On the blow-up criterion of periodic solutions for micropolar equations in homogeneous Sobolev spaces

Jens Lorenz[a], Wilberclay G. Melo[b,*]

[a] *Department of Mathematics and Statistics, The University of New Mexico, Albuquerque, NM 87131, United States*
[b] *Departamento de Matemática, Universidade Federal de Sergipe, São Cristóvão, SE 49100-000, Brazil*

A R T I C L E   I N F O

A B S T R A C T

We prove lower estimates for space periodic solutions $(\mathbf{u}, \mathbf{w})(t)$ of the micropolar equations in their maximal interval $[0, T^*)$ provided that $T^* < \infty$. For example, we show for $0 < \delta < 1$ that $\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}$ is at least of the order $(T^* - t)^{-(\delta s)/(1+2\delta)}$ for $s \geq 1/2 + \delta$. Moreover, we prove the inequality $\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)} \geq C(T^* - t)^{-1/2}$, which yields the blow-up rate $(T^* - t)^{-s/3}$ for $\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}$ for $s > 3/2$.

1. Introduction

In this paper we consider space periodic solutions for the following *micropolar system* in three dimensions:

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = (\mu + \chi)\Delta \mathbf{u} + \chi \nabla \times \mathbf{w}, \\ \mathbf{w}_t + \mathbf{u} \cdot \nabla \mathbf{w} = \gamma \Delta \mathbf{w} + \kappa \nabla \text{div } \mathbf{w} + \chi \nabla \times \mathbf{u} - 2\chi \mathbf{w}, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0(\cdot), \qquad \mathbf{w}(\cdot, 0) = \mathbf{w}_0(\cdot), \end{cases} \tag{1}$$

where $\mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$ denotes the velocity field, $\mathbf{w}(x, t) = (w_1(x, t), w_2(x, t), w_3(x, t)) \in \mathbb{R}^3$ describes the micro-rotational velocity, and $p(x, t) \in \mathbb{R}$ the hydrostatic pressure. The spatial domain is the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R} \bmod 2\pi)^3$. With $x \in \mathbb{T}^3$ we denote the space variable and $0 \leq t < T^*$ denotes the time variable. Here $[0, T^*)$ is the maximal interval of existence of the strong solution of (1) and we will always assume that T^* is finite. Our aim is to prove blow-up rates for various norms of the vector function $(\mathbf{u}, \mathbf{w})(t)$ and its Fourier coefficients as time approaches the blow-up time T^*.

---

* Corresponding author.
  E-mail address: wilberclay@gmail.com (W.G. Melo).

The positive constants μ, χ, κ, and γ are associated with the specific properties of the fluid. More precisely, μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities. The initial data for the velocity field, given by $\mathbf{u}_0$ in (1), is divergence-free, i.e., div $\mathbf{u}_0 = 0$. To make the pressure unique, we impose the condition

$$\int_{\mathbb{T}^3} p(x, t)\, dx = 0, \quad \forall 0 \le t < T^*.$$

We also assume, without loss of generality, that

$$\int_{\mathbb{T}^3} (\mathbf{u}_0, \mathbf{w}_0)(x)\, dx = 0. \tag{2}$$

There are many works in the literature that prove the existence and uniqueness of solutions for problems related to the micropolar system (1) as, for example, [1–7]. In particular, G.P. Galdi and S. Rionero [4] considered the existence of weak solutions of the following initial boundary-value problem for the micropolar system

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = (\mu + \chi)\varDelta \mathbf{u} + \chi \nabla \times \mathbf{w}, & x \in \Omega,\ t \in [0, T], \\ \mathbf{w}_t + \mathbf{u} \cdot \nabla \mathbf{w} = \gamma \varDelta \mathbf{w} + \kappa \nabla \text{div } \mathbf{w} + \chi \nabla \times \mathbf{u} - 2\chi \mathbf{w}, & x \in \Omega,\ t \in [0, T], \\ \text{div } \mathbf{u} = 0, & x \in \Omega,\ t \in [0, T], \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \qquad \mathbf{w}(x, 0) = \mathbf{w}_0(x), & x \in \Omega, \\ (\mathbf{u}(x, t), \mathbf{w}(x, t)) = 0, & (x, t) \in \partial\Omega \times [0, T]. \end{cases} \tag{3}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Also, for the system (3), J.L. Boldrini, M. Durán and M.A. Rojas-Medar [1] proved, by using the Galerkin method, the existence and uniqueness (local in time) of strong solution in $L^q(\Omega)$, for $q > 3$; here a compact C^2-boundary $\partial\Omega$ was assumed.

Considering the micropolar system (1) with spatial variable given in the whole space $\mathbb{R}^3$, J. Yuan [7] proved the next result, whose proof can be adapted to the space periodic case. For the definition of the operator $\varDelta_j$ used in the theorem, we refer to [7].

Theorem 1.1 (*See [7]*)**.**

1. *Local existence: Let $s_0 > 3/2$ and assume that $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{R}^3)$ with div $\mathbf{u}_0 = 0$. Then there exists a positive $T^* = T^*(\|(\mathbf{u}_0, \mathbf{w}_0)\|_{H^{s_0}(\mathbb{R}^3)})$, with $0 < T^* \le \infty$ so that a unique strong solution $(\mathbf{u}, \mathbf{w})(t) \in C^0([0, T^*); H^{s_0}(\mathbb{R}^3)) \cap C^1((0, T^*); H^{s_0}(\mathbb{R}^3)) \cap C^0((0, T^*); H^{s_0+2}(\mathbb{R}^3))$ for the system (1) exists;*
2. *Blow-up criterion: Assume that $s_0 > 3/2$ and let $(\mathbf{u}, \mathbf{w})(t) \in C^0([0, T^*); H^{s_0}(\mathbb{R}^3)) \cap C^1((0, T^*); H^{s_0}(\mathbb{R}^3)) \cap C^0((0, T^*); H^{s_0+2}(\mathbb{R}^3))$ denote the smooth solution for the system (1) in $0 \le t < T^*$. There is an absolute constant $M > 0$ with the following property: If*

$$\limsup_{\epsilon \to 0}\ \sum_{j \in \mathbb{Z}} \int_{T^*-\epsilon}^{T^*} \|\varDelta_j(\nabla \times \mathbf{u})(t)\|_\infty\, dt := \delta < M,$$

then $\delta = 0$ and the solution $(\mathbf{u}, \mathbf{w})(t)$ can be extended past time $t = T^*$. If

$$\limsup_{\epsilon \to 0}\ \sum_{j \in \mathbb{Z}} \int_{T^*-\epsilon}^{T^*} \|\varDelta_j(\nabla \times \mathbf{u})(t)\|_\infty\, dt \ge M,$$

then the solution $(\mathbf{u}, \mathbf{w})(t)$ blows-up at $t = T^*$.

It is important to point out that if $T^* < \infty$ is the blow-up instant for the solution $(\mathbf{u}, \mathbf{w})(t)$ given by Theorem 1.1, then one obtains $(\mathbf{u}, \mathbf{w}) \in C^\infty(\mathbb{T}^3 \times (0, T^*))$, with $(\mathbf{u}, \mathbf{w})(t) \in C^0((0, T^*); H^s(\mathbb{T}^3))$ for all

$s \geq s_0$. Furthermore, one has

$$\limsup_{t \to T^*} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^{s_0}(\mathbb{T}^3)} = \infty, \tag{4}$$

and also, by using elementary inequalities, one gets

$$\frac{d}{dt}\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^2 + 2\mu\|\nabla\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 + 2\gamma\|\nabla\mathbf{w}(t)\|_{L^2(\mathbb{T}^3)}^2 + 2\kappa\|\operatorname{div}\mathbf{w}(t)\|_{L^2(\mathbb{T}^3)}^2 + 2\chi\|\mathbf{w}(t)\|_{L^2(\mathbb{T}^3)}^2 \leq 0,$$

for all $0 < t < T^*$. As a result, one finds that

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)} \leq \|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2(\mathbb{T}^3)}, \quad \forall 0 \leq t_0 \leq t < T^*. \tag{5}$$

The inequality (5) implies that $(\mathbf{u}, \mathbf{w})(t) \in C^0([0, T^*); H^s(\mathbb{T}^3))$ for every $0 \leq s \leq s_0$ since

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)} \leq \|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{1-\frac{s}{s_0}} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^{s_0}(\mathbb{T}^3)}^{\frac{s}{s_0}},$$

for such s. Thus, all norms $\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)}$, $s \geq 0$, are finite for every $0 < t < T^*$.

For $\chi = 0$ and $\mathbf{w} = 0$ the micropolar system (1) reduces to the viscous incompressible Navier–Stokes equations. This classical system has been studied extensively (see, for example, [8–22] and references therein).

Our main results related to the blow-up of $\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}$ were inspired by particular cases for the Navier–Stokes equations. However, we not only extend the blow-up estimates from the Navier–Stokes equations to the micropolar system (1), but we also relax requirements on the parameter s.

In particular, we derive a lower bound for $\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)}$ for all $t \in [0, T^*)$ and $s > 3/2$. More precisely, Corollary 1.1(iii) extends to the system (1) the inequality

$$\|\mathbf{u}(t)\|_{H^s(\mathbb{T}^3)} \geq \frac{C_1(s)\mu^{\frac{s}{3}}\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}^{1-\frac{2s}{3}}}{(T^* - t)^{\frac{s}{3}}}, \quad \forall 0 \leq t < T^*, \tag{6}$$

where $3/2 < s \leq s_0$ (with $s_0 > 5/2$), obtained in Theorem 1.4 of [8] for the local strong space periodic solution $\mathbf{u}$ of the Navier–Stokes system. In the same paper, J. Benameur also derived the following inequalities:

$$\|\mathbf{u}(t)\|_{H^s(\mathbb{T}^3)} \geq \frac{C_1(s)\mu^{\frac{3s}{4}}\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}^{1-s}}{(T^* - t)^{\frac{s}{4}}}, \tag{7}$$

and

$$\|\widehat{\mathbf{u}}(t)\|_{l^1(\mathbb{Z}^3)} \geq \frac{2^{-\frac{1}{2}}\mu^{\frac{1}{2}}}{(T^* - t)^{\frac{1}{2}}}, \tag{8}$$

for all $0 \leq t < T^*$ and $1 \leq s \leq s_0$ (with $s_0 > 5/2$). We show in Theorem 1.2 and Corollary 1.1 that it is possible to extend all lower bounds obtained in Theorem 1.4 of [8] to the system (1). In addition, we also extended to the system (1) the inequality

$$\|\mathbf{u}(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s)}{(T^* - t)^{\frac{s}{2}-\frac{1}{4}}}, \quad \forall 0 \leq t < T^*, \tag{9}$$

where $1/2 < s < 3/2$. It was obtained in [19] for the local strong space periodic solution $\mathbf{u}$ of the Navier–Stokes equations.

Our main results are stated next.

Theorem 1.2. *Let $s_0 > 3/2$ and let $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{T}^3)$ with $\operatorname{div}\mathbf{u}_0 = 0$. Assume that $(\mathbf{u}, \mathbf{w})(t)$ is the strong space periodic solution of (1) defined in its maximal interval $[0, T^*)$. If $T^* < \infty$ and $\delta \in (0, 1)$, then the following inequalities hold:*

(i) *For each $s \geq 1/2 + \delta$ we have* $\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{(2s)/(1+2\delta)-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s,\delta)\alpha^{s(2-\delta)/(1+2\delta)}}{(T^*-t)^{(\delta s)/(1+2\delta)}}$;

(ii) $\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)} \geq \frac{(2\pi)^{3/2}\alpha^{1/2}}{6(T^*-t)^{1/2}}$,

for all $t \in [0, T^*)$. Here $C_1(s, \delta)$ is a positive constant depending only on s and δ, and $\alpha = \min\{\mu, \gamma\}$.

Notice that (ii) is not trivial since, by Lemma 2.2, the norm $\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)}$ is finite for every $t \in [0, T^*)$. In fact, one has that

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)} := \sum_{\xi} |(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\xi, t)| \leq C_1(s_0) \left(\sum_{\xi} |(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\xi, t)|^2 \right)^{\frac{1}{2} - \frac{3}{4s_0}} \left(\sum_{\xi} |\xi|^{2s_0} |(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\xi, t)|^2 \right)^{\frac{3}{4s_0}}$$

$$=: C_1(s_0) \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^2(\mathbb{Z}^3)}^{1 - \frac{3}{2s_0}} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^{s_0}(\mathbb{T}^3)}^{\frac{3}{2s_0}} \leq C_1(s_0) \|(\mathbf{u}, \mathbf{w})(t)\|_{H^{s_0}(\mathbb{T}^3)}^{1 - \frac{3}{2s_0}} \|(\mathbf{u}, \mathbf{w})(t)\|_{H^{s_0}(\mathbb{T}^3)}^{\frac{3}{2s_0}}$$

$$= C_1(s_0) \|(\mathbf{u}, \mathbf{w})(t)\|_{H^{s_0}(\mathbb{T}^3)} < \infty,$$

for every $t \in [0, T^*)$, and $C_1(s_0)$ is a positive constant depending only on $s_0 (>3/2)$. We have applied the discussion above; moreover, one has used the definitions and results established in Section 2.

Corollary 1.1. *Let $s_0 > 3/2$ and let $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{T}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. Assume that $(\mathbf{u}, \mathbf{w})(t)$ is the strong space periodic solution of (1) defined in its maximal interval $[0, T^*)$. If $T^* < \infty$ then the following inequalities hold:*

(i) *For each $s \geq 1$ we have* $\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{s-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s)\alpha^{(3s)/4}}{(T^*-t)^{s/4}}$;

(ii) *For each $1/2 < s < 3/2$ we have* $\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s)\alpha^{5/4-s/2}}{(T^*-t)^{s/2-1/4}}$;

(iii) *For each $s > 3/2$ we have* $\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{(2s)/3-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s)\alpha^{s/3}}{(T^*-t)^{s/3}}$,

for all $t \in [0, T^*)$. Here $C_1(s)$ is a positive constant that depends only on s, and $\alpha = \min\{\mu, \gamma\}$.

Even if we assume that $\mathbf{w} = 0$ and $\chi = 0$ in (1), Theorem 1.2(i) presents improvements of the estimates (7) and (9) given in [8,19] since we only require $s \geq 1/2+\delta$ with $0 < \delta < 1$. Furthermore, Theorem 1.2(ii) and Corollary 1.1 show explicitly how the inequalities (6)–(9) can be extended to the micropolar equations (1).

Remark 1.1. 1. For $s \geq 1/2+\delta$ with $\delta \in (0,1)$ it is easy to check, from Theorem 1.2(i) and (5), the inequality

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s,\delta)\alpha^{\frac{s(2-\delta)}{1+2\delta}} \|(\mathbf{u}_0, \mathbf{w}_0)\|_{L^2(\mathbb{T}^3)}^{1-\frac{2s}{1+2\delta}}}{(T^*-t)^{\frac{\delta s}{1+2\delta}}}, \quad \forall t \in [0, T^*),$$

where $C_1(s, \delta)$ is a positive constant depending only on s and δ, and $\alpha = \min\{\mu, \gamma\}$. This extends the estimates (7) and (9) given in [8,19] to the micropolar equations (1) since $\|\cdot\|_{H^s} \geq \|\cdot\|_{\dot{H}^s}$.

2. Corollary 1.1(i) is the particular case of Theorem 1.2(i) which is obtained for $\delta = 1/2$. Also, from Corollary 1.1(i), one has

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)} \geq \frac{C_1(s)\alpha^{\frac{3s}{4}} \|(\mathbf{u}_0, \mathbf{w}_0)\|_{L^2(\mathbb{T}^3)}^{1-s}}{(T^*-t)^{\frac{s}{4}}}, \quad \forall t \in [0, T^*),$$

provided that $s \geq 1$. For the Navier–Stokes system, the corresponding estimate was proved in [8] (see (7)). Specializing further and choosing $s = 1$ in Corollary 1.1(i), one obtains the classical Leray inequality

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^1(\mathbb{T}^3)} \geq \frac{C\alpha^{\frac{3}{4}}}{(T^*-t)^{\frac{1}{4}}}, \quad \forall t \in [0, T^*),$$

where C is an absolute positive constant and $\alpha = \min\{\mu, \gamma\}$. (See [8,14,15] for the Navier–Stokes equations).

3. Similarly, Corollary 1.1(ii) is also an immediate consequence of Theorem 1.2(i). The result is obtained if one chooses $\delta = s - 1/2$ in Theorem 1.2(i) provided that $1/2 < s < 3/2$. For the Navier–Stokes system, the corresponding result was obtained in [19] (see (9)).

4. We will show below that Corollary 1.1(iii) follows from Theorem 1.2(ii). As a result, we get

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)} \geq \frac{C_1(s)\alpha^{\frac{s}{3}} \|(\mathbf{u}_0, \mathbf{w}_0)\|_{L^2(\mathbb{T}^3)}^{1-\frac{2s}{3}}}{(T^* - t)^{\frac{s}{3}}}, \quad \forall t \in [0, T^*),$$

provided that $s > 3/2$. For the Navier–Stokes system, the corresponding result was obtained in [8] (see (6)).

In the next section, we list notations and definitions and also present two auxiliary results. In the last section, Theorem 1.2 and Corollary 1.1 are proved. As usual, $C_1(s, \delta), C_1(s), C_2$, and C_3 denote positive constants, which may change their value from line to line.

2. Notations, definitions and auxiliary results

In this section, we introduce notations and definitions used in the paper. We also present two inequalities which play an important role in the proof of Theorem 1.2 and Corollary 1.1.

2.1. Notations and definitions

Boldface letters denote vector fields. For example,

$$\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)),$$

where $x \in \mathbb{T}^3$ and $0 \leq t < T^*$, denotes the fluid velocity field. Similarly,

$$(\mathbf{u}, \mathbf{w}) = (\mathbf{u}, \mathbf{w})(x, t) = (\mathbf{u}(x, t), \mathbf{w}(x, t))$$

denotes the pair of the velocity field $\mathbf{u}$ and the micro-rotational velocity $\mathbf{w}$.

We define the Fourier coefficients of $\mathbf{u}$ and $(\mathbf{u}, \mathbf{w})$ by

$$\widehat{\mathbf{u}}(\xi, t) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} \exp(-i\xi \cdot x)\mathbf{u}(x, t)\, dx$$

and

$$(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\xi, t) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} \exp(-i\xi \cdot x)(\mathbf{u}, \mathbf{w})(x, t)\, dx,$$

respectively. Here $\xi \in \mathbb{Z}^3$, $\xi \cdot x := \sum_{j=1}^{3} \xi_j x_j$, with $\xi = (\xi_1, \xi_2, \xi_3)$ and $x = (x_1, x_2, x_3)$. Note that (2) yields

$$(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(0, t) = 0 \quad \text{for } 0 \leq t < T^*.$$

Let $\dot{H}^s(\mathbb{T}^3) = \left\{ \mathbf{u}(\cdot, t) \in S'(\mathbb{T}^3) : \widehat{\mathbf{u}}(0, t) = 0, \sum_{\xi \in \mathbb{Z}^3} |\xi|^{2s} |\widehat{\mathbf{u}}(\xi, t)|^2 < \infty \right\}$ denote the homogeneous Sobolev space, where $S'(\mathbb{T}^3)$ is the space of distributions on the torus $\mathbb{T}^3$. The $\dot{H}^s$-norm is defined by

$$\|\mathbf{u}(t)\|_{\dot{H}^s(\mathbb{T}^3)}^2 = \|\mathbf{u}(\cdot, t)\|_{\dot{H}^s(\mathbb{T}^3)}^2 := \sum_{\xi \neq 0} |\xi|^{2s} |\widehat{\mathbf{u}}(\xi, t)|^2,$$

and corresponds to the scalar product

$$\langle \mathbf{u}(t), \mathbf{w}(t) \rangle_{\dot{H}^s(\mathbb{T}^3)} = \langle \mathbf{u}(\cdot, t), \mathbf{w}(\cdot, t) \rangle_{\dot{H}^s(\mathbb{T}^3)} := \sum_{\xi \neq 0} |\xi|^{2s} \widehat{\mathbf{u}}(\xi, t) \cdot \widehat{\mathbf{w}}(\xi, t).$$

Here $x \cdot y := x_1 \overline{y}_1 + x_2 \overline{y}_2 + x_3 \overline{y}_3$ for complex vectors.

The standard L^2-inner product is

$$\langle \mathbf{u}(t), \mathbf{w}(t) \rangle_{L^2(\mathbb{T}^3)} = \langle \mathbf{u}(\cdot, t), \mathbf{w}(\cdot, t) \rangle_{L^2(\mathbb{T}^3)} := \int_{\mathbb{T}^3} \mathbf{u}(x, t) \cdot \mathbf{w}(x, t) \, dx$$

and

$$\|\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 = \|\mathbf{u}(\cdot, t)\|_{L^2(\mathbb{T}^3)}^2 := \int_{\mathbb{T}^3} |\mathbf{u}(x, t)|^2 \, dx,$$

defines the L^2-norm.

For $1 \leq q < \infty$, define the space

$$l^q(\mathbb{Z}^3) = \left\{ (a_\xi)_{\xi \in \mathbb{Z}^3} : a_\xi \in \mathbb{C}, \ \sum_{\xi \in \mathbb{Z}^3} |a_\xi|^q < \infty \right\}$$

endowed with the norm

$$\|a\|_{l^q(\mathbb{Z}^3)}^q := \sum_\xi |a_\xi|^q.$$

Similarly, we define

$$\|\widehat{\mathbf{u}}(t)\|_{l^q(\mathbb{Z}^3)}^q = \|\widehat{\mathbf{u}}(\cdot, t)\|_{l^q(\mathbb{Z}^3)}^q := \sum_\xi |\widehat{\mathbf{u}}(\xi, t)|^q,$$

for $1 \leq q < \infty$. For $q = 1$ we will also use

$$\|\widehat{\nabla \mathbf{u}}(t)\|_{l^1(\mathbb{Z}^3)} = \|\widehat{\nabla \mathbf{u}}(\cdot, t)\|_{l^1(\mathbb{Z}^3)} := \sum_{j=1}^3 \|\widehat{\nabla u_j}(\cdot, t)\|_{l^1(\mathbb{Z}^3)} := \sum_{j=1}^3 \sum_\xi |\widehat{\nabla u_j}(\xi, t)|,$$

where $\nabla \mathbf{u} = (\nabla u_1, \nabla u_2, \nabla u_3)$ with $\nabla u_j = (D_1 u_j, D_2 u_j, D_3 u_j)$, and $D_i = \partial/\partial x_i$, for $i, j = 1, 2, 3$.

The standard norm on the Sobolev space

$$H^s(\mathbb{T}^3) = \left\{ \mathbf{u}(\cdot, t) \in S'(\mathbb{T}^3) : \sum_{\xi \in \mathbb{Z}^3} (1 + |\xi|^2)^s |\widehat{\mathbf{u}}(\xi, t)|^2 < \infty \right\},$$

i.e., the $H^s(\mathbb{T}^3)$-norm, is defined by

$$\|\mathbf{u}(t)\|_{H^s(\mathbb{T}^3)}^2 = \|\mathbf{u}(\cdot, t)\|_{H^s(\mathbb{T}^3)}^2 := \sum_\xi (1 + |\xi|^2)^s |\widehat{\mathbf{u}}(\xi, t)|^2,$$

and corresponds to the $H^s(\mathbb{R}^3)$-inner product

$$\langle \mathbf{u}(t), \mathbf{w}(t) \rangle_{H^s(\mathbb{T}^3)} := \sum_\xi (1 + |\xi|^2)^s \widehat{\mathbf{u}}(\xi, t) \cdot \widehat{\mathbf{w}}(\xi, t).$$

For the tensor product and convolution we use the notations

$$\mathbf{u} \otimes \mathbf{w} = (\mathbf{u} \otimes \mathbf{w})(x, t) := (w_1(x, t)\mathbf{u}(x, t), w_2(x, t)\mathbf{u}(x, t), w_3(x, t)\mathbf{u}(x, t)),$$

$$u * w(x) = \int_{\mathbb{R}^3} u(x - y)w(y) \, dy \quad \text{and} \quad a * b(\xi) = \sum_\rho a_\rho b_{\xi - \rho}.$$

Here the sum is taken over $\rho \in \mathbb{Z}^3$.

The above definitions for vector fields $\mathbf{u}$ extend trivially to the vector fields $(\mathbf{u}, \mathbf{w})$. For example,

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}^2 := \|\mathbf{u}(t)\|_{\dot{H}^s(\mathbb{T}^3)}^2 + \|\mathbf{w}(t)\|_{\dot{H}^s(\mathbb{T}^3)}^2,$$

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{H^s(\mathbb{T}^3)}^2 := \|\mathbf{u}(t)\|_{H^s(\mathbb{T}^3)}^2 + \|\mathbf{w}(t)\|_{H^s(\mathbb{T}^3)}^2,$$

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^2 := \|\mathbf{u}(t)\|_{L^2(\mathbb{T}^3)}^2 + \|\mathbf{w}(t)\|_{L^2(\mathbb{T}^3)}^2,$$

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^q(\mathbb{Z}^3)}^q = \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\cdot, t)\|_{l^q(\mathbb{Z}^3)}^q := \sum_{\xi} |(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\xi, t)|^q,$$

$$\|(\widehat{\Delta\mathbf{u}}, \widehat{\Delta\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)} = \|(\widehat{\Delta\mathbf{u}}, \widehat{\Delta\mathbf{w}})(\cdot, t)\|_{l^1(\mathbb{Z}^3)} := \sum_{\xi} |(\widehat{\Delta\mathbf{u}}, \widehat{\Delta\mathbf{w}})(\xi, t)|,$$

where $1 \le q < \infty$. Here Δ denotes the Laplacian and $\Delta\mathbf{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$ with $\Delta u_j = \sum_{i=1}^3 D_i^2 u_j$ for all $j = 1, 2, 3$.

It will often be convenient to use shorter notations and write $\|\mathbf{u}\|_{\dot{H}^s}$ instead of $\|\mathbf{u}(t)\|_{\dot{H}^s(\mathbb{T}^3)}$, etc.

2.2. Two auxiliary lemmas

We present two Lemmas which are important for our main results. The first one is due to J.-Y. Chemin.

Lemma 2.1 (*See [10]*). *Let η, η' be two real numbers such that $\eta < 3/2$ and $\eta + \eta' > 0$. If $f, g \in \dot{H}^\eta(\mathbb{T}^3) \cap \dot{H}^{\eta'}(\mathbb{T}^3)$, then*

$$\|fg\|_{\dot{H}^{\eta+\eta'-\frac{3}{2}}(\mathbb{T}^3)} \le C(\eta, \eta')(\|f\|_{\dot{H}^\eta(\mathbb{T}^3)}\|g\|_{\dot{H}^{\eta'}(\mathbb{T}^3)} + \|f\|_{\dot{H}^{\eta'}(\mathbb{T}^3)}\|g\|_{\dot{H}^\eta(\mathbb{T}^3)})$$

where $C(\eta, \eta')$ is a positive constant that depends on η and η' only. If, in addition, we have $\eta' < 3/2$, then

$$\|fg\|_{\dot{H}^{\eta+\eta'-\frac{3}{2}}(\mathbb{T}^3)} \le C(\eta, \eta')\|f\|_{\dot{H}^\eta(\mathbb{T}^3)}\|g\|_{\dot{H}^{\eta'}(\mathbb{T}^3)}.$$

The second Lemma is due to J. Benameur and will be helpful in the proof of Corollary 1.1(iii). Because of a misprint in [8] we add our proof.

Lemma 2.2. *Let $a = (a_\xi)_{\xi \in \mathbb{Z}^3}$ and $s > 3/2$. Then,*

$$\sum_{\xi \ne 0} |a_\xi| \le C_1(s) \left(\sum_{\xi \ne 0} |a_\xi|^2 \right)^{\frac{1}{2} - \frac{3}{4s}} \left(\sum_{\xi \ne 0} |\xi|^{2s}|a_\xi|^2 \right)^{\frac{3}{4s}},$$

where $C_1(s)$ is a positive constant that depends on s only.

Proof. Let $\beta \ge 1$ denote a constant which will be chosen below. By the Cauchy–Schwarz inequality we have

$$\sum_{\xi \ne 0} |a_\xi| = \sum_{1 \le |\xi| \le \beta} |a_\xi| + \sum_{|\xi| > \beta} |a_\xi|$$

$$\le \left(\sum_{1 \le |\xi| \le \beta} |a_\xi|^2 \right)^{1/2} \left(\sum_{1 \le |\xi| \le \beta} 1 \right)^{1/2} + \left(\sum_{|\xi| > \beta} |\xi|^{2s}|a_\xi|^2 \right)^{1/2} \left(\sum_{|\xi| > \beta} |\xi|^{-2s} \right)^{1/2}.$$

For $[\beta] = \max\{m \in \mathbb{Z} : m \le \beta < m + 1\}$, we get

$$\sum_{1 \le |\xi| \le \beta} 1 \le C \int_1^{[\beta]+1} r^2 \, dr = C[([\beta] + 1)^3 - 1^3] \le C[(\beta + 1)^3 - 1] \le C[(2\beta)^3 - 1] \le C\beta^3,$$

since $1 \leq \beta$, and $[\beta] \leq \beta < [\beta] + 1$. Similarly, one obtains

$$\sum_{|\xi| > \beta} |\xi|^{-2s} \leq C \int_{[\beta]}^{\infty} r^{2-2s} dr \leq C_1(s) \beta^{-2s+3},$$

where $C_1(s)$ is a positive constant that depends on s. Consequently,

$$\sum_{\xi \neq 0} |a_\xi| \leq C_1(s) \left[\beta^{3/2} \left(\sum_{\xi \neq 0} |a_\xi|^2 \right)^{1/2} + \beta^{-s+3/2} \left(\sum_{\xi \neq 0} |\xi|^{2s} |a_\xi|^2 \right)^{1/2} \right].$$

Denote $A(s) = (\sum_{\xi \neq 0} |\xi|^{2s} |a_\xi|^2)^{1/2}$, for $s \geq 0$. Thus,

$$\sum_{\xi \neq 0} |a_\xi| \leq C_1(s) \left[\beta^{3/2} A(0) + \beta^{-s+3/2} A(s) \right].$$

We may assume $A(0) > 0$ and choose

$$\beta = (A(s)/A(0))^{1/s}$$

to obtain

$$\sum_{\xi \neq 0} |a_\xi| \leq C_1(s) A(0)^{1 - \frac{3}{2s}} A(s)^{\frac{3}{2s}}.$$

This proves Lemma 2.2.   ■

3. Proof of Theorem 1.2 and Corollary 1.1

In this section we prove three Propositions, which will imply Theorem 1.2 and Corollary 1.1. The first Proposition shows the results of Theorem 1.2(i) and Corollary 1.1(i) and (ii).

Proposition 3.1. *Let $s_0 > 3/2$ and let $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{T}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. Assume that $(\mathbf{u}, \mathbf{w})(t)$ is the strong space periodic solution of (1) defined in its maximal interval $[0, T^*)$. If $T^* < \infty$ then for each $s \geq 1/2 + \delta$, $0 < \delta < 1$, we have*

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{\frac{2s}{1+2\delta} - 1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s, \delta) \alpha^{\frac{s(2-\delta)}{1+2\delta}}}{(T^* - t)^{\frac{\delta s}{1+2\delta}}} \quad \text{for } 0 \leq t < T^*.$$

In particular, for each $s \geq 1$ it holds that

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{s-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s) \alpha^{\frac{3s}{4}}}{(T^* - t)^{\frac{s}{4}}} \quad \text{for } 0 \leq t < T^*,$$

and for $1/2 < s < 3/2$ we have

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \geq \frac{C_1(s) \alpha^{\frac{5}{4} - \frac{s}{2}}}{(T^* - t)^{\frac{s}{2} - \frac{1}{4}}} \quad \text{for } 0 \leq t < T^*.$$

Here $C_1(s, \delta)$ is a positive constant that depends on s and δ, $C_1(s)$ is a positive constant that depends on s; also, $\alpha = \min\{\mu, \gamma\}$.

Proof. Taking the inner product $\langle \mathbf{u}, \cdot \rangle_{\dot{H}^s}$ with the first equation of (1) we obtain

$$\langle \mathbf{u}, \mathbf{u}_t \rangle_{\dot{H}^s} = (\mu + \chi) \langle \mathbf{u}, \Delta \mathbf{u} \rangle_{\dot{H}^s} - \langle \mathbf{u}, \mathbf{u} \cdot \nabla \mathbf{u} \rangle_{\dot{H}^s} - \langle \mathbf{u}, \nabla p \rangle_{\dot{H}^s} + \chi \langle \mathbf{u}, \nabla \times \mathbf{w} \rangle_{\dot{H}^s}.$$

Here

$$\langle \mathbf{u}, \Delta \mathbf{u}\rangle_{\dot{H}^s} = \sum_{\xi \neq 0} |\xi|^{2s}\widehat{\mathbf{u}}(\xi) \cdot \widehat{\Delta \mathbf{u}}(\xi) = -\sum_{\xi \neq 0} |\xi|^{2s+2}|\widehat{\mathbf{u}}(\xi)|^2 = -\sum_{\xi \neq 0} |\xi|^{2s}|\widehat{\nabla \mathbf{u}}(\xi)|^2 = -\|\nabla \mathbf{u}\|_{\dot{H}^s}^2, \tag{10}$$

and

$$\langle \mathbf{u}, \mathbf{u}\cdot\nabla\mathbf{u}\rangle_{\dot{H}^s} = \sum_{\xi \neq 0} |\xi|^{2s}\widehat{\mathbf{u}}(\xi) \cdot \widehat{\mathbf{u}\cdot\nabla\mathbf{u}}(\xi) = -\sum_{\xi \neq 0} |\xi|^{2s}\sum_{j=1}^{3}[i\xi_j\widehat{\mathbf{u}}(\xi)]\cdot\widehat{u_j\mathbf{u}}(\xi)$$

$$= -\sum_{\xi \neq 0} |\xi|^{2s}\sum_{j=1}^{3}\widehat{D_j\mathbf{u}}(\xi)\cdot\widehat{u_j\mathbf{u}}(\xi) = -\langle\nabla\mathbf{u}, \mathbf{u}\otimes\mathbf{u}\rangle_{\dot{H}^s}.$$

To obtain the second equality we used that div $\mathbf{u} = 0$. We also have

$$\langle \mathbf{u}, \nabla p\rangle_{\dot{H}^s} = -\sum_{\xi \neq 0} |\xi|^{2s}\sum_{j=1}^{3}[i\xi_j\widehat{u_j}(\xi)]\widehat{p}(\xi) = -\sum_{\xi \neq 0} |\xi|^{2s}\widehat{\text{div }\mathbf{u}}(\xi)\widehat{p}(\xi) = 0. \tag{11}$$

Integrating by parts yields

$$\langle \mathbf{u}, \nabla\times\mathbf{w}\rangle_{\dot{H}^s} = \sum_{\xi \neq 0} |\xi|^{2s}\widehat{\mathbf{u}}(\xi)\cdot\widehat{\nabla\times\mathbf{w}}(\xi) = \sum_{\xi \neq 0} |\xi|^{2s}\widehat{\nabla\times\mathbf{u}}(\xi)\cdot\widehat{\mathbf{w}}(\xi) = \langle\nabla\times\mathbf{u}, \mathbf{w}\rangle_{\dot{H}^s}, \tag{12}$$

since $\widehat{\nabla\times\mathbf{u}}(\xi) = i\xi\times\widehat{\mathbf{u}}$. Therefore,

$$\langle \mathbf{u}, \mathbf{u}_t\rangle_{\dot{H}^s} = -(\mu+\chi)\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \langle\nabla\mathbf{u}, \mathbf{u}\otimes\mathbf{u}\rangle_{\dot{H}^s} + \chi\langle\nabla\times\mathbf{u}, \mathbf{w}\rangle_{\dot{H}^s}. \tag{13}$$

Taking the inner product $\langle\mathbf{w}, \cdot\rangle_{\dot{H}^s}$ with the second equation of (1) we obtain

$$\langle \mathbf{w}, \mathbf{w}_t\rangle_{\dot{H}^s} = -\gamma\|\nabla\mathbf{w}\|_{\dot{H}^s}^2 - \kappa\|\text{div }\mathbf{w}\|_{\dot{H}^s}^2 - 2\chi\|\mathbf{w}\|_{\dot{H}^s}^2 + \langle\nabla\mathbf{w}, \mathbf{u}\otimes\mathbf{w}\rangle_{\dot{H}^s} + \chi\langle\mathbf{w}, \nabla\times\mathbf{u}\rangle_{\dot{H}^s}, \tag{14}$$

since

$$\langle \mathbf{w}, \nabla\text{div }\mathbf{w}\rangle_{\dot{H}^s} = \sum_{\xi \neq 0} |\xi|^{2s}\widehat{\mathbf{w}}(\xi)\cdot\widehat{\nabla\text{div }\mathbf{w}}(\xi) = -\sum_{\xi \neq 0} |\xi|^{2s}\sum_{j=1}^{3}\widehat{D_j w_j}(\xi)\widehat{\text{div }\mathbf{w}}(\xi) = -\|\text{div }\mathbf{w}\|_{\dot{H}^s}^2. \tag{15}$$

Thus, using (13) and (14), one obtains

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u}, \mathbf{w})\|_{\dot{H}^s}^2 + (\mu+\chi)\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \gamma\|\nabla\mathbf{w}\|_{\dot{H}^s}^2 + \kappa\|\text{div }\mathbf{w}\|_{\dot{H}^s}^2 + 2\chi\|\mathbf{w}\|_{\dot{H}^s}^2$$
$$= \mathcal{R}e\left[\langle\nabla\mathbf{u}, \mathbf{u}\otimes\mathbf{u}\rangle_{\dot{H}^s}\right] + \chi\mathcal{R}e\left[\langle\nabla\times\mathbf{u}, \mathbf{w}\rangle_{\dot{H}^s}\right] + \mathcal{R}e\left[\langle\nabla\mathbf{w}, \mathbf{u}\otimes\mathbf{w}\rangle_{\dot{H}^s}\right] + \chi\mathcal{R}e\left[\langle\mathbf{w}, \nabla\times\mathbf{u}\rangle_{\dot{H}^s}\right]$$

where $\mathcal{R}e[z]$ denotes the real part of the complex number z. Using the estimate $|\widehat{\nabla\times\mathbf{u}}(\xi)| \leq |\xi||\widehat{\mathbf{u}}(\xi)|$ one obtains

$$|\langle\nabla\times\mathbf{u}, \mathbf{w}\rangle_{\dot{H}^s}| \leq \|\nabla\mathbf{u}\|_{\dot{H}^s}\|\mathbf{w}\|_{\dot{H}^s} \leq \frac{1}{2}\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \frac{1}{2}\|\mathbf{w}\|_{\dot{H}^s}^2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u}, \mathbf{w})\|_{\dot{H}^s}^2 + \mu\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \gamma\|\nabla\mathbf{w}\|_{\dot{H}^s}^2 + \kappa\|\text{div }\mathbf{w}\|_{\dot{H}^s}^2 + \chi\|\mathbf{w}\|_{\dot{H}^s}^2$$
$$\leq \|\nabla\mathbf{u}\|_{\dot{H}^s}\|\mathbf{u}\otimes\mathbf{u}\|_{\dot{H}^s} + \|\nabla\mathbf{w}\|_{\dot{H}^s}\|\mathbf{u}\otimes\mathbf{w}\|_{\dot{H}^s}. \tag{16}$$

By Lemma 2.1 with $\eta = 1/2 + \delta$ and $\eta' = 1 + s - \delta$ one gets

$$
\begin{aligned}
\|\mathbf{u} \otimes \mathbf{u}\|_{\dot{H}^s} &\leq C_1(s,\delta)\|\mathbf{u}\|_{\dot{H}^\eta}\|\mathbf{u}\|_{\dot{H}^{\eta'}} \\
&:= C_1(s,\delta)\|\mathbf{u}\|_{\dot{H}^\eta} \left(\sum_{\xi \neq 0} |\xi|^{2s+2-2\delta}|\widehat{\mathbf{u}}|^2 \right)^{\frac{1}{2}} \\
&\leq C_1(s,\delta)\|\mathbf{u}\|_{\dot{H}^\eta} \left(\sum_{\xi \neq 0} |\xi|^{2s}|\widehat{\mathbf{u}}|^2 \right)^{\frac{\delta}{2}} \left(\sum_{\xi \neq 0} |\xi|^{2s+2}|\widehat{\mathbf{u}}|^2 \right)^{\frac{1-\delta}{2}} \\
&=: C_1(s,\delta)\|\mathbf{u}\|_{\dot{H}^\eta}\|\mathbf{u}\|_{\dot{H}^s}^{\delta}\|\nabla\mathbf{u}\|_{\dot{H}^s}^{1-\delta},
\end{aligned}
$$

where Hölder's inequality was used in the second estimate. As above, $C_1(s,\delta)$ is a positive constant depending on s and δ. Similarly, one obtains the bound

$$
\|\mathbf{u} \otimes \mathbf{w}\|_{\dot{H}^s} \leq C_1(s,\delta)(\|\mathbf{u}\|_{\dot{H}^\eta}\|\mathbf{w}\|_{\dot{H}^s}^{\delta}\|\nabla\mathbf{w}\|_{\dot{H}^s}^{1-\delta} + \|\mathbf{w}\|_{\dot{H}^\eta}\|\mathbf{u}\|_{\dot{H}^s}^{\delta}\|\nabla\mathbf{u}\|_{\dot{H}^s}^{1-\delta}).
$$

Hence, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^s}^2 + \mu\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \gamma\|\nabla\mathbf{w}\|_{\dot{H}^s}^2 + \kappa\|\mathrm{div}\,\mathbf{w}\|_{\dot{H}^s}^2 + \chi\|\mathbf{w}\|_{\dot{H}^s}^2 \\
\leq C_1(s,\delta)\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^\eta}\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^s}^{\delta}\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{\dot{H}^s}^{2-\delta}
\end{aligned}
$$

and consequently

$$
\frac{d}{dt}\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^s}^2 + \min\{\mu,\gamma\}\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{\dot{H}^s}^2 + 2\kappa\|\mathrm{div}\,\mathbf{w}\|_{\dot{H}^s}^2 + 2\chi\|\mathbf{w}\|_{\dot{H}^s}^2 \leq C_2\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^\eta}^{\frac{2}{\delta}}\|(\mathbf{u},\mathbf{w})\|_{\dot{H}^s}^2.
$$

Here we applied Young's inequality ($ab \leq (a^c/c) + (b^d/d)$ with c and d conjugate exponents) and set $C_2 = C_2(s,\delta,\mu,\gamma) = C_1(s,\delta)\alpha^{\frac{\delta-2}{\delta}}$ with

$$
\alpha = \min\{\mu,\gamma\}. \tag{17}
$$

Gronwall's Lemma yields

$$
\|(\mathbf{u},\mathbf{w})(t)\|_{\dot{H}^s}^2 \leq \|(\mathbf{u},\mathbf{w})(t_0)\|_{\dot{H}^s}^2 \exp\left(C_2 \int_{t_0}^{t} \|(\mathbf{u},\mathbf{w})(\tau)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} d\tau \right), \tag{18}
$$

for all $0 \leq t_0 \leq t < T^*$. In particular,

$$
\|(\mathbf{u},\mathbf{w})(t)\|_{\dot{H}^{s_0}}^2 \leq \|(\mathbf{u},\mathbf{w})(t_0)\|_{\dot{H}^{s_0}}^2 \exp\left(C_2 \int_{t_0}^{t} \|(\mathbf{u},\mathbf{w})(\tau)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} d\tau \right), \tag{19}
$$

since $s_0 \geq 1/2 + \delta$. From (4) and (19), it follows that

$$
\int_{t_0}^{T^*} \|(\mathbf{u},\mathbf{w})(\tau)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} d\tau = \infty, \quad \forall \eta \in (1/2, 3/2). \tag{20}
$$

Thus, the limit relation

$$
\limsup_{t \to T^*} \|(\mathbf{u},\mathbf{w})(\tau)\|_{\dot{H}^\eta} = \infty
$$

holds for all $\eta \in (1/2, 3/2)$. Parseval identity yields for $s \geq \sigma$:

$$
\begin{aligned}
\|\theta\|_{\dot{H}^\sigma} &= \left(\sum_{\xi \neq 0} |\xi|^{2\sigma}|\widehat{\theta}(\xi)|^2 \right)^{1/2} \leq \left(\sum_{\xi \neq 0} |\widehat{\theta}(\xi)|^2 \right)^{\frac{1}{2} - \frac{\sigma}{2s}} \left(\sum_{\xi \neq 0} |\xi|^{2s}|\widehat{\theta}(\xi)|^2 \right)^{\frac{\sigma}{2s}} \\
&\leq \|\widehat{\theta}\|_{l^2}^{\frac{s-\sigma}{s}}\|\theta\|_{\dot{H}^s}^{\frac{\sigma}{s}} = \|\theta\|_{L^2}^{1-\frac{\sigma}{s}}\|\theta\|_{\dot{H}^s}^{\frac{\sigma}{s}},
\end{aligned} \tag{21}
$$

for all $\theta \in H^s$. Then, by (5), we obtain

$$\limsup_{t \to T^*} \|(\mathbf{u}, \mathbf{w})(\tau)\|_{\dot{H}^s} = \infty, \quad \forall s > 1/2, \tag{22}$$

since $s \geq \eta = 1/2 + \delta$. By (21) we conclude that

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} \leq \|(\mathbf{u}, \mathbf{w})(t)\|_{L^2}^{\frac{2(s-\eta)}{\delta s}} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s}^{\frac{2\eta}{\delta s}}, \quad \forall 0 \leq t < T^*.$$

Consequently, by (18) and (5):

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} \exp\left(-C_2 \int_{t_0}^t \|(\mathbf{u}, \mathbf{w})(\tau)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} d\tau\right) \leq \|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2}^{\frac{2(s-\eta)}{\delta s}} \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s}^{\frac{2\eta}{\delta s}}, \tag{23}$$

since $0 \leq t_0 \leq t < T^*$. Integrating the inequality (23) from t_0 to T, with $T < T^*$, one concludes that

$$-\frac{1}{C_2} \int_{t_0}^T \frac{d}{dt} \exp\left(-C_2 \int_{t_0}^t \|(\mathbf{u}, \mathbf{w})(\tau)\|_{\dot{H}^\eta}^{\frac{2}{\delta}} d\tau\right) dt \leq \|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2}^{\frac{2(s-\eta)}{\delta s}} \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s}^{\frac{2\eta}{\delta s}}(T - t_0).$$

Then, taking the limit as $T \to T^*$, we obtain:

$$\|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2}^{\frac{2(s-\eta)}{\delta s}} \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s}^{\frac{2\eta}{\delta s}} \geq \frac{1}{C_2(T^* - t_0)}, \quad \forall 0 \leq t_0 < T^*,$$

by (20). Finally,

$$\|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2}^{\frac{2s}{1+2\delta}-1} \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s} \geq \frac{C_1(s, \delta) \alpha^{\frac{s(2-\delta)}{1+2\delta}}}{(T^* - t_0)^{\frac{\delta s}{1+2\delta}}}, \quad \forall 0 \leq t_0 < T^*,$$

where $C_1(s, \delta)$ depends on s and δ, and α is given by (17). This proves Theorem 1.2(i).

In order to prove Corollary 1.1(i) it suffices to take $\delta = 1/2$ (i.e, $s \geq 1$). One obtains

$$\|(\mathbf{u}, \mathbf{w})(t_0)\|_{L^2}^{s-1} \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s} \geq \frac{C_1(s) \alpha^{\frac{3s}{4}}}{(T^* - t_0)^{\frac{s}{4}}}, \quad \forall 0 \leq t_0 < T^*.$$

Furthermore, if we choose $1/2 < s < 3/2$ and $\delta = s - 1/2$ we have that

$$\|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s} \geq \frac{C_1(s) \alpha^{\frac{5}{4}-\frac{s}{2}}}{(T^* - t_0)^{\frac{s}{2}-\frac{1}{4}}}, \quad \forall 0 \leq t_0 < T^*.$$

Consequently, Corollary 1.1(ii) follows and Proposition 3.1 is proved. ∎

A proof of Theorem 1.2(ii) is presented next.

Proposition 3.2. *Let $s_0 > 3/2$ and let $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{T}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. Assume that $(\mathbf{u}, \mathbf{w})(t)$ is the strong space periodic solution of (1) defined in its maximal interval $[0, T^*)$. If $T^* < \infty$ then*

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)} \geq \frac{(2\pi)^{3/2} \alpha^{\frac{1}{2}}}{6(T^* - t)^{\frac{1}{2}}}, \quad \forall 0 \leq t < T^*,$$

where $\alpha = \min\{\mu, \gamma\}$.

Proof. First note that

$$\|\mathbf{u} \otimes \mathbf{w}\|_{\dot{H}^s}^2 := \sum_{\xi \neq 0} |\xi|^{2s} |\widehat{\mathbf{u} \otimes \mathbf{w}}(\xi)|^2 \leq C \sum_{\xi \neq 0} |\xi|^{2s} \left[\sum_\rho |\widehat{\mathbf{u}}(\rho)| |\widehat{\mathbf{w}}(\xi - \rho)|\right]^2,$$

where C is a positive constant. Using the elementary inequalities

$$|\xi|^s \leq (|\xi - \rho| + |\rho|)^s \leq (2 \max\{|\xi - \rho|, |\rho|\})^s \leq 2^s (|\xi - \rho|^s + |\rho|^s)$$

one obtains that

$$\|\mathbf{u} \otimes \mathbf{w}\|_{\dot{H}^s}^2 \leq C_1(s)[\|(|\cdot|^s|\widehat{\mathbf{u}}|) * |\widehat{\mathbf{w}}|\|_{l^2}^2 + \|(|\cdot|^s|\widehat{\mathbf{w}}|) * |\widehat{\mathbf{u}}|\|_{l^2}^2].$$

By Young's inequality for convolutions one obtains that

$$\|\mathbf{u} \otimes \mathbf{w}\|_{\dot{H}^s} \leq C_1(s)[\|\mathbf{u}\|_{\dot{H}^s}\|\widehat{\mathbf{w}}\|_{l^1} + \|\mathbf{w}\|_{\dot{H}^s}\|\widehat{\mathbf{u}}\|_{l^1}].$$

Now (16) yields

$$\frac{1}{2}\frac{d}{dt}\|(\mathbf{u}, \mathbf{w})\|_{\dot{H}^s}^2 + \mu\|\nabla\mathbf{u}\|_{\dot{H}^s}^2 + \gamma\|\nabla\mathbf{w}\|_{\dot{H}^s}^2 + \kappa\|\text{div}\,\mathbf{w}\|_{\dot{H}^s}^2 + \chi\|\mathbf{w}\|_{\dot{H}^s}^2$$
$$\leq C_1(s)(\|\nabla\mathbf{u}\|_{\dot{H}^s}\|\mathbf{u}\|_{\dot{H}^s}\|\widehat{\mathbf{u}}\|_{l^1} + \|\nabla\mathbf{w}\|_{\dot{H}^s}\|\mathbf{u}\|_{\dot{H}^s}\|\widehat{\mathbf{w}}\|_{l^1} + \|\nabla\mathbf{w}\|_{\dot{H}^s}\|\mathbf{w}\|_{\dot{H}^s}\|\widehat{\mathbf{u}}\|_{l^1})$$

and we have

$$\frac{d}{dt}\|(\mathbf{u}, \mathbf{w})\|_{\dot{H}^s}^2 + \min\{\mu, \gamma\}\|(\nabla\mathbf{u}, \nabla\mathbf{w})\|_{\dot{H}^s}^2 + 2\kappa\|\text{div}\,\mathbf{w}\|_{\dot{H}^s}^2 + 2\chi\|\mathbf{w}\|_{\dot{H}^s}^2 \leq C_3\|(\mathbf{u}, \mathbf{w})\|_{\dot{H}^s}^2\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})\|_{l^1}^2.$$

Here $C_3 = C_3(s, \mu, \gamma) = C_1(s)\alpha^{-1}$ with α given in (17). By Gronwall's Lemma,

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s}^2 \leq \|(\mathbf{u}, \mathbf{w})(t_0)\|_{\dot{H}^s}^2 \exp\left(C_3 \int_{t_0}^t \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\tau)\|_{l^1}^2 \, d\tau\right),$$

for all $0 \leq t_0 \leq t < T^*$. Taking $s > 1/2$ one obtains that

$$\int_{t_0}^{T^*} \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\tau)\|_{l^1}^2 \, d\tau = \infty, \tag{24}$$

by (22).

On the other hand, applying Fourier expansion and then taking the scalar product of the first equation of (1) with $\widehat{\mathbf{u}}(\xi, t)$, we obtain

$$\widehat{\mathbf{u}} \cdot \widehat{\mathbf{u}}_t = -(\mu + \chi)|\widehat{\nabla\mathbf{u}}|^2 - \widehat{\mathbf{u}} \cdot \widehat{\mathbf{u} \cdot \nabla\mathbf{u}} + \chi\widehat{\nabla \times \mathbf{u}} \cdot \widehat{\mathbf{w}}.$$

(We refer to (10)–(12) for details.) Similarly, the second equation of (1) yields

$$\widehat{\mathbf{w}} \cdot \widehat{\mathbf{w}}_t = -\gamma|\widehat{\nabla\mathbf{w}}|^2 - \kappa|\widehat{\text{div}\,\mathbf{w}}|^2 - 2\chi|\widehat{\mathbf{w}}|^2 - \widehat{\mathbf{w}} \cdot \widehat{\mathbf{u} \cdot \nabla\mathbf{w}} + \chi\widehat{\mathbf{w}} \cdot \widehat{\nabla \times \mathbf{u}},$$

where (15) has been used. Therefore,

$$\frac{1}{2}\partial_t|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})|^2 + (\mu + \chi)|\widehat{\nabla\mathbf{u}}|^2 + \gamma|\widehat{\nabla\mathbf{w}}|^2 + \kappa|\widehat{\text{div}\mathbf{w}}|^2 + 2\chi|\widehat{\mathbf{w}}|^2$$
$$= -\mathcal{R}e\left[\widehat{\mathbf{u}} \cdot \widehat{\mathbf{u} \cdot \nabla\mathbf{u}}\right] + \chi\mathcal{R}e\left[\widehat{\nabla \times \mathbf{u}} \cdot \widehat{\mathbf{w}}\right] - \mathcal{R}e\left[\widehat{\mathbf{w}} \cdot \widehat{\mathbf{u} \cdot \nabla\mathbf{w}}\right] + \chi\mathcal{R}e\left[\widehat{\mathbf{w}} \cdot \widehat{\nabla \times \mathbf{u}}\right].$$

The Cauchy–Schwarz inequality yields

$$|\widehat{\mathbf{w}} \cdot \widehat{\nabla \times \mathbf{u}}| \leq \frac{1}{2}|\widehat{\mathbf{w}}|^2 + \frac{1}{2}|\widehat{\nabla\mathbf{u}}|^2$$

and we obtain

$$\frac{1}{2}\partial_t|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})|^2 + \mu|\widehat{\nabla\mathbf{u}}|^2 + \gamma|\widehat{\nabla\mathbf{w}}|^2 + \kappa|\widehat{\text{div}\,\mathbf{w}}|^2 + \chi|\widehat{\mathbf{w}}|^2 \leq |\widehat{\mathbf{u}}||\widehat{\mathbf{u} \cdot \nabla\mathbf{u}}| + |\widehat{\mathbf{w}}||\widehat{\mathbf{u} \cdot \nabla\mathbf{w}}|.$$

Let $\epsilon > 0$ be arbitrary. The previous estimate then yields that

$$\partial_t\sqrt{|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})|^2 + \epsilon} + \alpha\frac{|(\widehat{\nabla\mathbf{u}}, \widehat{\nabla\mathbf{w}})|^2}{\sqrt{|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})|^2 + \epsilon}} \leq |\widehat{\mathbf{u} \cdot \nabla\mathbf{u}}| + |\widehat{\mathbf{u} \cdot \nabla\mathbf{w}}|,$$

where α is given in (17). By integrating from t_0 to t, with $0 \leq t_0 \leq t < T^*$, one gets

$$\sqrt{|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)|^2 + \epsilon} - \sqrt{|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t_0)|^2 + \epsilon} + \alpha \int_{t_0}^t \frac{|(\widehat{\nabla\mathbf{u}}, \widehat{\nabla\mathbf{w}})(\tau)|^2}{\sqrt{|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\tau)|^2 + \epsilon}} \, d\tau \leq \int_{t_0}^t [|(\widehat{\mathbf{u} \cdot \nabla\mathbf{u}})(\tau)| + |(\widehat{\mathbf{u} \cdot \nabla\mathbf{w}})(\tau)|] \, d\tau.$$

Taking the limit as $\epsilon \to 0$ and then summing over $\xi \in \mathbb{Z}^3$ one obtains

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1} - \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t_0)\|_{l^1} + \alpha \int_{t_0}^t \|(\widehat{\Delta \mathbf{u}}, \widehat{\Delta \mathbf{w}})(\tau)\|_{l^1} d\tau \le (2\pi)^{-3/2} \int_{t_0}^t [\|\widehat{\mathbf{u}}\|_{l^1} \|\widehat{\nabla \mathbf{u}}\|_{l^1} + \|\widehat{\mathbf{u}}\|_{l^1} \|\widehat{\nabla \mathbf{w}}\|_{l^1}] d\tau.$$

Here the following has been used

$$
\begin{aligned}
\sum_\xi |(\widehat{\mathbf{u} \cdot \nabla \mathbf{u}})(\xi)| &\le \sum_\xi \sum_{k=1}^3 \sum_{j=1}^3 \left| \sum_{j=1}^3 \widehat{u_j D_j u_k}(\xi) \right| = (2\pi)^{-3/2} \sum_\xi \sum_{k=1}^3 \left| \sum_{j=1}^3 \widehat{u_j} * \widehat{D_j u_k}(\xi) \right| \\
&= (2\pi)^{-3/2} \sum_\xi \sum_{k=1}^3 \left| \sum_\rho \widehat{\mathbf{u}}(\rho) \cdot \widehat{\nabla u_k}(\xi - \rho) \right| \le (2\pi)^{-3/2} \sum_\xi \sum_{k=1}^3 \sum_\rho |\widehat{\mathbf{u}}(\rho)| |\widehat{\nabla u_k}(\xi - \rho)| \\
&= (2\pi)^{-3/2} \sum_{k=1}^3 \sum_\xi (|\widehat{\mathbf{u}}| * |\widehat{\nabla u_k}|)(\xi) \le (2\pi)^{-3/2} \sum_{k=1}^3 \|\widehat{\mathbf{u}}\|_{l^1} \|\widehat{\nabla u_k}\|_{l^1} \\
&= (2\pi)^{-3/2} \|\widehat{\mathbf{u}}\|_{l^1} \|\widehat{\nabla \mathbf{u}}\|_{l^1},
\end{aligned}
$$

where $\xi, \rho \in \mathbb{Z}^3$. By Cauchy–Schwarz:

$$\|\widehat{\nabla \mathbf{u}}\|_{l^1} = \sum_{k=1}^3 \|\widehat{\nabla u_k}\|_{l^1} = \sum_{k=1}^3 \sum_\xi |\widehat{\nabla u_k}(\xi)| \le \sum_{k=1}^3 \|\widehat{u_k}(\xi)\|_{l^1}^{1/2} \|\widehat{\Delta u_k}\|_{l^1}^{1/2} \le 3\|\widehat{\mathbf{u}}\|_{l^1}^{1/2} \|\widehat{\Delta \mathbf{u}}\|_{l^1}^{1/2},$$

thus

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1} + \frac{\alpha}{2} \int_{t_0}^t \|(\widehat{\Delta \mathbf{u}}, \widehat{\Delta \mathbf{w}})(\tau)\|_{l^1} d\tau \le \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t_0)\|_{l^1} + 18(2\pi)^{-3} \alpha^{-1} \int_{t_0}^t \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\tau)\|_{l^1}^3 d\tau.$$

Now Gronwall's Lemma yields the bound

$$\|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1}^2 \le \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t_0)\|_{l^1}^2 \exp \left\{ 36(2\pi)^{-3} \alpha^{-1} \int_{t_0}^t \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(\tau)\|_{l^1}^2 d\tau \right\},$$

for all $0 \le t_0 \le t < T^*$. Proposition 3.2 now follows using the same elementary arguments as in the proof of Proposition 3.1 together with (24). ∎

Corollary 1.1(iii) is an immediate consequence of Proposition 3.2 and Lemma 2.2.

Proposition 3.3. *Let $s_0 > 3/2$ and let $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{T}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$. Assume that $(\mathbf{u}, \mathbf{w})(t)$ is the strong periodic solution for (1) defined in its maximal interval $[0, T^*)$. If $T^* < \infty$ then for each $s > 3/2$ we have*

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2(\mathbb{T}^3)}^{\frac{2s}{3}-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)} \ge \frac{C_1(s)\alpha^{\frac{s}{3}}}{(T^* - t)^{\frac{s}{3}}}, \quad \forall 0 \le t < T^*,$$

where $C_1(s)$ is a positive constant depending on s, and $\alpha = \min\{\mu, \gamma\}$.

Proof. Since $s > 3/2$, we can apply Lemma 2.2. Using Proposition 3.2 and the equality $(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(0, t) = 0$ for $0 \le t < T^*$, we obtain that

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2}^{1-\frac{3}{2s}} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s}^{\frac{3}{2s}} \ge C_1(s) \|(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})(t)\|_{l^1} \ge \frac{C_1(s)\alpha^{\frac{1}{2}}}{(T^* - t)^{\frac{1}{2}}},$$

for all $0 \le t < T^*$. Here $C_1(s)$ is a positive constant depending on s and, as above, α is given in (17). Therefore,

$$\|(\mathbf{u}, \mathbf{w})(t)\|_{L^2}^{\frac{2s}{3}-1} \|(\mathbf{u}, \mathbf{w})(t)\|_{\dot{H}^s} \ge \frac{C_1(s)\alpha^{\frac{s}{3}}}{(T^* - t)^{\frac{s}{3}}}, \quad \forall 0 \le t < T^*.$$

Proposition 3.3 is proved. ∎

Acknowledgments

References

[1] J.L. Boldrini, M. Durán, M.A. Rojas-Medar, Existence and uniqueness of strong solution for the incompressible micropolar fluid equations in domains of $\mathbb{R}^3$, Ann. Univ. Ferrara Sez. VII Sci. Mat. 56 (2010) 37–51.

[2] Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, J. Differential Equations 252 (2012) 2698–2724.

[3] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966) 1–18.

[4] G.P. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations, Internat. J. Engrg. Sci. 15 (1977) 105–108.

[5] G. Lukaszewicz, Micropolar fluids, in: Theory and Applications. Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Boston, 1999, xvi+253pp.

[6] N. Yamaguchi, Existence of global strong solution to the micropolar fluid system in a bounded domain, Math. Methods Appl. Sci. 28 (2005) 1507–1526.

[7] J. Yuan, Existence theorem and blow-up criterion of the strong solutions to the magneto-micropolar fluid equations, Math. Methods Appl. Sci. 31 (2008) 1113–1130.

[8] J. Benameur, On the blow-up criterion of the periodic incompressible fluids, Math. Methods Appl. Sci. 36 (2013) 143–153.

[9] A.L. Bertozzi, A.J. Majda, Vorticity and Incompressible Flow, Cambridge U. Press, Cambridge, 2002.

[10] J.-Y. Chemin, About Navier–Stokes Equations, in: Publication du Laboratoire Jaques-Louis Lions, vol. VI, Université de Paris, 1996, p. R96023.

[11] L. Escauriaza, G.A. Serëgin, V. Šverák, $L_{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness, Uspekhi Mat. Nauk 58 (2003) 3–44.

[12] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, Math. Z. 187 (1984) 471–480.

[13] H.-O. Kreiss, J. Lorenz, Initial–boundary value problems and the Navier–Stokes equations, in: Pure and Applied Mathematics, vol. 136, Academic Press, Inc., Boston, MA, 1989.

[14] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1933) 22–25.

[15] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934) 193–248.

[16] J. Lorenz, P.R. Zingano, The Navier–Stokes Equations for Incompressible Flows: Solution Properties at Potential Blow-up Times, Federal University of Rio Grande do Sul, Porto Alegre, RS, 2010. Unpublished note.

[17] P. Marín-Rubio, J.C. Robinson, W. Sadowski, Solutions of the 3D Navier–Stokes equations for initial data in $\dot{H}^{1/2}$: robustness of regularity and numerical verification of regularity for bounded sets of initial data in $\dot{H}^1$, J. Math. Anal. Appl. 400 (2013) 76–85.

[18] K. Masuda, Weak solutions of Navier–Stokes equations, Tohoku Math. J. (2) 36 (1984) 623–646.

[19] J.C. Robinson, W. Sadowski, R.P. Silva, Lower bounds on blow up solutions of the three-dimensional Navier–Stokes equations in homogeneous Sobolev spaces, J. Math. Phys. 53 (2012) 15.

[20] G. Seregin, A certain necessary condition of potential blow up for Navier–Stokes equations, Comm. Math. Phys. 312 (2012) 833–845.

[21] G.A. Seregin, Necessary conditions of potential blow up for Navier–Stokes equations, J. Math. Sci. 178 (2011) 345–352.

[22] M. Wiegner, Decay results for weak solutions of the Navier–Stokes equations on $\mathbb{R}^n$, J. Lond. Math. Soc. (2) 35 (1987) 303–313.