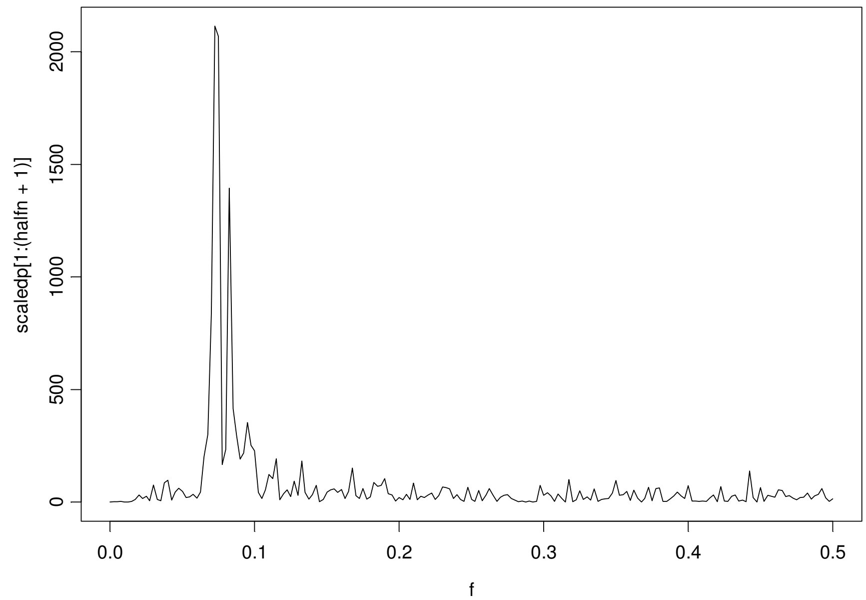
Time Series Homework #3 Solutions

1. (3 pts)

The graph below shows the frequencies versus the values of the periodogram for the 400 "short and central" EEG observations. The major peak is at 0.07 and a minor peak at about 0.08. The period of the EEG data is 1/0.07 ≈ 14.3 time units.



2. a. (2 pts)

$(1-B)X_t = (1-1.5B)\varepsilon_t$

$\Rightarrow \Phi(B) = 1 - B$ for the AR portion of the characteristic polynomial. Then the solution to the equation is $B = 1$ which is not strictly greater than 1, so the process is not stationary.

Similarly, the above equation implies $\Theta(B) = 1 - 1.5B$ for the MA portion of the characteristic polynomial. The solution to this portion of the equation is $B = \frac{2}{3}$ which is not greater than 1, so the process is not invertible.

b. (2 pts)

$(1-0.8B)X_t = (1-0.5B)\varepsilon_t$

$\Rightarrow \Phi(B) = 1 - 0.8B$ for the AR portion of the characteristic polynomial. Then the solution to the equation is $B = 1.25$ which is greater than 1, so the process is stationary.

Similarly, the above equation implies $\Theta(B) = 1 - 0.5B$ for the MA portion of the characteristic polynomial. The solution to this portion of the equation is $B = 2$ which is greater than 1, so the process is invertible.

c. (3 pts)

$(1 - 1.1B + 0.8B^2)X_t = (1 - 1.7B + 0.72B^2)\varepsilon_t$

$\Rightarrow \Phi(B) = 1 - 1.1B + 0.8B^2$ for the AR portion of the characteristic polynomial. Then the solution to the equation is

$$B = \frac{1.1 \pm 1.41i}{1.6}.$$

If you take the norm of *B*, it is always greater than 1, thus, the process is stationary.

Similarly, the above equation implies $\Theta(B) = 1 - 1.7B + 0.72B^2$ for the MA portion of the characteristic polynomial. The solution to this portion of the equation is

$$B = 1.11,\ 1.25,$$

which is greater than 1, so the process is invertible.

3. (4 pts)

From notes, we have that

$$\alpha_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$
$$\alpha_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

Given that the process is stationary, we have $-1 < \alpha_1 \le \alpha_2 < 1$. We need to look at $-1 < \alpha_1$, $\alpha_2 < 1$, and $\alpha_1 \le \alpha_2$.

<u>Case 1:</u>

$-1 < \alpha_1$

$$\Rightarrow -1 < \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

$$\Rightarrow -2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2}$$

$$\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < \phi_1 + 2$$

$$\Rightarrow \phi_1^2 + 4\phi_2 < \phi_1^2 + 4\phi_1 + 4$$

$$\Rightarrow \phi_2 - \phi_1 < 1$$

Case 2:

$$\alpha_2 < 1$$
$$\Rightarrow \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$
$$\Rightarrow \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$$
$$\Rightarrow \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$
$$\Rightarrow \phi_1^2 + 4\phi_2 < \phi_1^2 - 4\phi_1 + 4$$
$$\Rightarrow \phi_2 + \phi_1 < 1$$

Case 3:

$$\alpha_1 \le \alpha_2$$
$$\Rightarrow \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \le \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$
$$\Rightarrow \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} \le \phi_1 + \sqrt{\phi_1^2 + 4\phi_2}$$
$$\Rightarrow -\sqrt{\phi_1^2 + 4\phi_2} \le \sqrt{\phi_1^2 + 4\phi_2}$$

The last inequality is always true.

To find the partial autocorrelations, the autocorrelation must be found. Using

$$\rho_1 = \phi_1\rho_0 + \phi_2\rho_1$$
$$\rho_2 = \phi_1\rho_1 + \phi_2\rho_0$$

gives

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$
$$\rho_2 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 .$$

From here, we use Cramer's Rule to find P_1 and P_2.

$$P_1 = \phi_{11} = \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$P_2 = \phi_{22} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} = \frac{\frac{\phi_1^2}{1-\phi_2} + \phi_2 - \left(\frac{\phi_1}{1-\phi_2}\right)^2}{1 - \left(\frac{\phi_1}{1-\phi_2}\right)^2}$$

$$= \frac{\phi_1^2(1-\phi_2) + \phi_2(1-\phi_2)^2 - \phi_1^2}{(1-\phi_2)^2 - \phi_1^2}$$

$$= \frac{\phi_1^2 - \phi_1^2\phi_2 + \phi_2(1-\phi_2)^2 - \phi_1^2}{(1-\phi_2)^2 - \phi_1^2}$$

$$= \frac{\phi_2\left[(1-\phi_2)^2 - \phi_1^2\right]}{(1-\phi_2)^2 - \phi_1^2}$$

$$= \phi_2$$

For $k \geq 3$,

$$P_k = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_2 \\ \rho_2 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \rho_2 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_1 & 1 \end{vmatrix}} = 0$$

as the numerator is the determinant of a singular matrix.

4. (3 pts)

$E(X_t) = 0$ due to the fact that the process is stationary.

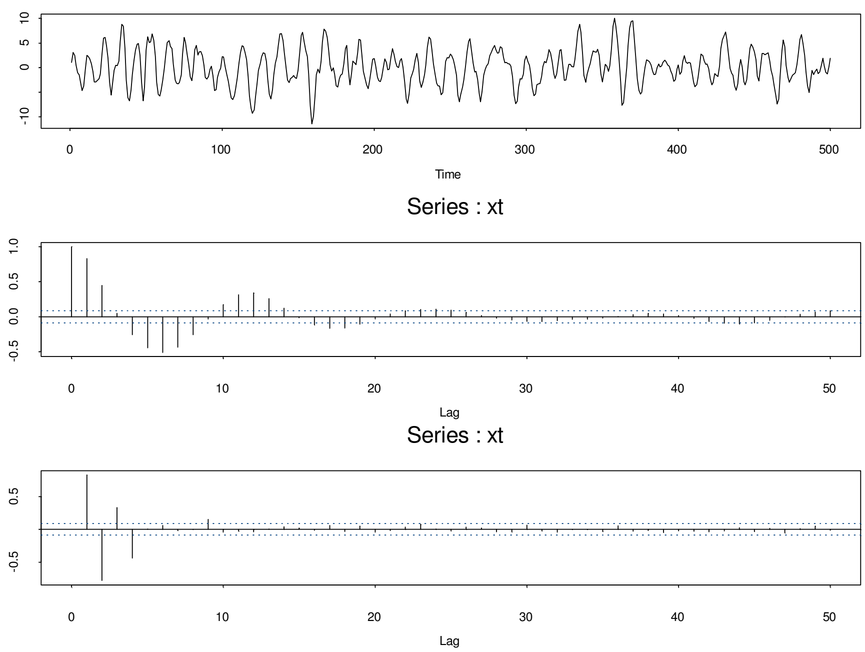
$$\lambda(h) = E(X_t X_{t+h})$$
$$= E((w_t - \theta_1 w_{t-1} - \theta_2 w_{t-2})(w_{t+h} - \theta_1 w_{t+h-1} - \theta_2 w_{t+h-2}))$$
$$= \begin{cases} \sigma^2(1+\theta_1^2+\theta_2^2) & h = 0 \\ \sigma^2\theta_1(\theta_2 - 1) & |h| = 1 \\ -\sigma^2\theta_2 & |h| = 2 \\ 0 & otherwise \end{cases}$$

$$\Rightarrow \rho(h)=\begin{cases} 1 & h=0 \\ \frac{\theta_1(\theta_2-1)}{1+\theta_1^2+\theta_2^2} & |h|=1 \\ -\frac{\theta_2}{1+\theta_1^2+\theta_2^2} & |h|=2 \\ 0 & otherwise \end{cases}$$

5. (4 pts)

The graph below is of the AR(4) process, its ACF and PACF. The data has a damped periodicity.

The coefficients are $\phi_1 = 1.908159$, $\phi_2 = -1.891430$, $\phi_3 = 1.154645$, and $\phi_4 = -0.455625$.



6. a. (2 pts)

Model: $X_t = w_t + C(w_{t-1} + w_{t-2} + \ldots)$ with C constant and $w_i \sim N(0,\sigma^2)$ iid.

$$\begin{aligned} E(X_t) &= E(w_t + C(w_{t-1} + w_{t-2} + \ldots)) \\ &= E(w_t) + CE(w_{t-1} + w_{t-2} + \ldots) \\ &= 0 \end{aligned}$$

$$\begin{aligned}Var(X_t) &= Var(w_t + C(w_{t-1} + w_{t-2} + \ldots)) \\ &= Var(w_t) + C^2 Var(w_{t-1} + w_{t-2} + \ldots) \\ &= \sigma^2 + C^2 \sum_{i=1}^{\infty} \sigma^2 \\ &= \infty\end{aligned}$$

The process is not stationary because the variance is not finite.

b. (3 pts)

Model: $Y_t = X_t - X_{t-1}$

$$\begin{aligned}Y_t &= w_t + C(w_{t-1} + w_{t-2} + \ldots) - w_{t-1} + C(w_{t-2} + w_{t-3} + \ldots) \\ &= w_t - (1-C)w_{t-1} \\ &= w_t \Theta(B) \\ &\Rightarrow \Theta(B) = 1 - (1-C) \\ &\Rightarrow \theta = 1 - C\end{aligned}$$

c. (2 pts)

The modulus of the reciprocal root must be less than 1.

$$\begin{aligned}&|1-C| < 1 \\ &\Rightarrow -1 < 1 - C < 1 \\ &\Rightarrow 0 < C < 2\end{aligned}$$

d. (2 pts)

$$E(Y_t) = 0$$

$$\begin{aligned}\gamma(h) &= E(Y_t Y_{t+h}) \\ &= E((w_t - (1-C)w_{t-1})(w_{t+h} - (1-C)w_{t+h-1})) \\ &= \begin{cases} \sigma^2 \left(1 + (1-C)^2\right) & h = 0 \\ -\sigma^2 (1-C) & |h| = 2 \\ 0 & otherwise \end{cases}\end{aligned}$$

$$\rho(h) = \begin{cases} 1 & h = 0 \\ -\dfrac{(1-C)}{1 + (1-C)^2} & |h| = 2 \\ 0 & otherwise \end{cases}$$

7. (2 pts)

For a white noise process, the confidence band to detect significant autocorrelations is $\frac{2}{\sqrt{n}}$. With $n = 400$, the confidence band is 0.1. The magnitude of the autocorrelations is increasing, which could indicate the process is not a white noise process.

8. a. (2 pt)

False.

A second order stationary process must also have a constant mean that does not depend on time.

b. (2 pts)

False.

The linear model is not the best model to fit all data that has an increasing mean. A dynamic linear model or quadratic model may be a better representation of the trend.

c. (2 pts)

False.

The first difference only needs to be taken if the process is not stationary to begin with.