The p–adic topologies on $\mathbf{Q}$

1. Suppose p is a prime number and $\frac{a}{b} \in \mathbf{Q}$. Write

$$\frac{a}{b} = p^r \frac{c}{d}$$

where neither c nor d is divisible by p. Define the p–adic absolute value of $\frac{a}{b}$ by

$$\left|\frac{a}{b}\right|_p = p^{-r},$$

with the additional provision that $|0|_p = 0$.

a. Show that $|\cdot|_p$ defines a metric on $\mathbf{Q}$, that is show that $d_p(x, y) = |x - y|_p$ is a distance function.

b. What if p is replaced by a composite number? Would d_p still be a distance function?

c. What if p^{-r} were replaced by p^r in the definition of $|\cdot|_p$: would this still give a distance function?

2. Show that $|\cdot|_p$ satisfies the stronger triangle inequality:

$$|x + y|_p \le \max\{|x|_p, |y|_p\}.$$

Show moreover that equality holds whenever $|x|_p \ne |y|_p$.

3. The *p–adic numbers* $\mathbf{Q}_p$ are the completion of $\mathbf{Q}$ with respect to $|\cdot|_p$, just like the real numbers are the completion of $\mathbf{Q}$ relative to the usual absolute value.

a. Show that $|\cdot|_p$ is discrete on $\mathbf{Q}$, that is it only takes values within a discrete set.

b. Show that $|\cdot|_p$ is discrete on $\mathbf{Q}_p$.

c. Suppose $z \in \mathbf{Q}_p$ and that $|z|_p = p^{-k}$. Show that there exists a unique integer x_k with $1 \le x_k \le p - 1$ so that

$$\left|z - x_k p^k\right| < p^k.$$

d. Conclude that z admits an expansion

$$z = \sum_{i=-N}^{\infty} a_i p^i$$

where $1 \le a_i \le p - 1$ and $|z|_p = p^N$. How does this compare to the decimal expansion of a real number? What is $|z|_p$ in terms of the expansion?

e. Show that in $\mathbf{Q}_p$ we have

$$\frac{1}{1-p} = \sum_{i=0}^{\infty} p^i$$

4. Show that there is a number $x \in \mathbf{Q}_7$ which satisfies $x^2 = 2$.

5. What does the topology on $\mathbf{Q}_p$ look like? Show, in particular, that every open set in $\mathbf{Q}_p$ is also closed.

6. The *p–adic integers* are defined by

$$\mathbf{Z}_p = \{z \in \mathbf{Q}_p : |z|_p \leq 1\}.$$

Show that $\mathbf{Z}_p$ is a ring whose units are those elements satisfying $|z|_p = 1$.