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NUMERICAL APPROXIMATION OF ROUGH INVARIANT CURVES OF PLANAR MAPS*

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Abstract. In this paper we describe a simple algorithm for the approximation of an invariant curve Γ of a planar map f. We are particularly interested in the case where Γ is attracting but not smooth. Results for the delayed logistic map illustrate the performance of the algorithm.

Key words. invariant curves, transition to chaos

AMS subject classifications. 65N35, 34C35, 58F27

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1. Introduction.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote a continuous map. By iterating the map

$$(x_n, y_n) \to f(x_n, y_n) = (x_{n+1}, y_{n+1}), \quad n = 0, 1, \ldots, \tag{1.1}$$

one obtains a discrete-time dynamical system in the plane. As is known from studies of the Henon attractor, the dynamics defined in this way can be very complicated. A simplification of the dynamics (1.1) may occur if f has an invariant, simply closed curve Γ, i.e., Γ is topologically a circle and $f(\Gamma) = \Gamma$. Then the restriction of (1.1) to Γ is one-dimensional dynamics, and an extensive theory (see, e.g., [1, 10]) becomes available. For this reason, the numerical approximation of invariant curves, and other invariant manifolds of diffeomorphisms, has received considerable attention in the literature; see, for example, [3, 4, 6, 8, 11, 12, 13, 14].

In this paper we are particularly interested in approximating *nonsmooth* invariant curves Γ. Therefore, we will not use any concepts involving derivatives, such as tangents or normals. Roughly speaking, our assumption is that the unknown invariant curve Γ is *attracting* and that a (crude) initial approximation for Γ is known. The initial approximation may be thought of as a polygon $\mathcal{P}^0$ with vertices p_j,

$$\mathcal{P}^0 : p_0, p_1, \ldots, p_N = p_0.$$

Since Γ is attracting, one expects an improved approximation if one applies f, i.e., if one forms the ordered set of points

$$\mathcal{P}^1 : f(p_0), f(p_1), \ldots, f(p_N) = f(p_0).$$

If one simply repeats the application of f to the points p_j, i.e., if one forms the sequence

$$\mathcal{P}^n : f^n(p_0), f^n(p_1), \ldots, f^n(p_N) = f^n(p_0), \quad n = 0, 1, \ldots,$$

then, in many cases, the points of $\mathcal{P}^n$ will cluster near a small, strongly attracting subset of Γ and will not at all approximate all of Γ. This fundamental difficulty of

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approximating invariant curves (or other attracting invariant sets) makes it necessary to supplement the process *application of f* by a process of *redistributing points*. At first, one might attempt to redistribute the points to achieve (approximate) equidistribution of arc-length. However, this idea is not good, at least for the application described below. One would completely miss the interesting, small-scale features, where Γ is not smooth. Instead of redistributing arc-length, we use some simple rules of adding points in almost empty regions and deleting points where they become too crowded. For details, see section 2.

Although the suggested algorithm is simple, its performance analysis for the approximation of nonsmooth curves has not been carried out and is expected to be far from trivial. We comment on this at the end of section 2.

To motivate our study, with its emphasis on nonsmooth curves, we consider in section 3 the delayed logistic map,

$$f_\lambda(x, y) = \Big(y, \lambda y(1 - x)\Big), \quad (x, y) \in \mathbb{R}^2, \tag{1.2}$$

depending on the parameter λ. For every $\lambda > 1$ the map f_λ has the trivial fixed point $P = (0, 0)$ and the nontrivial fixed point

$$Q_\lambda = \left(1 - \frac{1}{\lambda}, 1 - \frac{1}{\lambda}\right)$$

in the positive quadrant. For $1 < \lambda < 2$ the point Q_λ is asymptotically stable, but at $\lambda = 2$ it loses stability, and an attracting, smooth, invariant curve Γ_λ is born in a Neimark–Sacker bifurcation. See, for example, [9]. An extensive, computer-assisted study of the fate of Γ_λ, for increasing λ, has been given in [2]; see also [5] for the study of Lyapunov-type numbers related to the invariant curves Γ_λ. It appears that Γ_λ transforms itself from a smooth invariant curve (for $2 < \lambda < 2 + \varepsilon$) to a strange attractor for $\lambda \approx 2.2701$. However, a main conclusion of [2] is that there is no unique parameter value, $\lambda = \lambda^*$, which divides nice behavior (characterized by the existence of a smooth invariant curve Γ_λ) from strange behavior. Nevertheless, as λ increases and the curves Γ_λ disappear in a complicated fashion, they lose their smoothness on the way. This loss does not occur gradually, however. It appears that parameter intervals, for which Γ_λ is not differentiable but only topologically a circle, are interspersed with parameter intervals where Γ_λ is smooth. The analysis of [2] also shows why such a complicated behavior is to be expected for invariant curves of one-parameter families of diffeomorphisms of the plane.

The breakdown of Γ_λ is typically connected with the *transition from quasi periodic to chaotic behavior*, which is an interesting bifurcation that is not completely understood. To monitor the breakdown of Γ_λ numerically, one can try to approximate the invariant curves Γ_λ as λ changes. Then, clearly, one needs an algorithm that can approximate nonsmooth curves, which motivates our study.

Results of a case study for the map (1.2) are given in section 3. For $\lambda = 2.19$ the invariant curve Γ_λ contains two seven-periodic orbits of f_λ. One orbit is of saddle type; the other, more interesting one consists of seven sinks, i.e., of seven attracting fixed points of $(f_\lambda)^7$, the seventh power of f_λ. By linearizing $(f_\lambda)^7$ about any point of the sink orbit, one can show that the sinks are of spiral nature; i.e., the linearization has a pair of complex conjugate eigenvalues. Therefore, Γ_λ winds infinitely often about each point of the sink orbit and is not differentiable at the sinks. Also, the spirals are quite small in diameter, turn sharply, and shrink rapidly with every turn. Despite this, our algorithm approximates Γ_λ, including the spirals, very well. We emphasize that the algorithm does not use any a priori information about the existence of the spirals.

We believe that the simple algorithm suggested here is in many cases well suited to approximate nonsmooth, attracting invariant curves. A performance analysis, including error estimates, has not been carried out, however, and we anticipate such an analysis to be formidable. An application of the algorithm to the study of breakdown of the invariant curves Γ_λ, for increasing λ, will be presented in future work.

2. Description of the algorithm and comments on its performance. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a given continuous map and suppose that $\Gamma \subset \mathbb{R}^2$ is an invariant, attracting, simply closed curve for f; i.e., Γ is homeomorphic to a circle and $f(\Gamma) = \Gamma$.

The (unknown) curve Γ will be approximated by a sequence $\mathcal{S}^0, \mathcal{S}^1, \ldots$ of finite, ordered sets

$$\mathcal{S}^n = \left\{p_0^n, p_1^n, \ldots, p_{N(n)}^n = p_0^n\right\}. \tag{2.1}$$

Each set $\mathcal{S}^n$, which consists of $N(n)$ points $p_i^n \in \mathbb{R}^2$, determines a closed curve Γ^n obtained by drawing the straight lines from p_{i-1}^n to p_i^n for $1 \le i \le N(n)$. Of course, the curve Γ^n (instead of the finite set $\mathcal{S}^n$) may also be considered as approximation of Γ.

We assume that the set $\mathcal{S}^0$ is a given initial approximation of Γ, which is used to start an iterative process

$$\mathcal{S}^n \to \mathcal{S}^{n+1}, \quad n = 0, 1, \ldots. \tag{2.2}$$

For example, when treating the problem in the next section, we took $\mathcal{S}^0$ as a discretized circle of 12,000 evenly distributed points. Each step of the iteration (2.2) consists of three parts that we call (a) the map step $\mathcal{M}$, (b) the addition step $\mathcal{A}$, and (c) the deletion step $\mathcal{D}$. In the map step we simply map the points p_i of a set $\mathcal{S}$ forward by f. Note that we assume Γ to be attracting; thus the map step is expected to move the points closer to Γ. In the addition step, which depends on a parameter $\varepsilon_1 > 0$, we fill points into holes that may have developed in the map step. Correspondingly, in the deletion step we delete points if they get closer than a tolerance $\varepsilon_2 \ge 0$. In this way, details of a length scale less than ε_2 will be neglected in the approximation of Γ. For $\varepsilon_2 = 0$, the deletion step becomes trivial; i.e., no points will be deleted.

Let us formalize the simple processes.

(a) The map step $\boldsymbol{\mathcal{M}}$. Given a finite ordered set

$$\mathcal{S} = \left\{p_0, p_1, \ldots, p_N = p_0\right\}$$

in $\mathbb{R}^2$, let $\mathcal{S}' = \mathcal{M}\mathcal{S}$ consist of the points

$$\mathcal{S}' = \left\{q_0, q_1, \ldots, q_N = q_0\right\}$$

where

$$q_i = f(p_i), \quad 0 \le i \le N.$$

(b) The addition step $\boldsymbol{\mathcal{A}}$. Assume $\mathcal{S}' = \mathcal{M}\mathcal{S}$ is determined as described in (a). Compute the Euclidean distances

$$d_i = |q_i - q_{i-1}|, \quad 1 \le i \le N.$$

For every index i with $d_i > \varepsilon_1$ compute the integer part[1] of d_i/ε_1,

$$k = k(i) = \textit{integer part}\left(\frac{d_i}{\varepsilon_1}\right),$$

and then fill in k points

$$q_{i,j} = f\Big(p_{i-1} + \frac{j}{k+1}(p_i - p_{i-1})\Big), \quad j = 1, 2, \ldots, k, \tag{2.3}$$

between q_{i-1} and q_i. The outcome of this process is a new set, denoted by

$$\mathcal{S}'' = \mathcal{A}\mathcal{S}' = \Big\{r_0, r_1, \ldots, r_M = r_0\Big\}.$$

Remark. There is no guarantee that $|r_\nu - r_{\nu-1}| \le \varepsilon_1$ for all ν since we choose the r_ν as *images under* f of points between neighbors q_i and q_{i-1}. If one uses the formula

$$q_{i,j} = q_{i-1} + \frac{j}{k+1}(q_i - q_{i-1}) \tag{2.4}$$

instead of (2.3), then $|r_\nu - r_{\nu-1}| \le \varepsilon_1$ for all ν is guaranteed. However, we found that the algorithm performs slightly better with the choice (2.3) instead of (2.4).

(c) The deletion step $\boldsymbol{\mathcal{D}}$. Given a set $\mathcal{S}'' = \{r_0, r_1, \ldots, r_M = r_0\}$ as described in (b) and given a tolerance $\varepsilon_2 \ge 0$, we compute the Euclidean distances

$$d_\nu = |r_\nu - r_{\nu-1}|, \quad 1 \le \nu \le M.$$

If $d_\nu \ge \varepsilon_2$ for all ν, then no points will be deleted, and $\mathcal{D}\mathcal{S}'' = \mathcal{S}''$. Otherwise, let i denote the smallest index ν with $d_\nu < \varepsilon_2$. We replace the pair of points r_{i-1}, r_i with the single point $\frac{1}{2}(r_{i-1} + r_i)$ and re-index the points r_ν for $\nu > i$ accordingly. Formally, let

$$\begin{aligned} r'_\nu &= r_\nu, \quad 0 \le \nu \le i-1, \\ r'_i &= \frac{1}{2}(r_{i-1} + r_i), \\ r'_\nu &= r_{\nu+1}, \quad i+1 \le \nu \le M-1. \end{aligned}$$

The process of replacing one pair of points with a single point is then applied to the set $\{r'_0, r'_1, \ldots, r'_{M-1}\}$ and this is repeated until all distances of successive points are $\ge \varepsilon_2$. If the final outcome is a set

$$\mathcal{S}''' = \Big\{\bar{r}_0, \bar{r}_1, \ldots, \bar{r}_K\Big\} \tag{2.5}$$

with $\bar{r}_0 = \bar{r}_K$, then $\mathcal{S}''' = \mathcal{D}\mathcal{S}''$. It is possible, however, that $\bar{r}_0 \ne \bar{r}_K$. In this case we add the point $\bar{r}_{K+1} = \bar{r}_0$ to the above set $\mathcal{S}'''$ to obtain $\mathcal{D}\mathcal{S}''$.

Using the processes $\mathcal{M}, \mathcal{A}, \mathcal{D}$, we define

$$\mathcal{S}^{n+1} = \mathcal{D}\mathcal{A}\mathcal{M}\mathcal{S}^n, \quad n = 0, 1, 2, \ldots,$$

and obtain a sequence of finite, ordered sets $\mathcal{S}^n$ of the form (2.1).

[1]For example, the integer part of 2.99 is 2.

One would like to have a theoretically based criterion for stopping the iteration, possibly using the quantity

$$dist(\mathcal{S}^n, \mathcal{S}^{n+1}) = \max_{p\in\mathcal{S}^n} \min_{q\in\mathcal{S}^{n+1}} |p - q|.$$

However, this distance is expensive to compute. In our applications we found it satisfactory to iterate a preset number of times or "until the sets $\mathcal{S}^n$ settle in the picture norm."

Comments on the performance of the algorithm. To obtain some understanding of the expected performance of the algorithm, let us assume first that the unknown attracting invariant curve Γ is C^1. Under this assumption the Lyapunov-type numbers $\nu(p)$ and $\sigma(p)$ are defined for all $p \in \Gamma$. See [7] for the definition of these numbers in the general context of overflowing invariant manifolds and see, for example, [5] for the specific case of invariant curves in the plane. The number $\nu = \sup_p \nu(p)$ is assumed to be strictly less than 1, which expresses exponential attractivity of Γ. If $\nu = e^{-\beta}, \beta > 0$, then β can be thought of as the attractivity rate of Γ in normal directions.

If $\Gamma^{(n)}$ denotes the numerical approximation to Γ after n iterations, then, optimistically, one may expect the Hausdorff distance,

$$dist_H(\Gamma, \Gamma^{(n)}) = \max\Big(dist(\Gamma, \Gamma^{(n)}), dist(\Gamma^{(n)}, \Gamma)\Big),$$

to decrease by a factor $\nu = e^{-\beta}$ in each iteration. If this holds and if one wants an accuracy of $dist_H(\Gamma, \Gamma^{(n)}) < tol$, then the number n of required iterations is $n \sim \frac{1}{\beta}\log(1/tol)$. Thus one expects n to depend crucially on the attractivity rate of Γ in normal directions.

An extension of the above argument to nonsmooth invariant curves Γ is not directly possible since an attractivity rate in normal directions is not defined if Γ has no normal at even a single point. However, the nonsmooth regions of Γ are small, namely, confined to seven spirals that are small in diameter, at least in the example considered below. The computations in [5] indicate that an attractivity factor $\nu = e^{-\beta}$ is still numerically meaningful. In addition, the following observation is critical to understanding the performance of the suggested algorithm: The *rough* parts of Γ are precisely those parts that are dynamically *most strongly attracting*. Therefore, using the suggested algorithm, where points accumulate in regions of high attractivity, the rough parts automatically obtain the best numerical resolution. This observation—that the rough parts of Γ are dynamically most strongly attracting—holds in some generality. To explain this, let us embed the given map $f = f(x, y)$ into a family of maps $g = g(x, y, \lambda)$ depending on a parameter λ, and let $f = g(\cdot, \lambda_0)$. In typical applications, the map $g(\cdot, \lambda)$ has a smooth invariant curve, $\Gamma(\lambda)$, for λ in some interval. As λ changes, then, typically, rough parts of $\Gamma(\lambda)$ form when the attractivity rate of the dynamics *within* $\Gamma(\lambda)$ begins to exceed the attractivity rate *towards* $\Gamma(\lambda)$. (The ratio of attractivity *towards* and *within* $\Gamma(\lambda)$ is measured by the second Lyapunov-type number $\sigma(p)$ mentioned above.) Furthermore, the *rough* parts form in those places of $\Gamma(\lambda)$ where this *strong local attractivity* of the dynamics on $\Gamma(\lambda)$ takes place. This is a typical phenomenon, and our algorithm takes advantage of it. A precise analysis of the algorithm will require, however, a better quantitative understanding of the perturbation theory of nonsmooth invariant curves. As the dynamics is perturbed, how do these curves perturb? At present, no theory has been developed to answer this question.

3. Results for the delayed logistic map. A simple model for the discrete-time evolution of the size of a population is given by

$$N_{n+1} = \lambda N_n(1 - N_{n-1}),$$

where N_n is the scaled population size in the nth generation and $\lambda > 0$ is a parameter. If one defines $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_\lambda(x, y) = \Big(y, \lambda y(1 - x)\Big)$$

and sets $(x_n, y_n) = (N_{n-1}, N_n)$, then the evolution of N_n corresponds to the planar map

$$(x_n, y_n) \to f_\lambda(x_n, y_x) = (x_{n+1}, y_{n+1}).$$

The map $f_\lambda, \lambda > 0$, has the two fixed points

$$P = (0, 0), \quad Q_\lambda = \left(1 - \frac{1}{\lambda}, 1 - \frac{1}{\lambda}\right).$$

Of main interest is the fixed point Q_λ, which is asymptotically stable for $1 < \lambda < 2$ and unstable for $\lambda > 2$. At $\lambda = 2$, where Q_λ loses its stability, a Neimark–Sacker bifurcation (see, e.g., [9]) occurs, leading to a smooth, attracting, invariant curve Γ_λ of f_λ for $2 < \lambda < 2 + \varepsilon$. A detailed, computer-assisted study of the breakdown of the curves Γ_λ has been carried out in [2]. Breakdown occurs near $\lambda \approx 2.27$ but, as explained in [2], one cannot assign a precise λ value to the occurence of breakdown.

In the present paper we want to demonstrate the performance of our algorithm for approximating Γ_λ taking the particular value

$$\lambda_0 = 2.19.$$

We briefly summarize some known results; see, for example, [2, 5]. The value $\lambda_0 = 2.19$ lies in an interval of phase locking as follows: For $\lambda_1 < \lambda < \lambda_4$, where

$$\lambda_1 \approx 2.1763, \quad \lambda_4 \approx 2.2006,$$

the map f_λ has two seven-periodic orbits lying on Γ_λ. Denote these orbits by

$$\mathcal{O}^{sa}_\lambda = \Big\{q, f_\lambda q, f^2_\lambda q, \ldots, f^6_\lambda q\Big\}, \quad q = q_\lambda, \tag{3.1}$$

$$\mathcal{O}^{si}_\lambda = \Big\{r, f_\lambda r, f^2_\lambda r, \ldots, f^6_\lambda r\Big\}, \quad r = r_\lambda. \tag{3.2}$$

The saddle orbit, $\mathcal{O}^{sa}$, is attracting w.r.t. the normal directions of Γ_λ but repelling in tangential direction. The sink orbit, $\mathcal{O}^{si}$, attracts in all directions. The phase-locking λ interval $\lambda_1 < \lambda < \lambda_4$ contains a subinterval $\lambda_2 < \lambda < \lambda_3$, where

$$\lambda_2 \approx 2.181, \quad \lambda_3 \approx 2.196,$$

in which the sink orbit is of spiral type as follows: If r_λ is a point in $\mathcal{O}^{si}_\lambda$ and if

$$J_\lambda = (f^7_\lambda)'(r_\lambda)$$

denotes the Jacobian of the seventh iterate of f_λ, evaluated at r_λ, then J_λ has a complex conjugate pair of eigenvalues, $\mu_{1,2}(\lambda)$, lying inside the unit circle. Since the eigenvalues of J_λ are not real, one must expect the invariant curve Γ_λ to wind infinitely often around each of the points of the sink orbit $\mathcal{O}^{si}_\lambda$. Therefore, one must expect that the curve Γ_λ is not differentiable at any of the points of $\mathcal{O}^{si}_\lambda$.

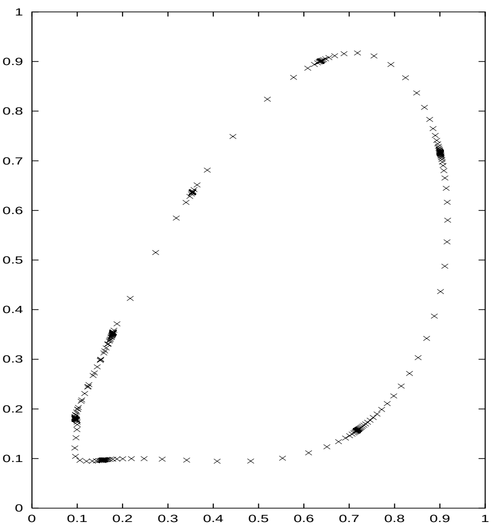

FIG. 1. *The set $\mathcal{S}^0$ after* 100 *applications of f_{λ_0} for $\lambda_0 = 2.19$.*

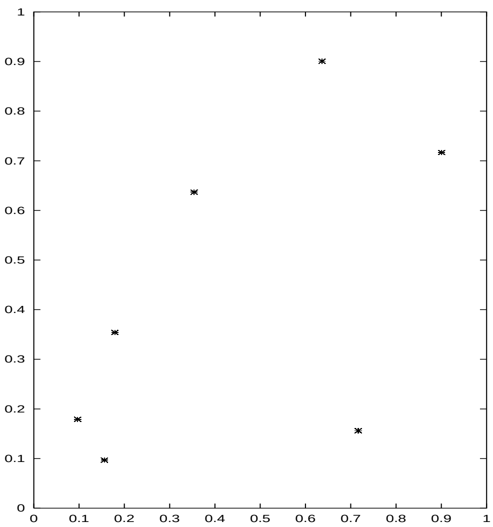

FIG. 2. *The set $\mathcal{S}^0$ after* 200 *applications of f_{λ_0} for $\lambda_0 = 2.19$.*

We present numerical approximations to the invariant curve,

$$\Gamma_{\lambda_0} \quad \text{for} \quad \lambda_0 = 2.19,$$

using the algorithm described in the previous section. Note that the value $\lambda_0 = 2.19$ lies inside the interval $\lambda_2 < \lambda < \lambda_3$; i.e., Γ_{λ_0} contains two seven-periodic orbits, (3.1) and (3.2), and the points of the sink orbit (3.2) consist of spiral fixed points of $f^7_{\lambda_0}$.

The seven sinks for $\lambda_0 = 2.19$ are

$$\begin{aligned}
P_1 &= (0.1790289, 0.3540937),\\
P_2 &= (0.0968686, 0.1790289),\\
P_3 &= (0.3540937, 0.6366345),\\
P_4 &= (0.6366345, 0.9005417),\\
P_5 &= (0.7166244, 0.1560906),\\
P_6 &= (0.9005417, 0.7166244),\\
P_7 &= (0.1560906, 0.0968687).
\end{aligned}$$

As a starting set $\mathcal{S}^0$ we chose $N = 12,000$ evenly distributed points on the circle of radius 0.2 centered at Q_{λ_0}. Precisely, $\mathcal{S}^0$ consists of the points p_i^0 with coordinates

$$p_i^0 = \left(1 - \frac{1}{\lambda_0} + 0.2\cos\left(\frac{2\pi i}{N}\right), 1 - \frac{1}{\lambda_0} + 0.2\sin\left(\frac{2\pi i}{N}\right)\right), \quad i = 0, 1, \ldots, N = 12{,}000.$$

To illustrate the importance of the addition step in our algorithm (to avoid the development of holes) we show in Figure 1 the set,

$$\mathcal{M}^{100}\mathcal{S}^0,$$

obtained by applying the map f_{λ_0} 100 times to the points of $\mathcal{S}^0$. One clearly sees the clustering of the iterates near the seven sinks. Further iteration will rapidly lead to a complete loss of the invariant curve, as shown in Figure 2.

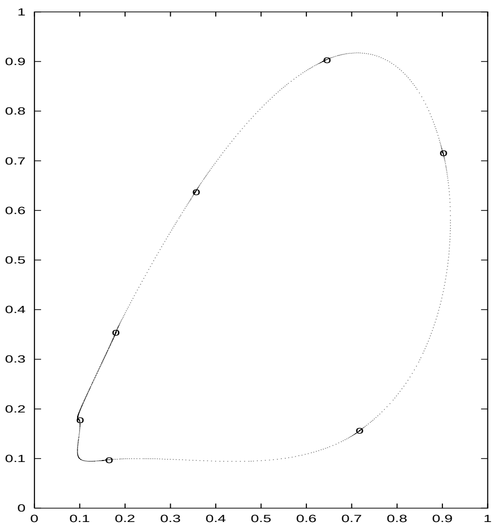

FIG. 3. *The invariant curve for* $\lambda_0 = 2.19$ *computed with our algorithm. The points* o *denote the seven sinks.*

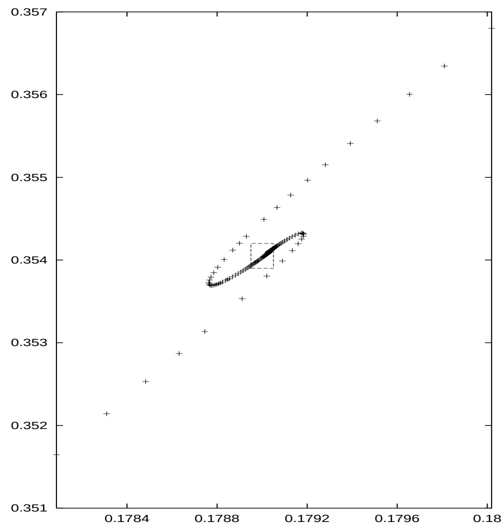

FIG. 4. *Blow-up of the computed curve near the sink* P_1.

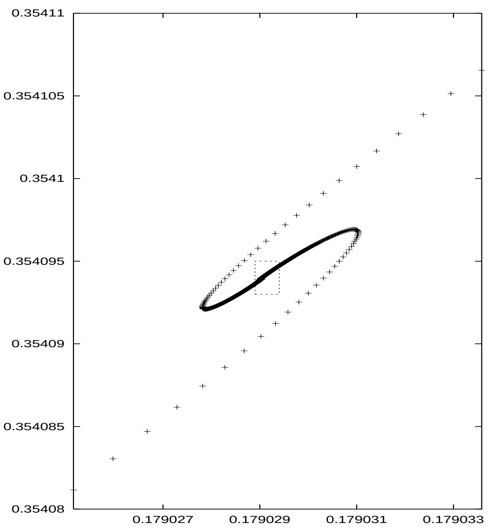

FIG. 5. *Blow-up of the region* [] *in Figure 4.*

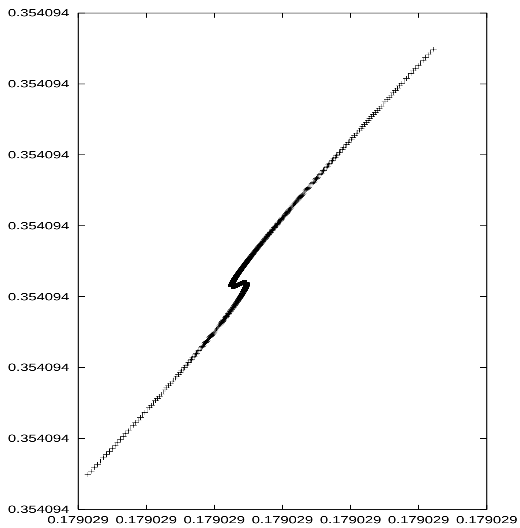

FIG. 6. *Blow-up of the region* [] *in Figure 5.*

In Figure 3 we plot the set,

$$\mathcal{S}^n = \left(\mathcal{DAM}\right)^n \mathcal{S}^0,$$

obtained with our algorithm for $\varepsilon_1 = 0.01, \varepsilon_2 = 1.0e - 8$, and $n = 100$. The location of the seven sinks is also indicated.

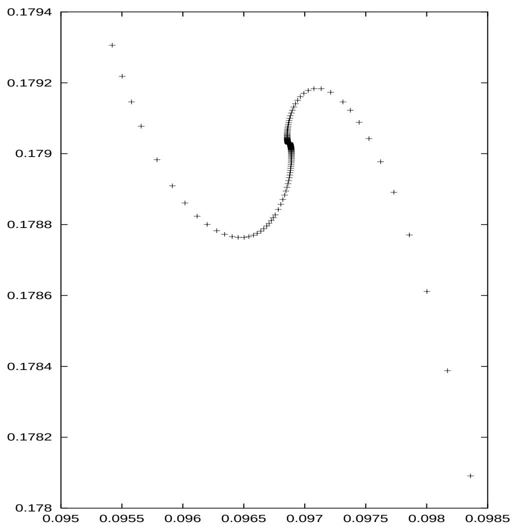

FIG. 7. *Blow-up of the computed curve near the sink* P_2.

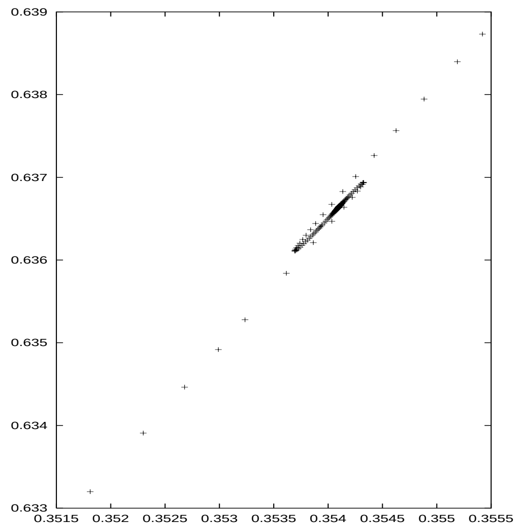

FIG. 8. *Blow-up of the computed curve near the sink* P_3.

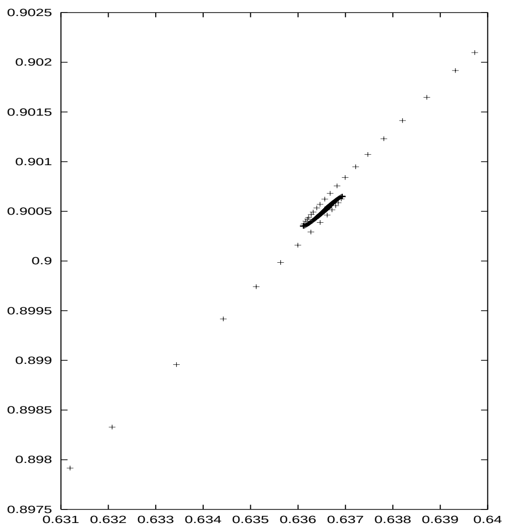

FIG. 9. *Blow-up of the computed curve near the sink* P_4.

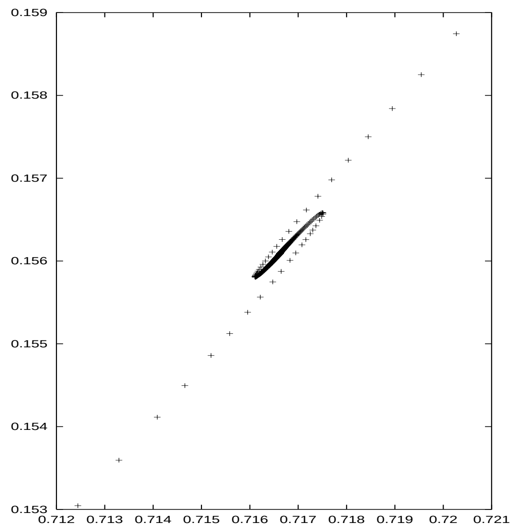

FIG. 10. *Blow-up of the computed curve near the sink* P_5.

In Figure 4 and Figures 7–12 we show blow-ups of the computed set $\mathcal{S}^{100}$ near the seven sinks. Also, Figure 5 shows a blow-up of a subregion of Figure 4 and, similarly, Figure 6 shows a blow-up of a subregion of Figure 5, but this computation was done with $\varepsilon_2 = 1.0e - 13$. Together, Figures 4–6 show three turns of the spiral near the sink P_2.

The graphs demonstrate that our simple algorithm can capture fine local structures of nonsmooth, attracting, invariant curves of maps.

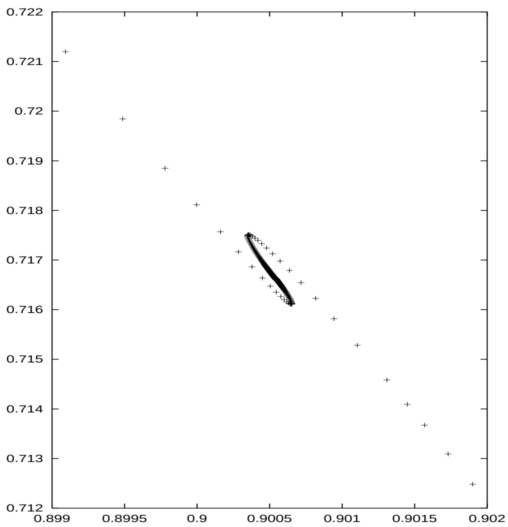

FIG. 11. *Blow-up of the computed curve near the sink P_6.*

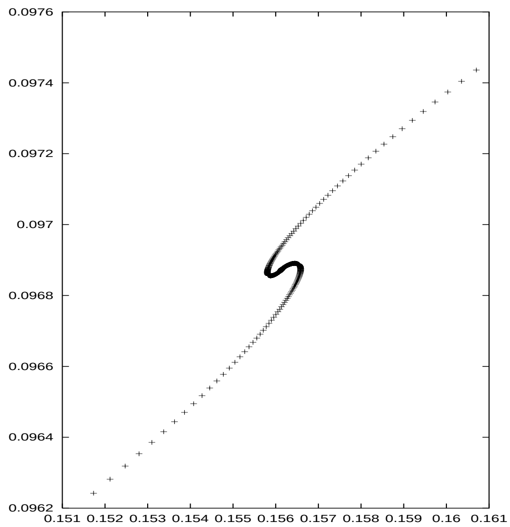

FIG. 12. *Blow-up of the computed curve near the sink P_7.*

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