LU AND PLU FACTORIZATION

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1. Products of Elementary matrices, revisited

In the posting "Expanding an Invertible to a Product of Elementary Matrices" there was this matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -8 & 8 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

which we expressed as a product of elementary matrices as

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose we multiply together the first two as

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and let L be the third matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

and multiply the last two as

$$U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have this description of A :

$$A = PLU$$

or

$$\begin{bmatrix} 0 & 1 & 0 \\ -8 & 8 & 1 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

- P is a so-called permutation matrix
- L is lower triangular

- U is upper triangular

This is called an LU or PLU decomposition of A.

Lower and upper triangular matrices are computationally easier than your typical invertible matrix. The matrix P is easy to deal with as well since it is mostly full of zeros. It is called a *permutation matrix* because it would equal the identity matrix if we could permute its rows.

In computer solutions of systems of equations, and elsewhere, it is easier to deal with three easier matrices than one hard one. We will deal with applications of this later, or not at all. For now, I want to show you how to use row and column ops to create an LU factorization.

2. ELEMENTARY, BUT DIFFERENT

Let's rework this example, but using different row operations. We still will do the operations in this order: type II, type III "clearing below," type II, them type III "clearing above."

$$\begin{bmatrix} 0 & 1 & 0 \\ -8 & 8 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

$\mathbf{R}1 \leftrightarrow \mathbf{R}2$

$$\sim \begin{bmatrix} -8 & 8 & 1 \\ 0 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$\mathbf{R}3 \leftarrow \mathbf{R}3 + \frac{1}{4}\mathbf{R}1$

$$\sim \begin{bmatrix} -8 & 8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\mathbf{R}1 \leftarrow -\frac{1}{8}\mathbf{R}1$

$$\sim \begin{bmatrix} 1 & -1 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\mathbf{R}3 \leftarrow 4\mathbf{R}3$

$$\sim \begin{bmatrix} 1 & -1 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R}1 \leftarrow \mathbf{R}1 + \mathbf{R}2$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{R}1 \leftarrow \mathbf{R}1 + \frac{1}{8}\mathbf{R}3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we look at the "undo" operations:

$$
\begin{array}{ccc}
\mathbf{R}1 \leftrightarrow \mathbf{R}2 & \dashrightarrow & \mathbf{R}1 \leftrightarrow \mathbf{R}2 \\
\mathbf{R}3 \leftarrow \mathbf{R}3 + \frac{1}{4}\mathbf{R}1 & \dashrightarrow & \mathbf{R}3 \leftarrow \mathbf{R}3 - \frac{1}{4}\mathbf{R}1 \\
\mathbf{R}1 \leftarrow -\frac{1}{8}\mathbf{R}1 & \dashrightarrow & \mathbf{R}1 \leftarrow -8\mathbf{R}1 \\
\mathbf{R}3 \leftarrow 4\mathbf{R}3 & \dashrightarrow & \mathbf{R}3 \leftarrow \frac{1}{4}\mathbf{R}3 \\
\mathbf{R}1 \leftarrow \mathbf{R}1 + \mathbf{R}2 & \dashrightarrow & \mathbf{R}1 \leftarrow \mathbf{R}1 - \mathbf{R}2 \\
\mathbf{R}1 \leftarrow \mathbf{R}1 + \frac{1}{8}\mathbf{R}3 & \dashrightarrow & \mathbf{R}1 \leftarrow \mathbf{R}1 - \frac{1}{8}\mathbf{R}3
\end{array}
$$

and now the elementary matrices:

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Let us now multiply together in three parts:

$$
P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

$$
L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix},
$$

$$
\begin{aligned}
U_1 &= \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & -1 & -\frac{1}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.
\end{aligned}
$$

That is, we have found

$$
A = P_1 L_1 U_1
$$

or

$$
\begin{bmatrix} 0 & 1 & 0 \\ -8 & 8 & 1 \\ 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} -8 & 8 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.
$$

3. ONCE MORE, USING MATLAB

In Matlab, the rather odd input

```
>> A = [ 2 1 0 1; 2 1 2 3; 0 0 1 2; -4 -1 0 -2];
>> [L,U,P] = lu(A)
```

gives the result

```
L =
    1.0000          0          0          0
   -0.5000     1.0000          0          0
   -0.5000     1.0000     1.0000          0
         0          0    -0.5000     1.0000
U =

   -4.0000    -1.0000          0    -2.0000
         0     0.5000     2.0000     2.0000
         0          0    -2.0000    -2.0000
         0          0          0     1.0000
P =
    0    0    0    1
    0    1    0    0
    1    0    0    0
    0    0    1    0
```

A weird thing is that Matlab promises that

    `[L,U,P] = lu(A)`

will set the three variables on the left in a way to ensure

$$A = P^{-1}LU.$$

In this case, that did not matter since $P = P^{-1}$.

4. PLU WITHOUT ELEMENTARY MATRICES.

In practice, one does not this list of elementary matrices. One can get directly the PLU factorization.

The new idea is that one can take an equation like

$$A = BC$$

and do a row operation on C and a balancing column operation on B to get

$$A = B_1 C_1.$$

This is because

$$BC = BE^{-1}EC$$

whenever E is an elementary matrix. For example

$$BC = B \begin{bmatrix} 1 & -\beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} C$$

tells us that if we do the row operation

$$\mathbf{R}1 \leftarrow \mathbf{R}1 + \beta\mathbf{R}2$$

on the right factor we can offset this with the column operation

$$\mathbf{C}2 \leftarrow \mathbf{C}2 - \beta\mathbf{C}2.$$

The offsets are as follows:

$$
\begin{array}{cc}
\text{on left factor} & \text{on right factor} \\
\frac{1}{\alpha}\mathbf{C}j \leftarrow \mathbf{C}j & \frac{1}{\alpha}\mathbf{R}j \leftarrow \mathbf{R}j \\
\mathbf{C}j \leftrightarrow \mathbf{C}k & \mathbf{R}j \leftrightarrow \mathbf{R}k \\
\mathbf{C}k \leftarrow \mathbf{C}k - \beta\mathbf{C}j & \mathbf{R}j \leftarrow \mathbf{R}j + \beta\mathbf{R}k
\end{array}
$$

We can start with the silly equality

$$A = IIA$$

and proceed to make the last factor upper triangular, the middle lower triangular and the left a permutation matrix. "nop" stands for "no operation." Each step involves two adjacent factors and leaves the other alone

For example:

$$
\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 2 & 13 \end{bmatrix}
$$

$$\text{nop} \quad \& \quad \mathbf{C}3 \leftarrow \mathbf{C}3 + 3\mathbf{C}1 \quad \& \quad \mathbf{R}3 \leftarrow \mathbf{R}3 - 3\mathbf{R}1$$

$$
\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}
$$

$$\text{nop} \quad \& \quad \mathbf{C}2 \leftrightarrow \mathbf{C}3 \quad \& \quad \mathbf{R}2 \leftrightarrow \mathbf{R}3$$

$$
\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$\mathbf{C}2 \leftrightarrow \mathbf{C}3 \quad \& \quad \mathbf{R}2 \leftrightarrow \mathbf{R}3 \quad \& \quad \text{nop}$$

$$
\begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 3 & 2 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

5. A 4-BY-4 PLU EXAMPLE

Now let's do a 4-by-4 factoring into PLU form.

$$
\begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix}
$$

$$
\text{nop} \quad \& \quad \mathbf{C}1 \leftarrow \mathbf{C}1 + \mathbf{C}2 \quad \& \quad \mathbf{R}2 \leftarrow \mathbf{R}2 - \mathbf{R}1
$$
$$
\text{nop} \quad \& \quad \mathbf{C}1 \leftarrow \mathbf{C}1 - 2\mathbf{C}4 \quad \& \quad \mathbf{R}4 \leftarrow \mathbf{R}4 + 2\mathbf{R}1
$$

$$
\begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
\text{nop} \quad \& \quad \mathbf{C}2 \leftrightarrow \mathbf{C}4 \quad \& \quad \mathbf{R}2 \leftrightarrow \mathbf{R}4
$$

$$
\begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}
$$

$$
\mathbf{C}2 \leftrightarrow \mathbf{C}4 \quad \& \quad \mathbf{R}2 \leftrightarrow \mathbf{R}4 \quad \& \quad \text{nop}
$$

$$
\begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}
$$

$$
\text{nop} \quad \& \quad \mathbf{C}3 \leftarrow \mathbf{C}3 + 2\mathbf{C}4 \& \mathbf{R}4 \leftarrow \mathbf{R}4 - 2\mathbf{R}3
$$

$$
\begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ -4 & -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}
$$

The Matlab session

```
>> A = [ 2 1 0 1; 2 1 2 3; 0 0 1 2; -4 -1 0 -2];
>> [L,U,P] = lu(A);
>> P = inv(P);
>> P
P =

     0     0     1     0
```

```
        0         1         0         0
        0         0         0         1
        1         0         0         0
>> L
L =

      1.0000             0             0             0
     -0.5000        1.0000             0             0
     -0.5000        1.0000        1.0000             0
           0             0       -0.5000        1.0000
>> U
U =
     -4.0000       -1.0000             0       -2.0000
           0        0.5000        2.0000        2.0000
           0             0       -2.0000       -2.0000
           0             0             0        1.0000
```

shows another of the possible PLU factorization of this is

$$
\begin{bmatrix}
2 & 1 & 0 & 1 \\
2 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 \\
-4 & -1 & 0 & -2
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & 0 & 0 \\
-\frac{1}{2} & 1 & 1 & 0 \\
0 & 0 & -\frac{1}{2} & 1
\end{bmatrix}
\begin{bmatrix}
-4 & -1 & 0 & 1 \\
0 & \frac{1}{2} & 2 & 2 \\
0 & 0 & -2 & -2 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

I prefer the one we found by hand, as we eschewed fractions.