1. Cauchy (1789-1857) - Riemann (1829-1866) Equations

*Theorem:* Suppose \( f(z) = u(x,y) + iv(x,y) \) is differentiable at \( z = z_0 \). Then \( u, v \) must satisfy the Cauchy-Riemann equations:

\[
\begin{align*}
u_x &= v_y , & u_y &= -v_x
\end{align*}
\]

Furthermore, \( f'(z) = u_x + iv_x = v_y - iu_y \).

*Proof:* in class

*Example 1:* \( f(z) = z^2 \) is differentiable and indeed, C-R are satisfied. Note \( f'(z) \) computed earlier using definition satisfies formula.

*Example 2:* \( f(z) = \bar{z} \) does not satisfy C-R. Thus it is not differentiable.

From the above theorem it does not follow that if C-R are satisfied, the function is differentiable. For that we need a stronger theorem.

*Theorem:* Suppose \( u_x, u_y, v_x, v_y \) exist in a neighbourhood of \( z_0 \), and are continuous at \( z_0 \), then \( f(z) \) is differentiable at \( z_0 \) \( \iff \) Cauchy-Riemann are satisfied

*Proof:* in class, using Taylor series. We reviewed Taylor series for functions of 1 and 2 variables.

From this second theorem it follows that if \( u, v \) are sufficiently nice, it is enough to check whether the Cauchy-Riemann equations are satisfied to determine whether \( f \) is differentiable.

2. Cauchy-Riemann Equations in polar coordinates

*Theorem:* Let \( f(z) = u(r,\theta) + iv(r,\theta) \) where \( z = x + iy \) and \( x = r \cos \theta, y = r \sin \theta \). If \( u_x, u_y, v_x, v_y \) exist in a neighbourhood of a nonzero point \( z_0 \neq 0 \), and are continuous at \( z_0 \), then \( f(z) \) is differentiable at \( z_0 \) \( \iff \) \( u_r = \frac{1}{r} v_\theta, \quad \frac{1}{r} u_\theta = -v_r \).

In that case, \( f'(z) = e^{-i\theta}(u_r + iv_r) \).

*Proof:* We outlined the equivalence of Cauchy-Riemann in Cartesian and polar coordinates in class. Full details are in HW.

*Example 3:* Show \( f(z) = |z|^2 = r^2 \) is not differentiable.

3. Analytic functions

*Definition:* \( f(z) \) is analytic at \( z_0 \) if it is differentiable in a nbhd of \( z_0 \)

*Definition:* \( f(z) \) is analytic (or holomorphic) in \( R \) if it is differentiable at all \( z \in R \)

*Definition:* \( f(z) \) is entire if it is differentiable in \( \mathbb{C} \)

*Definition:* \( z_0 \) is a singular point of \( f(z) \) if \( f \) is analytic at some point in every nbhd of \( z_0 \)

*Theorem:* Sums, products, quotients of analytic functions are analytic, as long as denominator not zero.