Lecture 4: Sets of points in the complex plane

Today we define several concepts we will use throughout the semester. Below, $S \subset \mathbb{C}$ is a set of points in the complex plane.

Definition: A **neighbourhood of a point** z_0 is a set $\{z : |z - z_0| < \epsilon$, for any $\epsilon > 0\}$. It is also referred to as ϵ-neighbourhood.

Definition: A point $z_0 \in S$ is an **interior point** of S if there exists a neighbourhood of z_0 contained in S.

Definition: A point $z_0 \in S$ is an **exterior point** of S if there exists a neighbourhood of z_0 which does not intersect S.

Definition: A point $z_0 \in S$ is a **boundary point** of S if every neighbourhood of z_0 contains points both in S and outside of S. That is, z_0 is neither an interior nor an exterior point.

Definition: The **boundary of a set S** is the set of all boundary points of S.

Definition: The **closure of a set S**, denoted by $\overline{S}$, is $S\cup$boundary of S.

Definition: A set S is **open** if every point in S is interior.

Definition: A set S is **closed** if S contains all its boundary points.

Definition: A point z_0 is an **accumulation point** of S if every neighbourhood of z_0 contains a point in S distinct from z_0.

Definition: A set S is **bounded** if every $z \in S$ is within a disk $|z| \leq R$ for some R.

Definition: A set S is **connected** if for every pair of points $z_1, z_2 \in S$ there is a smooth path in S connecting the points that is within S.

Definition: The **complement of a set S**, denoted by S^c, is $\mathbb{C}\backslash S$ (everything in $\mathbb{C}$ not in S).

Definition: The stereographic projection of the complex plane onto the Riemann sphere minus the north pole - define it geometrically. Infinity is projected onto the north pole.

Questions and examples:

1. Solve the inequalities $|x - c| < r$, $|x - c| \geq r$, where x, c, r are real.
2. Give examples of open sets, closed sets, and sets that are neither.
3. Is the set $\{1/n, n = 1, 2, 3, \ldots\}$ open, closed or neither? Explain.
4. What is the difference between an accumulation point and a boundary point?
5. Give examples of unbounded sets
6. Note: a neighbourhood of a point is an open disk centered at the point.
7. Where does the stereographic projection of points $z \in \mathbb{C}$ with $|z| > 1$ lie? What if $|z| < 1$? What if $|z| = 1$?
8. The complement of a closed set is open. Why?