Lectures 10-11: Cauchy-Riemann Equations. Analytic functions.

1. Cauchy (1789-1857) - Riemann (1829-1866) Equations

Theorem: Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = z_0$. Then u, v must satisfy the Cauchy-Riemann equations:

$$u_x = v_y \ , \quad u_y = -v_x$$

Furthermore, $f'(z) = u_x + iv_x = v_y - iu_y$.

Proof: in class

Example 1: $f(z) = z^2$ is differentiable and indeed, C-R are satisfied. Note $f'(z)$ computed earlier using definition satisfies formula.

Example 2: $f(z) = \overline{z}$ does not satisfy C-R. Thus it is not differentiable.

From the above theorem it does not follow that if C-R are satisfied, the function is differentiable. For that we need a stronger theorem.

Theorem: Suppose u_x, u_y, v_x, v_y exist in a neighbourhood of z_0, and are continuous at z_0, then

$$f(z) \text{ is differentiable at } z_0 \iff \text{Cauchy-Riemann are satisfied}$$

Proof: in class, using Taylor series. We reviewed Taylor series for functions of 1 and 2 variables.

From this second theorem it follows that if u, v are sufficiently nice, *it is enough* to check whether the Cauchy-Riemann equations are satisfied to determine whether f is differentiable.

2. Cauchy-Riemann Equations in polar coordinates

Theorem: Let $f(z) = u(r, \theta) + iv(r, \theta)$ where $z = x + iy$ and $x = r\cos\theta$, $y = r\sin\theta$. If u_x, u_y, v_x, v_y exist in a neighbourhood of a nonzero point $z_0 \neq 0$, and are continuous at z_0, then

$$f(z) \text{ is differentiable at } z_0 \iff u_r = \frac{1}{r}v_\theta \ , \quad \frac{1}{r}u_\theta = -v_r$$

In that case, $f'(z) = e^{-i\theta}(u_r + iv_r)$.

Proof: We outlined the equivalence of Cauchy-Riemann in Cartesian and polar coordinates in class. Full details are in HW.

Example 3: Show $f(z) = |z|^2 = r^2$ is not differentiable.

3. Analytic functions

Definition: $f(z)$ is analytic at z_0 if it is differentiable in a nbhd of z_0

Definition: $f(z)$ is analytic (or holomorphic) in R if it is differentiable at all $z \in R$

Definition: $f(z)$ is entire if it is differentiable in $\mathbb{C}$

Definition: z_0 is a singular point of $f(z)$ if f is analytic at some point in every nbhd of z_0

Theorem: Sums, products, quotients of analytic functions are analytic, as long as denominator not zero.