A numerical approach to Chern numbers and unbounded Fredholm modules

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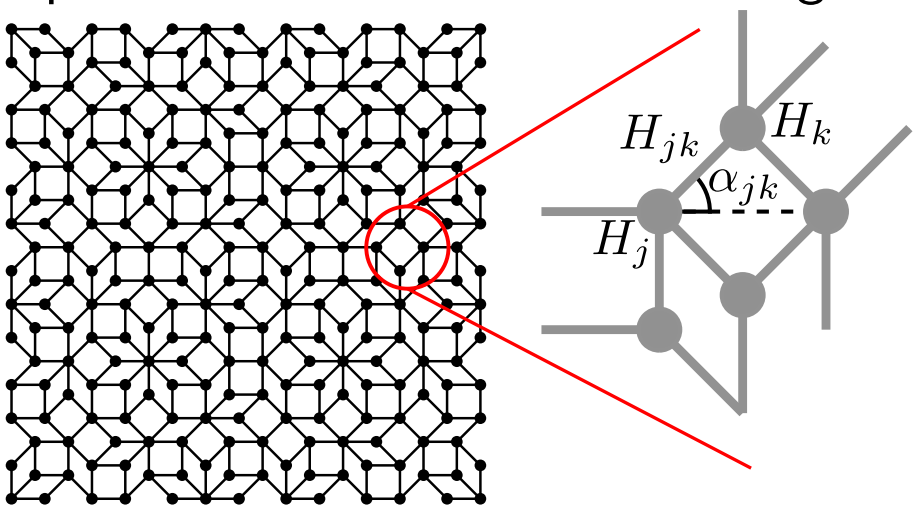
Quasicrystalline Chern insulator

Aperiodic Ammann-Beenker tiling.



For Chern number -1, we will use

$$\mu = 1, \ t = 1, \ \mu = 2.$$

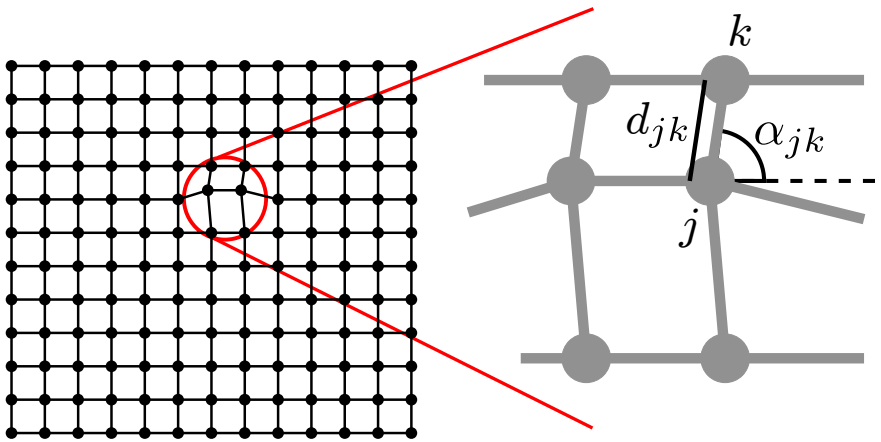
"$p_x + ip_y$" tight binding model, H:

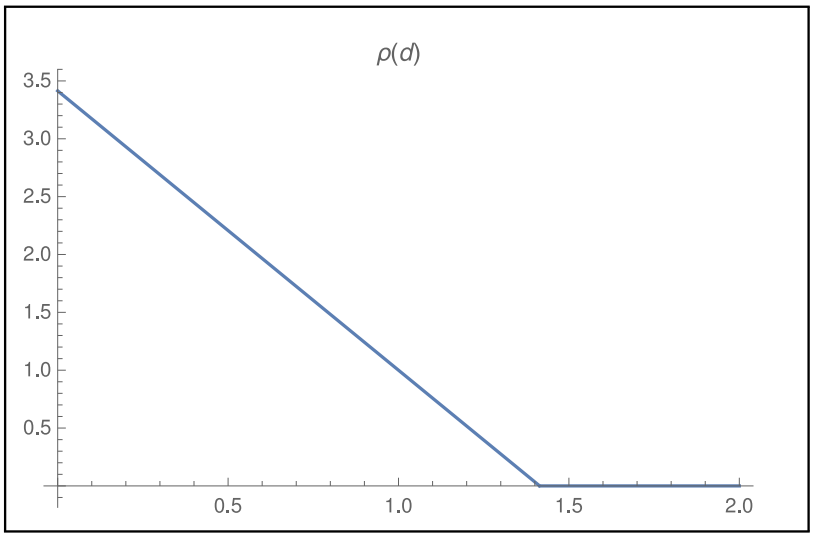
$$H_j = -\mu\sigma_z$$

$$H_{jk} = -t\sigma_z - \frac{i}{2}\Delta\sigma_x \cos(\alpha_{jk}) - \frac{i}{2}\Delta\sigma_y \sin(\alpha_{jk})$$

Fulga, Ion Cosma, Dmitry I. Pikulin, and L. "Aperiodic Weak Topological Superconductors." Physical Review Letters 116.25 (2016): 257002.

Chern insulator on regular lattice with defects





"$p_x + ip_y$" tight binding model, now adapted for variable length edges, H:

$$H_j = -\mu\sigma_z$$

$$H_{jk} = \rho(d_{jk})\left(-t\sigma_z - \frac{i}{2}\Delta\sigma_x\cos(\alpha_{jk}) - \frac{i}{2}\Delta\sigma_y\sin(\alpha_{jk})\right)$$

L., J. Michala, A. Pierson, A. Watson. "Wave-packet propagation and the spectral localizer index", in preparation.

Essential features of a Chern insulator

To validate the claim that these are Chern insulators, we must do at least three things.

1. Show $\sigma(H)$ has a gap at zero (or the Fermi level) and compute size of this spectral gap. Better yet, compute the entire spectrum of H.
2. Compute the K-theory of the infinite system. This is the index of the Fredholm operator

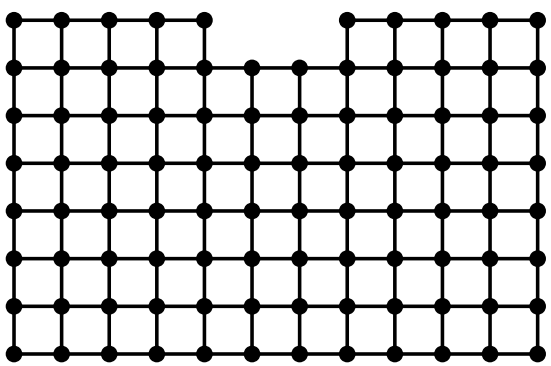
$$\mathrm{ind}(H, X, Y) = \mathrm{ind}\left(\Pi_v \left(\frac{X + iY}{|X + iY|}\right) \Pi_v + (I - \Pi_v)\right)$$

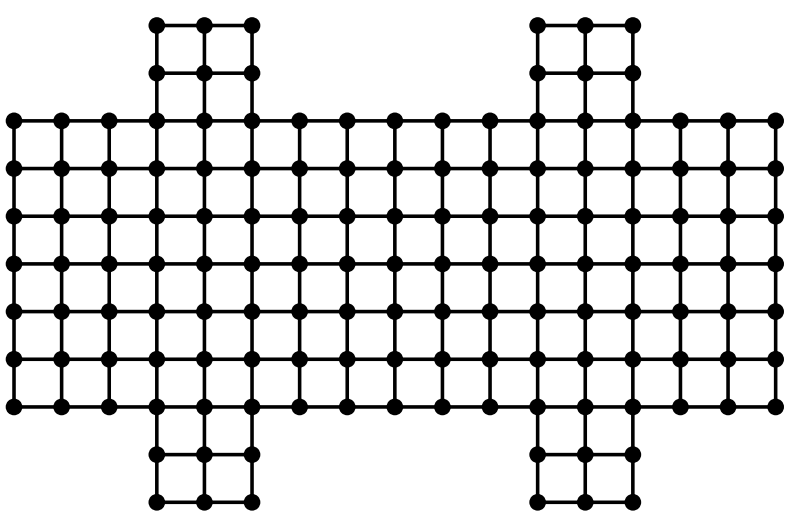
   where Π_v is the spectral projector of H for to the valence band and X and Y are the operators for position.
3. Establish the existence of robustness of edge modes for finite models based on a portion of the infinite system.

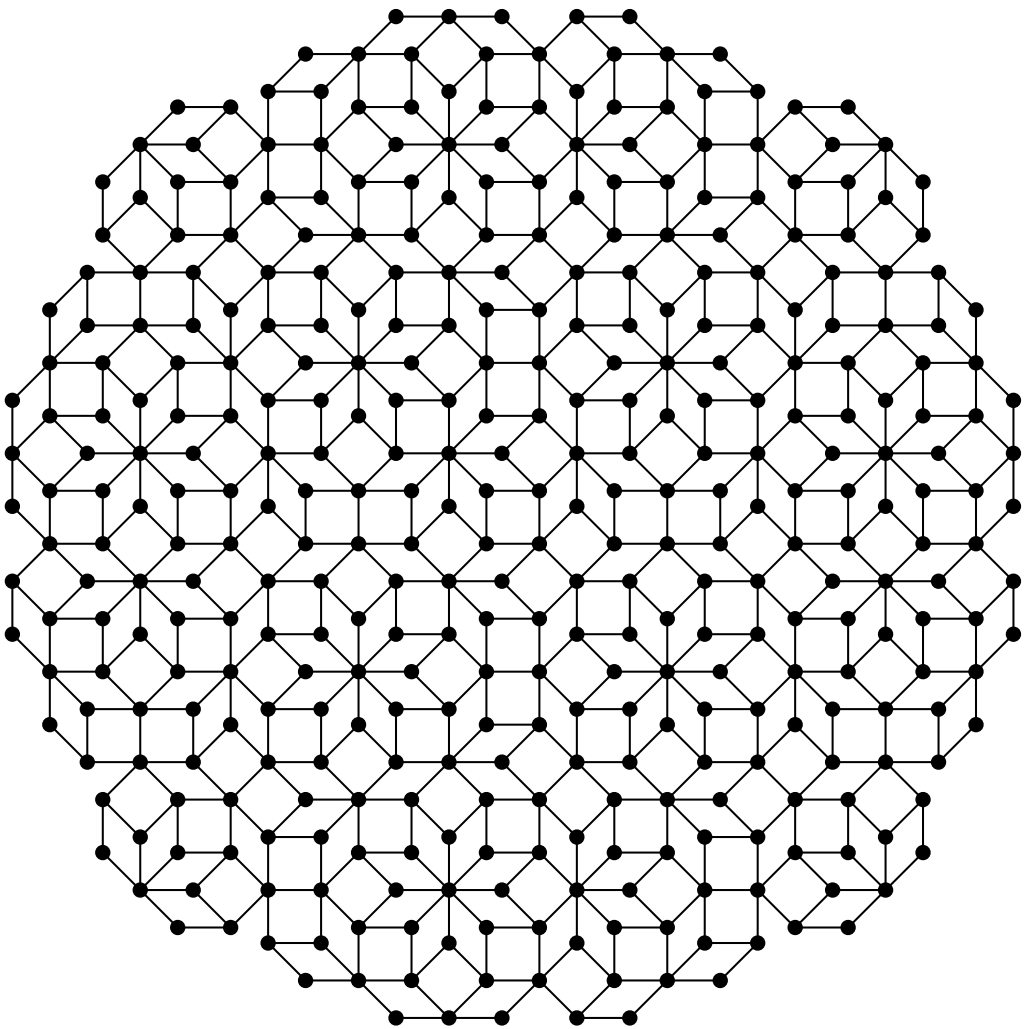
One can make progress on all three using the spectral localizer.

Finite systems with open boundaries

Given a region R and an infinite system (H, X, Y) as above we define a finite system with observables (H_R, X_R, Y_Y) using open boundaries — just compress by the projection determined by R.

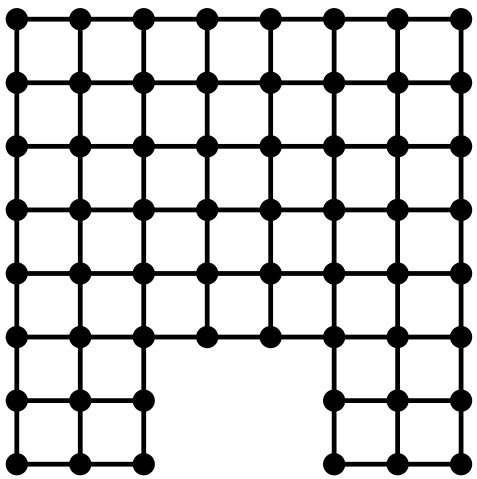






Due to edge modes, $\sigma(H_R)$ will not have the gap that is in $\sigma(H)$.

Edge mode, defective square



Play edge mode around defect video.

The spectral localizer

H and H_R will have related approximate eigenvectors, if we avoid edges.

Definition. Given Hermitian matrices (perhaps operators) X_1, X_2, X_3, define the *spectral localizer*.

$$L_\lambda(X_1, X_2, X_3) = \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix} - \begin{pmatrix} \lambda_3 I & \lambda_1 I - i\lambda_2 I \\ \lambda_1 I + i\lambda_2 I & -\lambda_3 I \end{pmatrix}$$

or

$$L_\lambda(X_1, X_2, X_3) = \sum (X_j - \lambda_j I) \otimes \gamma_j$$

for γ_j the Pauli spin matrices (or a representation of the Clifford relations).

This is a Dirac type operator. Some string theorists call it the Dirac operator.
It comes up in also in the context of almost commuting matrices.
So many name conflicts.....

Choi, Man Duen. "Almost commuting matrices need not be nearly commuting." Proceedings of the American Mathematical Society 102.3 (1988): 529-533..

Berenstein, David, and Eric Dzienkowski. "Matrix embeddings on flat $\mathbb{R}^3$ and the geometry of membranes." Physical Review D 86.8 (2012): 086001..

Approximate eigenvectors via spectral localizer

Suppose X_1, X_2, X_3 are almost commuting. For ψ in $\mathbb{C}^N \oplus \mathbb{C}^N$,

$$\psi = \left(\|\psi_1\|^2 + \|\psi_2\|^2\right)^{-\frac{1}{2}} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

we find for either $r = 1$ or $r = 2$ that

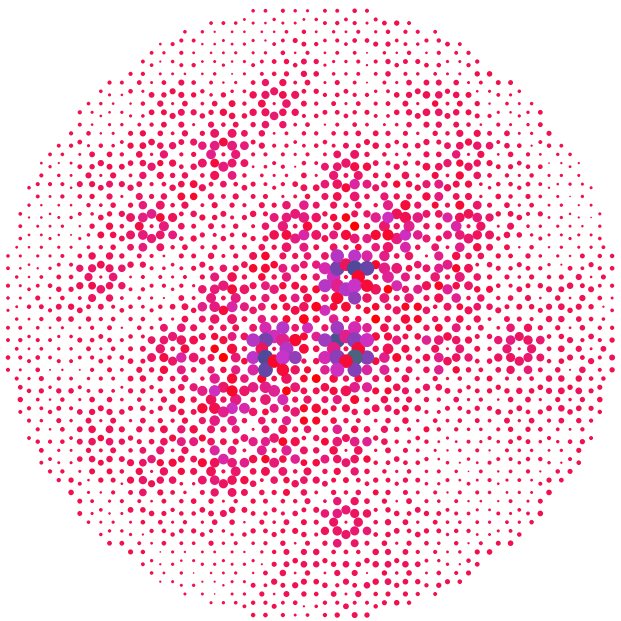
$$L_{\boldsymbol{\lambda}}(X_1, X_2, X_3)\psi = 0 \implies X_j \psi_r \approx \lambda_j \psi_r.$$

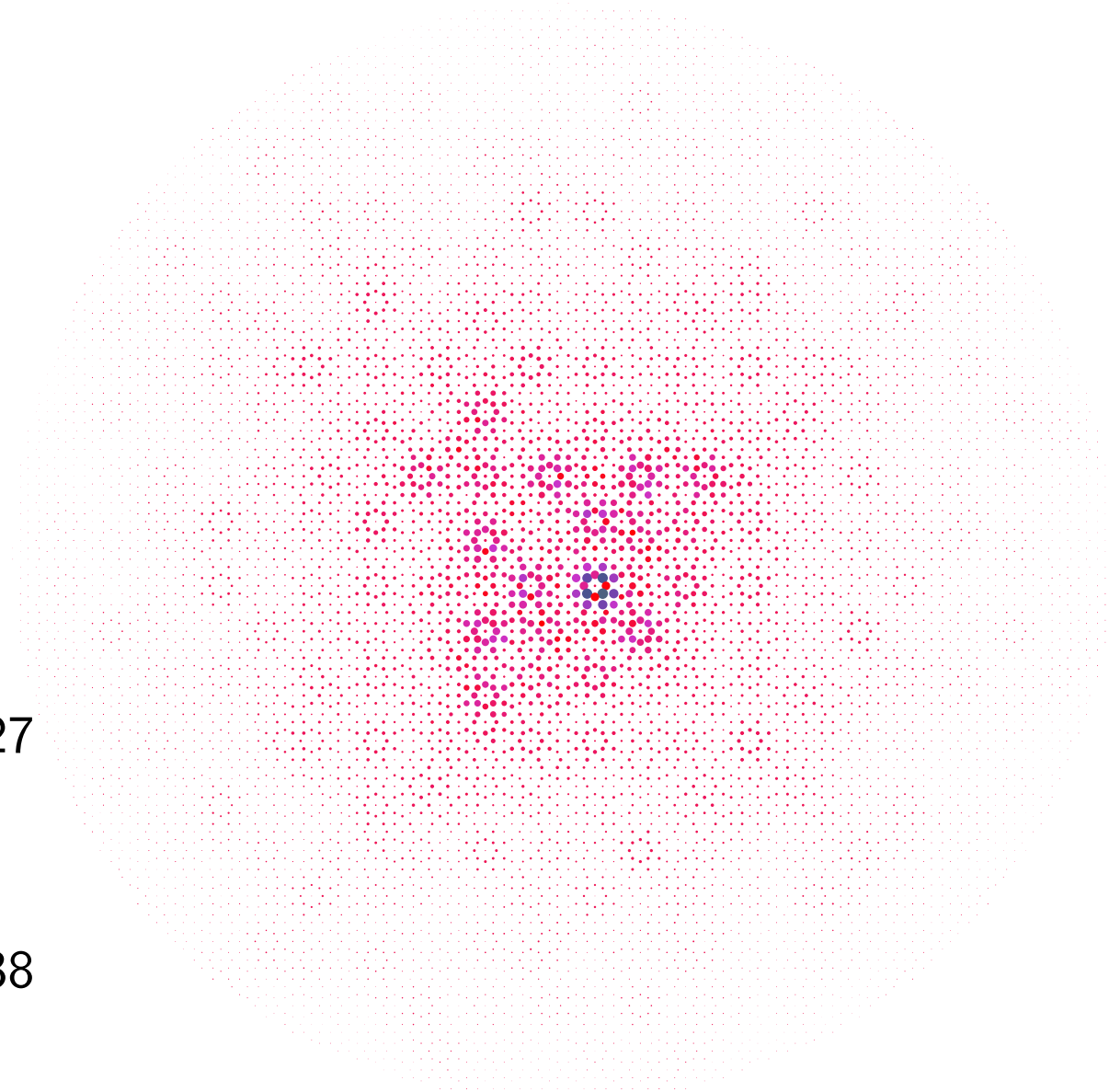
With an appropriate scaling parameter κ and $\boldsymbol{\lambda} = (0, 0, \lambda)$,

$$\begin{aligned} L_{\boldsymbol{\lambda}}(\kappa X_R, \kappa Y_R, H_R)\psi = 0 &\implies H_R \psi_r \approx \lambda \psi_r \text{ and } \psi \text{ is small near edges} \\ &\implies H \begin{bmatrix} \psi_r \\ 0 \end{bmatrix} \approx \lambda \begin{bmatrix} \psi_r \\ 0 \end{bmatrix} \end{aligned}$$

Note: an extra tapering step applied to ψ_r makes this method more effective for reasonable sized r.

Approximate eigenvectors of the QC system





$$\|H\psi_r - 0.605\psi_r\| \approx 0.127$$

versus

$$\|H\psi_r - 0.605\psi_r\| \approx 0.038$$

Computing $\sigma(H)$ for the Quasicrystalline system

Much computer time indicates that the spectrum of H is close to

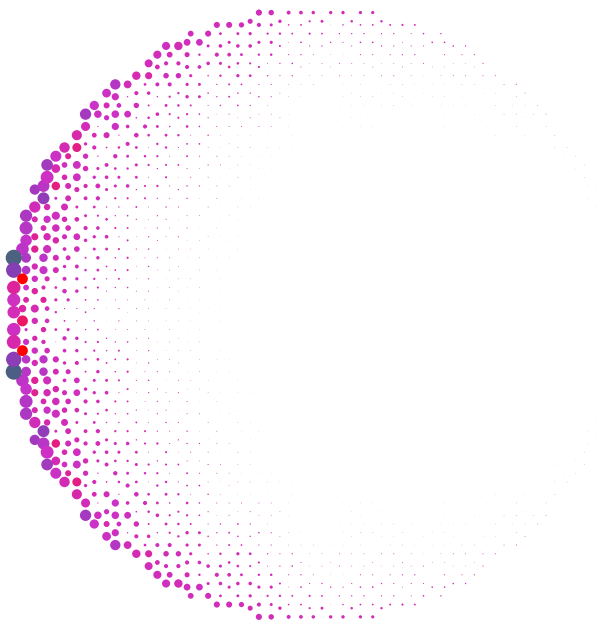
$$[-6.227, -0.604] \cup [0.604, 6.227]$$

with error in Hausdorff distance less that 0.044. (A semi-rigorous result.) It might be faster to compute using another recent method due to Colbrook, Roman, and Hansen.

L. "Bulk spectrum and K-theory for infinite-area topological quasicrystals." Journal of Mathematical Physics 60.8 (2019): 081903.

Colbrook, M. J., Roman, B., and Hansen, A C.. "How to compute spectra with error control." Physical review letters 122.25 (2019): 250201.

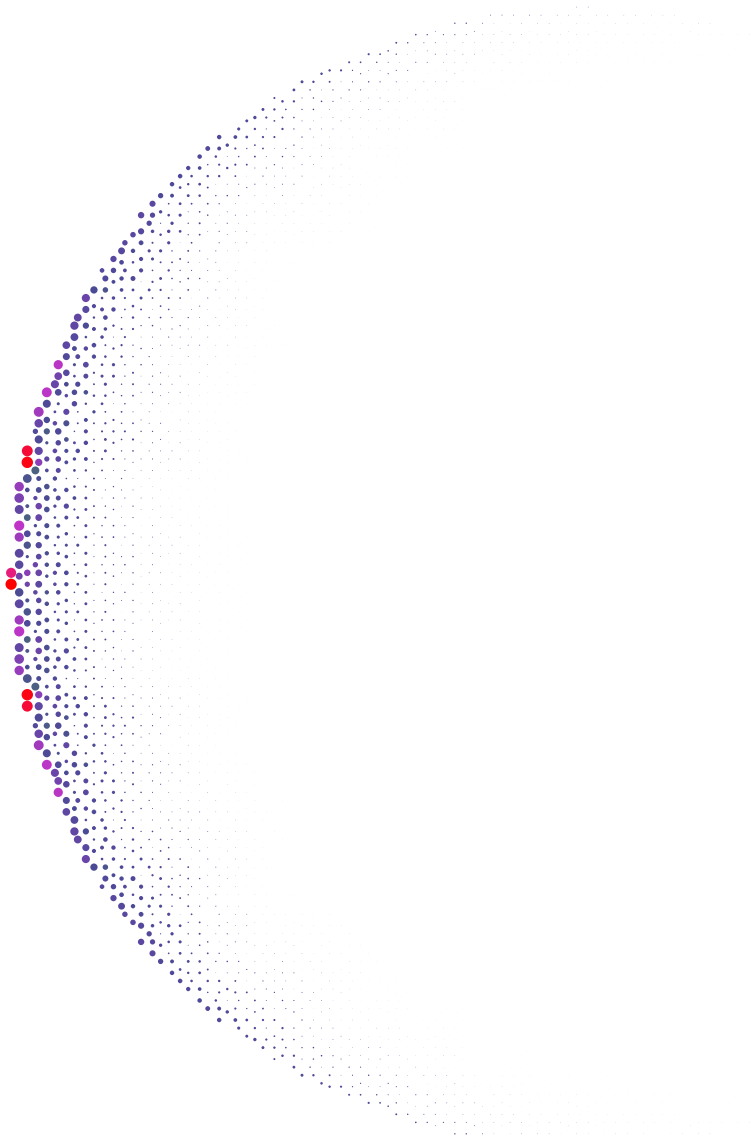
Edge modes for the Quasicrystalline system



$$\|H\psi_r\| \approx 0.024$$

versus

$$\|H\psi_r\| \approx 0.018$$



C*-relations via the localizer

Suppose

$$\|[X,Y]\| \leq \delta, \ \|[X,Z]\| \leq \delta, \ \|[Y,Z]\| \leq \delta$$

and

$$r \leq L_0(X,Y,Z) \leq R$$

for $r < 1 < R$. Since $L_0(X,Y,Z)^2$ equals

$$\begin{bmatrix} X^2+Y^2+Z^2 & 0 \\ 0 & X^2+Y^2+Z^2 \end{bmatrix} + i \begin{bmatrix} [X,Y] & [Y,Z]-i[Z,X] \\ [Y,Z]+i[Z,X] & -[X,Y] \end{bmatrix}$$

these relations imply

$$r^2 - 3\delta \leq X^2+Y^2+Z^2 \leq R^2 + 3\delta$$

and so

$$\left\|X^2+Y^2+Z^2 - I\right\| \leq \epsilon$$

for an appropriate ϵ.

Fuzzy spheres

We can define a nice variation on a fuzzy 2-sphere as representations of

$$\|[X,Y]\| \leq \delta, \ \|[X,Z]\| \leq \delta, \ \|[Y,Z]\| \leq \delta$$

and

$$r \leq L_0(X,Y,Z) \leq R$$

for small delta and r and R close to one.

The interesting K-theory class is represented by $\frac{1}{2}U + \frac{1}{2}I$ for

$$U = L_0(X,Y,Z) = \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix}.$$

The K-theory – finite and infinite models

Computing where $L_{\boldsymbol{\lambda}}\left(\kappa X_R, \kappa Y_R, H_R\right)$ is singular is difficult.

We settle for finding many $\boldsymbol{\lambda}$ where the "localizer gap"

$$\left\| L_{\boldsymbol{\lambda}}\left(\kappa X_R, \kappa Y_R, H_R\right)^{-1} \right\|^{-1}$$

is small.

For $\|\boldsymbol{\lambda}\|$ large $L_{\boldsymbol{\lambda}}\left(\kappa X_R, \kappa Y_R, H_R\right)$ will be invertible with the same number of positive as negative eigenvalues.

For smaller $\boldsymbol{\lambda}$ we can relatively easily also compute the imbalance between positive and negative eigenvalues.

Recall for Hermitian X we have

$$\mathrm{Sig}(X) = \#(\text{pos. e-values}) - \#(\text{neg. e-values}).$$

The K-theory – finite and infinite models

Theorem(L-Schulz-Baldes). If a finite region R contains a disk of radius ρ, and if large enough and κ is a certain range, then

$$\operatorname{ind}\left(\Pi_v\left(\frac{X+iY}{|X+iY|}\right)\Pi_v+(I-\Pi_v)\right)=\frac{1}{2}\operatorname{Sig}\left(L_{\boldsymbol{\lambda}}\left(\kappa X_R,\kappa Y_R,H_R\right)\right)$$

where $\boldsymbol{\lambda}=(x_0,y_0,E_F)$ where (x_0,y_0) is the center of the disk and E_F is the Fermi level.

- To know how large to make ρ one needs to know the size of the spectral gap as well as $\|H\|$, $\|[H,X]\|$, $\|[H,Y]\|$.
- For the QC system, $\rho\geq 10$ gives the right answer every time checked, but we can only prove the answer will be correct if $\rho\geq 1480$.
- For R a disk of radius 1480, much computer time found the finite index to be -1.

"Theorem" The index for the above quasicrystalline system is -1.

L., Schulz-Baldes, "The spectral localizer for even index pairings," Journal of Non-commutative Geometry, to appear (202X).

An unbounded even Fredholm module

For (H, X, Y) either of the models above, or related models based on hopping between points in the plane, consider $\mathcal{A}$ the C^*-algebra generated by H. The odd operator

$$D = \begin{bmatrix} 0 & D_0^* \\ D_0 & 0 \end{bmatrix}, \quad D_0 = X + iY$$

and the representation

$$\pi(f(H)) = \begin{bmatrix} f(H) & 0 \\ 0 & f(H) \end{bmatrix}$$

determine an even Fredholm module.

- The theorem above gives a method to compute the index pairing of this Fredholm module with a spectral projection of H.
- Our proof seems to depend on the fact that D_0 is normal.

Paths of fuzzy spheres

Lemma(L-Schulz-Baldes). Suppose we have a short exact sequence of C^*-algebras

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0.$$

For any unitary u in $\mathcal{B}$, there are x, y, z in $\widetilde{I}$ that form a representation of a fuzzy sphere and so that

$$\frac{1}{2} L_0(x, y, z) + \frac{1}{2} I$$

is a representation of $\partial\left([u]\right)$.

The bulk of the proof of the theorem involves tracking the localizer gap, keeping that nonzero, across many pathes of triples of operators.

Local K-theory

We can examine $L_\lambda\left(X_0, Y_0, H_0\right)$ where for simplicity we let $H_0 = H_R$, $X_0 = \kappa Y_R$ and $Y_0 = \kappa Y_R$. If we compute both

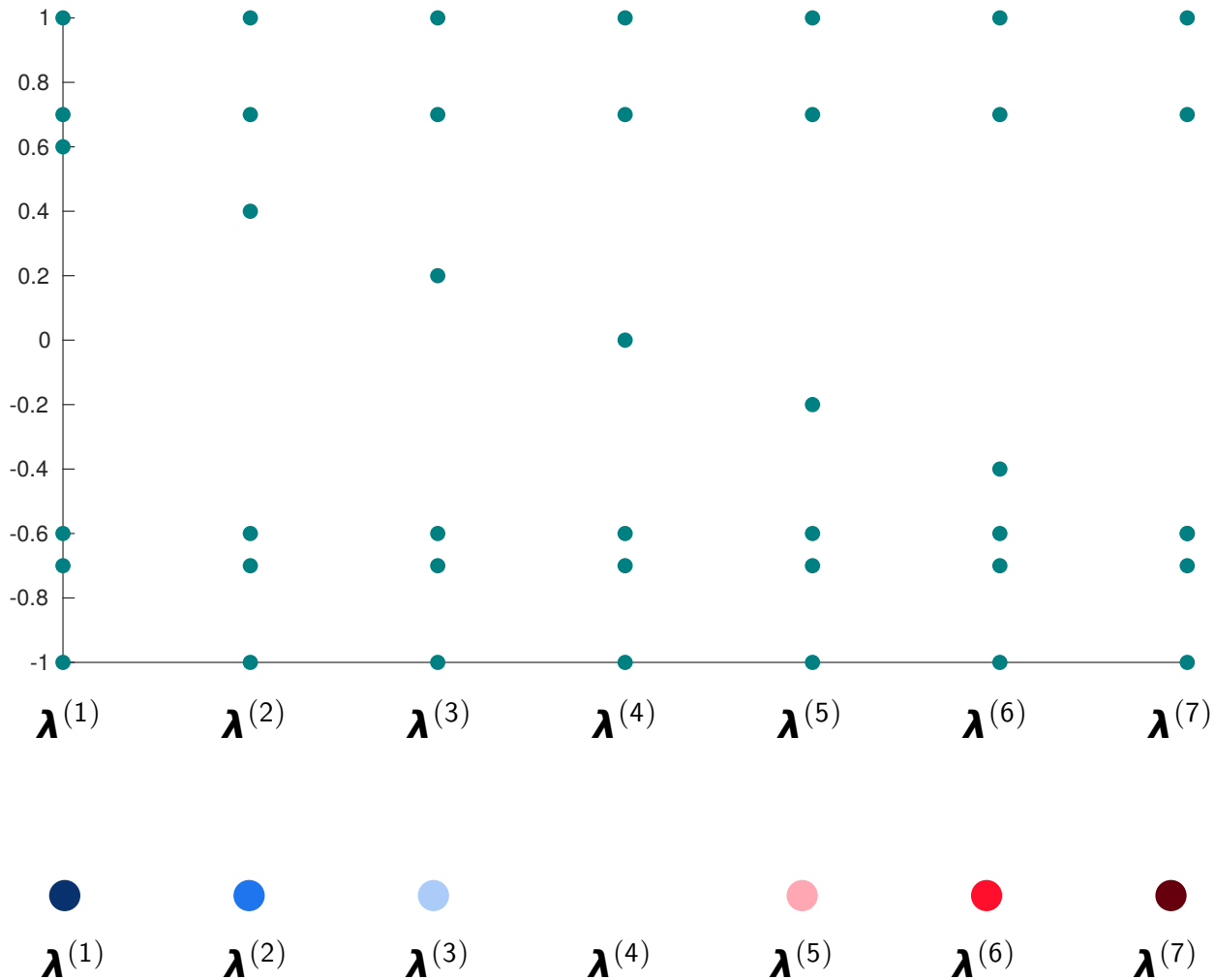
$$\mathrm{ind}_\lambda = \frac{1}{2}\mathrm{Sig}\left(L_\lambda\left(X_0, Y_0, H_0\right)\right)$$

and

$$\mathrm{gap}_\lambda = \left\| L_\lambda\left(X_0, Y_0, H_0\right)^{-1} \right\|^{-1}$$

we can mark the region R according to local K-theory and strength. Typically we keep λ_3 fixed at Fermi level.
We can visualize this as a color-map. Intensity will represent the gap sizer, and index the hue.

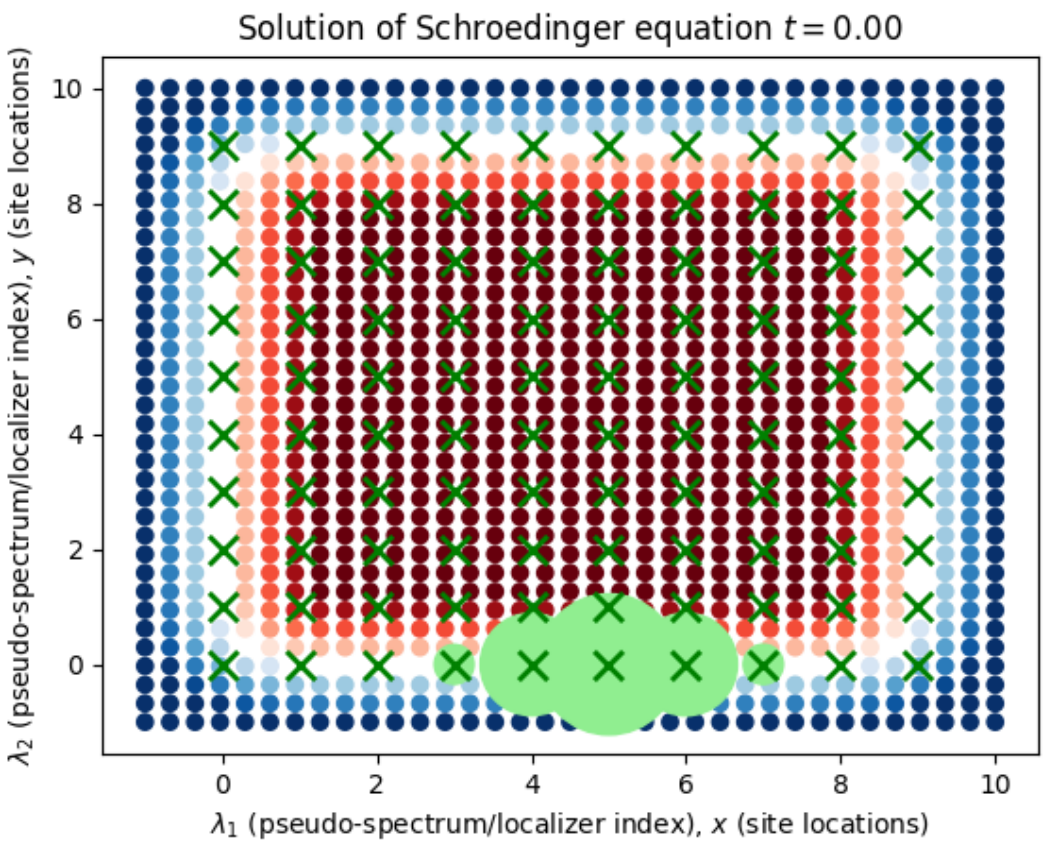
Local K-theory



We say $\boldsymbol{\lambda}^{(4)}$ is in the Clifford spectrum.

We say $\boldsymbol{\lambda}^{(3)}, \boldsymbol{\lambda}^{(4)}, \boldsymbol{\lambda}^{(5)}$ are in the Clifford 0.2-pseudospectrum.

Local K-theory and wave propagation



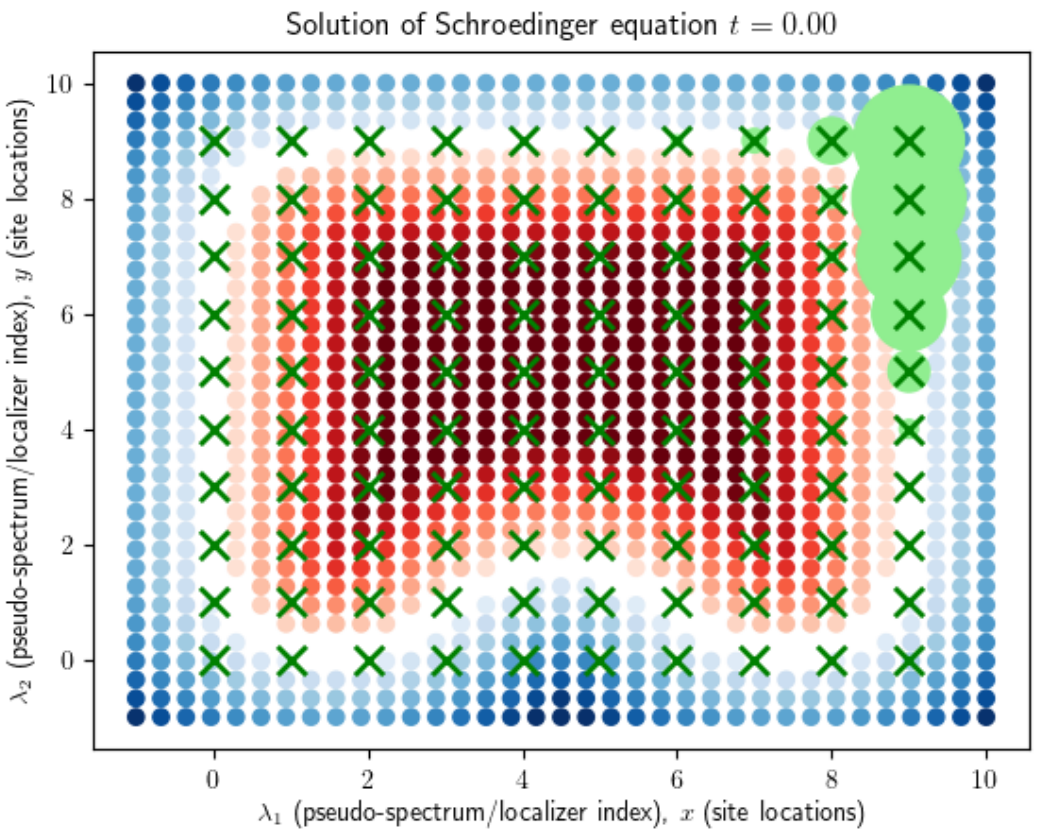
Play edge mode around clean system video.

 indicates a site defining the Hilbert space, or where a site was removed

 area indicates probability that the state is found at this site

 shows gap and index

Local K-theory and wave propagation



Play edge mode around defect video.

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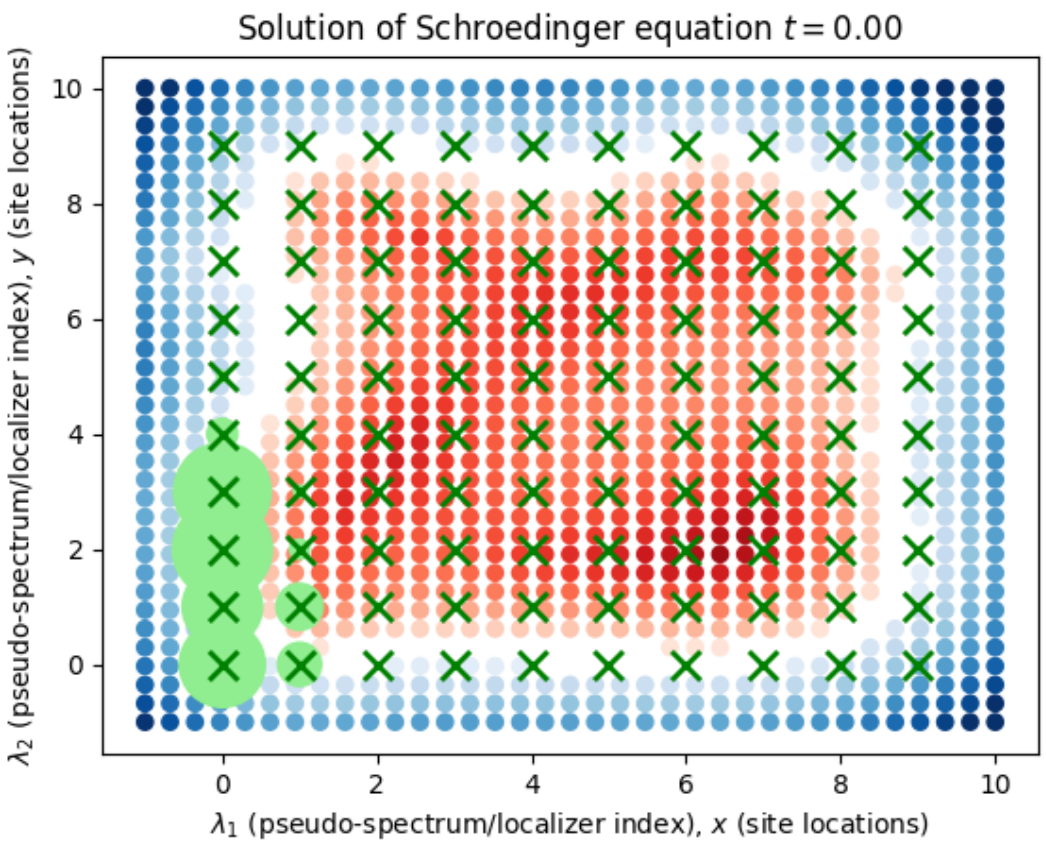
Local K-theory and wave propagation



Play edge mode for square with disorder video.

 indicates a site defining the Hilbert space, or where a site was removed

 area indicates probability that the state is found at this site

 shows gap and index