MATH 264 — R E V I E W   E X A M   2

TOPICS COVERED (Chapter 14)

1. Functions of several variables

Graphing

2D: Graph surfaces $z = f(x, y)$ in 3D.

Graph contour curves $f(x, y) = const$ in the x-y plane.

3D: Graph level surfaces of $f(x, y, z) = const$ in 3D.

Limits and continuity.

Derivatives and slopes.

Compute partial derivatives.

Compute the directional derivatives in any specified direction.

Tangent planes and normals.

Find equation of plane tangent to the surface $z = f(x, y)$ at a point

Find equation of plane tangent to the surface $F(x, y, z) = c$ at a point

Find a vector normal to any surface ($F(x, y, z) = c$, $z = f(x, y)$) at a point

Linearization.

Find the linear approximation $L(x, y)$ of $f(x, y)$ at a basepoint (x_0, y_0). What is the relation between L and the plane tangent to $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$?

Find the linear approximation of the change Δf of a function $f(x, y)$ as (x, y) changes from a base point (x_0, y_0) to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$.

Chain Rule. Implicit differentiation.

Properties of the Gradient: (be able to use them!)

Vector that points in direction of maximal increase.

Magnitude = maximal rate of change (derivative in direction of maximal increase)

functions of 2 variables: $\nabla f(x_0, y_0)$ points normal to level curve through (x_0, y_0).

functions of 3 variables: $\nabla f(x_0, y_0, z_0)$ points normal to level surface through (x_0, y_0, z_0)

2. Maxima and Minima

Find local max/min

Find critical points. Use second derivative test.

Find absolute max/min

1. Find local max/min

2. Investigate behaviour as $x, y \to \pm\infty$ if function is defined on an infinite domain.

2'. Investigate behaviour on boundary if function is defined on a closed, bounded domain (sample domains: squares, triangles or circles)

Remember: continuous functions on closed, bounded domains always have an absolute max/min

STUDY PROBLEMS

The following problems are a pretty comprehensive set, but make sure to also review homework problems you felt shaky on.

Chapter 14 Review, Concept Check: 1-17
Chapter 14 Review, True-False: 1-12
Chapter 14 Review, Exercises: 1-10,12,13,15,19,22,23,25,31,33,34,
35,39,43,44,45,47,48,51,52,55,56
Section 14.4: 42

A few other fun problems.

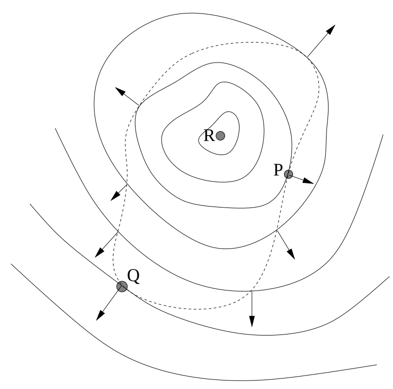
1. Find a unit vector tangent to the surface $z = x^2 + 3xy$ at the point $\mathrm{P}(1, 1, 4)$ on the surface, parallel to the $x - z$ plane.
   Find another unit vector tangent to the surface at P, but now parallel to the $y - z$ plane.

2. Let $f(x, y) = x^2 + 3xy$
   (a) Write down the equation for the tangent plane to the graph of f at $(1, 1, 4)$ in the form $z = L(x, y)$, where L is the linearization of f.
   (b) Find a normal to the surface by writing the surface as a level surface of some $F(x, y, z)$. Use the normal to write down the equation of a tangent plane.
   (c) Confirm that the results in (a) and (b) are the same.

3. (a) Graph the surface $r = a$ in 3D, where $r = \sqrt{x^2 + y^2}$.
   (b) Find a unit normal to the surface at an arbitrary point (x_o, y_o, z_o) on surface (Hint: proceed as in 2b).
   (c) Also sketch the unit normal in your graph, and make sure the answer makes sense.

4. Prove that the directional derivative in the direction of the gradient (the direction of maximal increase) is $|\nabla f|$.

5. The figure at right shows the curve $g(x, y) = k$ (dashed) and several level curves of the function $f(x, y)$ (solid). It also shows the gradient ∇f at several points. Does the absolute maximum of f under the constraint $g(x, y) = k$ occur at P? At Q? At R? Somewhere else?



6. Plot domain of $f(x, y) = \sqrt{x^2 - y^2}$. On the same plot, graph the contour curve through $(4, 2)$.

7. Use linear approximation to approximate $f(0.96, -1.002)$ if $f = x^3y^2$.

PARTIAL ANSWERS

Chapter 14 Review, Exercises.

12: $T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + \frac{86-72}{4}(x-6) + \frac{75-87}{4}(y-4) = 80 + \frac{7}{2}(x-6) - 3(y-4)$ $T(5,3.8) \approx 75.9$

34: (a) $A = xy/2$, $\Delta A \approx (y/2)\Delta x + (x/2)\Delta y$ $|\Delta A| \approx |(y/2)\Delta x + (x/2)\Delta y| \leq (12/2)0.002 + (5/2)0.002$ (here we used the Triangle Inequality)

(b) $d = \sqrt{x^2+y^2}$, $\Delta d \approx \frac{x\Delta x + y\Delta y}{\sqrt{x^2+y^2}}$ $|\Delta d| \approx |\frac{x\Delta x + y\Delta y}{\sqrt{x^2+y^2}}| \leq \frac{17(0.002)}{13}$

48: $\nabla f(P)$, $|\nabla f(P)|$

52: saddle at (0,0), local min at (1,1/2)

56: Critical points in interior $(0,0), (0,\pm 1), (\pm 1, 0)$ Critical points on boundary $(\pm 2, 0), (0, \pm 2)$ Abs max: $2/e$. Abs min: 0.

Section 14.4.

42: $V = \pi r^2 h$, $\Delta V \approx 2\pi r h \Delta r + \pi r^2 \Delta h$, where $h = 10$ cm, $r = 4$ cm, $\Delta r = 0.05$ cm, $\Delta h = 0.2$ cm. Therefore $\Delta V \approx 2.8\pi$ cm^3.

Other selected problems:

2: (a) $z = 4 + 5(x-1) + 3(y-1)$ (b) The surface is a level surface of $F(x,y,z) = x^2 + 3xy - z$ (since it is given by $F(x,y,z) = 0$). Therefore, a normal to the surface at P is $\nabla F(P) = \langle 5, 3, -1\rangle$. The equation for the tangent plane at P is $5(x-1) + 3(y-1) - (z-4)$.

3: (a) Cylinder of radius a. (b) Unit normal $\mathbf{n} = \dfrac{\langle x_o, y_o\rangle}{\sqrt{x_0^2 + y_0^2}} = \dfrac{\langle x_o, y_o\rangle}{a}$

4: See proof of Theorem 15.

5: At Q.