Banach Algebras: First Examples

A Banach algebra is a Banach space V with a law of multiplication which satisfies

$$|uv| \leq |u||v|$$

for all $u, v \in V$. It is called commutative if the multiplication is commutative.

1. Show that the set of continuous real–valued functions $C([0,1])$ with sup norm is a Banach algebra. Here addition is addition of functions and multiplication is multiplication of functions. Does this Banach algebra have a unit?

2. Suppose V is a Banach space. Let $\mathrm{End}(V)$ denote the ring of continuous linear maps $\phi : V \to V$.

- **a.** Show that $\mathrm{End}(V)$ is a Banach algebra with addition being defined as addition of linear maps and multiplication as *composition* of linear maps.
- **b.** Does $\mathrm{End}(V)$ have a unit?
- **c.** Identify $\mathrm{End}(V)$ when $V = \mathbf{R}^n$ with the Euclidean metric.

3. Let $C_{\text{per}}(\mathbf{R})$ denote the space of continuous real valued functions with a period of 2π, that is for each $f \in C_{\text{per}}(\mathbf{R})$ and each integer n we have $f(x + 2\pi n) = f(x)$ for all x. With addition defined as addition of functions and multiplication defined by convolution:

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)g(x - t)dt,$$

show that this is a Banach algebra. Is there a unit? Is this algebra commutative?

4. Show that a maximal ideal M of a commutative Banach algebra is closed in the metric topology. Show also that the closure of an ideal (relative to the topology on the Banach algebra) is an ideal.

5. Show that the map

$$e^x = 1 + x + \frac{x}{2} + \ldots$$

makes sense in a Banach algebra V (that is, the right hand side converges in V for all $x \in V$). What is the exponentiation map for $V = C([0,1])$ and $V = \mathrm{End}(\mathbf{R}^n)$?

6. A *character* λ of a complex Banach algebra V is a map $\lambda : V \to \mathbf{C}$ which satisfies $\lambda(xy) = \lambda(x)\lambda(y)$ for all $x, y \in V$ and which maps e, the multiplicative identity of V to 1. Show that if λ is a character of V then $\lambda^{-1}(0)$ is a maximal ideal of V. The converse is also true and is a fundamental result due to Gelfand and Mazur. This is a difficult result to show in general but we will consider here the case $V = C([0,1])$ with the sup norm.

a. Let $I \subset C([0,1])$ be an ideal. Let $Z(I) = \{x \in [0,1] : f(x) = 0 \ \forall f \in I\}$. Show that if $Z(I)$ is empty then $I = C([0,1])$.

b. Suppose $I \subset C([0,1])$ is a maximal ideal. Show that there is a point $x \in [0,1]$ so that $I = \{f \in C([0,1]) : f(x) = 0\}$. Conclude that the Gelfand–Mazur theorem holds for $C([0,1])$.

7. Suppose B is a Banach algebra and $f \in B$. Suppose $|1-f| < 1$. Show that f is invertible and that the inverse of f can be expressed as

$$f^{-1} = \sum_{n=0}^{\infty} (1-f)^n.$$

Show also that the map $f \to f^{-1}$ is continuous.

8. Let $C([0,1])$ be the complex valued continuous functions defined on the closed interval $[0,1]$. Let $I \subset C([0,1])$ be the set of functions with a multiplicative inverse in $C([0,1])$.

a. Show that I is an ideal, corresponding to those continuous functions which do not take the value 0.

b. Let $i : [0,1] \to \mathbf{C}$ be the map with $i(x) = 1$ for all x. Let $I_0 \subset I$ be the collection of functions $f : [0,1] \to \mathbf{C}^*$ which can be continuously deformed into i: that is, there exists a continuous map, called a *homotopy*,

$$F : [0,1] \times [0,1] \to \mathbf{C}^*$$

such that $F(0,t) = i(t)$ and $F(1,t) = f(t)$. The quotient I/I_0 is the *fundamental group* of $\mathbf{C}^*$, that is homotopy equivalence classes of maps $f : [0,1] \to \mathbf{C}^*$.

9. Suppose X is a compact Hausdorff space and $\phi : X \to X$ a homeomorphism. Show that ϕ induces an automorphism of the Banach algebra $C(X)$ by sending f to $f \circ \phi$. Conversely, every automorphism of $C(X)$ arises in this way.