CLASSIFYING SMALL GRAPHS

For graphs that have few vertices, it is practical to write down a list of all the possible "shapes" that graph can take. We say we have classified the graphs with n vertices if we have a list of graphs so that every graph with n vertices is isomorphic to just one graph on that list. Another way we say this is that, up to isomorphism, the list contains all the possible graphs with n vertices.

For very small n, it is possible to do the classification by brute force. We start with no edges, find all the places we can add an edge, find all the places we can add a second edge, and so forth.

For n just a bit larger, we need some help. We need to have a clear idea of what symmetry means, and we need the concept of the complement of a graph.

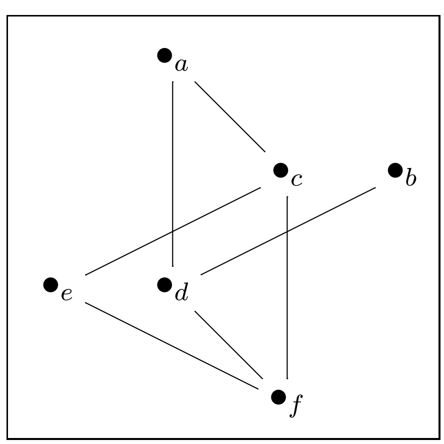
1. Complements

If a graph has a lot of edges, it is often easier to describe it in terms of the gaps.

Definition 1. If G is a graph, then the complement $\bar{G}$ is another graph, with the same set of vertices as G, but where for $v \neq w$ there is an edge between v and w in $\bar{G}$ if and only if there is not an edge between v and w in G.

So if this is G,

$G:$



then here is $\bar{G}$:

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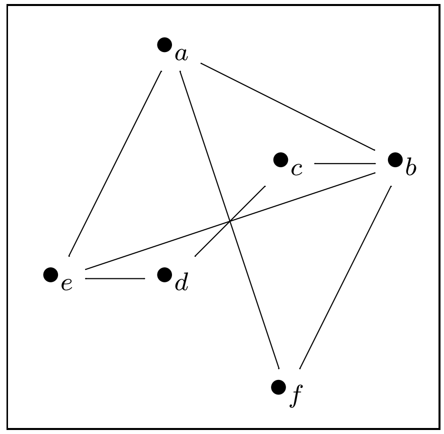


If G has n vertices, then the degree of v in $\bar{G}$ is $(n-1)$ minus the degree of v in G. This checks out in the example just given, where G has degree sequence

$$(1, 2, 2, 3, 3, 3)$$

and taking these values away from 5 we get

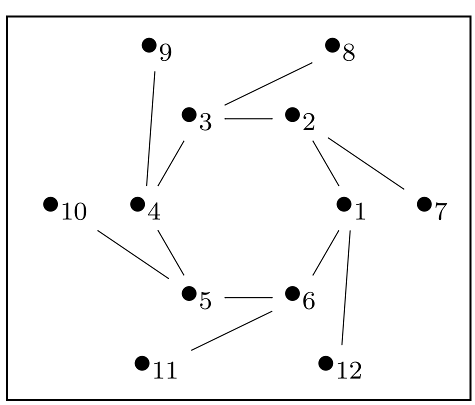
$$(4, 3, 3, 2, 2, 2),$$

and indeed, the degree sequence of $\bar{G}$ is

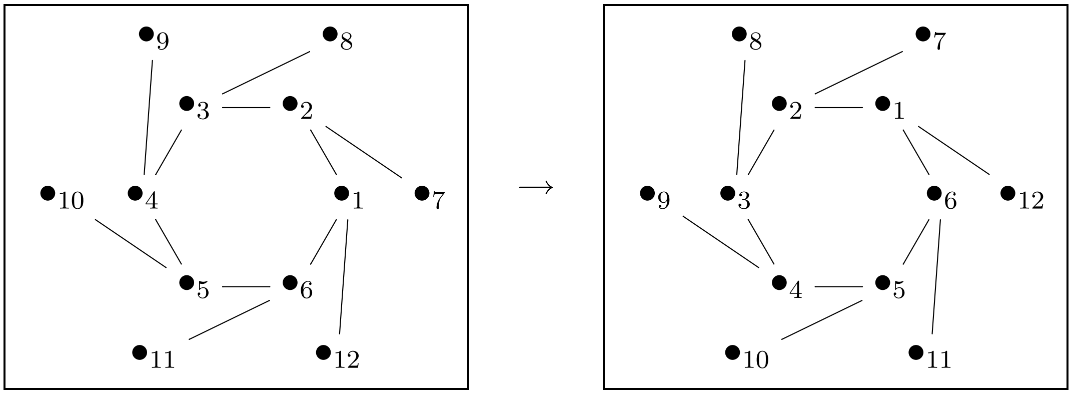
$$(2, 2, 2, 3, 3, 4).$$

2. Symmetries

In some cases, we can see a symmetry in a graph based on the way it is drawn. Consider the following graph:



We can rotate the picture by $\pi/3$ and the picture of the graph looks the same except the number on the vertices have changed:



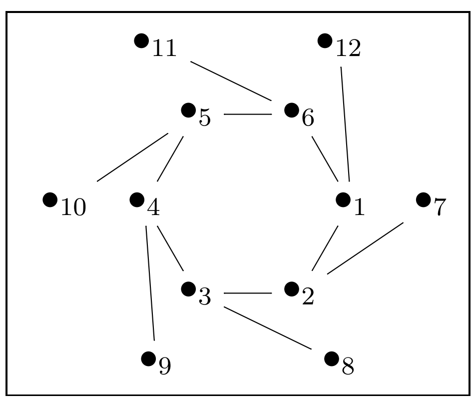
The symmetry we want is the symmetry of the graph, not the picture. A *symmetry* of a graph is an isomorphism of teh graph with itself. In this case, the symmetry we found by rotating the picture is this:

$$\varphi : \{1, 2, \ldots, 12\} \to \{1, 2, \ldots, 12\},$$

$$1 \to 2 \quad 2 \to 3 \quad 3 \to 4 \quad 4 \to 5 \quad 5 \to 6 \quad 6 \to 1$$

$$7 \to 8 \quad 8 \to 9 \quad 9 \to 10 \quad 10 \to 11 \quad 11 \to 12 \quad 12 \to 7.$$

If we flip the pciture, it looks different:



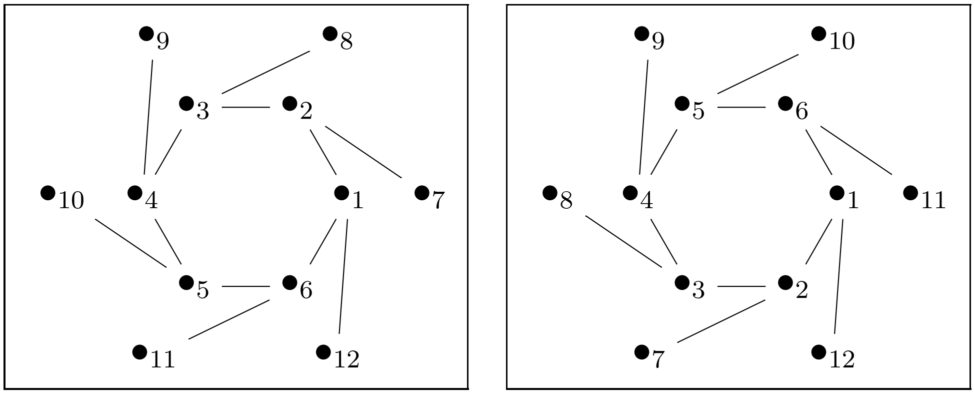
but there is another isomorphism here:

$$\psi : \{1, 2, \ldots, 12\} \to \{1, 2, \ldots, 12\},$$

$$1 \to 1 \quad 2 \to 6 \quad 3 \to 5 \quad 4 \to 4 \quad 5 \to 3 \quad 6 \to 2$$

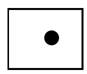
$$7 \to 11 \quad 8 \to 10 \quad 9 \to 9 \quad 10 \to 8 \quad 11 \to 7 \quad 12 \to 12.$$

We can see this on the diagram if we change the labels in one copy:

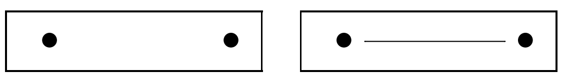


3. 1 and 2 Vertices

The most edges a graph with n vertices can have is $\frac{n(n-1)}{2}$. So a graph with one vertex cannot have any edges. up to isomorphism, the only graph possible is this:

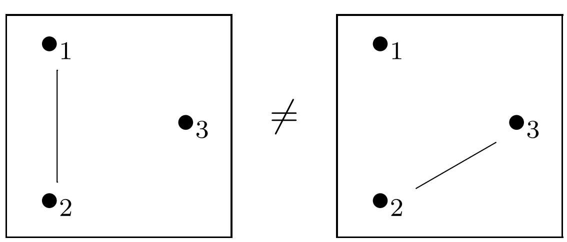


We saw that one the vertices $\{1, 2\}$, there $2 = 2^1$ possible graphs. Either there is one edge or none, and this are not isomorpic. Up to isomorphism, the possible graphs are:

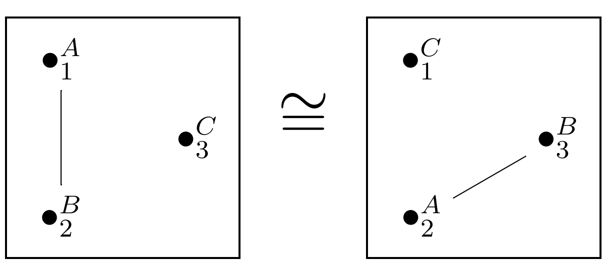


4. 3 Vertices

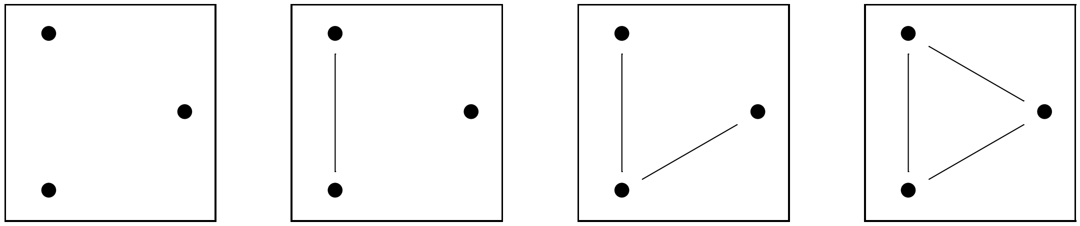
We saw that there are $8 = 2^3$ possible graphs on a set of 3 vertices. But how many of these are isomorphic to each other? (What are the isomorphism classes?) While



these are isomorphic, as the following labels show.



All the graphs on three vertices with two edges are isomorphic, so we have at least four isomorphism classes, represented by



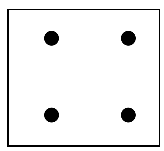
These have different numbers of edges, so no two cannot be isomorpic.

The point to this is that *any* graph with 3 vertices will be isomorpic to exactly one of these four graphs.

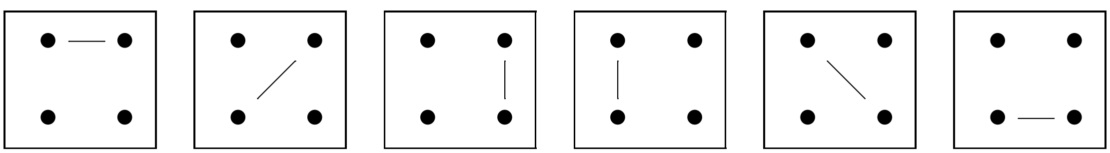
5. 4 Vertices

We now calculate all the possible 4-vertex graphs, up to isomorphism. We work our way up, starting with no edges and then finding all the places we can add an edge.

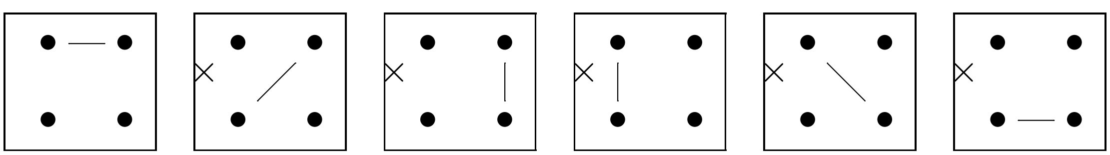
No edges, only one way to go:



There are 6 places to add an edge to this,

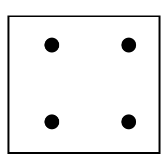


but these are all isomorphic. So we toss all but the first: ,

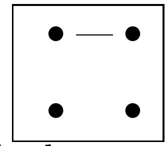


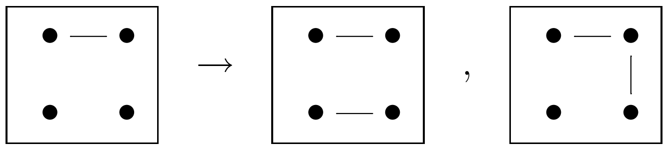
We could have save some effort if we had first noticed that there are lots of symmetries of the graph

E :

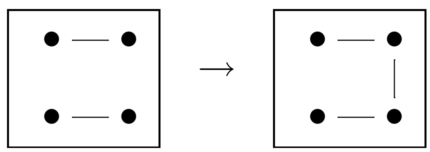


in fact there for every pair of vertices, there is a symmetry that moves that pair to the top two vertices. We can say *every pair of vertices is like any other* so it does not matter where we add the first edge.

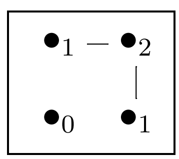
Now starting with our single one-edge graph, we find there are 5 places to add an edge. However, there is a symmetry of  that flips the top two vertices, and there is one that flips the bottom two vertices. Either the new edge goes between the two unattacted vertices, or it must attach a top vertex to a bottom vertex, and it does not matter which top vertex and which bottom vertex. So the single one-edge graph spawns two two-edge graphs:



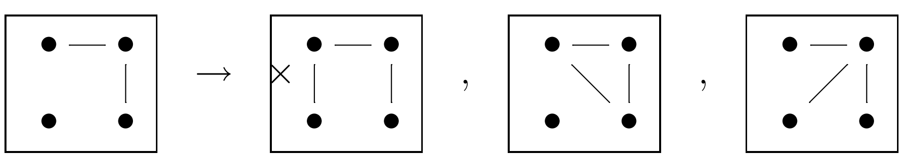
The first graph has symmetries flip the top two or the bottom two vertices, so there is really only one way to add an edge—connect the a top vertex to a bottom vertex.



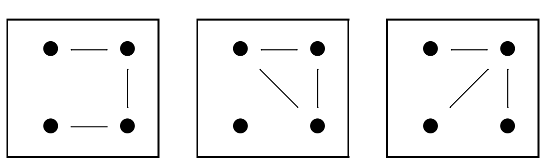
The second two-edge graph,



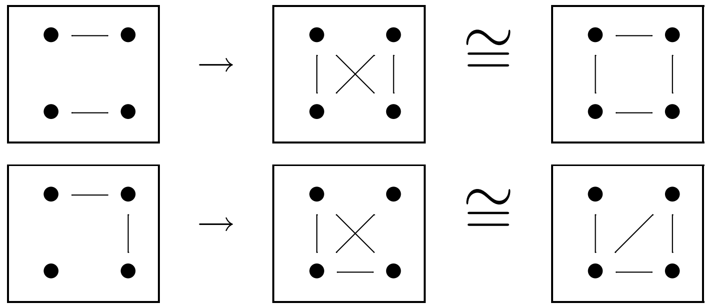
has only one symmetry, that which exchanges the two vertices of degree one. So we can add an edge in several essentially different ways: connect a degree-one to the degree zero; connect a degree-one to the other degree-one; connect a degree zero to a degree two:



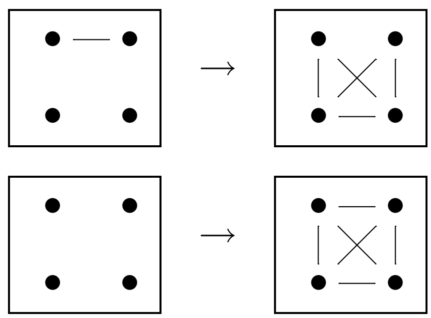
The first is a repeat of a graph we saw in the first list, so we drop it. There are, up to isomorphism, three graphs with four vertices and three edges:



Now we can take complements of the graphs we found with two-edges:



Taking complements of our single graph with one edge, and the single graph with no edges, finishes our search:



To summarize, every graph with 4 vertices is isomorphic to one of the following 11 graphs, no two of which are isomorphic: