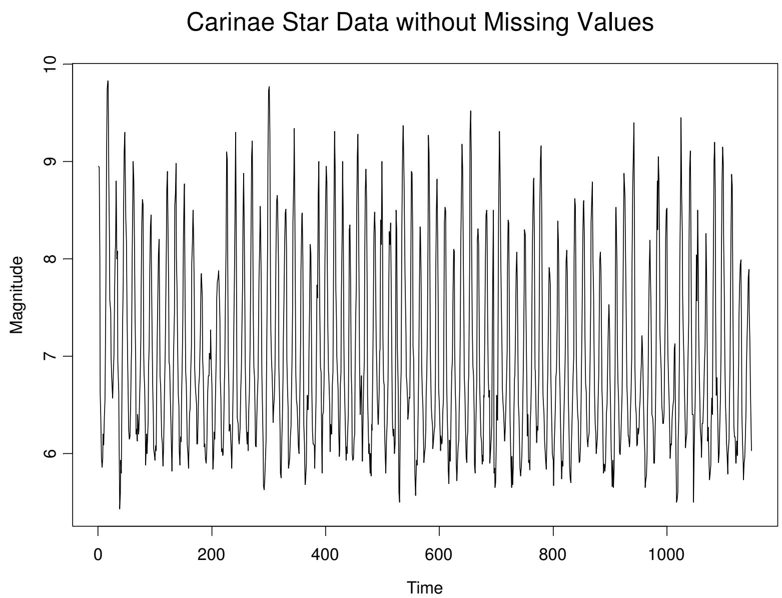
# Time Series Homework #1 Solutions

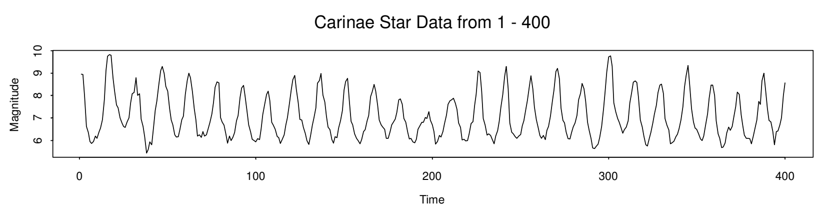
1. a. (4 pts)

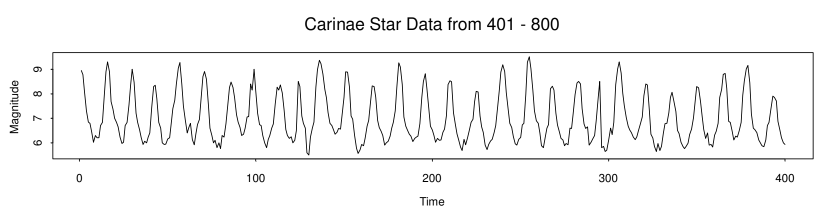
Below is the representation of the Carinae Star Data. There does not appear top be a trend, but it does appear stationary as the mean does not seem to be dependent on time.

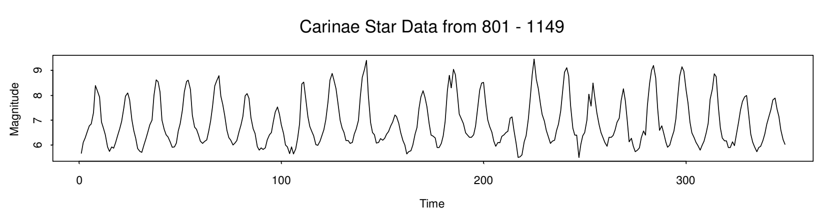



b. (2 pts)

Below is the representation of the Carinae Star Data in sections of 400 (except the last section, which only has 349). By breaking the data into smaller pieces, it is easier to see the stationarity of the process.



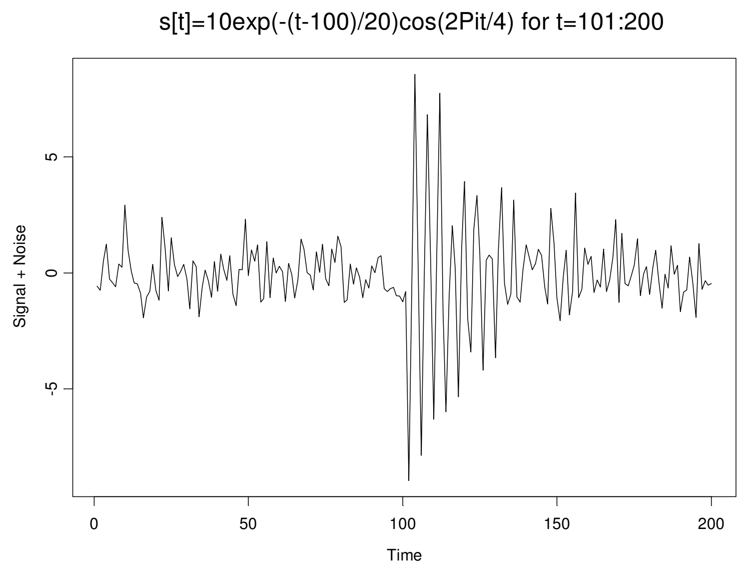



2. Model: $x_t = s_t + w_t$ where $w_t$ is Gaussian noise with $\sigma_w^2 = 1$.

a. (2 pts)

Below is a plot of $x_t = s_t + w_t$ for $t$ = 1, . . . . , 200, where

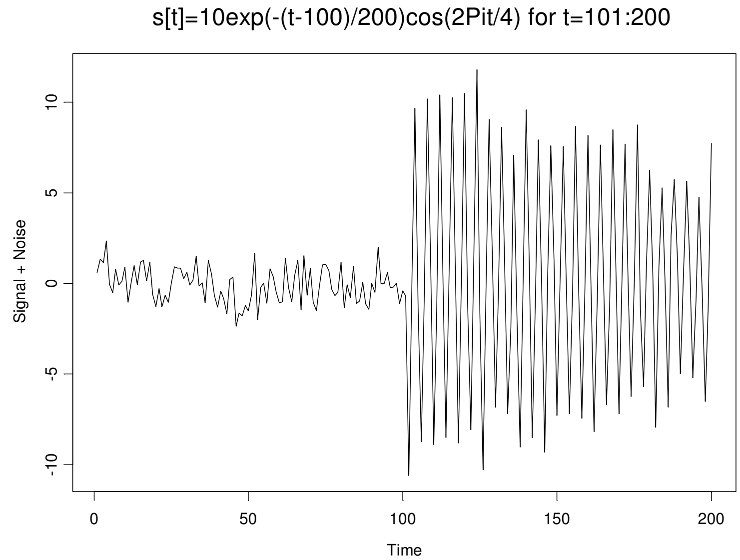
$$s_t = \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\dfrac{(t-100)}{20}\right\}\cos\left(\dfrac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$




b. (2 pts)

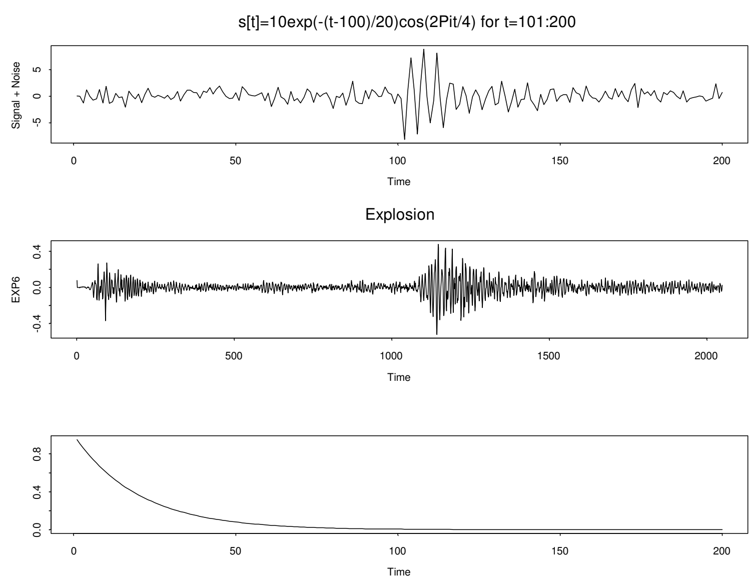
Below is a plot of $x_t = s_t + w_t$ for $t$ = 1, . . . . , 200, where

$$s_t = \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\dfrac{(t-100)}{200}\right\}\cos\left(\dfrac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$

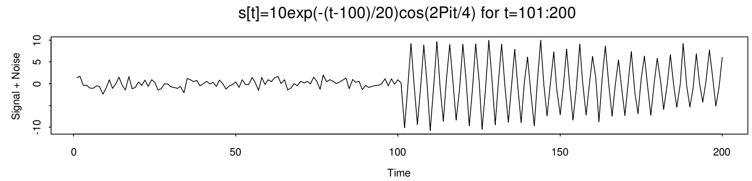


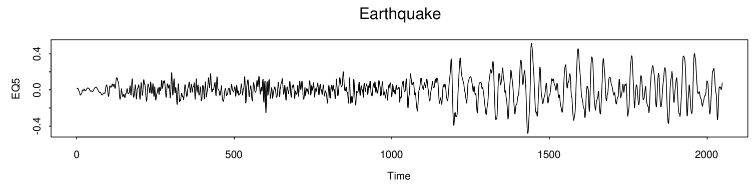

c. (2 pts)

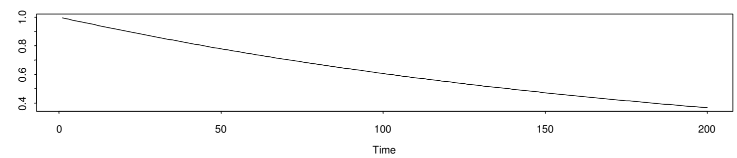
Below is a plot showing the series generated in a with the Earthquake series from Figure 1.7 and the signal modulator $\exp\{-t/20\}$ for $t = 1, \ldots, 100$. The series from (a) is most similar to the Earthquake series in the fact that there is fairly little noise from time 1 to 100 and then there is a jolt to the system after which the noise quickly dies down after that. The signal modulator shows the fact that the noise will die down quickly. It decreases fairly rapidly.




Below is a plot showing the series generated in a with the Explosion series from Figure 1.7 and the signal modulator $\exp\{-t/200\}$ for $t = 1, \ldots, 100$. The series from (a) is most similar to the Explosion series in the fact that there is fairly little noise from time 1 to 100 and then there is a jolt to the system after which the noise slowly decreases. The signal modulator shows the fact that the noise will die down slowly as its decrease is slow.



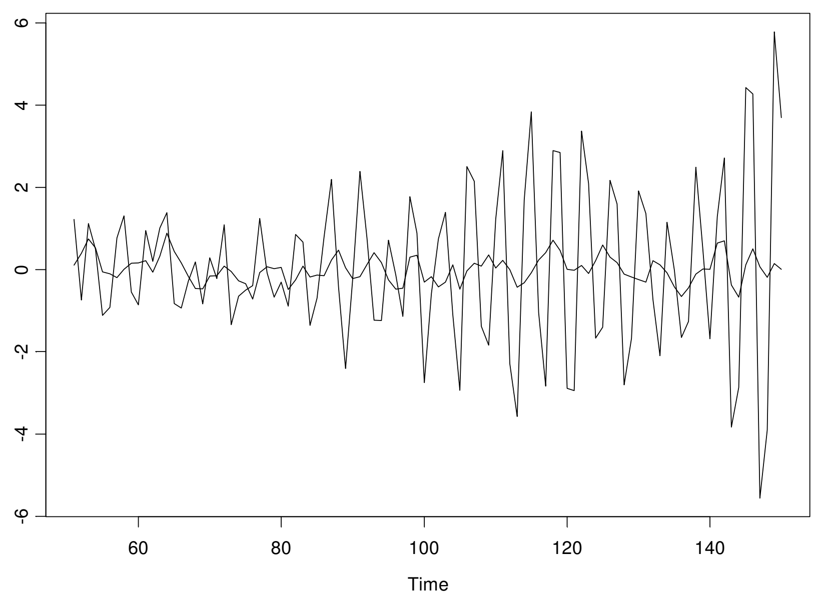




3. a. (3 pts)

Model: $x_t = -0.9x_{t-2} + w_t$ where $w_t$ is Gaussian noise with $\sigma_w = 1$.

Moving average filter: $v_t = (x_t + x_{t-1} + x_{t-2} + x_{t-3})/4$

Below is a plot of the series, generated by the model above, with the filter superimposed on the series. In general, it looks as though the variability in the data increases over time, implying that the series may not be stationary. However, when the filter is added, it decreases the noise (which should happen with averages), but it also looks like the moving average is stationary. The mean did not change after applying the MA filter (which should also happen with averages).

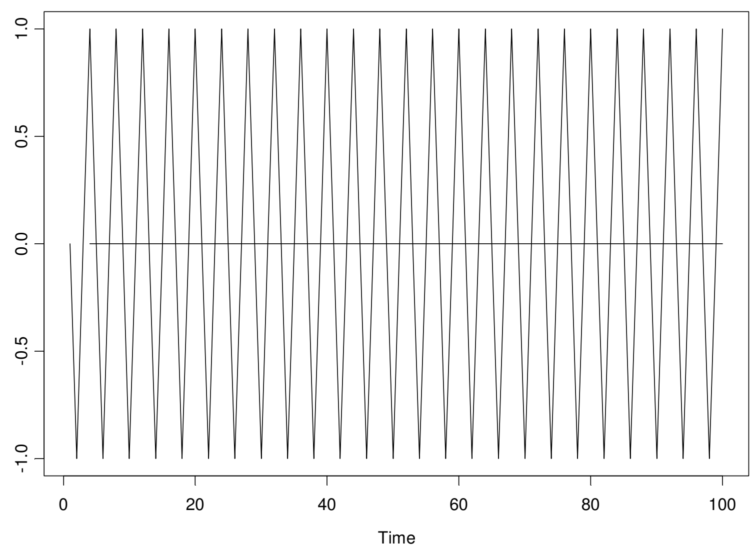


b. (2 pts)

Model: $x_t = \cos\left(\frac{2\pi t}{4}\right)$.

Moving average filter: $v_t = (x_t + x_{t-1} + x_{t-2} + x_{t-3})/4$

Below is a plot of the series, generated by the model above, with the filter superimposed on the series. Since this series has no random component, and it is based on cosine, the plot looks perfectly cyclical. When the filter is added, it does not change the mean or but changes the variability to 0 in the data.

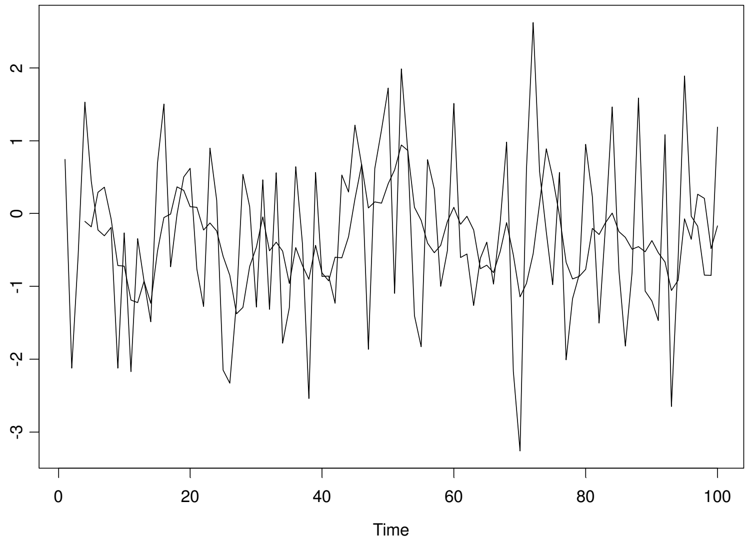



c. (2 pts)

Model: $x_t = \cos\left(\frac{2\pi t}{4}\right) + w_t$ .where $w_t \sim N(0,1)$.

Moving average filter: $v_t = (x_t + x_{t-1} + x_{t-2} + x_{t-3})/4$

Below is a plot of the series, generated by the model above, with the filter superimposed on the series. This series has a random component, and therefore, is not deterministic like the series in (b). The series may be stationary as the mean is constant over time and the variability does not seem to increase or decrease over time. When the filter is added, it does not change the mean, but does decrease the variability in the data slightly.




d. (2 pts)

In series (a), we see an increase in the variability of the data toward the end of the series (ranging from –6 to 6), and the MA filter reduces that variability quite a bit in the second half of the series. In contrast, (c) has smaller variability to start with (ranging from –2 to 2), so the filter does not smooth very much. The MA filter smoothes the data slightly, but not as significantly as when the MA is applied to a series with much larger variability. In (b), the MA filter basically removes the variability as the series is completely deterministic.

4. Model 1: $x_t = s_t + w_t$ where $w_t$ is Gaussian noise with $\sigma_w^2 = 1$ and

$$s_t = \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\frac{(t-100)}{20}\right\}\cos\left(\frac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$

Model 2: $x_t = s_t + w_t$ where $w_t$ is Gaussian noise with $\sigma_w^2 = 1$ and

$$s_t = \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\frac{(t-100)}{200}\right\}\cos\left(\frac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$

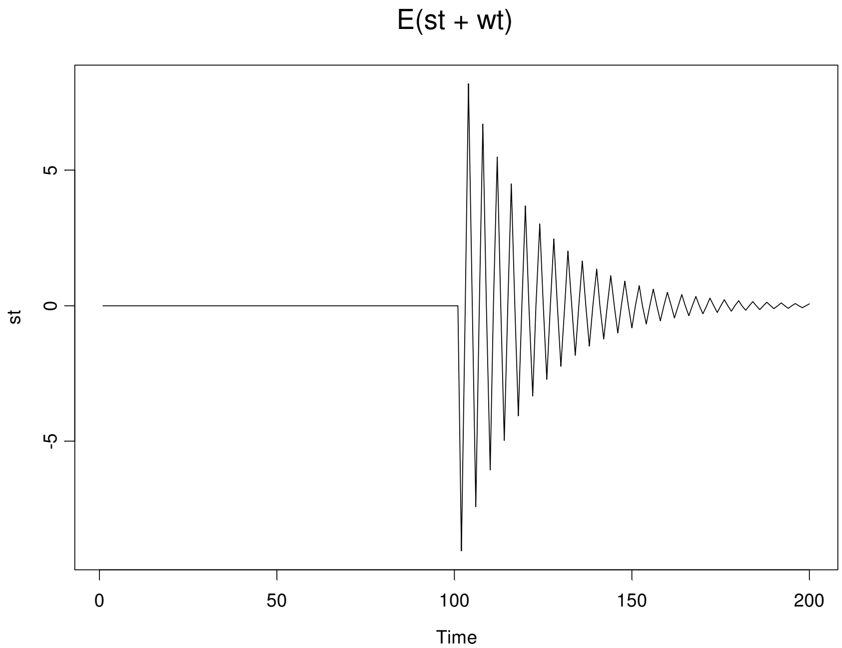
a. (4 pts)

Mean for Model 1:

$$E(x_t) = E(s_t + w_t)$$

$$= \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\frac{(t-100)}{20}\right\}\cos\left(\frac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$

Sketch of mean function for Model 1:

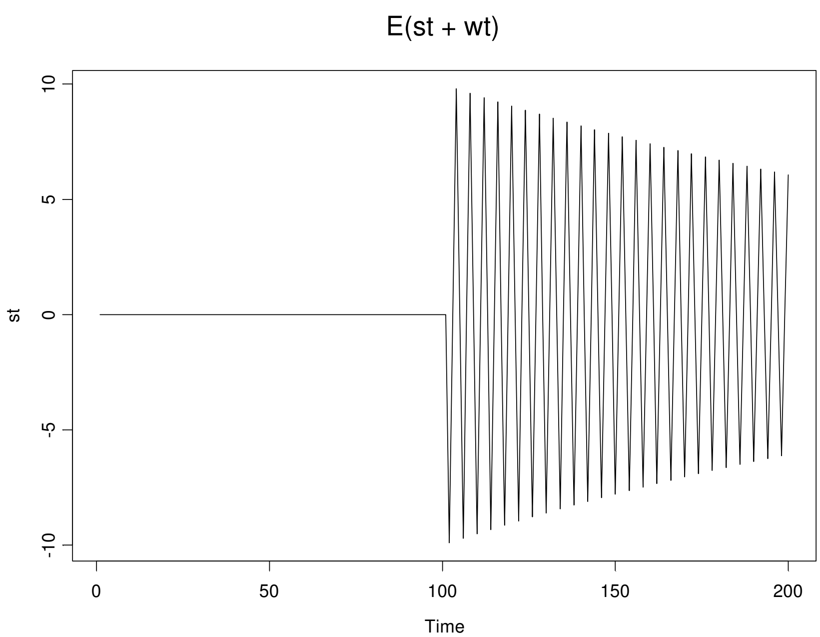



Mean for Model 2:

$$E(x_t) = E(s_t + w_t)$$

$$= \begin{cases} 0, & t = 1,\ldots,100 \\ 10\exp\left\{-\dfrac{(t-100)}{200}\right\}\cos\left(\dfrac{2\pi t}{4}\right), & t = 101,\ldots,200. \end{cases}$$

Sketch of mean function for Model 2:



b. (2 pts)

Autocovariance function for Model 1:

$$\gamma_x(s,t) = E[(x_s - \mu_s)(x_t - \mu_t)]$$
$$= E[(x_s - s_s)(x_t - s_t)]$$
$$= E[(w_s)(w_t)]$$
$$= \begin{cases} 0, \ s \neq t \\ 1, \ s = t \end{cases}$$

This is the same for Model 2.

5. (5 pts)

Model: $x_t = w_{t-1} + 2w_t + w_{t+1}$ where $w_t$ are ind. with zero means and variance $\sigma_w^2$.

$$\gamma_x(s,t) = E[(x_s - \mu_s)(x_t - \mu_t)]$$
$$= E[(x_s)(x_t)]$$
$$= E[w_{\text{s-1}}w_{\text{t-1}} + 2w_{\text{s-1}}w_{\text{t}} + w_{\text{s-1}}w_{\text{t+1}} + 2w_{\text{s}}w_{\text{t-1}} + 4w_{\text{s}}w_{\text{t}} + 2w_{\text{s}}w_{\text{t+1}} + w_{\text{s+1}}w_{\text{t-1}}$$
$$+ 2w_{\text{s+1}}w_{\text{t}} + w_{\text{s+1}}w_{\text{t+1}}]$$
$$= \begin{cases} 6\sigma_w^2, \ s = t \ \ \textit{or equivalently } h = 0 \\ 4\sigma_w^2, \ |s-t| = 1 \ \ \textit{or equivalently } |h| = 1 \\ \sigma_w^2, \ |s-t| = 2 \ \ \textit{or equivalently } |h| = 2 \\ 0, \ |s-t| \geq 3 \ \ \textit{or equivalently } |h| \geq 3 \end{cases}$$

$$\rho_{xy}(h) = \begin{cases} 1, \ s = t \ \ \textit{or equivalently } h = 0 \\ \frac{2}{3}, \ |s-t| = 1 \ \ \textit{or equivalently } |h| = 1 \\ \frac{1}{6}, \ |s-t| = 2 \ \ \textit{or equivalently } |h| = 2 \\ 0, \ |s-t| \geq 3 \ \ \textit{or equivalently } |h| \geq 3 \end{cases}$$

6. Model: $x_t = \delta + x_{t-1} + w_t$, $t = 1, 2, \ldots,$ $x_0 = 0$, $w_t$ is white noise with variance $\sigma_w^2$.

a. (2 pts)

NTS: $x_t = \delta t + \sum_{k=1}^{t} w_k$

$$x_1 = \delta + x_0 + w_1 = \delta + w_1$$
$$x_2 = \delta + x_1 + w_2 = \delta + \delta + w_1 + w_2 = 2\delta + w_1 + w_2$$
$$x_3 = \delta + x_2 + w_3 = \delta + 2\delta + w_1 + w_2 + w_3 = 3\delta + w_1 + w_2 + w_3$$
$$\vdots$$
$$x_t = t\delta + \sum_{k=1}^{t} w_k$$

b. (2 pts)

$$E(x_t) = E\left(t\delta + \sum_{k=1}^{t} w_k\right)$$
$$= t\delta + \sum_{k=1}^{t} E(w_k)$$
$$= t\delta$$

$$\gamma_x(s,t) = E\left[(x_s - \mu_s)(x_t - \mu_t)\right]$$
$$= E\left[\left(\sum_{k=1}^{s} w_k\right)\left(\sum_{k=1}^{t} w_k\right)\right]$$
$$= \begin{cases} t\sigma_w^2, & s > t \\ s\sigma_w^2, & s < t \end{cases}$$
$$= min\{s,t\}\sigma_w^2$$

c. (2 pts)

$$\rho_x(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(0)\gamma_y(0)}}$$
$$= \frac{min\{s,t\}\sigma_w^2}{\sqrt{s\sigma_w^2 t\sigma_w^2}}$$
$$= \frac{min\{s,t\}}{\sqrt{st}}$$

$$\Rightarrow \begin{aligned} \rho_x(t-1,t) &= \frac{t-1}{\sqrt{(t-1)t}} \\ &= \sqrt{\frac{t-1}{t}} \end{aligned}$$

$$\lim_{t\to\infty} \sqrt{\frac{t-1}{t}} = 1$$

This implies that as t becomes very large, the $t^{th}$ value can be perfectly predicted from the $(t-1)^{th}$.

d. (2 pts)

From part (b) we have already seen that the mean is dependent on time. This implies that the process is not stationary.

e. (2 pts)

A possible transformation to make the process is

$$y_t = x_t - x_{t-1}$$
$$= t\delta + \sum_{k=1}^{t} w_k - (t-1)\delta + \sum_{k=1}^{t-1} w_k$$
$$= \delta + w_t$$
$$E(y_t) = E(\delta + w_t)$$
$$= \delta$$

$$\begin{aligned}\gamma_y(s,t) &= E[(y_s-\mu_s)(y_t-\mu_t)]\\ &= E[(w_s)(w_t)]\\ &= \begin{cases}\sigma_w^2, & s=t\\ 0, & s\neq t\end{cases}\end{aligned}$$

7. (4 pts)

Model: $x_t = U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)$ where $U_1$ and $U_2$ are independent with zero means and variance = $\sigma^2$.

$$\begin{aligned}E(x_t) &= E(U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t))\\ &= E(U_1 \sin(2\pi w_0 t)) + E(U_2 \cos(2\pi w_0 t))\\ &= \sin(2\pi w_0 t)E(U_1) + \cos(2\pi w_0 t)E(U_2)\\ &= 0\end{aligned}$$

$$\begin{aligned}\gamma_x(s,t) &= E[(x_s-\mu_s)(x_t-\mu_t)]\\ &= E[(x_s)(x_t)]\\ &= E((U_1 \sin(2\pi w_0 s) + U_2 \cos(2\pi w_0 s))(U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)))\\ &= E(U_1^2 \sin(2\pi w_0 s)\sin(2\pi w_0 t) + U_1U_2 \sin(2\pi w_0 s)\cos(2\pi w_0 t)\\ &\qquad + U_1U_2 \sin(2\pi w_0 t)\cos(2\pi w_0 s) + U_2^2 \cos(2\pi w_0 s)\cos(2\pi w_0 t))\\ &= E(U_1^2 \sin(2\pi w_0 s)\sin(2\pi w_0 t) + U_2^2 \sin(2\pi w_0 s)\sin(2\pi w_0 t))\\ &= \sigma_w^2(\sin(2\pi w_0 s)\sin(2\pi w_0 t) + \sin(2\pi w_0 s)\sin(2\pi w_0 t))\\ &= \sigma_w^2 \cos(2\pi w_0(s-t))\\ &= \begin{cases}\sigma_w^2 \cos(2\pi w_0(s-t)), & s\neq t\\ 0, & s=t\end{cases}\\ &= \begin{cases}\sigma_w^2 \cos(2\pi w_0 h), & h\neq 0\\ 0, & h=0\end{cases}\end{aligned}$$

8. Model: $x_t = \sin(2\pi Ut)$ where $t = 1, 2, \ldots$ and $U \sim$ Uniform (0,1)

a. (2 pts)

$$\begin{aligned}E(x_t) &= E(\sin(2\pi Ut))\\ &= \int_0^1 f(u)\sin(2\pi ut)du\\ &= \int_0^1 1\sin(2\pi ut)du\\ &= -\left.\frac{\cos(2\pi ut)}{2\pi t}\right|_{u=0}^{1}\\ &= 0\end{aligned}$$

$$
\begin{aligned}
\gamma_x(s,t) &= E[(x_s-\mu_s)(x_t-\mu_t)] \\
&= E[(x_s)(x_t)] \\
&= E((sin(2\pi Us))(sin(2\pi Ut))) \\
&= \frac{1}{2}E(cos(2\pi U(s-t)) - cos(2\pi U(s+t))) \\
&= \frac{1}{2}\left(\int_0^1 f(u)cos(2\pi u(s-t)) - cos(2\pi u(s+t))du\right) \\
&= \frac{1}{2}\left(\int_0^1 1\,cos(2\pi u(s-t))du - \int_0^1 1\,cos(2\pi u(s+t))du\right) \\
&= \frac{1}{2}\left(\left.\frac{sin(2\pi u(s-t))}{2\pi(s-t)}\right|_{u=0}^{1} - \left.\frac{sin(2\pi u(s+t))}{2\pi(s+t)}\right|_{u=0}^{1}\right) \\
&= 0
\end{aligned}
$$

b. (2 pts)

Strict Stationarity: $P(x_t \le c_1, x_s \le c_2) = P(x_{t+h} \le c_1, x_{s+h} \le c_2) \ \forall h \in \mathrm{N}, \forall t$

Let $t = 1$, $s = 2$, $h = 1$.

NTS: $P(x_1 \le c_1, x_2 \le c_2) = P(x_2 \le c_1, x_3 \le c_2)$ for strict stationarity

$$
\begin{aligned}
P(x_1 \le c_1, x_2 \le c_2) &= P(sin(2\pi U) \le c_1, sin(4\pi U) \le c_2) \\
&= P\left(U \le \frac{arcsin(c_1)}{2\pi}, U \le \frac{arcsin(c_2)}{4\pi}\right)
\end{aligned}
$$

$$
\begin{aligned}
P(x_2 \le c_1, x_3 \le c_2) &= P(sin(4\pi U) \le c_1, sin(6\pi U) \le c_2) \\
&= P\left(U \le \frac{arcsin(c_1)}{4\pi}, U \le \frac{arcsin(c_2)}{6\pi}\right)
\end{aligned}
$$

$P(x_1 \le c_1, x_2 \le c_2) \ne P(x_2 \le c_1, x_3 \le c_2)$

$\Rightarrow$ The series is not strictly stationary.