CORRECTED NOTES FOR SEPT. 5

Let $C([0,1], \mathbf{R})$ be the following, infinite-dimensional vector space:

$$C([0,1], \mathbf{R}) = \{f : [0,1] \to \mathbf{R} \mid f \text{ is continuous}\}$$

For any f and g in $C([0,1], \mathbf{R})$, we define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

This is an inner product, a fact that is easy enough to prove except for the axiom that only the zero vector (zero function) satisfies $\langle f, f \rangle = 0$.

Lemma 1. *Suppose $f : [0,1] \to \mathbf{R}$ is continuous. If $\langle f, f \rangle = 0$ then $f = 0$. (I.e. $\forall x \in [0,1], f(x) = 0$.)*

Proof. We'll prove the contrapositive.

Suppose f is continuous and $f \neq 0$. This means at some x_0 in $[0,1]$, $f(x_0) \neq 0$. We can assume $0 < x_0 < 1$ since a continuous function cannot be zero on $(0,1)$ and nonzero at 0 or 1.

The square of f is continuous and, and $f(x)^2 \geq 0$ for all x. Let $\delta = f(x_0)^2$ so that $\delta > 0$. The continuity of f^2 at x_0 tells us there exists some positive η such that

$$x_0 - \eta < x < x_0 + \eta \Rightarrow f(x_0)^2 - \delta/2 < f(x)^2 < f(x_0)^2 + \delta/2$$

We don't care about the upper bound on $f(x)$. Let $\mu = f(x_0)^2 - \delta/2$.

At this point, we've found x_0, η and μ so that:

$$\begin{aligned} \eta &> 0 \\ \mu &> 0 \\ x_0 - \eta < x < x_0 + \eta &\Rightarrow f(x_0)^2 > \mu \end{aligned}$$

We can replace η with a smaller positive value and the above equations hold, and we do so if needed to get

$$0 < x_0 - \eta \text{ and } x_0 + \eta < 1.$$

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Using these facts, and the fact that $f(x)^2$ is always non-negative, we conclude

$$
\begin{aligned}
& \langle f, f \rangle = \\
= \;& \int_0^1 f(x)^2 dx \\
= \;& \int_0^{x_0-\eta} f(x)^2 dx + \int_{x_0-\eta}^{x_0+\eta} f(x)^2 dx + \int_{x_0+\eta}^{1} f(x)^2 dx \\
\geq \;& \int_0^{x_0-\eta} 0 dx + \int_{x_0-\eta}^{x_0+\eta} \mu dx + \int_{x_0+\eta}^{1} 0 dx \\
= \;& 0 + 2\eta\mu + 0 \\
> \;& 0
\end{aligned}
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