S(x, x_0)|$.

In the first part of the appendix we will prove a perturbation theorem for generalized exponential dichotomies under L_1 perturbation of the matrix function $M(x)$. For ordinary dichotomies this was already proved in [3]. In the following we give a somewhat simplified proof along the lines of [2, Appendix] for the generalized case. For our later applications it will be essential to provide precise estimates of the dichotomy data for the perturbed operator.

With any operator $Lz = z_x - Mz$ that has a generalized exponential dichotomy we associate a Green's function,

$$G(x, \xi) = \begin{cases} S(x, \xi)\pi(\xi), & x \geq \xi, \\ S(x, \xi)(\pi(\xi) - I), & x < \xi. \end{cases} \tag{1.7}$$

It will also be convenient to introduce the weight function

$$\varrho(x, \xi) = \begin{cases} e^{\alpha(x - \xi)}, & x \geq \xi \\ e^{\beta(x - \xi)}, & x < \xi. \end{cases} \tag{1.8}$$

Theorem A.1 *Let $Lz = z_x - Mz$ have a generalized exponential dichotomy on J with data $(\alpha, \beta, K, \pi(x))$ and let $\Delta \in C(J, \mathbb{R}^{N,N})$ be a matrix-function that satisfies, for some $q < 1$,*

$$K||\Delta||_{L_1(J)} \leq q < 1. \tag{1.9}$$

Then the perturbed operator $\tilde{L}z = z_x - (M + \Delta)z$ has a generalized exponential dichotomy on J with data $(\alpha, \beta, \tilde{K}, \tilde{\pi})$ where

$$\tilde{K} = \frac{K}{1 - q} \ . \tag{1.10}$$

In addition, the following estimate holds:

$$|\tilde{\pi}(x) - \pi(x)| \leq \frac{K^2}{1 - q}||\Delta||_{L_1(J)}, \quad x \in J. \tag{1.11}$$

Proof. As in [2] we consider the space of matrix–valued functions

$$X = \{H \in C(J \times J, \mathbb{R}^{N,N}) : ||H||_\varrho < \infty\}$$

where $|| \cdot ||_\varrho$ is the weighted norm defined by

$$||H||_\varrho = \sup\{\frac{|H(x, \xi)|}{\varrho(x, \xi)} : x, \xi \in J\}.$$

Using the variation–of–constants formula one finds that the difference $H = \tilde{G} - G$ between the (yet unknown) Green's function $\tilde{G}$ of the perturbed operator and the given G satisfies the fixed point equation

$$H = F(H) + F(G), \tag{1.12}$$

where F is defined by

$$F(H)(x,\xi) = \int_J G(x,\eta)\Delta(\eta)H(\eta,\xi)d\eta, \quad x,\xi \in J. \tag{1.13}$$

Note that F maps continuous kernels into continuous kernels and actually maps X into itself, as we will show below. We will also see that $F(G)$ is in X even though G has a jump on the diagonal. First, the exponential dichotomy of L implies

$$|G(x,\xi)| \leq K\varrho(x,\xi), \quad x,\xi \in J\ . \tag{1.14}$$

Then we claim that F satisfies the bound

$$||F(H)||_\varrho \leq q||H||_\varrho \quad \text{for } H \in X. \tag{1.15}$$

For $x,\xi \in J$ we estimate, using (1.14), as follows:

$$\begin{aligned} \frac{|F(H)(x,\xi)|}{\varrho(x,\xi)} &\leq ||H||_\varrho \int_J \frac{1}{\varrho(x,\xi)}|G(x,\eta)||\Delta(\eta)|\varrho(\eta,\xi)d\eta \\ &\leq K||H||_\varrho \int_J \frac{\varrho(x,\eta)\varrho(\eta,\xi)}{\varrho(x,\xi)}|\Delta(\eta)|d\eta \\ &\leq K||\Delta||_{L_1}||H||_\varrho \leq q||H||_\varrho. \end{aligned}$$

To obtain the estimate in the last line we have used that $\alpha \leq \beta$ implies $\varrho(x,\eta)\varrho(\eta,\xi) \leq \varrho(x,\xi)$ for all $x,\xi,\eta \in J$. This proves (1.15), and by applying the estimates above to the jump kernel G together with (1.14) we obtain

$$||F(G)||_\varrho \leq K^2||\Delta||_{L_1}. \tag{1.16}$$

By the contraction mapping theorem equation (1.12) has a unique solution $H \in X$ which satisfies

$$||H||_\varrho \leq \frac{1}{1-q}||F(G)||_\varrho \leq \frac{K^2||\Delta||_{L_1}}{1-q} \leq \frac{Kq}{1-q}\ . \tag{1.17}$$

We now define $\tilde{G} = G + H$ and by the same arguments as in [2] obtain that $\tilde{\pi}(x) := \tilde{G}(x,x)$ are projectors such that $\tilde{G}$ assumes the required form

$$\tilde{G}(x,\xi) = \begin{cases} S(x,\xi)\tilde{\pi}(\xi), & x \geq \xi \\ S(x,\xi)(\tilde{\pi}(\xi) - I), & x < \xi. \end{cases} \tag{1.18}$$

The dichotomy estimates (1.10) and (1.11) then follow from (1.17) and

$$||\tilde{G}||_\varrho \leq ||G||_\varrho + ||H||_\varrho \leq K + \frac{Kq}{1-q} = \frac{K}{1-q} = \tilde{K}.$$

■

Remark: We note that the dichotomy exponents α, β remain unchanged under L_1 perturbations and that (1.17) implies the following estimates, which are more general estimates than (1.11):

$$\begin{aligned} |S(x,\xi)\pi(\xi) - \tilde{S}(x,\xi)\tilde{\pi}(\xi)| &\leq K\tilde{K}||\Delta||_{L_1} e^{\alpha(x-\xi)}, \quad x \geq \xi \\ |S(x,\xi)(\pi(\xi) - I) - \tilde{S}(x,\xi)(\tilde{\pi}(\xi) - I)| &\leq K\tilde{K}||\Delta||_{L_1} e^{\beta(x-\xi)}, \quad x < \xi. \end{aligned}$$

Without further assumptions we cannot conclude that the projectors π and $\tilde{\pi}$ are of the same rank. Sufficient conditions are given in the following Proposition.

Proposition A.1 *Suppose that the assumptions of Theorem A.1 hold on an interval* $J = [x_0, \infty)$ *with either* $\alpha < \beta$ *or* $K^2||\Delta||_{L_1(J)} < 1 - q$. *Then the projectors* $\pi(x)$ *and* $\tilde{\pi}(x)$ *have the same rank for* $x \in J$ *and, in particular, the projectors constructed in the proof of Theorem A.1 satisfy*

$$ker(\pi(x)) = ker(\tilde{\pi}(x)), \; x \in J. \tag{1.19}$$

Proof. Let us first assume that $K^2||\Delta||_{L_1(J)} < 1 - q$. Then (1.11) implies $|\pi(x) - \tilde{\pi}(x)| < 1$, and hence the equality of ranks follows from Lemma B.1. For the proof of (1.19) note that equations (1.12) and (1.13) imply

$$\tilde{\pi}(x_0) = \pi(x_0) + H(x_0, x_0) = \pi(x_0) + \int_{x_0}^{\infty} G(x_0,\xi)r(\xi)d\xi$$

where $r(\xi) = \Delta(\xi)(H(\xi, x_0) - G(\xi, x_0))$. Using (1.2) and (1.7) we obtain

$$\tilde{\pi}(x_0) = \pi(x_0) + (\pi(x_0) - I)\int_{x_0}^{\infty} S(x_0,\xi)(\pi(\xi) - I)r(\xi)d\xi$$

and, therefore, $range(\tilde{\pi}(x_0) - I) \subset range(\pi(x_0) - I)$. Because both ranges have the same dimension, equation (1.19) follows at $x = x_0$ and then for a general $x \in J$ from relation (1.2).

In the case $\alpha < \beta$ we first determine $x_1 \geq x_0$ so that $K^2||\Delta||_{L_1[x_1,\infty)} < 1 - q$. Then $\tilde{L}$ has a generalized exponential dichotomy on $[x_1, \infty)$ with data $(\alpha, \beta, \hat{K}, \hat{\pi}(x))$ where $ker(\pi(x_1)) = ker(\hat{\pi}(x_1))$ and $\pi(x_1)$ and $\hat{\pi}(x_1)$ have the same rank. Since, in case $\alpha < \beta$, the ranges of $\tilde{\pi}(\xi), \xi \in [x_0, \infty)$ are uniquely determined (cf. (1.5)) we have $range(\hat{\pi}(x_1)) = range(\tilde{\pi}(x_1))$ and therefore $rank(\pi(x_1)) = rank(\tilde{\pi}(x_1))$. Again, (1.2) yields equality at each $x \in [x_0, \infty)$. Finally, (1.19) follows in the same way as in the first case. ■

Similarly, for $J = (-\infty, x_0]$, under the assumptions of the proposition one has

$$range(\tilde{\pi}(x_0)) = range(\pi(x_0)) \; , \tag{1.20}$$

and equality of ranks is implied.

As in [2, Appendix A] we need estimates for solutions of inhomogeneous boundary value problems on general intervals $J = [x_-, x_+]$, i.e., for problems of the form

$$Lz = h, \quad x \in J \tag{1.21}$$

$$\pi(x_-)z(x_-) = \gamma_-, \quad (I - \pi(x_+))z(x_+) = \gamma_+. \tag{1.22}$$

Here the boundary condition at x_- or x_+ is empty if $x_- = -\infty$ or $x_+ = \infty$. Since we need these estimates with L_1, L_2 and L_∞ norms for z and h, we formulate the result for general L_p-norms.

Theorem A.2 *Let L have an exponential dichotomy on J with data (β, K, π); let the Green's function G be defined by (1.7); and let $1 \le p \le \infty$. Then, for any $h \in L_p(J)$ and any $\gamma_- \in range(\pi(x_-)), \gamma_+ \in range((I - \pi(x_+)))$, the boundary value problem (1.21), (1.22) has a uniqe solution $z \in W^{1,p}(J)$, namely $z = z_{sp} + z_{hom}$, where*

$$z_{sp}(x) = \int_J G(x,\xi)h(\xi)d\xi, \tag{1.23}$$

$$z_{hom}(x) = S(x,x_-)\gamma_- + S(x,x_+)\gamma_+. \tag{1.24}$$

This solution z satisfies $z \in L_{p'}(J)$ for all $1 \le p \le p' \le \infty$, and the following estimates hold:

$$\beta^{1-\frac{1}{p}+\frac{1}{p'}}||z_{sp}||_{L_{p'}} + \beta^{1-\frac{1}{p}}|z_{sp}|_\Gamma \le 6K||h||_{L_p} \tag{1.25}$$

$$\beta^{\frac{1}{p'}}||z_{hom}||_{L_{p'}} + |z_{hom}|_\Gamma \le (K+1)(|\gamma_-| + |\gamma_+|). \tag{1.26}$$

Remark: In the estimate we have used the abbreviation $|z|_\Gamma = |z(x_-)| + |z(x_+)|$ for boundary terms. Moreover, $\frac{1}{p} = 0$ for $p = \infty$.

Proof. One easily shows that $z = z_{\rm sp} + z_{\rm hom}$ solves (1.21),(1.22), and uniqueness is proved as in [2]. From the exponential dichotomy of L we have $|G(x,\xi)| \le Ke^{-\beta(x-\xi)}$. Let $\frac{1}{p} + \frac{1}{q} = 1$ and use a Hölder estimate to obtain:

$$\begin{aligned}
|z_{\rm sp}(x)|^{p'} &\le K^{p'}\left(\int_J e^{-\frac{\beta}{q}|x-\xi|}e^{-\frac{\beta}{p}|x-\xi|}|h(\xi)|d\xi\right)^{p'} \\
&\le K^{p'}\left(\frac{2}{\beta}\right)^{\frac{p'}{q}}\left(\int_J e^{-\beta|x-\xi|}|h(\xi)|^p d\xi\right)^{\frac{p'}{p}} \\
&\le K^{p'}\left(\frac{2}{\beta}\right)^{\frac{p'}{q}}||h||_{L_p}^{p'-p}\int_J e^{-\beta|x-\xi|}|h(\xi)|^p d\xi.
\end{aligned}$$

Integration and Fubini's Theorem yield:

$$\begin{aligned}
||z_{\rm sp}||_{L_{p'}}^{p'} &\le K^{p'}\left(\frac{2}{\beta}\right)^{\frac{p'}{q}}||h||_{L_p}^{p'-p}\int_J \frac{2}{\beta}|h(\xi)|^p d\xi \\
&\le K^{p'}\left(\frac{2}{\beta}\right)^{1+\frac{p'}{q}}||h||_{L_p}^{p'} \le \left(4K\beta^{-\frac{1}{p'}-\frac{1}{q}}||h||_{L_p}\right)^{p'}.
\end{aligned}$$

For the second term on the left–hand side of (1.25) we obtain

$$\begin{aligned}
|z_{\rm sp}(x_+)|^{p'} &\le K^{p'}\left(\int_J e^{-\beta(x_+-\xi)}|h(\xi)|d\xi\right)^{p'} \\
&\le K^{p'}\left(\frac{1}{\beta q}\right)^{\frac{p'}{q}}||h||_{L_p}^{p'} \le \left(K\beta^{-\frac{1}{q}}||h||_{L_p}\right)^{p'}.
\end{aligned}$$

Together with the corresponding estimate at x_- we have proved (1.25). Finally, we integrate

$$|S(x,x_-)\gamma_-|^{p'} \le K^{p'} e^{-\beta p'(x-x_-)}|\gamma_-|^{p'}$$

with respect to x and find

$$||S(\cdot,x_-)\gamma_-||_{L_{p'}} \le K\left(\frac{1}{\beta p'}\right)^{\frac{1}{p'}}|\gamma_-| \le K\beta^{-\frac{1}{p'}}|\gamma_-|.$$

Combining this with a corresponding estimate at x_+ leads to the desired estimate (1.26). ■

Finally, we consider the case where exponential dichotomies hold on both semi-infinite intervals, $(-\infty,0]$ and $[0,\infty)$, and where the equation $Lz=0$ has a solution that decays in both directions.

Theorem A.3 *Suppose that L has an exponential dichotomy on $(-\infty,0]$ and on $[0,\infty)$ with data $(\beta,K,\pi_\pm)$, and let $k_-=rank(\pi_-)$, $k_+=rank(\pi_+)$. Then the operator $L: W^{1,p}(\mathbb{R}) \mapsto L_p(\mathbb{R})$ is Fredholm of index k_++k_--N. Assume, in addition, that*

$$\dim\left(range(\pi_+(0)) \cap ker(\pi_-(0))\right) = k_+ + k_- - N. \tag{1.27}$$

Then, for any $1\le p\le p'\le\infty$, there exists a constant $C>0$ such that all solutions $z\in W^{1,p}(\mathbb{R})$ of the inhomogeneous equation $Lz=h$ with $h\in L_p(\mathbb{R},\mathbb{R}^N)$ are of the form

$$z(x) = z_{sp}(x) + S(x,0)\eta \quad \textit{where} \quad \eta \in range(\pi_+(0)) \cap ker(\pi_-(0)) \tag{1.28}$$

and where the special solution z_{sp} satisfies a bound

$$\beta^{1-\frac{1}{p}+\frac{1}{p'}}||z_{sp}||_{L_{p'}} \le C||h||_{L_p}. \tag{1.29}$$

Proof. In $L_\infty-$spaces the assertion about the Fredholm index in proved in [9], [1]. For completeness, we indicate the main steps for the L_p–spaces considered here. In a first step we note that all bounded solutions of the homogeneous equation $Lz=0$ on $\mathbb{R}$ are given by $z(x)=S(x,0)\eta$, $\eta\in range(\pi_+(0))\cap ker(\pi_-(0))$, and that these solutions lie in any L_p. Therefore,

$$\dim ker(L) = \dim\left(range(\pi_+(0)) \cap ker(\pi_-(0))\right). \tag{1.30}$$

In a second step introduce the adjoint operator $L^*z = z' + M(x)^T z$ and note that L^* has exponential dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$ with data $(\beta,K,I-\pi_\pm^T)$. Then one can show that the range of L can be characterized as follows:

$$range(L) = \{h\in L_p : \int_{\mathbb{R}} \varphi^T(x)h(x)dx = 0 \;\forall \varphi\in ker(L^*)\}. \tag{1.31}$$

Applying formula (1.30) to L^* yields

$$\begin{aligned}
\operatorname{codim} range(L) = \dim ker(L^*) &= \dim\left(ker(I-\pi_-(0)^T)\cap range(I-\pi_+(0)^T)\right)\\
= \dim\left(range(\pi_-(0)^T)\cap ker(\pi_+(0)^T)\right) &= \dim\left(ker(\pi_-(0))^\perp \cap range(\pi_+(0)^\perp)\right)\\
= \dim\left((ker(\pi_-(0)) + range(\pi_+(0)))^\perp\right) &= N-(k_-+k_+-\dim(range(\pi_+(0))\cap ker(\pi_-(0)))).
\end{aligned}$$

By subtraction from (1.30) we obtain the formula for the Fredholm index. Under the additional assumption (1.27) it then follows that the operator L is onto and that the representation (1.28) holds provided we construct a special solution $z_{\rm sp}$ with the estimate (1.29).

We abbreviate $V_0 = range(\pi_+(0)) \cap ker(\pi_-(0))$ and choose decompositions

$$range(\pi_+(0)) = V_0 \oplus V_+, \quad ker(\pi_-(0)) = V_0 \oplus V_-, \tag{1.32}$$

where $\dim(V_+) = N - k_-, \dim(V_-) = N - k_+$. Counting dimensions one obtains the direct sum $\mathbb{R}^N = V_0 \oplus V_- \oplus V_+$. Now we modify the projectors π_+ and π_- such that $ker(\pi_+(0)) = V_-$ and $range(\pi_-(0)) = V_+$. According to Remark 2 following Definition A.1 the dichotomy properties still hold with the same exponents but a modified constant $\hat{K}$. We keep the same symbol $\pi_\pm$ for the modified projectors. By Theorem A.2 we have unique solutions z_-, z_+ of the one-sided boundary value problems

$$Lz_- = h, \ x \le 0, \quad (I - \pi_-(0))z_-(0) = 0 \tag{1.33}$$
$$Lz_+ = h, \ 0 \le x, \qquad \pi_+(0)z_+(0) = 0 \tag{1.34}$$

and both satisfy an estimate (1.25). By construction we have $z_+(0) \in V_-$ and $z_-(0) \in V_+$. Therefore we can continue z_+ to $x \le 0$ and z_- to $x \ge 0$ by solving the homogeneous equation, i.e., we define

$$z_{\rm sp}(x) = \begin{cases} z_-(x) + S(x,0)z_+(0), & \text{for } x \le 0, \\ z_+(x) + S(x,0)z_-(0), & \text{for } x \ge 0. \end{cases} \tag{1.35}$$

Then $z_{\rm sp}$ is continuous at 0 and satisfies $Lz_{\rm sp} = h$ on the whole line; hence $z_{\rm sp} \in W^{1,p}(\mathbb{R})$. Moreover, from (1.25) and (1.26) we obtain the desired estimate:

$$\begin{aligned} \beta^{1-\frac{1}{p}+\frac{1}{p'}}||z_{\rm sp}||_{L_{p'}} &\le C\left(\beta^{1-\frac{1}{p}+\frac{1}{p'}}\left(||z_+||_{L_{p'}[0,\infty)} + ||z_-||_{L_{p'}(-\infty,0}\right) + \beta^{1-\frac{1}{p}}(|z_+(0)| + |z_-(0)|)\right) \\ &\le C\left(||h||_{L_p(-\infty,0]} + ||h||_{L_p(0,\infty]}\right) \le C||h||_{L_p}. \end{aligned}$$

■

B Polychotomies and L_1-Perturbations

As originally suggested by Sacker and Sell (see [10]), it is often useful to split the exponential growth behavior into several intervals $[\alpha_j, \beta_j]$ with associated projectors $\pi_j(x), j = 1, \ldots, k-1$, for some $1 \le k \le n$. We assume that the intervals are arranged in increasing order,

$$\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \ldots \le \alpha_{k-1} \le \beta_{k-1}. \tag{2.1}$$

In the language of [10] the intervals (β_j, α_{j+1}) belong to the resolvent of the operator L while $[\alpha_j, \beta_j]$ may contain spectrum. Our intention is to derive a perturbation theorem that is analogous to Theorem A.1 for this more refined situation. Our result slightly differs from the general perturbation theorem for Sacker, Sell spectra [10] because we allow the intervals to have some endpoints in common and because we will treat L_1 perturbations in a quantitative way.

Let us introduce the vector notation

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_{k-1}), \vec{\beta} = (\beta_1, \ldots, \beta_{k-1}), \vec{\pi} = (\pi_1, \ldots, \pi_{k-1}).$$

Definition B.1 *The operator L has an* **exponential polychotomy on J with data** $(\vec{\alpha}, \vec{\beta}, K, \vec{\pi}(x))$ *if the α's and β's are arranged as in (2.1) and if the $\pi_j(x), x \in J$, are projectors in $\mathbb{R}^n$ with the following properties:*

(i) $\pi_i(x)\pi_j(x) = \delta_{i,j}\pi_j(x)$ holds for all $x \in J$ and $i, j = 1, \ldots, k-1$,

(ii) for $j = 1, \ldots, k-1$ the operator L has a generalized exponential dichotomy on J (in the sense of Definition A.1) with data $(\alpha_j, \beta_j, K, \sum_{\nu=1}^{j} \pi_\nu)$.

In the following it will be convenient to introduce the remaining projector

$$\pi_k = I - \sum_{j=1}^{k-1} \pi_j.$$

From property (ii) above we then have

$$\pi_i\pi_j = \delta_{i,j}\pi_j \quad \text{for all} \quad i, j = 1, \ldots, k, \quad \sum_{j=1}^{k} \pi_j = I. \tag{2.2}$$

Moreover, we denote the projectors occuring in (ii) by

$$Q_j = \sum_{\nu=1}^{j} \pi_\nu, \quad j = 1, \ldots, k-1. \tag{2.3}$$

This definition yields the estimates

$$\begin{array}{rcll} |S(x,\xi)\pi_j(\xi)| & = & |S(x,\xi)Q_j(\xi)\pi_j(\xi)| \le K^2 e^{\alpha_j(x-\xi)}, & x \ge \xi \\ |S(x,\xi)\pi_j(\xi)| & = & |S(x,\xi)(I - Q_{j-1}(\xi))\pi_j(\xi)| \le K^2 e^{\beta_{j-1}(x-\xi)}, & x < \xi. \end{array}$$

These inequalities show how initial data under the same projector can be bounded under forward and backward integration.

Theorem B.1 *Suppose that the differential operator L has an exponential polychotomy on $J = [x_0, \infty)$ with data $(\vec{\alpha}, \vec{\beta}, K, \vec{\pi})$ and assume*

$$\alpha_j < \beta_{j+1}, \quad j = 1, \ldots, k-2. \tag{2.4}$$

Let the matrix valued function $\Delta \in C(J, \mathbb{R}^{n,n})$ satisfy the same smallness assumption (1.9) as in Theorem A.1. Then the operator $\tilde{L}z = z_x - (M(x) + \Delta(x))z$ has an exponential polychotomy *on J with data $(\vec{\alpha},\vec{\beta},\tilde{K},\vec{\tilde{\pi}})$, where $\tilde{K}=\frac{K}{1-q}$ as in Theorem A.1, and the following estimates hold for $x\in J$:*

$$|\tilde{\pi}_j(x)-\pi_j(x)| \leq \frac{2K^2}{1-q}||\Delta||_{L_1}, j=1,\ldots,k \tag{2.5}$$

$$|\tilde{Q}_j(x)-Q_j(x)| \leq \frac{K^2}{1-q}||\Delta||_{L_1}, j=1,\ldots,k-1 \tag{2.6}$$

Remarks: Condition (2.4) requires that there is at least one strict inequality between 4 consecutive numbers in (2.1).

We conjecture this theorem to hold also in the all–line case, $J=\mathbb{R}$, but we have not pursued the details of a proof. In the one–sided case considered here, we will use that the kernels of the projectors remain constant under perturbations; see (1.19).

Proof. We proceed by induction in $k\geq 2$ and include the statement

$$ker(Q_j)=ker(\tilde{Q}_j),\quad j=1,\ldots,k-1\ , \tag{2.7}$$

in the induction. For $k=2$ our assertion follows from Theorem A.1. When we proceed from k to $k+1$ the induction hypothesis yields that we have an exponential polychotomy for $\tilde{L}$ on J with data $\tilde{K}$ and

$$\alpha_1,\ldots,\alpha_{k-1},\quad \beta_1,\ldots,\beta_{k-1},\quad \tilde{\pi}_1,\ldots\tilde{\pi}_{k-1},\quad \tilde{Q}_j=\sum_{\nu=1}^{j}\tilde{\pi}_\nu,\ j=1,\ldots,k-1.$$

Next we apply Theorem A.1 to L with the data $(\alpha=\alpha_k,\beta=\beta_k,K,\pi=Q_k=\sum_{\nu=1}^{k}\pi_\nu)$. This yields a generalized exponential dichotomy for $\tilde{L}$ on J with data $(\alpha_k,\beta_k,\tilde{K},\tilde{Q}_k)$ where

$$|\tilde{Q}_k(x)-Q_k(x)|\leq\frac{K^2}{1-q},\quad x\in J.$$

Let us first show

$$\tilde{Q}_k\tilde{Q}_{k-1}=\tilde{Q}_{k-1}. \tag{2.8}$$

For $x_0\leq\xi\leq x$ we have by (2.4)

$$\begin{aligned}|(\tilde{Q}_k(\xi)-I)\tilde{Q}_{k-1}(\xi)| &= |(\tilde{Q}_k(\xi)-I)\tilde{S}(\xi,x)\tilde{S}(x,\xi)\tilde{Q}_{k-1}(\xi)|\\ \leq |\tilde{S}(\xi,x)(\tilde{Q}_k(x)-I)||\tilde{S}(x,\xi)\tilde{Q}_{k-1}(\xi)| &\leq \tilde{K}e^{\beta_k(\xi-x)}\tilde{K}e^{\alpha_{k-1}(x-\xi)}\\ &= \tilde{K}^2e^{(\alpha_{k-1}-\beta_k)(x-\xi)}\to 0\quad\text{as } x\to\infty,\end{aligned}$$

which proves (2.8). We now define $\tilde{\pi}_k$ by

$$\tilde{\pi}_k=\tilde{Q}_k(I-\tilde{Q}_{k-1}) \tag{2.9}$$

and obtain from (2.8) the relation

$$\tilde{Q}_k=\tilde{Q}_{k-1}+\tilde{\pi}_k \tag{2.10}$$

Out proof is complete if we show that $\tilde{\pi}_k$ is a projector for which the following relations hold:

$$\tilde{\pi}_i \tilde{\pi}_j = \delta_{ij} \tilde{\pi}_j, \; i, j = 1, \ldots, k \; . \tag{2.11}$$

Note that the final estimate (2.5) is a consequence of (2.6), (2.10) and the triangle inequality,

$$|\tilde{\pi}_j - \pi_j| = |\tilde{Q}_j - Q_j - (\tilde{Q}_{j-1} - Q_{j-1})| \le |\tilde{Q}_j - Q_j| + |\tilde{Q}_{j-1} - Q_{j-1})|.$$

The definition (2.9) implies the first of the following identities,

$$\tilde{\pi}_k \tilde{Q}_{k-1} = 0 = \tilde{Q}_{k-1} \tilde{\pi}_k. \tag{2.12}$$

The relations (2.12) complete our proof because they imply that $\tilde{\pi}_k = \tilde{Q}_k - \tilde{Q}_{k-1}$ is a projector and that the equalities

$$\tilde{\pi}_k \tilde{\pi}_j = \tilde{\pi}_k \tilde{Q}_{k-1} \tilde{\pi}_j = 0, \quad \tilde{\pi}_j \tilde{\pi}_k = \tilde{\pi}_j \tilde{Q}_{k-1} \tilde{\pi}_k = 0$$

hold for $j = 1, \ldots, k-1$.

It remains to prove the second equality in (2.12). From (1.19) we have $ker(\tilde{Q}_k) = ker(Q_k) = range(I - Q_k) = range(\pi_{k+1})$ and, by the induction hypothesis (2.7), $ker(\tilde{Q}_{k-1}) = ker(Q_{k-1}) = range(I - Q_{k-1}) = range(\pi_k + \pi_{k+1})$. Therefore we obtain the relations

$$range(I - \tilde{Q}_{k-1}) = ker(\tilde{Q}_{k-1}) \supset ker(\tilde{Q}_k) = range(I - \tilde{Q}_k).$$

From these we finally conclude that $(I - \tilde{Q}_{k-1})(I - \tilde{Q}_k) = I - \tilde{Q}_k$ as well as $\tilde{Q}_{k-1}(I - \tilde{Q}_k) = 0$ and

$$\tilde{Q}_{k-1} \tilde{\pi}_k = \tilde{Q}_{k-1}(\tilde{Q}_k - \tilde{Q}_{k-1}) = \tilde{Q}_{k-1}(\tilde{Q}_k - I) = 0.$$

■

As an application of this theorem, we treat the s–dependent operators from (3.11).

Proposition B.1 *Under the assumptions of Theorem 3.1 the operator* $L(s) = \frac{\partial}{\partial x} - M(\cdot, s)$, $0 < |s| < \delta_R, \mathrm{Re}\, s \ge 0$ *has an exponential polychotomy on* $[l-1, \infty)$ *with* $k = 4$ *and data* $(\vec{\alpha}(s), \vec{\beta}(s), 2K_R, \vec{\pi}(s))$. *The constants* $\alpha_j(s), \beta_j(s), j = 1, 2, 3,$ *are given by (3.20) and the projectors* $\vec{\pi}(x, s) = (\pi_1(x, s), \pi_2(x, s), \pi_3(x, s))$ *satisfy*

$$\sup_{x \ge l-1} |\pi_j(x, s) - \pi_j^R(s)| \le 4K^2 \varepsilon(l-1), \quad j = 1, \ldots, 4. \tag{2.13}$$

The projectors $\pi_j^R(s)$ *are defined in (3.21), and we set* $\pi_4(\cdot, s) = I - \sum_{j=1}^{3} \pi_j(\cdot, s)$.

Proof. We simply note that the diagonalization in Lemma 3.1 shows the polychotomy for $L^R(s)$ with data as in (3.20). Then an application of Theorem B.1 proves the assertion. Therefore, the decomposition of fundamental solutions illustrated in Figures 1 and 2 persists for the variable coefficient operator in the tail regions. ■

In our final lemma we treat perturbations of projectors. We show how the condition that an element vanishes under the perturbed projector can be expressed as an inhomogenous equation with the unperturbed projector and a small right–hand side. This lemma is used when solving finite interval problems in Section 4.

Lemma B.1 *Let P and Q be projectors in $\mathbb{R}^n$ that satisfy $|P-Q|<1$ for some subordinate matrix norm $|\cdot|$. Then P and Q have the same rank and the equation $Qz=0$ is equivalent to*

$$Pz = P(I-(P-Q))^{-1}(P-Q)(I-P)z. \tag{2.14}$$

Proof. First note that $(I-(P-Q))^{-1}$ exists since $|P-Q|<1$. Next we show that

$$range(Q) \cap range(I-P) = \{0\}. \tag{2.15}$$

To this end, note that $x \in range(Q) \cap range(I-P)$ implies $Qx = x = (I-P)x$, and then the estimate

$$2|x| = |2x| = |(I-P+Q)x| \leq (1+|P-Q|)|x|$$

yields $x=0$. Since our assumptions are symmetric in P and Q we also obtain $range(P) \cap range(I-Q) = \{0\}$. A count of the dimensions then shows that $rank(P) = \dim range(P) = \dim range(Q) = rank(Q)$.

Let us assume $Qz=0$ for some $z \in \mathbb{R}^n$. Since P is a projector we find $(I-(P-Q))Pz = (P-Q)(I-P)z$ and, therefore, $Pz = (I-(P-Q))^{-1}(P-Q)(I-P)z$. Multiplying by P proves (2.14).

Conversely, let us assume (2.14). Writing $P-Q = P-Q-I+I$ we obtain

$$Pz = P(-I+(I-(P-Q))^{-1})(I-P)z = P(I-(P-Q))^{-1}(I-P)z.$$

Similarly, writing $I-P = I-(P-Q)-Q$ on the right–hand side, we have

$$0 = P(I-(P-Q))^{-1}Qz. \tag{2.16}$$

Finally, note that $(I-P)(I-(P-Q))^{-1}Qz = (Q-Q(I-(P-Q))^{-1}Q)z \in range(Q) \cap range(I-P)$. Therefore, due to (2.15), the term vanishes. Combining this with (2.16) leads to $(I-(P-Q))^{-1}Qz = 0$, and hence $Qz=0$. ■

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