THE MOORE-OSGOOD THEOREM ON EXCHANGING LIMITS

Theorem 5.11 in our book can be stated a little more simply as follows. There is also a version for complex-valued functions.

Theorem 0.1. *Suppose X is a metric space and E is a subset of X and x is a limit-point of E. Suppose $f_n : X \to \mathbb{R}$ (for each $n \in \mathbb{N}$) and $f : X \to \mathbb{R}$ are functions and A_n are numbers. If*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{uniformly on } X$$

and

$$\lim_{y\to x} f_n(y) = A_n \quad \textit{pointwise over } \mathbb{N}$$

then the double limits exists and

$$\lim_{y\to x}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{y\to x} f_n(x).$$

A few remarks are in order. The "pointwise limit" assertion is just that for each n, the limit $\lim\limits_{y\to x} f_n(y)$ exists. These limits are "along E" and you probably want to think of something like $E = [x, +\infty)$ inside $\mathbb{R}$ and so the double limits would be

$$\lim_{y\to x^+}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{y\to x^+} f_n(x)$$

in that case.

An important special case of this theorem comes from considering

$$X = \mathbb{N} \cup \{\infty\}$$

with the metric we considered last semester:

$$d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|, \quad d(\infty, n) = d(n, \infty) = \frac{1}{n}.$$

You can just as well consider that we have embedded $\mathbb{N}$ into $\mathbb{R}$ in the nonstandard fashion,

$$n \mapsto \frac{1}{n}$$

and are extending this to

$$\infty \mapsto 0.$$

A real sequence $a = \langle a_k \rangle$ is then just a function $a : \mathbb{N} \to \mathbb{R}$ where now

$$a(k) = a_k.$$

With this switch, and considering $E = \mathbb{N} \subseteq X$ and ∞ a limit point of E, we have two types of limit that really mean the same thing:

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} a(k).$$

So we can figure out exactly what is meant by uniform convergence of a double sequence to a sequence: Given $a_{n,k}$ in $\mathbb{R}$ and A_k in $\mathbb{R}$, we say

$$\lim_{n\to\infty} a_{n,k} = A_k \quad \text{uniformly in } k$$

if,

$$\forall \epsilon > 0, \exists N < \infty \text{ s.t. } \forall k, \quad n \geq N \implies |a_{n,k} - A_k| < \epsilon.$$

So a special case of Theorem 5.11 in Rudin is:

Theorem 0.2. *Suppose we are given $a_{n,k}$ in $\mathbb{R}$ and A_n and B_k in $\mathbb{R}$. If*

$$\lim_{n\to\infty} a_{n,k} = B_k \quad \textit{uniformly in } k$$

and for each n we have

$$\lim_{k\to\infty} a_{n,k} = A_n$$

then the double limits exists and

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{n,k} = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,k}.$$

The Moore-Osgood Theorem goes the other direction, and allows for a general metric space in both variables. For example, it allows one to conclude

$$\lim_{x\to a^+} \lim_{y\to b^+} f(x,y) = \lim_{y\to b^+} \lim_{x\to a^+} f(x,y)$$

so long as both $\lim_{x\to a^+} f(x,y)$ and $\lim_{y\to b^+} f(x,y)$ exist pointwise and one of these limits happens uniformly. We have no need for this in this class, but now you know the context. Also, you can quote "the Moore-Osgood Theorem" to impress friends and intimidate enemies.

An important special case of Theorem 0.2 involves exchanging order of summation. It is in Rudin as Theorem 8.3. Here it is in the complex case, which subsumes the real case.

Theorem 0.3. *Given a complex double sequence $a_{m,n}$, if*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty \tag{0.1}$$

then the iterated sums of the $a_{m,n}$ exist and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}.$$

I'll not prove this, but have a few remarks. There is no problem interpreting (0.1) should it happen that for some m the sum $\sum_{n=1}^{\infty} |a_{m,n}|$ is infinite. We just declare the double sum in (0.1) to be $+\infty$ and so the hypothesis is false. The essence of the proof is to fiddle with finite sums until the real problem is exposed. We must show

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{m,n}.$$

To to this, we need uniform convergence in either m or n. This will be easier on one variable than the other, as one order was selected in (0.1). *A postiori* we know

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}| < \infty \implies \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a_{m,n}| = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|.$$