Compact Operators and Spectral Theory

1. A linear map $L : V \to W$ of Banach spaces is called *compact* if for each bounded set $S \subset V$ the closure of the image $L(S)$ is compact.

a. Show that the identity map $I : V \to V$ is compact if and only if V is a finite dimensional vector space.

b. Suppose $K(x, y)$ is a continuous function on $[0, 1] \times [0, 1]$ Consider the operator $S : C([0, 1]) \to C([0, 1])$ given by

$$Sf(y) = \int_a^b K(x, y)f(x)dx.$$

Show that S is a compact operator.

c. For the Hilbert space $l^2(\mathbf{R})$ consider the shift operator defined by

$$L((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots).$$

Is L compact?

d. Consider the map $L : C([0, 1]) \to C([0, 1])$ given by

$$Lf(x) = xf(x).$$

Show that L is not compact.

2. A linear map $L : V \to W$ of Banach spaces is called *Fredholm* if

i. the kernel lf L is finite dimensional

ii. the cokernel of L is finite dimensional

a. Show that the Hilbert shift operator of problem **1c** is a Fredholm operator.

b. Is a compact operator ever Fredholm?

c. If u is a compact operator and I is the identity then $I - u$ is Fredholm.

3. Suppose $L : V \to V$ is a linear operator on a Banach space V. Here we will show that, unlike in the finite dimensional vector space theory, L can fail to have an inverse even if 0 is not an eigenvalue of L. Let $V = C([0, 1])$.

a. Let $L : V \to V$ be the linear map from **1d** above and let I denote the identity. Show that $L - \alpha I$ is not an invertible linear map for any $\alpha \in \mathbf{R}$.

b. Show that α is not an eigenvalue of L for any $\alpha \in \mathbf{R}$.

4. Here is a further example along the lines of Problem **3** above. Consider the shift operator L for the Hilbert space $l^2(\mathbf{R})$ from Problem **1c** above.

a. Show that every value $-1 \leq \lambda \leq 1$ is in the spectrum of L, that is $L - \lambda I$ is *not* invertible. Show also that $L - \lambda I$ is injective for each value of λ.

b. Show that if λ is an *isolated* point in the spectrum of a linear operator $L : V \to V$ then λ must be an eigenvalue of L.

5. Here is one more example exhibiting the type of behavior that occurs for eigenvalues and spectra of operators on banach spaces. Let $T : l^2(\mathbf{R}) \to l^2(\mathbf{R})$ be defined by

$$T((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2/2, x_3/3, \ldots)$$

a. Show that T, unlike the shift operator, is compact.

b. Show that, like the shift operator, T has no eigenvalues.

c. Suppose one considers instead the truncated operator $T_n : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$T_n((x_1, \ldots, x_n)) = (0, x_1, \ldots, x_{n-1}/(n-1)).$$

Does T_n have eigenvalues? What is the difference bewteen T_n and T?

6. For a Fredholm operator $L : V \to W$ the *index* of L is defined as

$$\mathrm{ind}(L) = \dim(\ker(L)) - \dim(\mathrm{coker}(L)).$$

a. Show that if $\dim(V)$ and $\dim(W)$ are finite the index of any linear map is $\dim(W) - \dim(V)$.

b. Show that if L is Fredholm and u compact then $L + tu$ is Fredholm.

c. Show that $\mathrm{ind}(L + tu)$ is a constant function, whenever L is Fredholm and u is compact.

7. Suppose $u : V \to V$ is a compact operator and $\alpha \in \mathbf{R}$. Then either $u - \alpha I$ is invertible or α is an eigenvalue of u.

a. Suppose that α is not an eigenvalue of u. Using Problem 6 the function $t \to \mathrm{ind}(\alpha I - tu)$ is continuous. Conclude, plugging in $t = 0$ and $t = 1$, that $\mathrm{Image}(\alpha I - u) = V$.

b. By the Open Mapping Theorem, $(\alpha I - u)^{-1}$ is continuous and so $\alpha I - u$ is invertible.

c. If α is an eigenvalue of u then of course $\alpha I - u$ is not invertible.

8. (spectral theorem for a compact operator) Suppose V is a Banach space, $u : V \to V$ a compact operator.

a. Suppose α is an eigenvalue of u. Show that the α–eigenspace is finite dimensional (remember that $u - \alpha I$ is Fredholm).

b. For all n sufficiently large, $\ker(u - \alpha A)^n$ is a fixed subpace of V (if not, then for infinitely many n, find x_n, of norm 1, which is contained in $\ker(u - \alpha A)^n$ but not in $\ker(u - \alpha A)^{n-1}$ and which is not close to $\ker(u - \alpha A)^{n-1}$).

c. Show that for any positive real number c there are only finitely many eigenvalues λ for u which are greater than c.

Selecting Jordan basis for each λ–eigenspace gives a partial basis for V with respect to which u can be represented as a possibly infinite upper triangular matrix with the eigenvalues of u on the diagonal and some 1's immediately above the diagonal.