NAME: MATH 1512 – HW 14 Due: Tuesday Apr 25, 2023

HOMEWORK DAY 38 – *Average Value §5.5*

1. §5.5: 2

2. §5.5: 15

3. An aluminum rod is 8cm long. The temperature of the rod at a point x cm from one end is given by

$$T(x) = \begin{cases} x^3 - x^2 + 32 \ , & 0 \leq x \leq 2 \\ 36 - 2x + x^2 \ , & 2 < x \leq 8 \end{cases}$$

Find the average temperature of the rod.

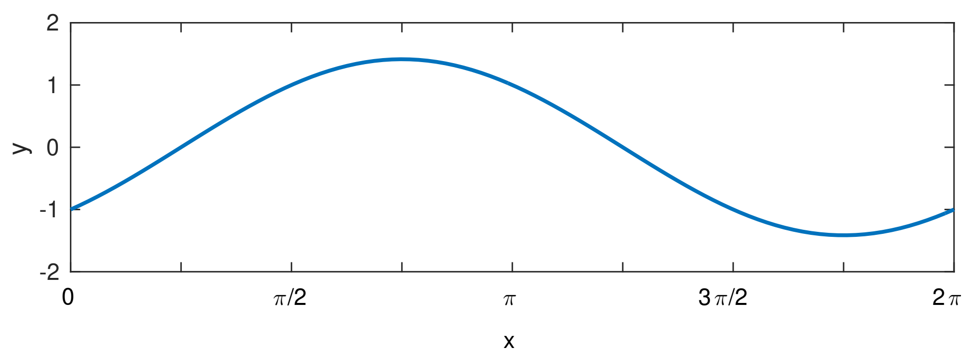
4. §5.5: 20 (avge vs maximal blood flow)

5. Consider the function $f(x) = \sin(x) - \cos(x)$.

   (a) Find the average f_{av} of f over the interval $[a, a+\pi]$?

(b) For what value of $a \in [0, \pi]$ is the average on $[a, a + \pi]$ largest?

(c) The figure shows a graph of f. Show the interval $[a, a + \pi]$ over which the average is maximal and the line $y = f_{av}$ for that interval. Explain why for any other a the average would clearly be less.



Mean Value Theorem. The MVT states that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

6. §3.2: 3

7. §3.2: 4

8. §3.2: 15

Applications of the MVT. From the above exercises it is not apparent why we care about the MVT. Below are two practical uses. The next exercise uses the MVT to obtain upper bounds for functions, or differences of functions.

9. Use the Mean Value Theorem to show that $|\sin(a) - \sin(b)| \leq |a - b|$ for all a and b.

The MVT is also necessary to rigourously prove the Fundamental Theorem of Calculus, or to derive one of calculus students favorite rule: L'Hôpitals rule. Here is a simple outline.
Suppose f and g are continuously differentiable on an interval (b, c) containing the number a, and that $f(a) = 0$ and $g(a) = 0$. What is the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} \ ?$$

Since every derivative $H'(a) = \lim_{x\to a} \dfrac{H(x) - H(a)}{x - a}$ is of this form, you can guess that the answer is not determined from the values of $f(a)$ and $g(a)$. We say that this 0/0 form is *indeterminate*.

Here comes the MVT to the rescue

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{f'(c_x)(x - a)}{g'(d_x)(x - a)} = \lim_{x\to a} \frac{f'(c_x)}{g'(d_x)} \tag{1}$$

where c_x and d_x are some points between x and a. As $x \to a$, it must be that $c_x \to a$ and $d_x \to a$. Thus the limit equals

$$\lim_{x\to a} \frac{f'(a)}{g'(a)}$$

This is L'Hôpitals rule. (Note: more details are needed for complete rigor.)

Other related theorems are the **Mean Value Theorem for Integrals**, stated in the book, and the **generalized MVT for integrals**, which states that if f and $g(x)$ are continuous on $[a, b]$, then there exists a number c in (a, b) such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

These are necessary to prove, for example, how good the Trapezoid Rule is in approximating an integral.