MATH 1512 — F I N A L E X A M R E V I E W

Topics and sample problems

A. Definitions

Continuity.

1. State the definition of continuity.
2. Let

$$F(r) = \begin{cases} GMr/R^3 & if \quad 0 \le r < R \\ GM/r^2 & if \quad r \ge R \end{cases}$$

   where G, M are positive constants. Use the definition to show that F is continuous at $r = R$.

The derivative.

3. State the definition of the derivative. Sketch a graph that illustrates
4. Use the definition to find the derivative of

   (a) $f(x) = \sqrt{x+2}$ (b) $f(x) = 1/x^2$ (c) $f(x) = x^3$

5. True or False: If f is continuous at $x = a$, then it is differentiable at a. If false, give counterexample.
6. Recognize when a limit is a derivative.

The definite integral.

7. State the definition as a limit of a sum.
8. *Rewriting the limit of a sum as an integral.*

   Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\Delta x$, where $\{x_i\}$ is a partition of $[0,1]$. (Hint: write as an integral.)

B. Using the Rules

Limits. Find limits as $x \to c$, onesided limits, infinite limits (vertical asymptotes), limits at infinity. Including functions whose graphs have holes, functions defined piecewise.

9. Find the following limits. Show all work.

   (a) $\lim_{x\to\infty} \frac{x^3+3x+1}{3x^3+5}$

   (b) $\lim_{x\to 1} \frac{x^3-3x^2+3x-1}{x-1}$

Derivatives. Use power rule, product rule, quotient rule, chain rule, to find derivatives. Implicit differentiation. Include derivatives of x^p, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$

10. Chapter 2 Review: odd 13-44
11. Find the derivatives of the following functions. Simplify your answer.

    (a) $g(s) = \sqrt{s} + \dfrac{1}{\sqrt[3]{s^4}}$ (b) $f(x) = \sin(\pi x)\displaystyle\int_{x^2}^{2} \sqrt{t^2+1}\,dt$

    (c) $g(s) = \displaystyle\int_{2s}^{3s} \frac{u}{u^2+1}\,du$ (d) $P(R) = \dfrac{E^2R}{(R+r)^2}$, where E, r are constants

12. Find dy/dx. Simplify your answer.

    (a) $y = 2x\sqrt{x^2+1}$ (b) $y = \dfrac{3x-2}{\sqrt{2x+1}}$

    (c) $y = \tan^2 x$ (d) $\sin(xy) = x^2 - y^2$

13. If f and g are differentiable, find

(a) $\frac{d}{dx}\left[\sqrt{f(x)}\right]$ (b) $\frac{d}{dx}\left[f(\sqrt{x})\right]$ (c) $\frac{d}{dx}\left[\sqrt{x}f(x)\right]$

(d) $\frac{d}{dx}\left[f(g(x))\right]$ (e) $\frac{d}{dx}\left[f(f(x))\right]$ (f) $\frac{d}{dx}\left[\sqrt{\frac{f(x)}{g(x)}}\right]$

14. Is the function in # 2 above differentiable at $r = R$? Explain. ("it has a corner" is not good enough). Sketch a graph of the function.

Indefinite integrals. Find antiderivatives (when possible) either directly or using substitution. Always check your answer!

15. §3.9: 53,55 (graphing antiderivative, including of piecewise)

Definite integrals. (Fundamental Theorem of Calculus - Part II). Find definite integrals. When using substitution, change the bounds of integration.

16. §4.5: odd 9-32.
17. Chapter 4 Review: odd 11-32

The function $g(x) = \int_a^x f(t)\,dt$. (Fundamental Theorem of Calculus - Part I)

18. (a) What is the difference between $\int f(x)\,dx$, $\int_a^x f(t)\,dt$, and $\int_a^b f(t)\,dt$.

    (b) What is the difference between $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$ and $\frac{d}{dx}\left[\int_b^x f(t)\,dt\right]$, where $a \neq b$, if any?

19. Find a function f and a value of the constant a such that $2\int_a^x f(t)\,dt = 2\sin x - 1$.

20. Given the equation $x\sin(\pi x) = \int_0^{x^2} f(t)\,dt$, find $f(4)$.

21. Evaluate

    (a) $\int_0^1 \frac{d}{dx}\left[\frac{1}{1+x^2}\right]dx$

    (b) $\frac{d}{dx}\left[\int_0^1 \frac{1}{1+x^2}\,dx\right]$

    (c) $\frac{d}{dx}\left[\int_0^x \frac{1}{1+t^2}\,dt\right]$

C. Applications of the derivative

Geometric interpretation of the derivative,

22. Chapter 2 Review: 50 (tangent/normal lines, implicit)

Graphing.

23. Sketch the graphs of the following functions. Clearly label axes, intercepts, local max/min.

    (a) $f(x) = \tan(\pi x)$ (b) $g(s) = 2\sin(3s)$ (c) $h(t) = 1 - \cos t$ (d) $h(t) = \sec t$

    (e) $f(x) = x^3 - x$ (f) $g(x) = x^2 + 4x + 2$ (g) $h(x) = x(x+1)^2(2-x)^3$

24. Let $f(x) = \frac{4-x}{3+x}$.

    (a) Find the intercepts of f and all its asymptotes. Also find the limiting behaviour of f near its vertical asymptotes ($\lim_{x\to a^{\pm}} f(x)$).

    (b) Use the above information to sketch a guess for the graph of f.

(c) Confirm details of your guess by finding the intervals where the function is increasing/decreasing and concave up/down.

(d) Use algebra (long division in this case) to rewrite the function as a polynomial plus a proper quotient. Now sketch a graph of f using translations/dilations of $y = \frac{1}{x}$. Which way of obtaining the sketch do you prefer?

25. Find all roots (if possible), asymptotes, intervals of increase/decrease, intervals of concave up/down to sketch a clear graph of the given function on their domain. Use symmetry when possible.
    (a) $f(x) = 10 + 27x - x^3$
    (b) $f(x) = x^2 - x^4$
    (c) $f(x) = x + \dfrac{2}{x}$
    (d) $f(x) = \dfrac{1}{1+x^2}$
    (e) $f(x) = \dfrac{x}{1+x^2}$
26. The reaction rate V of a common enzyme reaction is given in terms of substrate level S by

$$V = \frac{V_* S}{K + S}, \quad S \geq 0$$

where V_* and K are positive constants.

(f) Show that V is a strictly increasing function of S. Explain why it follows that V has no absolute maximum value.
(g) What is $\lim_{S\to\infty} V$?
(h) Determine the concavity of the graph of V (where is it concave up? where concave down?).
(i) Sketch a graph of V as a function of S.

Rate-of-change.

27. §2.7: # 28 (vibrating string)
28. If $Q = Q(p)$ is the quantity (in pounds) of a ground coffee that is sold by a coffee compay at a price of p dollars per pound,
    (i) What is the meaning of $Q'(8)$? What are its units?
    (ii) Is $Q'(8)$ positive or negative? Explain.
29. If the units of x are feet and the units for $a(x)$ are pounds per foot, what are the units for da/dx? what units does $\int_2^8 a(x)\,dx$ have?

Related Rates. Given a relation between quantities, find a relation between their derivatives.

Linearization. (1) Find the linearization $L(x)$ of a function $f(x)$ at a base point $x = a$. Use it to approximate $f(x) \approx L(x)$. (2) Approximate the change of a function Δf near a base point $x = a$ by the change in its linearization, $\Delta L = f'(a)\Delta x$. Equivalently, $\Delta f \approx f'(a)\Delta x$.

30. Find the linear approximation of $f(x) = (1 + x)^k$ at $x = 0$, where k is any real number. Use your result to approximate $\sqrt{0.9}$.
31. Use linearization to approximate the volume of a cylindrical shell of average radius r, height h, and thickness Δr.
32. Use linear approximations to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50 m.

33. The dosage D of diphenhydramine for a dog of body mass w kg, is $D = kw^{2/3}$ mg, where k is a constant. A cocker spaniel has mass $w = 10$ kg according to a veterinarians scale. Use linear approximations to estimate the maximum allowable error Δw in w if the percentage error $\Delta D/D$ in the dosage D must be less than 5%.

Optimization.
Find absolute maxima/minima for continuous functions on bounded domains.
Find all absolute and local maxima/minima for any given function. JUSTIFY your answer (either by graph, or using first derivative test or second derivative test) Note: the fact that the function has a local minimum at one point is not a justification for absolute minimum.

34. Find the local and absolute maximum and minimum values of the function on the given interval
    (a) $f(x) = x - \sqrt{x}$, $[0, 4]$
    (b) $f(x) = \dfrac{x}{x^2 + x + 1}$, $[-2, 1]$
    (c) $f(x) = x - \sqrt{2}\sin x$, $[0, \pi]$
    (d) $f(x) = \cos^2 x - 2\sin x$, $[0, 2\pi]$
35. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are positive constants. What is the length of the wave that gives the minimum velocity?

Differential equations and Initial Value Problems. Solve first and second order differential equations of the form $y' = f(x)$ or $y" = f(x)$ with or without initial conditions.

36. Solve the differential equation $f'(x) = 8x - 3\sec^2 x$
37. Solve the initial value problem $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}$, $f(1) = 3$.

D. Applications of the definite integral.

Total change.

38. §4.4: 70 (given graph of rate of change of volume of water in tank, find amount of water)
39. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3, t \geq 0$.
    (a) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
    (b) Find the particle's total displacement in the time interval $0 \leq t \leq 3$.
40. A particle moves on a line with velocity $v(t) = 3t^2 - 3t - 6$, $t \geq 0$.
    (c) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
    (d) Find the particle's total displacement in the time interval $0 \leq t \leq 3$.
41. A rabbit population starts with 80 rabbits and increases at a rate on dP/dt rabbits per week. What does

$$80 + \int_0^{12} \frac{dP}{dt}(t)\, dt$$

represent?

Areas

42. Use geometry to evaluate $\int_0^2 3x - 2\sqrt{4 - x^2}\, dx$.

Solids of revolution

43. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines:
    (a) the x-axis (b) the y-axis (c) $y = 2$

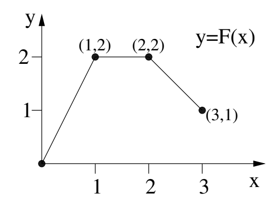
44. (a) Use calculus to find the volume of a sphere of radius R.
    (b) Find the volume of the sphere after a hole of radius $R/2$ has been drilled through its center.
    (c) What percentage of the volume of the whole sphere is left after drilling the hole?

Work

45. §5.4:5

Average of a function.

46. §5.5: 20 (avge vs maximal blood flow)
47. (a) Find a function f such that $F(x) = \int_0^x f(t)\,dt$ for the function F whose graph is shown in the figure at right.
    (b) What is the average of f over [0,3]?
    (c) What is the average of F over [0,3]?



E. Combination

48. For the function $f(x) = |x^2 + x|$,
    (a) Sketch the graph of $g(x) = x^2 + x$.
    (b) Sketch the graph of f.
    (c) Use your graph above to sketch the graph of $f'(x)$
    (d) Find a formula for $f'(x)$. Confirm that it is consistent with your graph in (a).
    (e) Is f continuous everywhere? Explain.
    (f) Is f differentiable everywhere? Explain.
    (g) Find $\int_{-1}^{2} f(x)\,dx$