Abstract

We study arbitrary-order Hermite difference methods for the numerical solution of initial-boundary values problems for symmetric hyperbolic systems. These differ from standard difference methods in that derivative data (or equivalently local polynomial expansions) are carried at each grid point. Time-stepping is achieved using staggered grids and Taylor series. We prove that methods using derivatives of order m in each coordinate direction are stable under m-independent CFL constraints and converge at order $2m+1$. The stability proof relies on the fact that the Hermite interpolation process generally decreases a seminorm of the solution. We present numerical experiments demonstrating the resolution of the methods for large m as well as illustrating the basic theoretical results.

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Hermite Methods for Hyperbolic Initial–Boundary Value Problems

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1 Introduction

Challenging problems in large scale scientific computing typically involve multiple spatial and temporal scales. For hyperbolic problems this entails the propagation of short waves over many wavelengths. In such cases, one expects high-order methods to be particularly efficient. In recent years, a number of high-order finite element methods have been developed and applied to wave propagation problems [2, 3, 9]. In contrast, the application of high-order nodal based finite-difference methods has lagged. A possible explanation for this is difficulties associated with the application of boundary conditions. (See, e.g., [13] for a somewhat complicated approach to dealing with this issue.)

In this work we develop the basic theory for a different class of node-based methods of arbitrary order. The ingredients of our method, which could be applied independently, are:

i. The association of derivative data, or equivalently local polynomials, with each grid point. This allows us to derive methods of arbitrary order using $2s$-point stencils in s space dimensions.

ii. The use of high-order Taylor series in time to evolve the solution on a staggered space-time mesh. Thus we evolve the solution at a point where a smooth, high-degree Hermite interpolant can be used to represent the approximate solution.

Certainly these ingredients have been used elsewhere in different contexts: high-order Taylor series methods for ordinary differential equations are discussed in [11], and the use of staggered grids to facilitate time-integration of hyperbolic equations is a key part of the so-called central difference schemes for hyperbolic conservation laws [10]. However, we believe their combination with Hermite interpolation to solve hyperbolic systems and the convergence analysis we provide is new.

At first glance Hermite methods might appear less efficient than high-order methods requiring only function data; after all we are introducing many additional degrees of freedom. However, we

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will demonstrate that the Hermite methods allow significant sub-grid resolution, so that the number of degrees of freedom required per wavelength is comparable to other techniques. An advantage of our approach is that the time step stability restrictions are determined by the cell size independent of order. Thus much larger time steps can be taken.

The outline of the remainder of the paper is as follows. In section 2 we describe and analyze our method for the simplest possible problem: a constant coefficient scalar equation with periodic inital data. The convergence proof is based on the observation that Hermite interpolation generally decreases a seminorm of the solution. In sections 3 and 4 we generalize the analysis to systems with variable coefficients and nonperiodic boundary conditions, which is conceptually straightforward but technically detailed. Again we emphasize that the boundary treatment is completely natural, requiring no changes in the basic discretization. In section 5 we consider systems in multiple space dimensions using tensor-product grids. Here we must generalize the seminorm in an interesting way to obtain the convergence theorem. Finally, in section 6 we present a number of simple numerical experiments which demonstrate the resolving power of the methods as well as the basic theoretical results.

We note that the methods are applicable to nonlinear hyperbolic–parabolic systems. Extension of the theory to cover that case as well as applications of the method to the study of compressible flows will be reported elsewhere.

2 Convergence Analysis for a Model Problem

To begin we consider the model problem

$$u_t = u_x, \quad u(x,0) = f(x) \ , \tag{2.1}$$

where $f(x)$ is a smooth 2π–periodic function. The exact solution is $u(x,t) = f(x+t)$. In subsequent sections a more general convergence analysis will be presented which includes this one as a special case. However, consideration of this model problem in detail clarifies the main points.

2.1 Discretization

The Grid. We introduce staggered, not necessarily smooth, periodic spatial grids. Let

$$0 < x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < x_2 < \ldots < x_{N-\frac{1}{2}} < x_N < x_{\frac{1}{2}} + 2\pi = x_{N+\frac{1}{2}},$$

and set

$$\begin{aligned} \tilde{G}_h &= \left\{ x_{j+\frac{1}{2}} : j \in \mathbb{Z} \right\}, \quad x_{j+\frac{1}{2}+N} = x_{j+\frac{1}{2}} + 2\pi \\ G_h &= \left\{ x_j : j \in \mathbb{Z} \right\}, \quad x_{j+N} = x_j + 2\pi. \end{aligned}$$

Set

$$h_{j+\frac{1}{2}} = x_{j+1} - x_j, \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \tag{2.2}$$

We assume for some fixed positive constants c_1, c_2

$$c_1 \max(h_j, h_{j+1}) \leq h_{j+\frac{1}{2}} \leq c_2 \min(h_j, h_{j+1}). \tag{2.3}$$

Set

$$h = \max_j (h_j, h_{j+\frac{1}{2}}), \quad \tilde{h} = 2 \min_j (x_{j+\frac{1}{2}} - x_j, x_{j+1} - x_{j+\frac{1}{2}}), \tag{2.4}$$

and denote the time step by k. Our space-time mesh uses G_h at times $t = nk$ and $\tilde{G}_h$ at times $t = (n + \frac{1}{2})k$.

The Numerical Process. Let m be a fixed nonnegative integer, which is the number of space derivatives that we carry at each mesh point. The numerical approximation and the interpolants will depend on m, but this dependence is suppressed in our notation.

Set $D = \partial/\partial x$. For $0 \leq \nu \leq m$,

$$(U_j^n)_\nu = D^\nu u(x_j, nk)$$

and

$$(U_{j+\frac{1}{2}}^{n+\frac{1}{2}})_\nu = D^\nu u(x_{j+\frac{1}{2}}, (n + \frac{1}{2})k)$$

are the exact values at the grid points. Let

$$V_j^n, V_{j+\frac{1}{2}}^{n+\frac{1}{2}} \in \mathbb{R}^{m+1}$$

denote the numerical approximations to the vectors U_j^n and $U_{j+\frac{1}{2}}^{n+\frac{1}{2}}$.

For $n = 0$ these vectors are provided by the initial data $f(x)$, i.e.,

$$(V_j^0)_\nu = D^\nu f(x_j), \quad 0 \leq \nu \leq m.$$

We now proceed to evolve the discrete data in half time steps. For definiteness, suppose the current time is $t = nk$, so we are given the periodic $\mathbb{R}^{m+1}$–valued grid function $V^n = (V_j^n)_{j \in \mathbb{Z}}$ on the grid G_h. There exists a unique Hermite interpolant, $p^n(x)$, satisfying:

1. $p^n \in C^m(\mathbb{R})$;
2. $D^\nu p(x_j) = (V_j^n)_\nu, \ 0 \leq \nu \leq m$;
3. $p^n(x + 2\pi) \equiv p^n(x)$;
4. restricted to any interval $[x_j, x_{j+1}]$, the function $p^n(x)$ is a polynomial of degree $\leq 2m + 1$.

We denote this interpolant by

$$p^n = IV^n \ .$$

(Later on we will apply I to functions, in which case we mean the interpolant of the data obtained by evaluating the function and its derivatives on the grid.)

For the simple equation under consideration, $u_t = u_x$, the evolution step can be carried out exactly. We have

$$D^\nu u(x_{j+\frac{1}{2}}, (n + \frac{1}{2})k) = D^\nu u(x_{j+\frac{1}{2}} + \frac{k}{2}, nk), \quad 0 \leq \nu \leq m \ .$$

For the discrete scheme we replace $u(x, nk)$ by the interpolant $p^n(x)$. Correspondingly, we obtain

$$(V_{j+\frac{1}{2}}^{n+\frac{1}{2}})_\nu = D^\nu p^n(x_{j+\frac{1}{2}} + \frac{k}{2}), \quad 0 \leq \nu \leq m \ .$$

Thus we have defined the mapping $V^n \to V^{n+\frac{1}{2}}$. The mapping $V^{n+\frac{1}{2}} \to V^{n+1}$ proceeds similarly via the Hermite interpolation on the dual grid $\tilde{G}_h$:

$$p^{n+\frac{1}{2}} = \tilde{I} V^{n+\frac{1}{2}},$$

$$(V_j^{n+1})_\nu = D^\nu p^{n+\frac{1}{2}}(x_j + \frac{k}{2}), \quad 0 \le \nu \le m \ .$$

We note that due to periodicity only the $N(m+1)$ numbers V_j^n, $V_{j-\frac{1}{2}}^{n+\frac{1}{2}}$, $1 \le j \le N$, are evolved. Also, note that if we restrict the time step so that:

$$\frac{k}{2} < \min_j \{(x_{j+1} - x_{j+\frac{1}{2}}), (x_{j+\frac{1}{2}} - x_j)\},$$

then the evolution of the data over one time step at the new mesh can be exactly represented by a degree $2m+1$ Taylor polynomial in time. This will be used to generalize the method to more complicated equations.

Introducing S to denote the exact solution operator with time step $k/2$,

$$Sg(x) = g(x + \frac{k}{2}),$$

then the numerical process can be succinctly expressed in terms of I, $\tilde{I}$, and S as follows:

1. Let $p^0 = If$;
2. For $n = 0, 1, \ldots$ let

$$p^{n+\frac{1}{2}} = \tilde{I} S p^n, \tag{2.5}$$

$$p^{n+1} = I S p^{n+\frac{1}{2}}. \tag{2.6}$$

In the subsequent analysis we find it convenient to work with the description of the method in terms of the evolution of the interpolants. However, in practice we only evolve the Hermite data at the nodes.

2.2 Results on Hermite Interpolation

Obviously, the stability and accuracy of the proposed method are determined by the properties of Hermite interpolation. Denote the L_2–inner product and norm of 2π-periodic functions by,

$$(f, g) = \int_{-\pi}^{\pi} f(x) g(x)\, dx, \quad \|g\|^2 = (g, g) \ ,$$

and the Sobolev norms on H^q_{per}, the space of 2π-periodic functions with q weak derivatives in L_2, by:

$$\|f\|^2_{H^q} = \sum_{\nu=0}^{q} \|D^\nu f\|^2.$$

In some instances we will use these norms, as well as the uniform norms $\|\cdot\|_{C^q}$, on subintervals (a, b). In such cases the subinterval will be explicitly included in the index: $\|f\|_{H^q(a,b)}$, $\|f\|_{C^q(a,b)}$.

We begin with standard results on the interpolation error which follow directly from the Peano Kernel representation. (See [8] and the appendix). Note that interpolation on each of the meshes has the same properties so we state results for I only.

Lemma 2.1 *The Hermite interpolation operators, I and $\tilde{I}$, satisfy:*

$$\|g - Ig\| \leq Ch^{2m+2}\|D^{2m+2}g\| \quad for \quad g \in H_{per}^{2m+2} \tag{2.7}$$

$$\|D^{m+1}(g - Ig)\| \leq Ch^{m+1}\|D^{2m+2}g\| \quad for \quad g \in H_{per}^{2m+2} \tag{2.8}$$

$$\|g - Ig\| \leq Ch^{m+1}\|D^{m+1}g\| \quad for \quad g \in H_{per}^{m+1} \tag{2.9}$$

If one sets $g = q - Iq$ in the last equation and observes that $Ig = 0$ for this g, then one obtains

Corollary 2.1

$$\|q - Iq\| \leq Ch^{m+1}\|D^{m+1}(q - Iq)\| \quad for \quad q \in H_{per}^{m+1} \tag{2.10}$$

Stability of the algorithm depends on the stability of the interpolation process. A direct analysis in L_2 fails because the operator norms $\|I\|$, $\|\tilde{I}\|$, are *not* bounded by $1 + O(h)$. Instead we make use of the following crucial **Orthogonality Lemma**.

Lemma 2.2 *For all $f, g \in H_{per}^{m+1}$,*

$$\Big(D^{m+1}If, D^{m+1}(g - Ig)\Big) = 0 \ .$$

Proof: As this lemma is central to the stability of the algorithm, we give its simple proof. Noting that If restricted to each interval (x_j, x_{j+1}) is a polynomial of degree $2m+1$ and integrating by parts we have:

$$\begin{aligned}\Big(D^{m+1}If, D^{m+1}(g - Ig)\Big) &= \sum_{j=0}^{N}\int_{x_j}^{x_{j+1}} D^{m+1}If \cdot D^{m+1}(g - Ig)dx \qquad (2.11)\\ &= \sum_{j=0}^{N}\sum_{k=0}^{m}(-1)^{(m-k)}D^{2m+1-k}If \cdot D^k(g - Ig)|_{x_j}^{x_{j+1}}\\ &\quad +(-1)^{(m+1)}\int_{x_j}^{x_{j+1}} D^{2m+2}If \cdot (g - Ig)dx.\end{aligned}$$

However, since by interpolation,

$$D^k(g - Ig) = 0, \quad x = x_j, x_{j+1}, \quad 0 \leq k \leq m,$$

and since If is a polynomial of degree no greater than $2m+1$,

$$D^{2m+2}If = 0,$$

all terms on the right-hand side of equation (2.11) are zero. Thus the lemma is proved.$\diamond$

If we define the semi–inner product:

$$(p, q)_{m+1} = (D^{m+1}p, D^{m+1}q), \quad p, q \in H_{per}^{m+1} \ ,$$

then the Orthogonality Lemma says that any interpolant, If, is semi–orthogonal to any interpolation error, $g - Ig$. Clearly, for every $f \in H_{per}^{m+1}$,

$$\begin{aligned} f &= If + (f - If)\\ D^{m+1}f &= D^{m+1}If + D^{m+1}(f - If)\end{aligned}$$

Applying Lemma 2.2 with $g = f$ we obtain

$$\|D^{m+1} f\|^2 = \|D^{m+1} I f\|^2 + \|D^{m+1}(f - If)\|^2, \quad f \in H_{per}^{m+1} \ . \tag{2.12}$$

In particular,

$$\|D^{m+1} I f\| \leq \|D^{m+1} f\|, \quad f \in H_{per}^{m+1} \ . \tag{2.13}$$

In this sense the interpolation process is stable w.r.t. the seminorm, $\|\cdot\|_{m+1}$, corresponding to the above semi–inner–product. This is the key fact which allows us to prove convergence.

Lemma 2.1, Corollary 2.1 and Lemma 2.2 are all that is needed to complete the convergence proof in this special case. However, for generalizations to variable coefficient problems, we will also need to consider the stability of truncated temporal Taylor series, which in turn requires the following pointwise estimates.

Lemma 2.3 *The Hermite interpolation operators, I and $\tilde{I}$, satisfy for $0 \leq q \leq 2m+1$:*

$$\max_{x_k \leq x \leq x_{k+1}} |D^q I g| \leq C \|g\|_{C^{2m+2}(x_k, x_{k+1})}, \tag{2.14}$$

$$\max_{x_k \leq x \leq x_{k+1}} |D^q I g| \leq C h_{k+\frac{1}{2}}^{\kappa} \|Ig\|_{H^{m+1}(x_k, x_{k+1})}, \quad \kappa = \min\left(-\frac{1}{2}, m + \frac{1}{2} - q\right), \tag{2.15}$$

$$\|D^q I g\|^2_{L_2(x_k, x_{k+1})} \leq C \sum_{j=0}^{m} h_{k+\frac{1}{2}}^{2j-2q+1} \left((D^j g(x_k))^2 + (D^j g(x_{k+1}))^2 \right), \tag{2.16}$$

2.3 Convergence Theorem

Let u be the solution to (2.1) and set:

$$u^n = u(\cdot, nk) \quad \text{and} \quad u^{n+\frac{1}{2}} = u(\cdot, (n + \frac{1}{2})k).$$

Denote the corresponding approximations by $p^n(x)$ and $p^{n+\frac{1}{2}}(x)$. We measure the error in the L_2–norm. Thus we consider

$$\|u^n - p^n\| \quad \text{and} \quad \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \ .$$

Note that in practice we only evolve the Hermite data, but we find it more convenient to work with the interpolants when estimating the error. Pointwise estimates then follow from Sobolev's inequality, though they may not be optimal.

Theorem 2.1 *Let $\lambda = k/h > 0$ and let $T > 0$ be fixed. Suppose $f \in H_{per}^{2m+2}$. Then there is a constant C, independent of h, so that*

$$\|u^n - p^n\| + \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \leq C h^{2m+1} \|D^{2m+2} f\| \quad \text{for} \quad 0 \leq nk \leq T \ .$$

Proof: Substituting the exact solution into the equations (2.5), (2.6) defines the local truncation errors, η^n and $\eta^{n+\frac{1}{2}}$,

$$u^{n+\frac{1}{2}} = \tilde{I} S u^n + \eta^n \ , \tag{2.17}$$

$$u^{n+1} = I S u^{n+\frac{1}{2}} + \eta^{n+\frac{1}{2}} \ . \tag{2.18}$$

Since $Su^n = u^{n+\frac{1}{2}}$ and $Su^{n+\frac{1}{2}} = u^{n+1}$, the local truncation errors are in fact interpolation errors,

$$\eta^n = u^{n+\frac{1}{2}} - \tilde{I}u^{n+\frac{1}{2}} \tag{2.19}$$
$$\eta^{n+\frac{1}{2}} = u^{n+1} - Iu^{n+1}, \tag{2.20}$$

and we obtain from Lemma 2.1,

$$\|\eta^n\| + \|\eta^{n+\frac{1}{2}}\| \le C_m h^{2m+2} \|D^{2m+2} f\| \ . \tag{2.21}$$

Let

$$e^n = u^n - p^n, \quad e^{n+\frac{1}{2}} = u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}},$$

denote the errors. By subtracting the equations (2.5) and (2.6) from the equations (2.19) and (2.20), we obtain the error equations

$$e^{n+\frac{1}{2}} = \tilde{I}Se^n + \eta^n \ , \tag{2.22}$$
$$e^{n+1} = ISe^{n+\frac{1}{2}} + \eta^{n+\frac{1}{2}} \ . \tag{2.23}$$

Also,

$$e^0 = u^0 - p^0 = f - If$$

is an interpolation error, thus

$$\|e^0\| \le C_m h^{2m+2} \|D^{2m+2} f\| \ . \tag{2.24}$$

As mentioned above, we can not directly use these error equations due to the fact that the L_2–operator norms of I and $\tilde{I}$ can not be bounded by $1 + O(h)$. Therefore, the above considerations are not sufficient to prove the theorem.

If instead one measures the local truncation error in the seminorm, $\|\cdot\|_{m+1}$, one obtains an estimate $\|\eta^n\|_{m+1} = \|D^{m+1}\eta^n\| \le Ch^{m+1}$. Combining this fact with (2.13) we can prove convergence, but we do not get sharp bounds on $\|e^n\|$. For this reason, the remainder of the proof uses a combination of the L_2–norm, $\|\cdot\|$, in which the local truncation error is $\mathcal{O}(h^{2m+2})$, and the seminorm, $\|\cdot\|_{m+1}$, in which the interpolation is stable.

Estimates in the Seminorm $\|\cdot\|_{m+1}$. Apply D^{m+1} to the error equation (2.22) and obtain:

$$D^{m+1}e^{n+\frac{1}{2}} = D^{m+1}\tilde{I}Se^n + D^{m+1}\eta^n \ .$$

Since $\eta^n = u^{n+\frac{1}{2}} - \tilde{I}u^{n+\frac{1}{2}}$ we can apply Lemma 2.2 and obtain

$$\|D^{m+1}e^{n+\frac{1}{2}}\|^2 = \|D^{m+1}\tilde{I}Se^n\|^2 + \|D^{m+1}\eta^n\|^2 \ . \tag{2.25}$$

For simplicity, if we estimate the local truncation error, we will only write the correct powers of h, but we will not write how the estimate depends on f. Thus we have:

$$\|D^{m+1}e^{n+\frac{1}{2}}\|^2 = \|D^{m+1}\tilde{I}Se^n\|^2 + \mathcal{O}(h^{2m+2}) \ . \tag{2.26}$$

Next we consider the first term on the right–hand side of the above equation. Write

$$D^{m+1}Se^n = D^{m+1}\tilde{I}Se^n + D^{m+1}(Se^n - \tilde{I}Se^n) \ .$$

Using Lemma 2.2 again, one obtains

$$\|D^{m+1}Se^n\|^2 = \|D^{m+1}\tilde{I}Se^n\|^2 + \|D^{m+1}(Se^n - \tilde{I}Se^n)\|^2 \ . \tag{2.27}$$

Let us set

$$\varepsilon_n := \|D^{m+1}e^n\|, \quad \delta_n := \|D^{m+1}(Se^n - \tilde{I}Se^n)\| \ .$$

Then, using (2.26), (2.27), and the fact that S preserves the seminorm we find:

$$\varepsilon^2_{n+\frac{1}{2}} = \varepsilon^2_n - \delta^2_n + \mathcal{O}(h^{2m+2}) \ . \tag{2.28}$$

From the above equation one obtains the recursive error estimate

$$\varepsilon^2_{n+\frac{1}{2}} \leq \varepsilon^2_n + \mathcal{O}(h^{2m+2}) \ .$$

It turns out that the resulting error estimate, $\|D^{m+1}e^n\| \leq Ch^{m+\frac{1}{2}}$ for $nk \leq T$, is not good enough to prove the theorem. Therefore we proceed differently and first obtain an estimate of the quantities $\delta_n = \|D^{m+1}(Se^n - \tilde{I}Se^n)\|$.

From (2.28) we obtain:

$$\delta^2_n = \varepsilon^2_n - \varepsilon^2_{n+\frac{1}{2}} + \mathcal{O}(h^{2m+2}) \ .$$

Therefore, noting that inequalities analogous to those we derived also apply at the even half-steps,

$$\begin{aligned} \sum_{n=0}^{J}(\delta^2_n + \delta^2_{n+\frac{1}{2}}) &\leq \varepsilon^2_0 + CJh^{2m+2} \\ &\leq Ch^{2m+1} \quad \text{for} \quad Jk \leq T \ . \end{aligned} \tag{2.29}$$

Error Estimates in the L_2–norm. Having established the bound (2.29) we now treat the error equations (2.22),(2.23) in the L_2–norm:

$$\begin{aligned} \|e^{n+\frac{1}{2}}\| &\leq \|\tilde{I}Se^n\| + Ch^{2m+2} \\ &\leq \|Se^n\| + \|Se^n - \tilde{I}Se^n\| + Ch^{2m+2} \\ &= \|e^n\| + \|Se^n - \tilde{I}Se^n\| + Ch^{2m+2} \end{aligned}$$

Here we have, using Corollary 2.1,

$$\begin{aligned} \|Se^n - \tilde{I}Se^n\| &\leq Ch^{m+1}\|D^{m+1}(Se^n - \tilde{I}Se^n)\| \\ &= Ch^{m+1}\delta_n \end{aligned}$$

Therefore,

$$\|e^{n+\frac{1}{2}}\| \leq \|e^n\| + Ch^{m+1}\delta_n + Ch^{2m+2},$$

and, consequently, using the analogous inequality for the even half-steps:

$$\|e^{J+1}\| + \|e^{J+\frac{1}{2}}\| \leq Ch^{m+1}\sum_{n=0}^{J}(\delta_n + \delta_{n+\frac{1}{2}}) + Ch^{2m+1} \quad \text{for} \quad 0 \leq (J+1)k \leq T \ . \tag{2.30}$$

Finally,

$$\begin{aligned}
\sum_{n=0}^{N}(\delta_n + \delta_{n+\frac{1}{2}}) &\le C\Big(\sum_{n=0}^{N} 1\Big)^{1/2}\Big(\sum_{n=0}^{N}(\delta_n^2 + \delta_{n+\frac{1}{2}}^2)\Big)^{1/2} \\
&\le Ch^{-1/2}h^{m+\frac{1}{2}} \\
&= Ch^m
\end{aligned}$$

Using this bound in (2.30), the theorem is proved.$\diamond$

3 Variable Coefficient Symmetric Systems

We now consider the spatially-periodic, forced, symmetric hyperbolic system:

$$u_t = A(x,t)u_x + B(x,t)u + F(x,t), \quad A = A^T, \quad u(x,0) = f(x), \tag{3.1}$$

$$f(x+2\pi) = f(x), \quad F(x+2\pi,t) = F(x,t), \quad A(x+2\pi,t) = A(x,t), \quad B(x+2\pi,t) = B(x,t). \tag{3.2}$$

Here $u, f, F \in R^d$, $A, B \in R^{d\times d}$.

Theoretically, the natural generalization of the method described in the preceding sections is to:

(i) Compute the componentwise Hermite interpolant of the approximate solution at the current time level on the mesh;

(ii) Evolve the interpolant one half time step;

(iii) Interpolate the new data on the dual mesh;

(iv) Evolve another half time step.

We will see that the convergence analysis of this method is easily generalized from the scalar case. The problem is to practically carry out or approximate the exact evolution steps, (ii) and (iv).

3.1 Constant Coefficient Cauchy Problem

We begin with the case $B, F \equiv 0$, $A(x,t) = \bar{A} =$ constant. Then an exact solution for piecewise polynomial initial data may be locally constructed by a Taylor series in time. Precisely, setting $\rho(\bar{A})$ to be the spectral radius of $\bar{A}$, we have the lemma:

Lemma 3.1 *Suppose $p(x,t_n)$ is a degree $2m+1$ piecewise polynomial function of x with breaks at $x_0 < x_1 < \ldots < x_N$. Then if p satisfies $p_t = \bar{A}p_x$ for $t \ge t_n$ and $\min_k |\bar{x} - x_k| > \rho(\bar{A})(t_{n+1/2} - t_n)$:*

$$p(\bar{x}, t_{n+1/2}) = \sum_{j=0}^{2m+1} \frac{k^j}{2^j j!}\bar{A}^j \frac{\partial^j p}{\partial x^j}(\bar{x}, t_n).$$

Proof: Noting the local polynomial structure of p, it is clear that the space-time polynomial obtained by replacing $t_{n+1/2}$ by t in the right hand side of the expression above solves $p_t = \bar{A} p_x$ with the same initial data as p at $t = t_n$. The result then follows from standard domain of dependence considerations [12, Ch. 3,6].$\diamond$

Recalling the definition of $\tilde{h}$ as twice the minimum distance between nodes on the dual meshes (see (2.4)) we have the following:

Theorem 3.1 *Let $0 < \lambda = k/\tilde{h} < \rho(\bar{A})$ and $T > 0$ be fixed. Let u solve (3.1) with $B, F = 0$ and $A = \bar{A}$, let $f \in (H^{2m+2}_{per})^d$ and let p be the numerical approximation based on degree $2m+1$ Hermite interpolation and degree $2m+1$ Taylor evolution in time as defined in Lemma 3.1. Then there is a constant C, independent of h, so that*

$$\|u^n - p^n\| + \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \leq C h^{2m+1} \|D^{2m+2} f\| \quad \textit{for} \quad 0 \leq nk \leq T \ .$$

Proof: We simply repeat the proof of Theorem 2.1, noting that $\bar{A}$ can be diagonalized and that by Lemma 3.1 the Taylor formula to advance p is exact at the nodes.$\diamond$

3.2 Variable Coefficients

We now return to the full problem. The key difference is that it is no longer possible to evolve polynomial data exactly. We begin with standard estimates of the solution. (Again see [12, Ch. 3,6] and the appendix for proofs.) For simplicity we set:

$$c = \max_{x,t} |A(x,t)|. \tag{3.3}$$

Lemma 3.2 *Suppose:*

$$A, B \in C^r([0,T]; (C^r_{\text{per}})^{d \times d}), \quad F \in C^r([0,T]; (C^r_{\text{per}})^d), \quad f \in (C^r_{\text{per}})^d.$$

Then equation (3.1) has a solution, $u \in C^r([0,T]; (C^r_{\text{per}})^d)$. Moreover, for any $0 \leq q \leq r$ there exists $\alpha(q)$, $C(q)$ independent of the initial data and F such that for any $x_0 \leq x_1$, $0 \leq t_0 \leq t_1 \leq T$:

$$\begin{aligned}
\|\frac{\partial^q u}{\partial x^q}(\cdot, t_1)\|_{(x_0,x_1)} \quad \leq \quad & e^{\alpha \Delta t} \|\frac{\partial^q u}{\partial x^q}(\cdot, t_0)\|_{(x_0 - c\Delta t, x_1 + c\Delta t)} \\
& + C\Delta t \sum_{j=0}^{q-1} \|\frac{\partial^j u}{\partial x^j}(\cdot, t_0)\|_{(x_0 - c\Delta t, x_1 + c\Delta t)} \\
& + C\Delta t \max_{t_0 \leq t \leq t_1} \sum_{j=0}^{q} \|\frac{\partial^j F}{\partial x^j}(\cdot, t)\|_{(x_0 - c\Delta t, x_1 + c\Delta t)}.
\end{aligned}$$

$$\begin{aligned}
|\frac{\partial^q u}{\partial x^q}(x, t_1)| \quad \leq \quad & e^{\alpha \Delta t} \max_{|z-x| \leq c\Delta t} |\frac{\partial^q u}{\partial x^q}(z, t_0)| \\
& + C\Delta t \sum_{j=0}^{q-1} \max_{|z-x| \leq c\Delta t} |\frac{\partial^j u}{\partial x^j}(z, t_0)| \\
& + C\Delta t \max_{t_0 \leq t \leq t_1} \sum_{j=0}^{q} \max_{|z-x| \leq c\Delta t} |\frac{\partial^j F}{\partial x^j}(z, t)|.
\end{aligned}$$

and for $q_1 \geq 1$*:*

$$
\begin{aligned}
\| \frac{\partial^q u}{\partial t^{q_1} \partial x^{q-q_1}}(\cdot, t_1) \|_{(x_0,x_1)} \quad &\leq \quad C \sum_{j=0}^{q} \| \frac{\partial^j u}{\partial x^j}(\cdot, t_0) \|_{(x_0 - c\Delta t, x_1 + c\Delta t)} \\
&\quad + C \sum_{j=0}^{q} \sum_{l=0}^{\min(q_1-1,j)} \max_{t_0 \leq t \leq t_1} \| \frac{\partial^j F}{\partial t^l \partial x^{(j-l)}}(\cdot, t) \|_{(x_0 - c\Delta t, x_1 + c\Delta t)}.
\end{aligned}
$$

$$
\begin{aligned}
| \frac{\partial^q u}{\partial t^{q_1} \partial x^{q-q_1}}(x, t_1) | \quad &\leq \quad C \sum_{j=0}^{q} \max_{|z-x| \leq c\Delta t} \| \frac{\partial^j u}{\partial x^j}(z, t_0) | \\
&\quad + C \sum_{j=0}^{q} \sum_{l=0}^{\min(q_1-1,j)} \max_{t_0 \leq t \leq t_1} \max_{|z-x| \leq c\Delta t} | \frac{\partial^j F}{\partial t^l \partial x^{(j-l)}}(z, t) |.
\end{aligned}
$$

Here $\Delta t = t_1 - t_0$.

We again use Taylor series to evolve the solution in time. However, note that even for polynomial data and zero forcing the series does not terminate. Thus we must specify the degree of the Taylor series we wish to use and account for the additional truncation error. To be precise, denote by $p(x,t)$, $t_n \leq t \leq t_{n+\frac{1}{2}}$ the solution of (3.1) with initial data $p(x, t_n)$ which is a piecewise polynomial function of degree $2m+1$ with breaks x_i as before. Let $x_i < x_{i+1/2} < x_{i+1}$. Then if the CFL number:

$$
\lambda = c \frac{k}{\bar{h}} < 1, \tag{3.4}
$$

we define for $q \geq 2m+1$, $0 \leq l \leq m$:

$$
T^{(q,l)}(p)(x_{i+1/2}, t_{n+1/2}) = \sum_{j=0}^{q-l} \frac{k^j}{2^j j!} \frac{\partial^{(j+l)} p}{\partial t^j \partial x^l}(x_{i+1/2}, t_n). \tag{3.5}
$$

Then by differentiating (3.1) in space and time and applying Leibniz' rule we derive the following identity which is used to compute the terms appearing in (3.5) recursively:

$$
\begin{aligned}
\frac{\partial^{(j+l)} p}{\partial t^j \partial x^l} \quad &= \quad \sum_{i_1=0}^{j-1} \sum_{i_2=0}^{l} \begin{pmatrix} j-1 \\ i_1 \end{pmatrix} \cdot \begin{pmatrix} l \\ i_2 \end{pmatrix} \frac{\partial^{j+l-1-i_1-i_2} A}{\partial t^{j-1-i_1} \partial x^{l-i_2}} \frac{\partial^{i_1+i_2+1} p}{\partial t^{i_1} \partial x^{l+1+i_2}} \\
&\quad + \sum_{i_1=0}^{j-1} \sum_{i_2=0}^{l} \begin{pmatrix} j-1 \\ i_1 \end{pmatrix} \cdot \begin{pmatrix} l \\ i_2 \end{pmatrix} \frac{\partial^{j+l-1-i_1-i_2} B}{\partial t^{j-1-i_1} \partial x^{l-i_2}} \frac{\partial^{i_1+i_2} p}{\partial t^{i_1} \partial x^{l+i_2}} \\
&\quad + \frac{\partial^{(j-1+l)} F}{\partial t^{j-1} \partial x^l}.
\end{aligned} \tag{3.6}
$$

Here, all functions are evaluated at $(x_{i+\frac{1}{2}}, t_n)$ and we have used the standard notation for the binomial coefficients. We note that the right-hand side only involves t-derivatives of order up to $j-1$, and thus the recursion may be initiated by space derivatives of p alone. Note also that we may choose $q > 2m+1$ in which case space derivatives of p of order greater than $2m+1$ appear. Although these are zero initially, their time derivatives generally are not. We also note that the degree of the Taylor series in (3.5) decreases as the order of the spatial derivative increases. In some implementations for nonlinear systems it may be advantageous to use the same degree for all derivatives. Such issues will be discussed in a later work.

We denote the method defined by degree $2m+1$ Hermite interpolation and degree q Taylor expansion as the degree $(2m+1, q)$ Hermite-Taylor method. Our notation will be:

$$p^{n+\frac{1}{2}} = \tilde{I}T^{(q)}p^n. \tag{3.7}$$

This is a slight abuse of notation as the Taylor series in time is not defined at the breakpoints. However, in all arguments the operator $T^{(q)}$ is followed by an interpolation. We will now prove a generalization of Theorem 2.1 to cover this case.

Theorem 3.2 *Suppose $q \geq 2m+1$, A, B and F satisfy the assumptions of (3.2) with $r = q+1$, and $f \in C^{2m+2}_{\text{per}}$. Let u be the solution of (3.1), and p the numerical approximation to u computed via the degree $(2m+1, q)$ Hermite-Taylor method with $q \geq 2m+1$. Then for any $T > 0$ there exists C depending on T, F, f, A and B but independent of the mesh such that:*

$$\|u^n - p^n\| + \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \leq Ch^\kappa \quad \textit{for} \quad 0 \leq nk \leq T \ ,$$

where

$$\kappa = \min(2m+1, q - \frac{1}{2})$$

Remark: Clearly if we choose the minimal value of $q = 2m+1$ our error estimate is one-half order lower than in the constant coefficient case. We recover the earlier estimate by taking $q \geq 2m+2$. However in practice we have seen an advantage to choosing $q > 2m+1$ only for marginally resolved cases with m large. Some comparisons will be made in the final section.

Proof: We copy the proof of Theorem 2.1 but with the additional time truncation error term. Precisely we write:

$$\begin{aligned} e^{n+1/2} &= \tilde{I}(Su^n - Sp^n) + \tilde{I}(Sp^n - T^{(q)}p^n) + Su^n - \tilde{I}Su^n \\ &= \tilde{I}S_h e^n + \gamma^n + \eta^n, \end{aligned} \tag{3.8}$$

where S_h denotes the solution operator for the homogeneous problem ($F = 0$) and γ^n, η^n are the truncation error terms:

$$\gamma^n = \tilde{I}(Sp^n - T^{(q)}p^n),$$

and η^n is defined as in (2.17). We begin with the equality for the error seminorm:

$$\|D^{m+1}e^{n+1/2}\|^2 = \|D^{m+1}\tilde{I}S_h e^n + D^{m+1}\gamma^n\|^2 + \|D^{m+1}\eta^n\|^2. \tag{3.9}$$

From Lemmas 2.1 and 3.2 we then have for any $a > 0$:

$$\|D^{m+1}e^{n+1/2}\|^2 \leq (1+ah)\|D^{m+1}\tilde{I}S_h e^n\|^2 + (1+(ah)^{-1})\|D^{m+1}\gamma^n\|^2 + Ch^{2m+2}. \tag{3.10}$$

To proceed further we must estimate the Taylor truncation error term, which we accomplish in the next lemma.

Lemma 3.3 *Under the assumptions of the Theorem we have for some constant C independent of h:*

$$\|D^{m+1}\gamma^n\|^2 \le Ch^{2q-4m}\left(h^{2m} + \|D^{m+1}e^n\|^2 + \|e^n\|^2\right),$$

$$\|\gamma^n\|^2 \le Ch^{2q-2m+2}\left(h^{2m} + \|D^{m+1}e^n\|^2 + \|e^n\|^2\right),$$

Proof: By Taylor's Theorem we have for any space derivative of order l, $0 \le l \le m$, evaluated at $(x_{i+\frac{1}{2}}, t_{n+\frac{1}{2}})$:

$$|\left(D_x^l Sp^n - T^{(q,l)}p^n\right)|^2 \le Ch^{2q+2-2l} \max_{t_n \le t \le t_{n+1/2}} |\frac{\partial^{q+1} Sp^n}{\partial t^{q+1-l}\partial x^l}(x_{i+1/2}, t)|^2. \tag{3.11}$$

From (3.2) and using the fact that p^n is piecewise polynomial of degree $2m+1$ we bound the right-hand side of (3.11) by:

$$Ch^{2q+2-2l}\left(1 + \sum_{j=0}^{2m+1} \max_{x \in (x_i, x_{i+1})}\left(\frac{\partial^j p^n}{\partial x^j}(x, t_n)\right)^2\right).$$

We write:

$$p^n = Ip^n = Iu^n - Ie^n.$$

Then by Lemma 2.3 we bound the right-hand side of (3.11) by:

$$Ch_{k+\frac{1}{2}}^{2q+2-2l}\left(1 + h_{i+\frac{1}{2}}^{-(2m+1)}\|e^n\|^2_{H^{m+1}(x_i,x_{i+1})}\right).$$

Again applying Lemma 2.3 to bound $\|D^{m+1}\gamma^n\|^2$ and recalling (2.3) we obtain:

$$\begin{aligned}
\|D^{m+1}\gamma^n\|^2 &\le C\sum_{i=1}^{N}\sum_{l=0}^{m} h_i^{2l-2m-1}\sum_{\nu=0}^{1}|\left(D^l Sp^n - T^{(q,l)}p^n\right)(x_{i+\nu-1/2}, t_{n+1/2})|^2 \\
&\le C\sum_{i=0}^{N-1} h_{i+\frac{1}{2}}^{2q-2m+1}\left(1 + h_{i+\frac{1}{2}}^{-(2m+1)}\|e^n\|^2_{H^{m+1}(x_i,x_{i+1})}\right) \\
&\le Ch^{2q-4m}\left(h^{2m} + \|e^n\|^2_{H^{m+1}}\right).
\end{aligned}$$

Applying the well-known inequality:

$$\|e^n\|^2_{H^{m+1}} \le C\left(\|D^{m+1}e^n\|^2 + \|e^n\|^2\right),$$

we reach the first result. Similarly

$$\begin{aligned}
\|\gamma^n\|^2 &\le C\sum_{i=1}^{N}\sum_{l=0}^{m} h_i^{2l+1}\sum_{\nu=0}^{1}|\left(D^l Sp^n - T^{(q,l)}p^n\right)(x_{i+\nu-1/2}, t_{n+1/2})|^2 \\
&\le C\sum_{i=0}^{N-1} h_{i+\frac{1}{2}}^{2q+3}\left(1 + h_{i+\frac{1}{2}}^{-(2m+1)}\|e^n\|^2_{H^{m+1}(x_i,x_{i+1})}\right) \\
&\le Ch^{2q-2m+2}\left(h^{2m} + \|D^{m+1}e^n\|^2 + \|e^n\|^2\right).
\end{aligned}$$

This yields the final result.$\diamond$

Returning to the proof of Theorem 3.2 we now have:

$$\|D^{m+1}e^{n+1/2}\|^2 \le (1+ah)\|D^{m+1}\tilde{I}S_h e^n\|^2 + Ch^{2q-4m-1}\left(\|D^{m+1}e^n\|^2 + \|e^n\|^2\right) + Ch^{\bar{\kappa}},$$

where $\bar{\kappa} = \min(2q-2m-1, 2m+2)$. Defining as before:

$$\epsilon_n = \|D^{m+1}e^n\|, \quad \delta_n = \|D^{m+1}(S_h e^n - \tilde{I}S_h e^n)\|$$

we use Lemma 2.2 to deduce:

$$\epsilon^2_{n+\frac{1}{2}} \le (1+ah)\|D^{m+1}S_h e^n\|^2 - \delta_n^2 + Ch^{2q-4m-1}\left(\epsilon_n^2 + \|e^n\|^2\right) + Ch^{\bar{\kappa}}.$$

By Lemma 3.2 we conclude that

$$\|D^{m+1}S_h(e^n)\|^2 \le (1+Ch)\epsilon_n^2 + Ch\|e^n\|^2.$$

Since $q \ge 2m+1$ we finally have:

$$\epsilon^2_{n+\frac{1}{2}} \le (1+Ch)\epsilon_n^2 - \delta_n^2 + Ch\|e^n\|^2 + Ch^{\bar{\kappa}}. \tag{3.12}$$

From (3.12) we conclude:

$$\epsilon_n^2 + \epsilon^2_{n+\frac{1}{2}} \le Ch^{\bar{\kappa}-1} + Ch\sum_{j=0}^{n}\left(\|e^j\|^2 + \|e^{j+\frac{1}{2}}\|^2\right), \tag{3.13}$$

$$\sum_{j=0}^{n}\left(\delta_j^2 + \delta^2_{j+\frac{1}{2}}\right) \le Ch^{\bar{\kappa}-1} + Ch\sum_{j=0}^{n}\left(\|e^j\|^2 + \|e^{j+\frac{1}{2}}\|^2\right). \tag{3.14}$$

Finally we estimate the L_2–norm directly. In particular we have:

$$\|e^{n+\frac{1}{2}}\| \le \|\tilde{I}S_h e^n\| + \|\gamma^n\| + \|\eta^n\|.$$

Estimating the truncation errors as above we have:

$$\|\eta^n\| \le Ch^{2m+2},$$

and by Lemma 3.3 and (3.13):

$$\|\gamma^n\| \le Ch^{q+1} + Ch^{q-m+\frac{3}{2}}\left(\sum_{j=0}^{n}\left(\|e^j\|^2 + \|e^{j+\frac{1}{2}}\|^2\right)\right)^{1/2} + Ch^{q-m+1}\|e^n\|.$$

Finally we write as in the proof of Theorem 2.1:

$$\begin{aligned}\|\tilde{I}S_h e^n\| &\le \|S_h e^n\| + Ch^{m+1}\delta_n \\ &\le (1+Ch)\|e^n\| + Ch^{m+1}\delta_n.\end{aligned}$$

Set

$$E^n = \max_{j\le n}(\|e^j\|, \|e^{j-\frac{1}{2}}\|), \quad E^{n+\frac{1}{2}} = \max_{j\le n}(\|e^j\|, \|e^{j+\frac{1}{2}}\|).$$

Then combining the last few inequalitites and recalling that $q \geq 2m+1$ we find:

$$E^{n+\frac{1}{2}} \leq (1+Ch)E^n + Ch^{m+1}\delta_n + Ch^{2m+2}.$$

Using (3.14) this implies for $nk \leq T$:

$$\begin{aligned} E^{n+1} + E^{n+\frac{1}{2}} &\leq Ch^{2m+1} + Ch^{m+1}\sum_{j=0}^{n}\left(\delta_j + \delta_{j+\frac{1}{2}}\right) \\ &\leq Ch^{2m+1} + Ch^{m+\frac{1}{2}}\left(\sum_{j=0}^{n}\left(\delta_j + \delta_{j-\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \\ &\leq Ch^{m+\frac{\bar{\kappa}}{2}} + Ch^{m+\frac{1}{2}}E^{n+\frac{1}{2}}. \end{aligned}$$

Choosing h sufficiently small we can clearly eliminate the term involving $E^{n+\frac{1}{2}}$ from the right-hand side. Noting that:

$$\kappa = m + \frac{\bar{\kappa}}{2},$$

we have completed the proof of Theorem 3.2.$\diamond$

4 Initial–Boundary Value Problems

We now consider the initial-boundary value problem:

$$u_t = Au_x + Bu + F, \quad x \in (0,1), \quad u(x,0) = f(x) \tag{4.1}$$

$$u^R(0,t) = M^L u^L(0,t) + \gamma^L(t), \tag{4.2}$$

$$u^L(1,t) = M^R u^R(1,t) + \gamma^R(t). \tag{4.3}$$

We assume, for simplicity, that A is diagonal:

$$A = \begin{pmatrix} \Lambda^R & O \\ O & \Lambda^L \end{pmatrix}, \tag{4.4}$$

where

$$\Lambda^R \in R^{d_R \times d_R}, \quad \Lambda^R < O,$$
$$\Lambda^L \in R^{d_L \times d_L}, \quad \Lambda^L > O.$$

The partitioning

$$u = \begin{pmatrix} u^R \\ u^L \end{pmatrix},$$

thus corresponds to a splitting into right-going and left-going characteristic variables, and the boundary conditions (4.2),(4.3) explicitly express the incoming waves in terms of outgoing waves and inhomogeneous data. We note that any symmetric system can be diagonalized and the incoming and outgoing characteristic variables identified, which is all that is needed to construct the discrete approximations. For variable coefficient problems it is possible for the number of right-going and left-going waves to be different at each boundary, but for simplicity we ignore this possibility. The results extend to that case. It would also be possible for the number of ingoing and outgoing waves at a boundary to change as a function of time. Again we exclude this possibility, which would require further analysis.

We note that the problem (4.1)-(4.3) is strongly well-posed [12, 7.6]. Away from the boundaries the local estimates of Lemma 3.2 still hold. Near them, however, they must be modified to take account of the boundary data, so that additional inhomogeneous terms appear. Global estimates thus take the form [12, 7.6]:

$$\begin{aligned} &\sum_{j=0}^{m} \left(\|D^j u(\cdot,t)\|^2 + \int_0^t \left(|D^j u(0,\tau)|^2 + |D^j u(1,\tau)|^2\right) d\tau\right) \\ \leq\ & Ce^{\alpha_m t} \sum_{j=0}^{m} \left(\|D^j f\|^2 + \int_0^t \left(\|D^j F(\cdot,\tau)\|^2 + |\tfrac{\partial^j \gamma_L}{\partial t}(\tau)|^2 + |\tfrac{\partial \gamma_R}{\partial t}(\tau)|^2\right) d\tau\right) \end{aligned} \tag{4.5}$$

Time derivatives and mixed derivatives can be expressed by space derivatives; thus analogous estimates hold for them.

4.1 Description of the Method

We introduce staggered grids satisfying (2.3) as before:

$$0 = x_0 < x_{\frac{1}{2}} < x_1 < x_{\frac{3}{2}} < x_2 < \ldots < x_{N-\frac{1}{2}} < x_N = 1,$$

$$\begin{aligned} \tilde{G}_h &= \left\{x_{j+\frac{1}{2}},\ \ 0 \leq j \leq N-1\right\} \bigcup \{x_0, x_N\}, \\ G_h &= \left\{x_j,\ \ 0 \leq j \leq N\right\}. \end{aligned}$$

Note that the boundary points are included in each mesh.

Given Hermite data of order up through m on one of the grids, the (m,q) Hermite-Taylor update on the interior of the dual grid is carried out precisely as in the periodic cases described earlier. Thus we need only describe how to update the solution and its space derivatives at the points x_0 and x_N.

For definiteness we look at an even half-step where we propagate data associated with the piecewise polynomial p^n from G_h to $\tilde{G}_h$. Applying the update directly at $x = x_0$ and $x = x_N$ we obtain values:

$$T^{(q,l)}(p)(0, t_{n+\frac{1}{2}}),\ \ T^{(q,l)}(p)(1, t_{n+\frac{1}{2}}),$$

via (3.5). Note that these values only use one-sided spatial information and typically do not satisfy the boundary conditions. Thus we will use the boundary conditions to update the incoming characteristic variable and its space derivatives. Specializing to the left boundary $x = 0$, we will derive recursion relations allowing us to effectively express $D^l u^R$ in terms of $D^l u^L$ alone. We use the one-sided Taylor updates to compute this outgoing data.

Starting with function values we compute the approximation to u^L via (3.5):

$$\left(V_0^{n+\frac{1}{2},L}\right)_0 = (T^{(q,0)}(p)(0, t_{n+\frac{1}{2}}))^L.$$

The boundary condition itself (4.2) provides us with our approximation to $u^R(0,t_{n+\frac{1}{2}})$ directly:

$$\left(V_0^{n+\frac{1}{2},R}\right)_0 = M^L\left(V_0^{n+\frac{1}{2},L}\right)_0 + \gamma^L(t_{n+\frac{1}{2}}).$$

For ease of writing in what follows we drop the space-time indices. That is, it is to be understood that all updates are at the point $(x,t)=(0,t_{n+\frac{1}{2}})$. Also we introduce the notation:

$$(V)_{lj} \approx \frac{\partial^{l+j}u}{\partial x^l \partial t^j}(0,t_{n+\frac{1}{2}}),$$

so that the approximations computed above are redesignated $(V)_{00}$.

Proceeding to first space derivatives we again use the one-sided update to compute the approximation to Du^L;

$$(V^L)_{10} = (T^{(q,1)}(p))^L.$$

Using this data and (4.1), we compute an approximation to u_t^L:

$$(V^L)_{01} = \Lambda^L(V^L)_{10} + (B(V)_{00})^L + F^L,$$

which can be substituted into the time derivative of (4.2) to produce an approximation to u_t^R:

$$(V^R)_{01} = M^L(V^L)_{01} + M_t^L(V^L)_{00} + \gamma_t^L.$$

Finally, we compute an approximation to Du^R using (4.1):

$$(V^R)_{10} = (\Lambda^R)^{-1}\left((V^R)_{01} - (B(V)_{00})^R - F^R\right).$$

This process, though involved, can be repeated as desired. Proceeding recursively, suppose we know all space-time derivatives of order up through $\nu-1$. That is we have values for $(V)_{lj}$, $l+j\leq\nu-1$. We compute the derivatives of order ν as follows.

i. The approximate νth space derivative of u^L is computed via the one-sided Taylor update:

$$(V^L)_{\nu 0} = (T^{(q,\nu)}(p))^L. \tag{4.6}$$

ii. Approximate order ν space-time derivatives of u^L are computed using derivatives of (4.1). In particular, for $j=1,\ldots,\nu$ we set:

$$\begin{aligned}(V^L)_{\nu-j,j} &= \frac{\partial^{\nu-1}F^L}{\partial x^{\nu-j}\partial t^{j-1}} + \sum_{l_1=0}^{\nu-j}\begin{pmatrix}\nu-j\\ l_1\end{pmatrix}\sum_{l_2=0}^{j-1}\begin{pmatrix}j-1\\ l_2\end{pmatrix} \\ &\quad \left(\frac{\partial^{\nu-1-l_1-l_2}\Lambda^L}{\partial x^{\nu-j-l_1}\partial t^{j-1-l_2}}(V^L)_{l_1+1,l_2} + (\frac{\partial^{\nu-1-l_1-l_2}B}{\partial x^{\nu-j-l_1}\partial t^{j-1-l_2}}(V)_{l_1,l_2})^L\right).\end{aligned} \tag{4.7}$$

Note that if we proceed sequentially in j all terms on the right-hand side are known.

iii. Take the νth time derivative of (4.2) to compute $(V^R)_{0\nu}$:

$$(V^R)_{0\nu} = \sum_{l=0}^{\nu}\begin{pmatrix}\nu\\ l\end{pmatrix}\frac{d^{\nu-l}M^L}{dt^{\nu-l}}(V^L)_{0l} + \frac{\partial^\nu\gamma^L}{dt^\nu}. \tag{4.8}$$

iv. Finally, using (4.1) compute sequentially for $j = 1, \ldots, \nu$:

$$\Lambda^R (V^R)_{j,\nu-j} = (V^R)_{j-1,\nu+1-j} - \frac{\partial^{\nu-1} F^R}{\partial x^{j-1} \partial t^{\nu-j}} - \sum_{l_1=0}^{j-2} \begin{pmatrix} j-1 \\ l_1 \end{pmatrix} \sum_{l_2=0}^{\nu-j} \begin{pmatrix} \nu-j \\ l_2 \end{pmatrix} \left(\frac{\partial^{\nu-1-l_1-l_2} \Lambda^L}{\partial x^{j-1-l_1} \partial t^{\nu-j-l_2}} (V^R)_{l_1+1,l_2} + \left(\frac{\partial^{\nu-1-l_1-l_2} B}{\partial x^{j-1-l_1} \partial t^{\nu-j-l_2}} (V)_{l_1,l_2} \right)^R \right). \tag{4.9}$$

At its termination, this process produces all the required derivatives of V^R. At $x = 1$ we carry out the analogous process with the roles of V^L and V^R reversed and (4.2) replaced by (4.3). The update process for even time steps is identical. We note that this process defines an affine map between the outgoing Hermite data and the incoming Hermite data. We represent this map via the following notation:

$$(V_0^\mu)^R = P^{L,\mu} (V_0^\mu)^L + \Gamma^{L,\mu}, \tag{4.10}$$

$$(V_N^\mu)^L = P^{R,\mu} (V_0^\mu)^R + \Gamma^{R,\mu}, \tag{4.11}$$

where $\mu = n$ or $\mu = n + \frac{1}{2}$.

4.2 Stability Analysis

We first note that the Orthogonality Lemma 2.2 still holds in this case. Also, we note that equations (4.7)-(4.9) are satisfied by the exact solution u. A simplified stability and convergence analysis can then be constructed for problems where the Taylor time evolution is exact, that is when:

$$A(x,t) = \bar{A}, \quad M^L(t) = \bar{M}^L, \quad M^R(t) = \bar{M}^R, \quad B, F, \gamma = 0. \tag{4.12}$$

Precisely we have:

Lemma 4.1 *Suppose p^n is a C^m piecewise degree $2m+1$ polynomial relative to the grid G_h, and Sp^n denotes the exact solution of (4.1)-(4.3) for $t_n \leq t \leq t_{n+\frac{1}{2}}$ equal to p^n at $t = t_n$. Suppose further that (4.12) holds and that p^n is compatible; that is p^n satisfies (4.10)-(4.11). Then if (3.4) holds, Sp^n satisfies (4.10)-(4.11) at $t = t_{n+\frac{1}{2}}$.*

Proof: For systems satisfying (4.12) $u^L(0,t)$ is given by:

$$(u^L)_i(0,t) = (u^L)_i(\Lambda_i^L \Delta t, t - \Delta t).$$

Thus if the data, $p^n(x)$, is a degree $2m+1$ polynomial on $x \in (0, \Lambda_i^L \Delta t + \epsilon)$ then:

$$\begin{aligned} D^l (Sp^n)_i^L (0, t_{n+\frac{1}{2}}) &= D^l (p^n)_i^L (\Lambda_i^L \Delta t) \\ &= \sum_{j=0}^{2m+1-l} \frac{(\Lambda_i^L \Delta t)^j}{j!} D^{j+l} (p^n)_i^L (0) \\ &= (T^{(2m+1,l)} p^n)_i^L. \end{aligned}$$

Thus Sp^n satisfies the computational update formula (4.6). As our assumption that the compatibility conditions hold ensures a smooth solution, (4.7)-(4.9), which are consequences of the equation and boundary condition, are also satisfied.$\diamond$

Now we must consider the seminorm $\|D^{m+1}p^n\|$. We consider the simplest case when the reflection matrices are zero; that is $\bar{M}^L = O$, $\bar{M}^R = O$. Then the boundary conditions imply:

$$D^j u^R(0,t) = 0, \quad D^j u^L(1,t) = 0. \tag{4.13}$$

Moreover, the system decouples into scalar equations:

$$u^L_{i,t} = \Lambda^L_i u^L_{i,x}, \quad u^R_{i,t} = \Lambda^R_i u^R_{i,x}. \tag{4.14}$$

Combining this with (4.13) we find:

$$(Sp^n)^L_i = \begin{cases} p^n(x + \Lambda^L_i t), & 0 \leq x + \Lambda^L_i t \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.15}$$

$$(Sp^n)^R_i = \begin{cases} p^n(x + \Lambda^R_i t), & 0 \leq x + \Lambda^R_i t \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.16}$$

which implies:

$$\|D^{m+1}Sp^n\| \leq \|D^{m+1}p^n\|. \tag{4.17}$$

Combining (4.17) with Lemma 2.2 we deduce the basic stability inequality:

$$\begin{aligned} \|D^{m+1}p^{n+\frac{1}{2}}\| &\leq \|D^{m+1}S(p^n)\| \\ &\leq \|D^{m+1}p^n\|. \end{aligned} \tag{4.18}$$

In this special case we can now repeat the arguments leading to Theorems 2.1 and 3.1 to prove:

Theorem 4.1 *Let $0 < \lambda = k/h < \rho(\bar{A})$ and $T > 0$ be fixed. Let u solve (4.1)-(4.3) under the assumptions of Lemma 4.1 and suppose in addition that $\bar{M}^L = O$, $\bar{M}^R = O$. Let $f \in (H^{2m+2})^d$ be compatible with the boundary conditions and let p be the numerical approximation based on degree $2m+1$ Hermite interpolation and degree $2m+1$ Taylor evolution. Then there is a constant C, independent of h, so that*

$$\|u^n - p^n\| + \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \leq Ch^{2m+1}\|D^{2m+2}f\| \quad \textit{for} \quad 0 \leq nk \leq T \ .$$

Theorem 4.1 can be generalized to problems with variable coefficents and nonzero reflections using the techniques of Section 3 and the estimates of (4.5). The idea is again to estimate the error in truncating the Taylor series. Of course (4.18) is replaced in the argument by an inequality of the form (3.12). We have not carried out the details, but present some simple numerical experiments to illustrate the method's performance.

5 Generalization to Multiple Dimensions

We now consider a symmetric hyperbolic system in R^s:

$$u_t = \sum_{j=1}^{s} A_j(x,t)u_{x_j} + B(x,t)u + F(x,t), \quad A_j = A_j^T, \quad u(x,0) = f(x), \tag{5.1}$$

under the assumption that all functions are 2π-periodic d-vectors or $d \times d$ matrices in all space variables, denoting Sobolev spaces of such functions as $(H^q_{s,per})^d$. We assume tensor product grids:

$$\tilde{G}_h = \otimes_{i=1}^s \tilde{G}_{h,i}, \quad G_h = \otimes_{i=1}^s G_{h,i} \tag{5.2}$$

$$\begin{aligned} \tilde{G}_{h,i} &= \left\{ x_{i,j+\frac{1}{2}} : j \in \mathbb{Z} \right\}, \quad x_{i,j+\frac{1}{2}+N} = x_{i,j+\frac{1}{2}} + 2\pi \\ G_{h,i} &= \left\{ x_{i,j} : j \in \mathbb{Z} \right\}, \quad x_{i,j+N} = x_{i,j} + 2\pi. \end{aligned}$$

We set

$$h_i = \max_j (x_{i,j+1} - x_{i,j}), \quad \tilde{h}_i = 2 \min_j (x_{i,j+\frac{1}{2}} - x_{i,j}, x_{i,j+1} - x_{i,j+\frac{1}{2}}), \tag{5.3}$$

and

$$h = \max_i h_i, \quad \tilde{h}(e) = \min_i \frac{\tilde{h}_i}{|e_i|}, \tag{5.4}$$

where $e \in R^s$ is a unit vector.

On these grids one can define tensor-product Hermite interpolants of degree $(2m+1)^s$, that is polynomials of degree $(2m+1)^s$ which are products of polynomials of degree $2m+1$ in each variable, x_i. The nodal data used to define the interpolants is:

$$\left(\prod_{i=1}^s D_i^{j_i} \right) v, \quad 0 \leq j_i \leq m. \tag{5.5}$$

(Here we introduce the notation $D_i = \partial / \partial x_i$.) To evolve these polynomials in time, we look at the special case $A_j =$ constant, $B, F = 0$. Then, if we insist on exact evolution, the Taylor series in time must in general be of degree $(2m+1)^s$, which is of significantly higher degree than required simply to attain the order of accuracy $2m+1$. We note, however, that carrying the series to higher degree does not require any additional spatial data. However, for variable coefficient problems it does lead to additional derivatives of coefficients to be evaluated. In the final section we will present some numerical experiments related to the degree of the temporal Taylor series.

We must also consider the domain of dependence of the updated solution at a mesh point over a half-step. Given a unit vector $e \in R^s$ the maximum wave speed in the direction e is given by:

$$c(e) = \rho \left(\sum_{j=1}^s e_j A_j \right), \tag{5.6}$$

leading to the CFL constraint:

$$\max_{|e|=1} \lambda(e) \equiv c(e) \frac{k}{\tilde{h}(e)} < 1. \tag{5.7}$$

Note that for an isotropic problem on a uniform mesh this reduces to the one-dimensional CFL condition (3.4).

In the Theorem below we derive stability and convergence results for the constant coefficient case. We introduce:

$$\|f\|_{s,[r]} = \sum_{\alpha \in \mathcal{T}_r} \|D^\alpha f\|, \tag{5.8}$$

$$|f|_{s,[r]} = \|f\|_{s,[r]} - \|f\|, \tag{5.9}$$

where $\mathcal{T}_r$ is the set of s-dimensional multiindices defined by:

$$\mathcal{T}_r = \{\alpha | \alpha_j \in \{0, r\}\}. \tag{5.10}$$

In other words, we consider all partial derivatives of f involving derivatives in the coordinate directions of order r or 0. We also denote by $\mathcal{H}_s^{[r]}$ the space of all functions for which $\|\cdot\|_{s,[r]}$ is finite. Obviously, $\mathcal{H}_s^{[r]}$ contains the Sobolev space $H^{rs}_{s,per}$. However, the degree $(2m+1)^s$ tensor product interpolants we are using are, in fact, elements of $\mathcal{H}_s^{[m+1]}$ but are typically *not* elements of $H_s^{(m+1)s}$ for $s > 1$.

We also inductively define a new semi-inner-product for $s > 1$ by:

$$(f,g)_{s,[m+1]} = (\prod_{j=1}^{s} D_j^{m+1} f, \prod_{j=1}^{s} D_j^{m+1} g) + \sum_{i=1}^{s} h_{i,j_i} \sum_{j_i=1}^{N_i} (f,g)_{s-1,[m+1]}|_{x_i = x_{i,j_i}}. \tag{5.11}$$

Note that if we expand this out we see that each term involves inner products of derivatives $\prod_{j=i_1}^{i_r} D_j^{m+1} f(g)$ on r-dimenional faces. On one-dimensional faces we simply use the inner product from the previous sections. An inner-product associated with the dual mesh is defined analogously.

Using these we prove the following Theorem. Note that the proven order of convergence is the same as in the one-dimensional constant coefficient cases.

Theorem 5.1 *Let u solve (5.1) with $B, F = 0$ and $A_j = \bar{A}_j$, constant. Let $f \in \left(\mathcal{H}_{per}^{[2m+2]}\right)^d$ and let p be the numerical approximation based on degree $2m+1$ Hermite tensor-product interpolation and degree $(2m+1)^s$ Taylor evolution with time step k satisfying (5.7). Finally, let $T > 0$ be fixed. Then there is a constant C, independent of the grids, so that*

$$\|u^n - p^n\| + \|u^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}\| \leq C h^{2m+1} |f|_{s,[2m+2]} \quad \text{for} \quad 0 \leq nk \leq T \ .$$

Proof: As in the proof of Theorem 3.1 we use the fact that the Taylor formula to advance p^n in time is exact at the nodes. Thus to repeat the arguments of Theorem 2.1 we must generalize the basic results on Hermite interpolation, namely the analogues of Lemmas 2.1 - 2.2.

We first note the following estimates of the interpolation error which are discussed in the appendix. A crucial point is that the convergence rate for errors in approximating $\|g\|_{s,[m+1]}$ are the same as for approximations to $D_i^{m+1} g$.

Lemma 5.1 *The Hermite interpolation operators, I and $\tilde{I}$, satisfy:*

$$\begin{aligned} \|g - Ig\| &\leq C h^{2m+2} |g|_{s,[2m+2]} \quad \text{for} \quad g \in \mathcal{H}_s^{[2m+2]} && (5.12) \\ \|g - Ig\|_{s,[m+1]} &\leq C h^{m+1} |g|_{s,[2m+2]} \quad \text{for} \quad g \in \mathcal{H}_s^{[2m+2]} && (5.13) \\ \|g - Ig\| &\leq C h^{m+1} |g|_{s,[m+1]} \quad \text{for} \quad g \in \mathcal{H}_s^{[m+1]} && (5.14) \end{aligned}$$

$$\|q - Iq\| \leq C h^{m+1} |q - Iq|_{s,[m+1]} \quad \text{for} \quad q \in \mathcal{H}_s^{[m+1]} \tag{5.15}$$

We also have the generalization of Lemma 2.2:

Lemma 5.2 *For all $f,g \in \mathcal{H}_s^{[m+1]}$:*

$$\Big(If, g - Ig\Big)_{s,[m+1]} = 0\ .$$

Proof: We proceed by induction on s and note that the lemma holds for $s = 1$. Also, restricted to the face $x_i = x_{i,j}$, the function If is an $s-1$-dimensional interpolant and $g-Ig$ is an $s-1$-dimenional interpolation error. Thus the second term in (5.11) vanishes by the induction hypothesis. Hence we need only consider the first term. Looking at the integral on an arbitrary cell, integrating by parts the derivative D_s^{m+1} from $g - Ig$ to If and noting that in the cell interior $D_s^{2m+2}If = 0$ we obtain for the integral:

$$\begin{aligned}\sum_{i=0}^{m} \quad & (-1)^{m-i} \textstyle\int_{F_+} D_s^{m+1+i} \prod_{j=1}^{s-1} D_j^{m+1} I(f) \cdot D_s^{m-i} \prod_{j=1}^{s-1} D_j^{m+1}(g - I(g)) \quad + \\ & (-1)^{m+1-i} \textstyle\int_{F_-} D_s^{m+1+i} \prod_{j=1}^{s-1} D_j^{m+1} I(f) \cdot D_s^{m-i} \prod_{j=1}^{s-1} D_j^{m+1}(g - I(g)),\end{aligned}$$

where $F_\pm$ are the cell faces with normal in the s-direction. Let I_s denote the interpolation operator in x_s and $I^{(s-1)}$ denote $(s-1)$-dimensional interpolation in the other variables. Then:

$$I = I^{(s-1)} \circ I_s. \tag{5.16}$$

Using the fact that $D_s^{m-i} I_s g = D_s^{m-i} g$ on $F_\pm$, we can rewrite the typical integral on a given face as:

$$\int \left[\prod_{j=1}^{s-1} D_j^{m+1} I^{(s-1)} D_s^{m+1+i} I_s f\right] \cdot \left[\prod_{j=1}^{s-1} D_j^{m+1}\left(D_s^{m-i} g - I^{(s-1)} D_s^{m-i} g\right)\right].$$

Summing over all faces with $x_s = x_{s,i}$ we can write this as:

$$\left(I^{(s-1)}\tilde{f}, \tilde{g} - I^{(s-1)}\tilde{g}\right)_{s-1,[m+1]},$$

where

$$\tilde{f} = D_s^{m+1+i} I_s f, \quad \tilde{g} = D_s^{m-i} g.$$

By the induction hypothesis this inner product is zero, completing the proof of the lemma.$\diamond$

Secondly, we show that the new inner product can be used to control the seminorm appearing in the error estimates:

Lemma 5.3 *There exist positive constants C_1,C_2 independent of the grid G_h and the function $g \in \mathcal{H}_s^{[m+1]}$ such that:*

$$C_1 |g|_{s,[m+1]}^2 \le (g,g)_{s,[m+1]} \le C_2 \|g\|_{s,[m+1]}^2.$$

Proof: The second inequality follows directly from the standard Sobolev inequality in one dimension:

$$|f(x)| \le C\left(\|f\| + \|D^{m+1} f\|\right), \tag{5.17}$$

followed by repeated integration. (Note that the scaling of the terms in (5.11) involving lower-dimenional inner products is necessary here for $C_{1,2}$ to be grid-independent.) For the first inequality, we first prove by induction on s the following auxiliary inequality:

$$\|f\|^2 \le Ch^s \left((f,f)_{s,[m+1]} + \sum_{j_1=1}^{N_1} \cdots \sum_{j_s=1}^{N_s} f^2(x_{j_1}, \ldots, x_{j_s})\right). \tag{5.18}$$

We assume $h < \frac{1}{2}$. Taking $s = 1$ and considering an interval (x_j, x_{j+1}) we have:

$$\begin{aligned}
\max_{x_j \le x \le x_{j+1}} f^2(x) &\le f^2(x_j) + 2\int_{x_j}^{x_{j+1}} |f| \cdot |Df| dx \\
&\le f^2(x_j) + \|f\|^2_{(x_j,x_{j+1})} + \|Df\|^2_{(x_j,x_{j+1})} \\
&\le f^2(x_j) + h_{j+\frac{1}{2}} \max_{x_j \le x \le x_{j+1}} f^2(x) + \|Df\|^2_{(x_j,x_{j+1})},
\end{aligned}$$

which implies

$$\max_{x_j \le x \le x_{j+1}} f^2(x) \le 2f^2(x_j) + 2\|Df\|^2_{(x_j,x_{j+1})}. \tag{5.19}$$

Integrating (5.19) yields:

$$\|f\|^2_{(x_j,x_{j+1})} \le 2h_{j+\frac{1}{2}} \left(f^2(x_j) + \|Df\|^2_{(x_j,x_{j+1})} \right). \tag{5.20}$$

Summing over j and estimating $\|Df\|$ by $\|D^{m+1}f\|$ yields (5.18) for $s = 1$.

Now assume the analogue of (5.20) holds in dimensions up through $s-1$. That is, for a cell:

$$\mathcal{C}_{j_1,\ldots,j_{s-1}} = \otimes_{i=1}^{s-1}(x_{i,j_i}, x_{i,j_i+1}), \tag{5.21}$$

$$\|f\|^2_{\mathcal{C}_{j_1,\ldots,j_{s-1}}} \le Ch^{s-1} \left(f^2(x_{1,j_1},\ldots,x_{s-1,j_{s-1}}) + (f,f)_{s-1,[1],\mathcal{C}_{j_1,\ldots j_{s-1}}} \right), \tag{5.22}$$

where the inner product is restricted to the cell. Now consider an s-dimensional cell, $\mathcal{C}_{j_1,\ldots,j_s}$. For any fixed $x_s \in (x_{s,j_s}, x_{s,j_s+1})$ (5.22) holds. Integrating in x_s then yields:

$$\|f\|^2_{\mathcal{C}_{j_1,\ldots,j_s}} \le Ch^{s-1} (I_1 + I_2), \tag{5.23}$$

where:

$$I_1 = \int_{x_{s,j_s}}^{x_{s,j_s+1}} f^2(x_{1,j_1},\ldots,x_{s-1,j_{s-1}},x_s) dx_s, \tag{5.24}$$

$$I_2 = \int_{x_{s,j_s}}^{x_{s,j_s+1}} (f(\cdot,x_s), f(\cdot,x_s))_{s-1,[1],\mathcal{C}_{j_1,\ldots j_{s-1}}} dx_s. \tag{5.25}$$

Now we apply (5.20) to estimate I_1 and I_2:

$$I_1 \le 2h_{s,j_s+\frac{1}{2}} \left(f^2(x_{1,j_1},\ldots,x_{s,j_s}) + \int_{x_{s,j_s}}^{x_{s,j_s+1}} (D_s f)^2(x_{1,j_1},\ldots,x_{s-1,j_{s-1}},x_s) dx_s \right), \tag{5.26}$$

$$\begin{aligned}
I_2 \le\ & 2h_{s,j_s+\frac{1}{2}} (f(\cdot,x_{s,j_s}), f(\cdot,x_{s,j_s}))_{s-1,[1],\mathcal{C}_{j_1,\ldots j_{s-1}}} \\
&+ 2h_{s,j_s+\frac{1}{2}} \int_{x_{s,j_s}}^{x_{s,j_s+1}} (D_s f(\cdot,x_s), D_s f(\cdot,x_s))_{s-1,[1],\mathcal{C}_{j_1,\ldots j_{s-1}}} dx_s.
\end{aligned} \tag{5.27}$$

Noting that all of the integral terms are bounded by $(f,f)_{s,[1],\mathcal{C}_{j_1,\ldots,j_s}}$ we conclude that (5.22) holds in dimension s. Equation (5.18) follows from summing over the cells and bounding $(f,f)_{s,[1]}$ by $(f,f)_{s,[m+1]}$.

Now we prove the lemma by induction on s, again noting that we have already proven it for $s = 1$. Note that:

$$|g|^2_{s,[m+1]} = \left(\sum_i \int |D_i^{m+1} g|^2_{s-1,[m+1]} dx_i + \sum_i \|D_i^{m+1} g\|^2 \right). \tag{5.28}$$

Using the induction hypothesis we have:

$$\int |D_i^{m+1} g|^2_{s-1,[m+1]} dx_i \leq C \int (D_i^{m+1} g, D_i^{m+1} g)_{s-1,[m+1]} dx_i \leq C(g,g)_{s,[m+1]}, \tag{5.29}$$

and by (5.18) with s replaced by $s-1$ and f replaced by $D_i^{m+1} g$:

$$\begin{aligned} \|D_i^{m+1} g\|^2 &= \int \|D_i^{m+1} g(\ldots, x_i, \ldots)\|^2 dx_i \\ &\leq Ch^{s-1} \int (D_i^{m+1} g, D_i^{m+1} g)_{s-1,[m+1]} dx_i \\ &\quad + Ch^{s-1} \sum \cdots \sum \int (D_i^{m+1} g)^2 (x_{j_1}, \ldots, x_i, \ldots, x_{j_s}) dx_i \\ &\leq C(g,g)_{s,[m+1]}. \end{aligned} \tag{5.30}$$

This completes the proof of Lemma 5.3.◇

Now the proof of the main theorem follows the same pattern as the proof of Theorem 2.1. Set:

$$\varepsilon_n^2 = (e^n, e^n)_{s,[m+1]}, \quad \delta_n^2 = (Se^n - \tilde{I}Se^n, Se^n - \tilde{I}Se^n)_{s,[m+1]}, \tag{5.31}$$

and, by Lemmas 5.1 and 5.3,

$$\eta^n = u^{n+1} - Iu^{n+1}, \quad (\eta^n, \eta^n)_{s,[m+1]} = O(h^{2m+2}). \tag{5.32}$$

Then using Lemma 5.2 we derive (2.28) and (2.29). To estimate the error in the L_2–norm we again follow the arguments from the scalar case and use (5.15) in conjunction with Lemma 5.3 to conclude:

$$\|Se^n - \tilde{I}Se^n\| \leq Ch^{m+1} \delta_n, \tag{5.33}$$

which leads to the final result.◇

6 Numerical Experiments

In this section we present the results of a series of simple numerical experiments which demonstrate the resolution capabilities of the Hermite methods and illustrate the theoretical results.

An accepted framework within which to compare the efficiency of methods of different orders is to find the minimal number of points per wavelength required to propagate a wave a given number of periods with a fixed accuracy (e.g. [6, Ch. 3]). For standard central difference methods and periodic problems, this number is always greater than two, approaching two points per wavelength in the limit of infinite order, which yields the Fourier pseudospectral method [4]. As Hermite methods require more work and storage per point, to be competitive they must have superior resolution; that is they must be capable of using fewer than two points per wavelength. This will be demonstrated below. Note that more extensive experiments can be found in [5]. Also, the methods are used to discretize high-order radiation boundary conditions for the linearized Euler equations in [7].

Consider the initial value problem (2.1) with:

$$f(x) = e^{i\omega x}. \tag{6.1}$$

and ω is an integer. We assume a uniform mesh of width h and evolve the discrete solution with the m-derivative Hermite method. To study subgrid resolution we allow:

$$|\omega| h > \pi \ , \tag{6.2}$$

i.e., the function $f(x) = e^{i\omega x}$ may oscillate between consecutive grid points x_j, x_{j+1}. We set:

$$\xi = \omega h \ , \tag{6.3}$$

and note that the wavelength of the wave $u = e^{i\omega(x+t)}$ is

$$\Lambda = \frac{2\pi}{|\omega|} \ . \tag{6.4}$$

Therefore,

$$\frac{\Lambda}{h} = \frac{2\pi}{|\omega h|} = \frac{2\pi}{|\xi|} \tag{6.5}$$

where Λ/h is the number of points per wavelength Λ. For example, $\xi = \pi$ corresponds to two meshpoints per wavelength, $\xi = 2\pi$ corresponds to one meshpoint per wavelength, and $\xi = \pi/10$ corresponds to 20 meshpoints per wavelength.

Define the error function (the maximum is taken over all $x_{j+\frac{1}{2}} \in \tilde{G}_h$),

$$error(\xi, m, \lambda, n) = \max\Big\{ |u(x_{j+\frac{1}{2}}, nk) - p^n(x_{j+\frac{1}{2}})| \Big\} \tag{6.6}$$

We show some representative results for the error function (6.6). We have fixed the CFL number as $\lambda = .8$ and the number of half time steps as $n = 200$. For these values the functions

$$\xi \to \eta(\xi, m) = error(\xi, m, \lambda = .8, n = 200) \tag{6.7}$$

are plotted in Figure 1 for $|\xi| \le \pi$ and $m = 1, 2, 3$ and for $|\xi| \le 2\pi$ and $m = 1$ to 5. Note that the functions vanish to high order at $\xi = 0$, which is to be expected for any high–order method.

Again fixing $\lambda = .8$ and $n = 200$, we have maximized these functions over ξ and have varied m from 1 to 10. In Figure 2 we plot the functions

$$m \to \log_{10} \max\{\eta(\xi, m) \ : \ |\xi| \le \pi\} \tag{6.8}$$

and

$$m \to \log_{10} \max\{\eta(\xi, m) \ : \ |\xi| \le 2\pi\} \ . \tag{6.9}$$

From these figures it is clear that, for m large, subgrid resolution is achieved. Thus the number of degrees of freedom required is comparable to that required by high-order difference methods, while we may take significantly larger time steps.

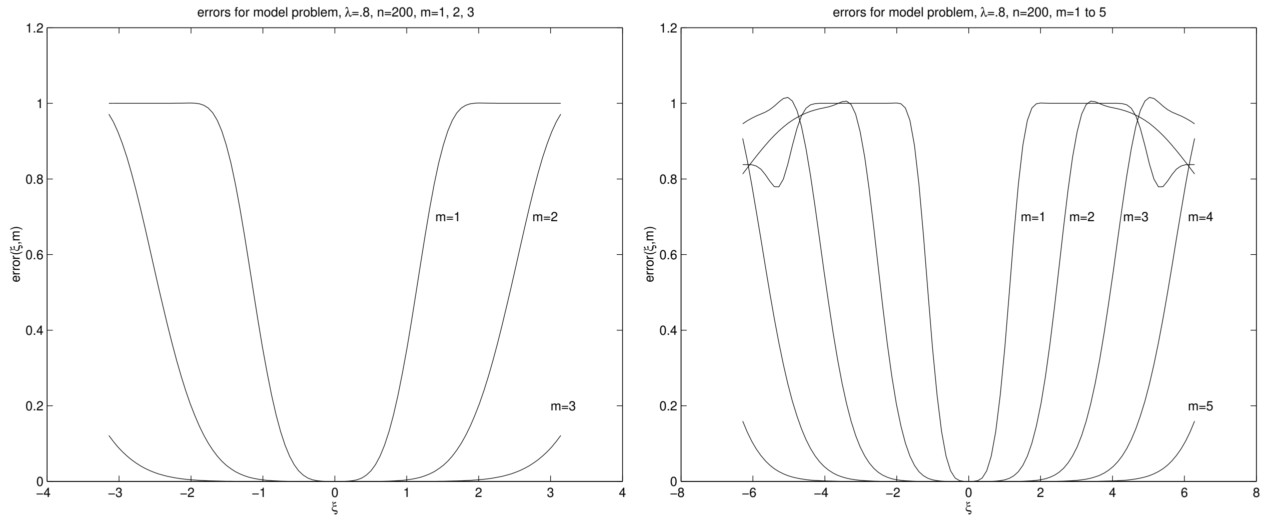

Figure 1: Error functions $\eta(\xi, m)$ for $|\xi| \leq \pi$ and $|\xi| \leq 2\pi$

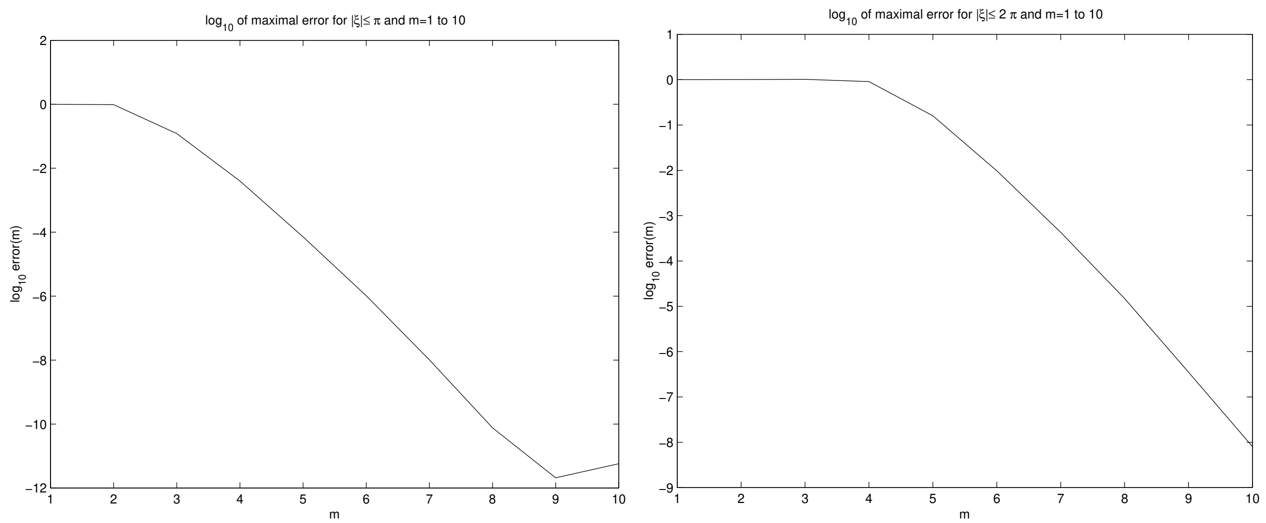

Figure 2: Logarithm of error functions maximized over $|\xi| \leq \pi$ and $|\xi| \leq 2\pi$

It is also of interest to study the convergence of the method in the limit $m \to \infty$ when the number of wavelengths per cell is large. Ainsworth [1] has considered this problem for finite element methods applied to the Helmholtz equation.

Following that work, we fix $n = 1$ and plot the error after two full time steps for $\xi = 50$ and $\xi = 100$. The graphs are qualitatively similar to those shown in [1], exhibiting, in particular, nonmonotonic behavior in increasing m followed by a superexponential convergence. Note that our calculations include the effects of time-stepping.

We now consider full-blown implementations of the method in one and two space dimensions and study the order of convergence under mesh refinement. In all cases we take $m = 2, 4, 7, 10$. In our first experiment we solve:

$$u_t = \left(1 + \frac{\sin x}{2}\right) u_x + f(x,t), \tag{6.10}$$

with periodic boundary conditions on the domain:

$$(x,t) \in [0, 2\pi] \times [0, 100], \tag{6.11}$$

where f is chosen so that the solution is:

$$u(x,t) = \cos\left(16(x+t)\right). \tag{6.12}$$

We varied both the number of mesh points, n, and the number of terms, q, in the temporal Taylor series. In most cases we chose a CFL number of 0.8 and various numbers of mesh points. For the larger values of m and minimal q this was sometimes unstable. Then we found that the method could be stabilized either by decreasing the CFL number or by increasing q. (The latter method is typically more efficient.) We compute an estimated convergence rate by comparing errors for different meshes and computing:

$$p_{\text{obs}} = \frac{\log\left(\frac{err_2}{err_1}\right)}{\log\left(\frac{h_2}{h_1}\right)}. \tag{6.13}$$

In our first experiment we use a uniform mesh, with results shown in Table 1. (The errors are relative l_2 errors at $t = 100$.) Clearly, the observed convergence rates are consistent with the theory. Lower rates are observed only when errors are at a level where roundoff plays a role. We see that increasing q had little or no effect on the accuracy, but for m large had significant effects on the stability.

In our second experiment we solve the same problem, but introduce random perturbations to the mesh. Precisely, we set:

$$x_j = (j + \tau_j)h, \quad x_{j+\frac{1}{2}} = (j + \frac{1}{2} + \tau_{j+\frac{1}{2}})h, \tag{6.14}$$

with τ chosen randomly from a uniform distribution on $(-.2, .2)$. Thus the mesh is not smooth. Note that the CFL constraint in this case is based on the smallest cell. Here it was never necessary to decrease the CFL number to achieve stability. The results are given in Table 2. We use average mesh width to compute p_{obs}. Of course this is somewhat imprecise. Given this imprecision, the agreement of the observed and theoretical convergence rates is reasonable. The results are essentially independent of q.

m	q	n	CFL	error	p_{obs}
2	5	40	0.8	$4.36E-2$	
2	5	80	0.8	$1.64E-3$	4.7
2	5	120	0.8	$2.90E-4$	4.3
2	6	40	0.8	$4.52E-2$	
2	6	80	0.8	$1.63E-3$	4.8
2	6	120	0.8	$3.00E-4$	4.2
4	9	25	0.8	$1.36E-3$	
4	9	50	0.8	$3.98E-6$	8.4
4	9	75	0.8	$1.16E-7$	8.7
4	10	25	0.8	$1.11E-3$	
4	10	50	0.8	$4.20E-6$	8.0
4	10	75	0.8	$1.22E-7$	8.7
7	15	10	0.6	$2.37E-2$	
7	15	20	0.6	$5.10E-7$	15.5
7	15	30	0.6	$9.73E-10$	15.4
7	16	10	0.6	$8.92E-3$	
7	16	20	0.6	$4.68E-7$	14.2
7	16	30	0.6	$9.21E-10$	15.4
7	17	10	0.8	$2.32E-1$	
7	17	20	0.8	$7.77E-7$	18.2
7	17	30	0.8	$9.78E-10$	16.5
10	21	8	0.4	$2.37E-3$	
10	21	16	0.4	$1.35E-9$	20.7
10	21	24	0.4	$5.31E-12$	13.7
10	22	8	0.4	$2.34E-3$	
10	22	16	0.4	$1.38E-9$	20.7
10	22	24	0.4	$5.23E-12$	13.7
10	25	10	0.8	$3.05E-4$	
10	25	15	0.8	$9.13E-8$	20.0
10	25	20	0.8	$9.78E-10$	26.3

Table 1: Errors at $t = 100$ in 1d with a uniform mesh.

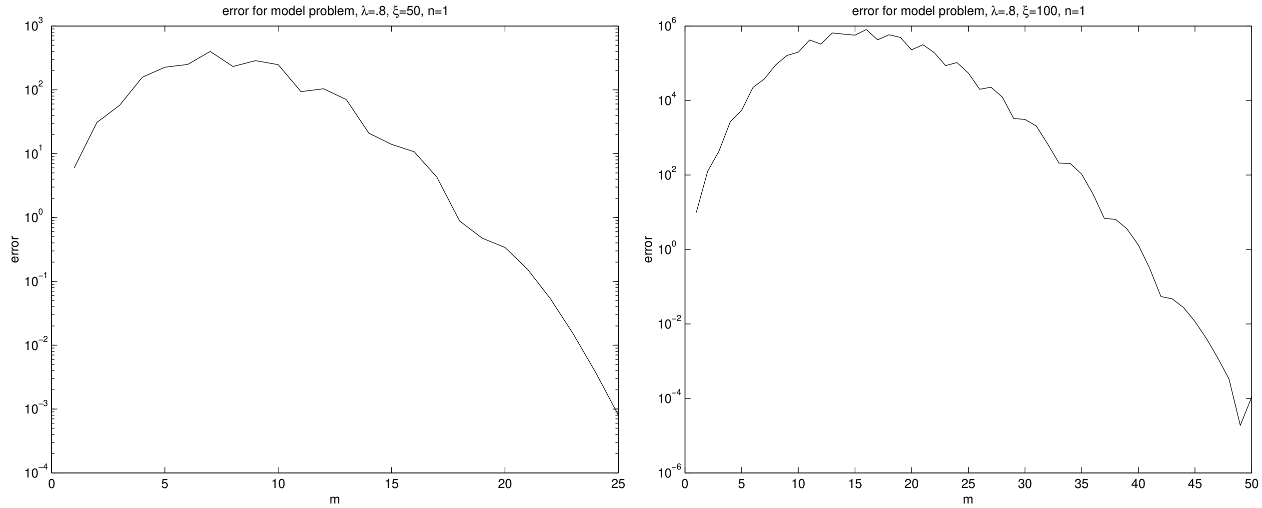

Figure 3: Logarithm of error functions with increasing m.

m	q	n	CFL	error	p_{obs}
2	5	55	0.8	$5.34E-2$	
2	5	110	0.8	$1.95E-3$	4.8
2	5	165	0.8	$3.02E-4$	4.6
2	6	50	0.8	$9.28E-2$	
2	6	100	0.8	$3.33E-3$	4.8
2	6	150	0.8	$5.08E-4$	4.6
4	9	25	0.8	$1.26E-2$	
4	9	50	0.8	$5.81E-5$	7.8
4	9	75	0.8	$1.23E-6$	9.5
4	10	25	0.8	$1.26E-2$	
4	10	50	0.8	$5.82E-5$	7.8
4	10	75	0.8	$1.23E-6$	9.5
7	15	12	0.8	$9.50E-3$	
7	15	24	0.8	$1.35E-6$	12.8
7	15	36	0.8	$5.46E-9$	13.6
7	16	12	0.8	$9.47E-3$	
7	16	24	0.8	$1.35E-6$	12.8
7	16	36	0.8	$5.46E-9$	13.6
10	21	10	0.8	$1.31E-3$	
10	21	20	0.8	$3.47E-10$	21.8
10	22	10	0.8	$1.34E-3$	
10	22	20	0.8	$3.48E-10$	21.9

Table 2: Errors at $t=100$ in 1d with a random mesh.

In our third set of experiments we test our implementation of nonperiodic boundary conditions as described in Section 4. Here we solve the system:

$$u_{1,t} = \frac{\sin x}{2} u_{1,x} + u_{2,x} + f_1, \tag{6.15}$$

$$u_{2,t} = u_{1,x} + \frac{\sin x}{2} u_{2,x} + f_2, \tag{6.16}$$

where $x \in [0, 2\pi]$ and f_1, f_2 are chosen so that:

$$u_1 = \sin(16x)\cos(16t), \quad u_2 = \cos(16x)\sin(16t). \tag{6.17}$$

We impose the boundary conditions:

$$u_1 = 0, \quad x = 0, 2\pi. \tag{6.18}$$

Note that at the boundaries the local characteristic variables are:

$$u^R = u_1 - u_2, \quad u^L = u_1 + u_2, \tag{6.19}$$

so that the boundary condition takes the form $u^R = -u^L$. We use a uniform mesh.

The results, shown in Table 3, are similar to the periodic case, except that we needed to decrease the CFL number somewhat. We believe that this is due to the particular implementation of the boundary condition. We plan to carefully examine the stability characteristics of different implementations in the future. Note that again increasing q sometimes allowed us to compute with larger time steps.

In our final experiment we consider a periodic problem in two space dimensions:

$$u_t = .6u_{x_1} + .8u_{x_2}, \quad (x_1, x_2) \in [0, 2\pi] \times [0, 2\pi], \tag{6.20}$$

$$u = \cos(7x_1 + 9x_2 + 11.4t). \tag{6.21}$$

Note that the theory we developed assumes $q = 4m + 2$. The experiments show that q can in practice be taken much smaller.

7 Conclusions

In conclusion, we have demonstrated that minimal stencil methods using derivatives of order m in each coordinate direction combined with Taylor series in time yield approximations to hyperbolic initial-boundary value problems which:

i. Are of order $2m + 1$ in space and time;

ii. Are stable under an order-independent stability constraint;

iii. Allow a natural implementation of boundary conditions to the order of the method;

iv. Achieve sub-grid resolution for m large.

In future work we will extend our analysis to hyperbolic-parabolic systems and discuss the efficient implementation of the method for problems with simple nonlinearities. We believe the methods have great potential for applications to problems in scientific computing, and we are currently developing codes which utilize them to solve the compressible Navier-Stokes equations.

m	q	n	CFL	error	p_{obs}
2	5	40	0.7	$5.04E-2$	
2	5	80	0.7	$1.85E-3$	4.8
2	5	120	0.7	$4.51E-4$	3.5
2	6	40	0.7	$4.32E-2$	
2	6	80	0.7	$1.26E-3$	5.1
2	6	120	0.7	$2.02E-4$	4.5
4	9	25	0.6	$4.74E-3$	
4	9	50	0.6	$5.41E-6$	9.8
4	9	75	0.6	$1.23E-7$	9.3
4	10	25	0.6	$1.12E-3$	
4	10	50	0.6	$4.05E-6$	8.1
4	10	75	0.6	$1.07E-7$	9.0
7	15	10	0.4	$1.82E-2$	
7	15	20	0.4	$4.74E-7$	15.2
7	15	30	0.4	$9.24E-10$	15.3
7	16	10	0.4	$9.37E-3$	
7	16	20	0.4	$4.18E-7$	14.5
7	16	30	0.4	$7.48E-10$	15.5
7	17	10	0.5	$5.26E-2$	
7	17	20	0.5	$3.36E-7$	17.3
7	17	30	0.5	$8.66E-10$	14.7
10	21	7	0.3	$8.38E-3$	
10	21	14	0.3	$1.14E-8$	19.5
10	22	7	0.3	$1.15E-2$	
10	22	14	0.3	$1.31E-8$	19.7
10	24	7	0.4	$8.17E-2$	
10	24	14	0.4	$1.36E-7$	19.2

Table 3: Errors in 1d at $t = 100$ with a reflective boundary condition.

m	q	n	CFL	error	p_{obs}
2	5	50	0.8	$5.60E-2$	
2	5	100	0.8	$1.10E-3$	5.7
2	7	55	0.8	$4.09E-2$	
2	7	110	0.8	$1.32E-3$	5.0
2	10	60	0.8	$2.66E-2$	
2	10	120	0.8	$8.53E-4$	5.0
4	9	20	0.7	$4.08E-3$	
4	9	40	0.7	$5.77E-6$	9.5
4	13	20	0.7	$3.65E-3$	
4	13	40	0.7	$7.72E-6$	8.9
4	18	20	0.7	$3.65E-3$	
4	18	40	0.7	$7.72E-6$	8.9
7	15	6	0.6	$9.90E-2$	
7	15	12	0.6	$5.27E-6$	14.2
7	22	6	0.6	$7.15E-2$	
7	22	12	0.6	$6.99E-6$	13.3
7	30	6	0.6	$7.15E-2$	
7	30	12	0.6	$6.99E-6$	13.3
10	21	4	0.6	$4.02E-2$	
10	21	8	0.6	$9.46E-8$	18.7
10	31	4	0.6	$2.92E-2$	
10	31	8	0.6	$6.69E-8$	18.7
10	42	4	0.6	$2.92E-2$	
10	42	8	0.6	$6.69E-8$	18.7

Table 4: Errors in 2d at $t = 100$.

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8 Appendix: Auxiliary Results

In this section we give or outline proofs for the basic lemmas on Hermite interpolation and hyperbolic systems used earlier. The results are in essence standard, but occasionally we need them in a nonstandard form.

Let g be a smooth funcion on $[x_i, x_{i+1}]$ and let Ig be the degree $2m+1$ Hermite interpolant of g constructed from the data $D^j g(x_i)$ and $D^j g(x_{i+1})$, $j = 0, \ldots, m$. Set $h_{i+\frac{1}{2}} = x_{i+1} - x_i$. We define the Hermite-Lagrange basis functions, $Q_j(z)$, $j = 0, \ldots, m$, on the standard interval $z \in [0,1]$ by:

1. Q_j is a polynomial of degree $2m+1$;

2. Q_j satisfies for $0 \leq i, j \leq m$:

$$\frac{d^i Q_j}{dz^i}(1) = \delta_{ij}, \quad \frac{d^i Q_j}{dz^i}(0) = 0.$$

Setting $G(z) = g(x_i + h_{i+\frac{1}{2}} z)$ and constructing the Hermite interpolant, $I_S G$ of G on the standard interval we find that Ig may be written:

$$Ig(x) = \sum_{j=0}^{m} \left((-h_{i+\frac{1}{2}})^j D^j g(x_i) Q_j(1-z) + h_{i+\frac{1}{2}}^j D^j g(x_{i+1}) Q_j(z) \right), \tag{8.1}$$

where

$$z = \frac{x - x_i}{h_{i+\frac{1}{2}}}.$$

For error estimation we use the well-known Peano kernel representation [8, Ch. 5]. For $g \in H^p$ and integers $0 \leq q < p \leq 2m+2$ we have:

$$\begin{aligned} D^q(g - Ig) &= h_{i+\frac{1}{2}}^{-q} \frac{d^q}{dz^q}(G - I_S G) \\ &= h_{i+\frac{1}{2}}^{-q} \int_0^1 K_{q,m,p}(z,s) \frac{d^p G}{dz^p}(s) ds \\ &= h_{i+\frac{1}{2}}^{p-1-q} \int_{x_i}^{x_{i+1}} K_{q,m,p}(h_{i+\frac{1}{2}}^{-1}(x - x_i), h_{i+\frac{1}{2}}^{-1}(y - x_i)) D^p g(y) dy. \end{aligned} \tag{8.2}$$

Note that the kernel, $K_{q,m,p}$, depends only on q, m, p, and its arguments but not on g. It is also bounded. Using the shorthand,

$$\bar{K}^{(i)}(x,y) \equiv K_{q,m,p}(h_{i+\frac{1}{2}}^{-1}(x - x_i), h_{i+\frac{1}{2}}^{-1}(y - x_i)),$$

a simple application of the Cauchy-Schwarz inequality yields:

$$\begin{aligned} \int_{x_i}^{x_{i+1}} (D^q(g - Ig))^2 dx &= h_{i+\frac{1}{2}}^{2p-2-2q} \int_{x_i}^{x_{i+1}} \left(\int_{x_i}^{x_{i+1}} \bar{K}^{(i)}(x,y) D^p g(y) dy \right)^2 dx \\ &\leq h_{i+\frac{1}{2}}^{2p-2-2q} \int_{x_i}^{x_{i+1}} \left(\int_{x_i}^{x_{i+1}} \left(\bar{K}^{(i)}(x,y) \right)^2 dy \right) \cdot \left(\int_{x_i}^{x_{i+1}} (D^q g(y))^2 dy \right) dx \\ &\leq C h_{i+\frac{1}{2}}^{2p-2q} \int_{x_i}^{x_{i+1}} (D^q g(y))^2 dy. \end{aligned} \tag{8.3}$$

Lemma 2.1 now follows from the appropriate choices for q and p followed by summation over i.$\diamond$

To prove the first statement in Lemma 2.3 we write:

$$|D^q Ig| \leq |D^q(g - Ig)| + |D^q g|.$$

By (8.2) the first term is bounded by $Ch_{i+\frac{1}{2}} \max |D^{2m+2}g|$. Thus (2.14) is immediate. To prove the remaining results we use norm equivalence arguments on the vector space of degree $2m+1$ polynomials of z. Suppose that $0 \leq q \leq 2m+1$ and $0 \leq p \leq q$. Then

$$\max_{0 \leq z \leq 1} |\frac{d^q I_S G}{dz^q}(z)|^2 \leq C(q,m,p) \| \frac{d^p I_S G}{dz^p} \|^2 .$$

Thus

$$\max_{x_i \leq x \leq x_{i+1}} |D^q I g(x)|^2 \leq C(q,m,p) h_{i+\frac{1}{2}}^{2p-1-2q} \int_{x_i}^{x_{i+1}} (D^p I g(x))^2 dx .$$

Now (2.15) follows from choosing $p = q$ when $q \leq m+1$ and $p = m+1$ for $q > m+1$.

Finally we also have for any q:

$$\int_0^1 \left(\frac{d^q I_S G}{dz^q}(z) \right)^2 dz \leq C(q,m) \sum_{j=0}^{m} \left(\left(\frac{d^j I_S G}{dz^j}(0) \right)^2 + \left(\frac{d^j I_S G}{dz^j}(1) \right)^2 \right) .$$

Transforming back to x coordinates as before yields (2.16).$\diamond$

These arguments can also be extended to tensor-product interpolants in multiple dimensions, leading to the results of Lemma 5.1. Beginning with (5.12)-(5.14) we proceed by induction on s, noting that the result holds for $s = 1$ by Lemma 2.1. Denote by I_i interpolation in the ith coordinate. That is, $I_i g$ is a function of s variables which is a piecewise degree $2m+1$ polynomial in x_i. Set:

$$I^{(r)} = I_r \circ I_{r-1} \circ \ldots \circ I_1 . \tag{8.4}$$

Then the full interpolation operator is $I^{(s)}$. Introduce:

$$e_s(x_1, \ldots, x_{s-1}, \tau) = (g - I_s g)(x_1, \ldots, x_{s-1}, \tau) . \tag{8.5}$$

We then have:

$$g - I^{(s)} g = -(e_s - I^{(s-1)} e_s) + e_s + (g - I^{(s-1)} g) . \tag{8.6}$$

First, using the induction hypothesis we will show that the last two terms on the right of (8.6) satisfy the relations in the lemma. For the L_2–norms we have:

$$\|e_s\|^2 = \int \cdots \int \|e_s(x_1, \ldots, x_{s-1}, \cdot)\|^2 dx_1 \cdots dx_{s-1} , \tag{8.7}$$

$$\|g - I^{(s-1)} g\|^2 = \int \|(g - I^{(s-1)} g)(\cdots, x_s)\|^2 dx_s . \tag{8.8}$$

The bounds in (5.12),(5.14) follow from the fact that all terms appearing in the lower dimensional norms in (5.8) are also present in the dimension s norm. Now differentiate (8.5),(8.6) to obtain (5.13) for e_s and $g - I^{(s-1)} g$. We simply use the fact that:

$$\|D_i^{m+1} f\| \leq \|D_i^{2m+2} f\| . \tag{8.9}$$

Then:

$$\| \prod_i D_i^{m+1} e_s \| \leq C h^{m+1} \| D_s^{2m+2} \prod_{i \neq s} D_i^{m+1} g \| \leq C h^{m+1} |g|_{s,[2m+2]} , \tag{8.10}$$

$$\|\prod_i D_i^{m+1}(g - I^{(s-1)}g)\| \leq Ch^{m+1}\|D_s^{m+1}g\|_{s-1,[2m+2]} \leq Ch^{m+1}|g|_{s,[2m+2]} \tag{8.11}$$

This shows that (5.13) holds for the last two terms in (8.6).

We now consider the first term on the right-hand side of (8.6). Working on the strip:

$$x_{s,j_s} < x_s < x_{s,j_s+1},$$

we denote $h_s = x_{s,j_s+1} - x_{s,j_s}$. Using the Peano kernel we have:

$$e_s = h_s^{p-1}\int_{x_{s,j_s}}^{x_{s,j_s+1}} K_{0,m,p}\left(\frac{(\tau - x_{s,j_s})}{h_s}, \frac{(y - x_{s,j_s})}{h_s}\right) D_s^p g(x_1, \ldots, x_{s-1}, y)dy. \tag{8.12}$$

Setting $p = 2m+2$ and using the induction hypothesis and the Cauchy-Schwarz inequality as above we deduce:

$$\begin{aligned} \|e_s - I^{(s-1)}e_s\|^2 \quad &= \\ \sum_{j_s} h_s^{4m+2}\int\cdots\int\int_{x_{j_s}}^{x_{j_s+1}}\left(\int_{x_{j_s}}^{x_{j_s+1}} K(D_s^{2m+2}g - I^{(s-1)}D_s^{2m+2}g)\right)^2 d\tau dx_1\cdots dx_{s-1} \quad &\leq \\ \sum_{j_s} h_s^{4m+4}\int\cdots\int\int_{x_{j_s}}^{x_{j_s+1}}(D_s^{2m+2}g - I^{(s-1)}D_s^{2m+2}g)^2 d\tau dx_1\cdots dx_{s-1} \quad &\leq \\ h^{8m+8}|g|_{s,[2m+2]}^2. \end{aligned}$$

This proves (5.12). The second and third relations are shown analogously using the more general form of (8.2) to represent e_s and its derivatives. Equation (5.15) follows again from replacing g by $g - Ig$ and noting that $I(g - Ig) = 0.\diamond$

We now turn to the proof of Lemma 3.2. The proof of the first two statements is based on standard energy arguments and domain of dependence considerations. In particular, consider first:

$$w = \frac{\partial^q u}{\partial x^q},$$

which satisfies an equation of the form:

$$\frac{\partial w}{\partial t} = A\frac{\partial w}{\partial x} + B_q w + F_q, \tag{8.13}$$

where F_q depends linearly on space derivatives of u of order 1 through $q-1$ as well as on the qth space derivative of F. An energy inequality for the entire domain takes the form:

$$\frac{d}{dt}\|w\| \leq \alpha(q)\|w\| + \beta\|F_q\|.$$

This inequality implies:

$$\|w(\cdot, t_1)\| \leq e^{\alpha(t_1-t_0)}\|w(\cdot, t_0)\| + \beta\frac{e^{\alpha(t_1-t_0)} - 1}{\alpha}\max_{t_0\leq t\leq t_1}\|F_q(\cdot, t)\|.$$

Combined with a simple induction argument on q this inequality yields the desired result when the integrals are taken over the entire domain. The second inequality follows from using derivatives of the equation to express time derivatives in terms of space derivatives. Our final form follows from domain of dependence considerations [12, Ch. 3,6].

To derive maximum norm estimates we again look at (8.13), but now use the method of characteristics. Diagonalizing the system by a unitary transformation, $Q(x,t)$, we obtain for $v = Q^T w$:

$$\frac{\partial v}{\partial t} = \Lambda \frac{\partial v}{\partial x} + \tilde{B}_q v + \tilde{F},$$

where

$$\Lambda = \mathrm{diag}(\lambda_i(x,t)).$$

Thus along the characteristic $X_i(\zeta, t)$ defined by:

$$\frac{dX_i}{dt} = -\lambda_i(X_i, t), \quad X_i(t_0) = \zeta,$$

we have

$$\frac{dv_i}{dt} = \left(\tilde{B}_q v + \tilde{F}\right)_i.$$

From these we easily derive the inequality:

$$\max_x |v(x,t)| \leq \max_x |v(x,t_0)| + C \int_{t_0}^{t} \left(\max_x |v(x,s)| + \max_x |\tilde{F}(x,s)|\right) ds.$$

Application of Gronwall's lemma implies:

$$\max_x |v(x,t_1)| \leq e^{\alpha \Delta t} \max_x |v(x,t_0)| + C\Delta t \max_{t_0 \leq t \leq t_1} \max_x |\tilde{F}(x,t)|.$$

The desired inequalities for all space derivatives then follow by induction and domain of dependence considerations. The inequalities for the mixed space-time derivatives follow from expressing time derivatives in terms of space derivatives.$\diamond$