On two new inequalities for Leray solutions of the Navier-Stokes equations in $\mathbb{R}^n$

T. Hagstrom, J. Lorenz, J. P. Zingano and P. R. Zingano

$^1$Department of Mathematics
Southern Methodist University
Dallas, TX 75275-0235, USA

$^2$Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131-0001, USA

$^3$Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509-900, Brazil

Abstract

We derive two asymptotic inequalities, not previously observed, that are valid for spatial derivatives (of arbitrary order) of Leray solutions to the incompressible Navier-Stokes equations in $\mathbb{R}^n$, $n \leq 4$. They are important to the decay investigation of solution derivatives.

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1. Introduction

In this note we derive two new inequalities for arbitrary Leray solutions of the incompressible Navier-Stokes equations (in dimension $2 \leq n \leq 4$), that is, global solutions $u(\cdot, t) \in L^\infty((0, \infty), L^2_\sigma(\mathbb{R}^n)) \cap L^2((0, \infty), H^1(\mathbb{R}^n)) \cap C^0([0, \infty), L^2(\mathbb{R}^n))$ of the fluid flow system

$$u_t + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad (1.1a)$$
$$\nabla \cdot u(\cdot, t) = 0, \quad (1.1b)$$
$$u(\cdot, 0) = u_0 \in L^2_\sigma(\mathbb{R}^n), \quad (1.1c)$$

that satisfy the strong energy inequality
for a.e. \( s \geq 0 \), including \( s = 0 \). (For the construction of Leray solutions, see e.g. [12, 15, 20, 21].) In (1.1) above, \( \nu > 0 \) is constant, \( \mathbf{u} = \mathbf{u}(x, t) \) and \( p = p(x, t) \) are the unknowns (the flow velocity and pressure, respectively), with condition (1.1c) satisfied in \( L^2(\mathbb{R}^n) \), i.e., \( \| \mathbf{u}(\cdot, t) - \mathbf{u}_0 \|_{L^2(\mathbb{R}^n)} \to 0 \) as \( t \searrow 0 \). In the present work, we assume \( 2 \leq n \leq 4 \) throughout.

A well known property of Leray solutions is that they are eventually very regular: there is always some \( t_* \geq 0 \) such that one has \( \mathbf{u} \in C^\infty(\mathbb{R}^n \times (t_*, \infty)) \) and, moreover,

\[
\mathbf{u}(\cdot, t) \in C((t_*, \infty), H^m(\mathbb{R}^n)), \quad \forall \ m \geq 0,
\]

(1.3) see e.g. [5, 9, 12, 19, 20]. It is also well established that \( \lim_{t \to 0} \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0 \) and, more generally,\(^2\)

\[
\lim_{t \to 0} t^{m/2} \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0
\]

(1.4) for every \( m \geq 1 \), and for all Leray solutions to the system (1.1) [1, 9, 13, 14, 16, 19]. With additional assumptions on the initial data, stronger decay properties of \( \mathbf{u}(\cdot, t) \) are observed. A remarkable example are Leray solutions with exponential decay, i.e., \( \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t}) \) for some \( \alpha > 0 \), see [2, 17, 18] and the excellent review [4].

Like the familiar heat equation, their derivatives will then all decay at a similar rate, as the following result shows. (For the proof of Theorem A, see Section 3.) Thus, for such solutions, we have, for any \( m, k \) and large \( t \): \( \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t}) \), \( \| D^k D^m_x \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(e^{-\alpha t}) \), \( \| D^k D^m_x \mathbf{p}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = O(e^{-\alpha t}) \), \( \| D^m \mathbf{p}(\cdot, t) \|_{L^2} = O(\nu^{-2\alpha t}) \), \( \| D^k D^m_x \mathbf{p}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(\nu^{-2\alpha t}) \), and so on. These estimates have apparently not been obtained before.

**Theorem A.** Let \( n \leq 4 \), \( \mathbf{u}_0 \in L^2_n(\mathbb{R}^n) \), and let \( \mathbf{u}(\cdot, t) \) be any given Leray solution to the Navier-Stokes equations (1.1). Then we have, for every \( \alpha > 0 \):

\[
\limsup_{t \to \infty} e^{\alpha t} \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} \leq K(m, \alpha) \nu^{-m/2} \limsup_{t \to \infty} e^{\alpha t} \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)}
\]

(1.5)

for every \( m \geq 1 \), where \( K(m, \alpha) = 2^m \alpha^{m/2} \).

\(^1\)Moreover, it is known that \( t_* = 0 \) if \( n = 2 \), \( t_* \leq 0.000465 \nu^{-5} \| \mathbf{u}_0 \|_{L^2(\mathbb{R}^3)}^4 \) if \( n = 3 \), and \( t_* \leq 0.002728 \nu^{-2} \| \mathbf{u}_0 \|_{L^2(\mathbb{R}^3)}^2 \) if \( n = 4 \); see e.g. [5], Theorem A.

\(^2\)For the definition of \( \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} \), \( \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} \) and other similar norms, see (1.7).
A more usual situation is having only algebraic decay: \( \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha}) \) with \( \alpha \leq n/4 + 1/2 \), cf. [4, 8, 11, 16, 17, 22, 23]. For larger values of \( \alpha \), see [3, 4, 6, 7]. In this case, the following fundamental inequality (derived in Section 4) is of use.

**Theorem B.** Let \( n \leq 4 \), \( \mathbf{u}_0 \in L^2_0(\mathbb{R}^n) \), and let \( \mathbf{u}(\cdot, t) \) be any given Leray solution to the Navier-Stokes equations (1.1). Then we have, for every \( \alpha \geq 0 \):

\[
\limsup_{t \to \infty} t^{\alpha + m/2} \| D^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} \leq K(\alpha, m) \nu^{-m/2} \limsup_{t \to \infty} t^{\alpha} \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)}
\]

(1.6a)

for every \( m \geq 1 \), where

\[
K(\alpha, m) = \min_{\delta > 0} \left\{ \delta^{-1/2} \prod_{j=0}^{m} (\alpha + j/2 + \delta)^{1/2} \right\}.
\]

(1.6b)

In particular, if \( \| \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha}) \) for large \( t \), then we will have, for any \( m, k \) and \( t \) big enough: \( \| D^k D_x^m \mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(t^{-\alpha - k - m/2}) \), \( \| D^k D_x^m \mathbf{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = O(t^{-\alpha - k - m/2 - n/4}) \), \( \| D^k D_x^m p(\cdot, t) \|_{L^2(\mathbb{R}^n)} = O(t^{-2\alpha - k - m/2 - n/4}) \), \( \| D^k D_x^m p(\cdot, t) \|_{L^\infty} = O(t^{-2\alpha - k - m/2 - n/2}) \), etc. Thus, we quickly retrieve important results in [1, 14, 19].

**Notation.** As already shown, boldface letters are used for vector quantities, as in \( \mathbf{u}(x, t) = (u_1(x, t), \ldots, u_n(x, t)) \). Also, \( \nabla \mathbf{p} \equiv \nabla \mathbf{p}(\cdot, t) \) denotes the spatial gradient of \( \mathbf{p}(\cdot, t); D_j = \partial_\partial x_j; \nabla \cdot \mathbf{u} = D_1 u_1 + \ldots + D_n u_n \) is the (spatial) divergence of \( \mathbf{u}(\cdot, t) \). \( L^2_0(\mathbb{R}^n) \) denotes the space of solenoidal fields \( \mathbf{v} = (v_1, \ldots, v_n) \in L^2(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)^n \) with \( \nabla \cdot \mathbf{v} = 0 \) in the distributional sense; \( \mathcal{H}^1(\mathbb{R}^n) = \mathcal{H}'(\mathbb{R}^n)^n \) with \( \mathcal{H}'(\mathbb{R}^n) \) being the homogeneous \( L^2 \) Sobolev space of order 1; \( \mathcal{H}^m(\mathbb{R}^n) = \mathcal{H}'(\mathbb{R}^n)^n \), where \( \mathcal{H}^m(\mathbb{R}^n) \) is the space of functions \( v \in L^2(\mathbb{R}^n) \) whose \( m \)-th order derivatives are also square integrable. \( C_w(I, L^q(\mathbb{R}^n)) \) denotes the set of mappings from a given interval \( I \subseteq \mathbb{R} \) to \( L^q(\mathbb{R}^n) \) that are \( L^2 \)-weakly continuous at each \( t \in I \). \( \| \cdot \|_{L^q(\mathbb{R}^n)}, 1 \leq q \leq \infty \), are the standard norms of the Lebesgue spaces \( L^q(\mathbb{R}^n) \), with the vector counterparts

\[
\| \mathbf{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i=1}^n \left( \int_{\mathbb{R}^n} |u_i(x, t)|^q \, dx \right)^{1/q} \right\}^{1/q}
\]

(1.7a)

\[
\| D^m \mathbf{u}(\cdot, t) \|_{L^q(\mathbb{R}^n)} = \left\{ \sum_{i, j_1, \ldots, j_m = 1}^n \left( \int_{\mathbb{R}^n} \left| D_{j_1} \cdots D_{j_m} u_i(x, t) \right|^q \, dx \right)^{1/q} \right\}^{1/q}
\]

(1.7b)

(also denoted by \( \| D^m \mathbf{x}(\cdot, t) \|_{L^q(\mathbb{R}^n)} \)) if \( 1 \leq q < \infty \); if \( q = \infty \), then \( \| \mathbf{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \left\{ \| u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} ; 1 \leq i \leq n \right\} \), \( \| D \mathbf{u}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} = \max \left\{ \| D_j u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} ; 1 \leq i, j \leq n \right\} \), and so forth.
We briefly recall some basic material that will be needed in the derivation of (1.5) and (1.6) in the next sections. In dimension $n = 2, 3$, pointwise values can be estimated in terms of $H^2$ norms: one has\(^3\)

$$\| u \|_{L^\infty(\mathbb{R}^2)} \leq \| u \|_{L^2(\mathbb{R}^2)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^2)}^{1/2}$$  \hspace{1cm} (2.1a)

for arbitrary $u \in H^2(\mathbb{R}^2)$; likewise,

$$\| u \|_{L^\infty(\mathbb{R}^3)} \leq \| u \|_{L^2(\mathbb{R}^3)}^{1/4} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{3/4}$$  \hspace{1cm} (2.1b)

for $u \in H^2(\mathbb{R}^3)$. These can be shown by Fourier transform, which also gives

$$\| D u \|_{L^2(\mathbb{R}^n)} \leq \| u \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^n)}^{1/2}$$  \hspace{1cm} (2.2a)

(for any $n$), or, more generally,

$$\| D^\ell u \|_{L^2(\mathbb{R}^n)} \leq \| u \|_{L^2(\mathbb{R}^n)}^{1-\theta} \| D^m u \|_{L^2(\mathbb{R}^n)}^\theta, \quad \theta = \frac{\ell}{m}$$  \hspace{1cm} (2.2b)

for all $0 \leq \ell \leq m$. Combining (2.1) and (2.2), we get the following basic inequalities: for any $m \geq 1$, $0 \leq \ell \leq m - 1$,

$$\| D^\ell u \|_{L^\infty(\mathbb{R}^2)} \| D^{m-\ell} u \|_{L^2(\mathbb{R}^2)} \leq \| u \|_{L^2(\mathbb{R}^2)} \| D^{m+1} u \|_{L^2(\mathbb{R}^2)}$$  \hspace{1cm} (2.3a)

$$\| D^\ell u \|_{L^\infty(\mathbb{R}^3)} \| D^{m-\ell} u \|_{L^2(\mathbb{R}^3)} \leq \| u \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^{2} u \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^{m+1} u \|_{L^2(\mathbb{R}^3)}$$  \hspace{1cm} (2.3b)

In dimension $n = 4$, we begin instead with the fundamental Sobolev inequality

$$\| u \|_{L^4(\mathbb{R}^4)} \leq \| D u \|_{L^2(\mathbb{R}^4)}$$  \hspace{1cm} (2.4)

(for arbitrary $u \in \dot{H}^1(\mathbb{R}^4)$), and obtain, using (2.2) again,

$$\| D^\ell u \|_{L^4(\mathbb{R}^4)} \| D^{m-\ell} u \|_{L^2(\mathbb{R}^4)} \leq \| u \|_{L^2(\mathbb{R}^4)} \| D^{2} u \|_{L^2(\mathbb{R}^4)} \| D^{m+1} u \|_{L^2(\mathbb{R}^4)}$$  \hspace{1cm} (2.5)

for all $m \geq 0$, $0 \leq \ell \leq m$, which plays the role of (2.3) in this case. Finally, because of (2.3) and (2.5) above, we will need the following basic facts about Leray solutions of the Navier-Stokes system (1.1):

$$\lim_{t \to \infty} \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0$$  \hspace{1cm} (2.6)

\(^3\)We actually have $\| u \|_{L^\infty(\mathbb{R}^2)} \leq K \| u \|_{L^2(\mathbb{R}^2)}^{1/2} \| D u \|_{L^2(\mathbb{R}^2)}^{1/2}$ with $K = 1/2$, and similarly for (2.1b). All interpolation inequalities we use have constants $K \leq 1$, and so we simply take $K = 1$ throughout.
(actually, (2.6) will be needed only if \( n = 2 \)), and

\[
\lim_{t \to \infty} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^n)} = 0,
\tag{2.7}
\]

which will be required if \( n = 3, 4 \). The following elementary proofs of (2.6) and (2.7) (taken from [10, 11, 23]) are noteworthy: because (by (1.2)) \( \int_0^\infty \| Du(\cdot, t) \|_{L^2(\mathbb{R}^n)}^2 dt \) is finite and \( \| Du(\cdot, t) \|_{L^2(\mathbb{R}^n)} \) is monotone for large \( t \) (see e.g. [5], Theorem B), it follows that \( t \| Du(\cdot, t) \|_{L^2(\mathbb{R}^n)}^2 \to 0 \) as \( t \to \infty \). This not only shows (2.7) above but, using some simple heat kernel estimates, yields (2.6) too (for details, see [10, 11, 23]).

### 3. Proof of Theorem A

Let \( u_0 \in L^2_2(\mathbb{R}^n) \), and let \( u(\cdot, t) \) be any given Leray solution of the Navier-Stokes system (1.1) such that we have, for some \( \alpha > 0 \),

\[
\limsup_{t \to \infty} e^{\alpha t} \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)} =: \lambda_0(\alpha) < \infty.
\tag{3.1}
\]

We now prove (1.5), for every \( m \). As in [10], the strategy is to use induction in \( m \). The argument is similar for \( n = 2, 3, 4 \), so we will show the details for \( n = 3 \) only.

Let \( t_* \geq 0 \) be chosen so that (1.3) holds, and let, for convenience:

\[
f(t) := e^{2\beta t}, \quad g(t) := e^{2(\beta - \alpha)t}
\tag{3.2}
\]

(for some \( \beta > \alpha \) that we will choose later). Given \( 0 < \epsilon < 1 \), we begin with \( m = 1 \). First, taking the dot product of (1.1a) with \( f(t) u(x, t) \) and integrating the result on \( \mathbb{R}^3 \times [t_0, t] \), for \( t \geq t_0 > t_* \), we get, because of (1.1b),

\[
f(t) \| u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \nu \int_{t_0}^t f(\tau) \| Du(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau = f(t_0) \| u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^2 + 2 \beta \int_{t_0}^t f(\tau) \| u(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau.
\]

By (3.1), we can then choose \( t_0 > t_* \) sufficiently large so that

\[
f(t) \| u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \nu \int_{t_0}^t f(\tau) \| Du(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^2 d\tau \leq g(t_0) (\lambda_0(\alpha) + \epsilon)^2 + 2 \beta \int_{t_0}^t g(\tau) (\lambda_0(\alpha) + \epsilon)^2 d\tau
\leq g(t_0) (\lambda_0(\alpha) + \epsilon)^2 + \frac{\beta}{\beta - \alpha} (\lambda_0(\alpha) + \epsilon)^2 \{ g(t) - g(t_0) \}
\leq \frac{\beta}{\beta - \alpha} (\lambda_0(\alpha) + \epsilon)^2 e^{2(\beta - \alpha)t}
\]

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for all $t \geq t_0$. This gives, in particular,

$$
\int_{t_0}^{t} f(\tau) \| D\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau \leq \frac{1}{2\nu} \frac{\beta}{\beta - \alpha} (\lambda_0(\alpha) + \epsilon)^2 e^{2(\beta - \alpha)t}
$$  \hspace{1cm} (3.3)

for all $t \geq t_0$. Now, we proceed as follows. Differentiating the equation (1.1a) with respect to $x$, taking the dot product with $(f(t) - f(t_0)) D_t \mathbf{u}(x, t)$, and integrating the result on $\mathbb{R}^3 \times [t_0, t]$, $t \geq t_0$, we get, summing over $1 \leq \ell \leq 3$,

$$
(f(t) - f(t_0)) \| D\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_{t_0}^{t} (f(\tau) - f(t_0)) \| D^2\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau
$$

\begin{align*}
&\leq 2\beta \int_{t_0}^{t} f(\tau) \| D\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau + \\
&\quad K_1 \int_{t_0}^{t} (f(\tau) - f(t_0)) \| \mathbf{u}(\cdot, \tau) \|_{L^\infty(\mathbb{R}^3)} \| D\mathbf{u}(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \| D^2\mathbf{u}(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \, d\tau
\end{align*}

\begin{align*}
&\leq 2\beta \int_{t_0}^{t} f(\tau) \| D\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau + \\
&\quad K_1 \int_{t_0}^{t} (f(\tau) - f(t_0)) \| \mathbf{u}(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D\mathbf{u}(\cdot, \tau) \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2\mathbf{u}(\cdot, \tau) \|_{L^2(\mathbb{R}^3)} \, d\tau
\end{align*}

by (1.1b) and (2.3b), where $K_1 = 2\sqrt{3}$. Therefore, by (2.7) and (3.3), and increasing $t_0$ if necessary, we get

$$
(f(t) - f(t_0)) \| D\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} + (2 - \epsilon) \nu \int_{t_0}^{t} (f(\tau) - f(t_0)) \| D^2\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau
$$

\begin{align*}
&\leq \frac{\beta}{\nu} \frac{\beta}{\beta - \alpha} (\lambda_0(\alpha) + \epsilon)^2 e^{2(\beta - \alpha)t}
\end{align*}

for all $t \geq t_0$. Writing $f(t) - f(t_0) = f(t)(1 - e^{-2\beta(t-t_0)})$ and taking $t_1 > t_0$ large enough that $e^{-2\beta(t-t_0)} < \epsilon$, this gives

$$
f(t) \| D\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} \leq A_1 (\lambda_0(\alpha) + \epsilon)^2 g(t), \quad A_1 := \frac{1}{1 - \epsilon} \frac{\beta^2}{\beta - \alpha} \nu^{-1}
$$  \hspace{1cm} (3.4)

for all $t \geq t_1$. We now proceed by induction: having obtained that, for some $m \geq 1$,

$$
f(t) \| D^m\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} \leq A_m (\lambda_0(\alpha) + \epsilon)^2 g(t), \quad \forall \ t \geq t_m
$$  \hspace{1cm} (3.5a)

(for some $A_m > 0$, and for some $t_m > t_1$ sufficiently large), we proceed as follows to estimate $f(t) \| D^{m+1}\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)}$: increasing $t_m$ if necessary, we obtain, as before,

$$
\begin{align*}
f(t) \| D^m\mathbf{u}(\cdot, t) \|^2_{L^2(\mathbb{R}^3)} + (2 - \epsilon) \nu \int_{t_m}^{t} f(\tau) \| D^{m+1}\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau
&\leq f(t_m) \| D^m\mathbf{u}(\cdot, t_m) \|^2_{L^2(\mathbb{R}^3)} + 2\beta \int_{t_m}^{t} f(\tau) \| D^{m+1}\mathbf{u}(\cdot, \tau) \|^2_{L^2(\mathbb{R}^3)} \, d\tau
\end{align*}
$$

\hspace{1cm} (3.5a)
for all \( t \geq t_m \). Recalling (3.5a), this then gives

\[
f(t) \| D^m u(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 + (2 - \epsilon) \nu \int_{t_m}^t f(\tau) \| D^{m+1} u(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau \leq \]

\[
\leq A_m (\lambda_0(\alpha) + \epsilon)^2 g(t_m) + 2 \beta \int_{t_m}^t A_m (\lambda_0(\alpha) + \epsilon)^2 g(\tau) \, d\tau \quad \text{[by (3.5a)]}
\]

\[
= A_m (\lambda_0(\alpha) + \epsilon)^2 g(t_m) + \frac{\beta}{\beta - \alpha} A_m (\lambda_0(\alpha) + \epsilon)^2 (g(t) - g(t_m))
\]

\[
\leq \frac{\beta}{\beta - \alpha} A_m (\lambda_0(\alpha) + \epsilon)^2 g(t)
\]

for all \( t \geq t_m \), so that we have

\[
\int_{t_m}^t f(\tau) \| D^{m+1} u(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau \leq \frac{1}{2 - \epsilon} \frac{\beta}{\beta - \alpha} \nu^{-1} A_m (\lambda_0(\alpha) + \epsilon)^2 g(t) \quad (3.5b)
\]

for all \( t \geq t_m \). Since, by (2.3b) and (2.7) again, we get (increasing \( t_m \) if necessary):

\[
(f(t) - f(t_m)) \| D^{m+1} u(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 + (2 - \epsilon) \nu \int_{t_m}^t (f(\tau) - f(t_m)) \| D^{m+2} u(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau \]

\[
\leq 2 \beta \int_{t_m}^t f(\tau) \| D^{m+1} u(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau
\]

for all \( t \geq t_m \), we then obtain, by (3.5b),

\[
(f(t) - f(t_m)) \| D^{m+1} u(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 + (2 - \epsilon) \nu \int_{t_m}^t (f(\tau) - f(t_m)) \| D^{m+2} u(\cdot,\tau) \|_{L^2(\mathbb{R}^3)}^2 \, d\tau \]

\[
\leq \frac{2}{2 - \epsilon} \frac{\beta^2}{\beta - \alpha} \nu^{-1} A_m (\lambda_0(\alpha) + \epsilon)^2 g(t)
\]

for all \( t \geq t_m \). Writing \( (f(t) - f(t_m)) = f(t) (1 - e^{-2\beta(t-t_m)}) \) and taking \( t_{m+1} > t_m \) such that \( e^{-2\beta(t_{m+1}-t_m)} < \epsilon \), this gives

\[
f(t) \| D^{m+1} u(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{1 - \epsilon} \frac{2}{2 - \epsilon} \frac{\beta^2}{\beta - \alpha} \nu^{-1} A_m (\lambda_0(\alpha) + \epsilon)^2 g(t) \quad (3.6)
\]

for all \( t \geq t_{m+1} \). That is, by induction in \( m \), we have shown that, for every \( m \geq 1 \):

\[
f(t) \| D^m u(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 \leq A_m (\lambda_0(\alpha) + \epsilon)^2 g(t) \quad \forall \ t \geq t_m \quad (3.7a)
\]

for \( t_m > t_* \) sufficiently large, and where the constant \( A_m > 0 \) satisfies the recurrence

\[
A_m = \frac{1}{1 - \epsilon} \frac{2}{2 - \epsilon} \frac{\beta^2}{\beta - \alpha} \nu^{-1} A_{m-1} \quad (3.7b)
\]

for all \( m \geq 2 \), with \( A_1 > 0 \) given in (3.4) above. This gives
for all $m \geq 1$. Choosing $\beta = 2\alpha$, we then obtain, by (3.7a) and (3.8),
\[
\limsup_{t \to \infty} \epsilon^{2\alpha t} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \leq \left( \frac{1}{1-\epsilon} \right)^m \left( \frac{2}{2-\epsilon} \right)^{m-1} 4^m \alpha^m \lambda_0(\alpha) = 2 \epsilon^{2\alpha t} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \]
for every $m \geq 1$, where $\epsilon \in (0, 1)$ is arbitrary. Letting $\epsilon \to 0$, this gives the result. 

\[\square\]

### 4. Proof of Theorem B

We now show (1.6). Again, since the proofs for $n = 2, 3, 4$ are entirely similar, we will present the details for one case only — say, $n = 4$. Let then $u(\cdot, t)$ be any given Leray solution to (1.1), in $\mathbb{R}^4$, such that we have, for some $\alpha \geq 0$,
\begin{equation}
\limsup_{t \to \infty} t^\alpha \| u(\cdot, t) \|_{L^2(\mathbb{R}^4)} =: \lambda_0(\alpha) < \infty. \tag{4.1}
\end{equation}

Let $\delta > 0$, $0 < \epsilon < 2$ be given, and let $t_*$ be the solution’s regularity time as defined in (1.3). Taking the dot product of $u(\cdot, t)$ with $(t - t_0)^{2\alpha + \delta} u(x, t)$ and integrating over $\mathbb{R}^4 \times [t_0, t]$, for $t \geq t_0 > t_*$, we obtain, because of (1.1b),
\[
(t - t_0)^{2\alpha + \delta} \| u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2 \nu \int_{t_0}^t (\tau - t_0)^{2\alpha + \delta} \| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau = (2\alpha + \delta) \int_{t_0}^t (\tau - t_0)^{2\alpha + \delta - 1} \| u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau
\]
for $t \geq t_0 > t_*$. This promptly gives, by (4.1), taking $t_0 > t_*$ sufficiently large, that
\begin{equation}
\int_{t_0}^t (\tau - t_0)^{2\alpha + \delta} \| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau \leq \frac{1}{2\nu} \frac{2\alpha + \delta}{\delta} (\lambda_0(\alpha) + \epsilon)^2 (t - t_0)^\delta \tag{4.2}
\end{equation}
for all $t \geq t_0$. Next, for $m = 1$, we similarly have
\[
(t - t_0)^{2\alpha + 1 + \delta} \| D u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 + 2 \nu \int_{t_0}^t (\tau - t_0)^{2\alpha + 1 + \delta} \| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau \leq (2\alpha + 1 + \delta) \int_{t_0}^t (\tau - t_0)^{2\alpha + \delta} \| D u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)}^2 d\tau + K_1 \int_{t_0}^t (\tau - t_0)^{2\alpha + 1 + \delta} \| u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} \| D u(\cdot, \tau) \|_{L^4(\mathbb{R}^4)} \| D^2 u(\cdot, \tau) \|_{L^2(\mathbb{R}^4)} d\tau
\]
(where $K_1 = 8 \sqrt{2}$), which gives, by (2.5):

\[\square\]
for all \(t \geq t_0\). By \((2.7)\) and \((4.2)\), we then get (increasing \(t_0\) if necessary)

\[
(t - t_0)^{2a + 1 + \delta} \| Du(\cdot, t) \|^2 _{L^2(\mathbb{R}^4)} + 2 \nu \int_{t_0}^t (t - \tau)^{2a + 1 + \delta} \| D^2 u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau \\
\leq (2a + 1 + \delta) \int_{t_0}^t (t - \tau)^{2a + \delta} \| D u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau +
K_1 \int_{t_0}^t (\tau - t_0)^{2a + 1 + \delta} \| D u(\cdot, \tau) \| L^2(\mathbb{R}^4) \| D^2 u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau
\]

for \(t \geq t_0\). Proceeding in this way \((m = 2, 3, \ldots)\), we obtain, at the \(m\)th step:

\[
(t - t_0)^{2a + m + \delta} \| D^m u(\cdot, t) \|^2 _{L^2(\mathbb{R}^4)} + 2 \nu \int_{t_0}^t (t - \tau)^{2a + m + \delta} \| D^{m+1} u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau \\
\leq (2a + m + \delta) \int_{t_0}^t (t - \tau)^{2a + m - 1 + \delta} \| D^m u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau +
K_m \int_{t_0}^t (\tau - t_0)^{2a + m + \delta} \| D^{m+1} u(\cdot, \tau) \| L^2(\mathbb{R}^4) \| D^m u(\cdot, \tau) \| L^4(\mathbb{R}^4) \| D^{m-\ell} u(\cdot, \tau) \| L^4(\mathbb{R}^4) d\tau
\]

for \(t \geq t_0\), and some constant \(K_m > 0\), where \([m/2]\) denotes the integer part of \(m/2\).

This gives, by \((2.5)\):

\[
(t - t_0)^{2a + m + \delta} \| D^m u(\cdot, t) \|^2 _{L^2(\mathbb{R}^4)} + 2 \nu \int_{t_0}^t (t - \tau)^{2a + m + \delta} \| D^{m+1} u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau \\
\leq (2a + m + \delta) \int_{t_0}^t (t - \tau)^{2a + m - 1 + \delta} \| D^m u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau +
(1 + [m/2]) \cdot K_m \int_{t_0}^t (\tau - t_0)^{2a + m + \delta} \| Du(\cdot, \tau) \| L^2(\mathbb{R}^4) \| D^{m+1} u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau.
\]

At this stage, we would have already obtained from the previous steps that

\[
(t - t_0)^{2a + k} \| D^k u(\cdot, t) \|^2 _{L^2(\mathbb{R}^4)} \leq \frac{1}{\delta \cdot [(2 - \epsilon) \nu]^k} \left\{ \prod_{j=0}^{k} (2a + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2
\]

and

\[
\int_{t_0}^t (t - \tau)^{2a + k + \delta} \| D^{k+1} u(\cdot, \tau) \|^2 _{L^2(\mathbb{R}^4)} d\tau \leq \frac{\delta^{-1}}{[(2 - \epsilon) \nu]^{k+1}} \left\{ \prod_{j=0}^{k} (2a + j + \delta) \right\} \times (\lambda_0(\alpha) + \epsilon)^2 \cdot (t - t_0)^\delta
\]

(4.5a) and

(4.5b)
for all $t \geq t_0$, and each $0 \leq k < m$. By (2.7) and (4.4), and increasing $t_0$ if necessary, we would then obtain (4.5) for $k = m$ as well, completing the induction step.

The argument above established that, for each $m \geq 1$, we have

$$(t - t_0)^{2\alpha + m} \| D^m u(\cdot, t) \|_{L^2(\mathbb{R}^4)}^2 \leq \frac{1}{\delta \cdot [(2 - \epsilon) \nu]^{m}} \left\{ \prod_{j=0}^{m} (2\alpha + j + \delta) \right\} (\lambda_0(\alpha) + \epsilon)^2$$

for all $t$ sufficiently large. Since $\delta > 0$, $0 < \epsilon < 2$ are arbitrary, this gives the result. $\square$

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