NONLINEAR DYNAMICS AND CHAOS

TOPICS COVERED

1. Introduction

- Dynamics: how things change in time
  - Differential equations vs. difference equations
  - ODE vs PDE, autonomous coupled systems of ODE's
- Nonlinear
  - Linearity and the Superposition Principle
  - Linearization of nonlinear problems
- Chaos
  - Long-time behaviour in 1D, 2D, 3D (brief survey)

2. Flows in 1D

- Exact solutions: separation of variables
- Phase Portrait: fixed points and stability
- Linear stability analysis
- Existence and Uniqueness
- Potential and Kinetic Energy; the limit of infinite damping

3. Bifurcations in 1D

- Saddle-Node (or turning point or blue-sky) Bifurcation
- Transcritical Bifurcation
- Subcritical and Supercritical Pitchfork Bifurcation
- Imperfections and Catastrophe (example: budworm population)
- Nondimensionalization

4. Flows on the circle (periodic 1D)

- Plotting phase portrait on the circle
- One and two uniform oscillators
- Bifurcations on the circle (illustrated by overdamped pendulum with torque)

5. Flows in 2D : Linear systems

- Examples: nodes, saddles, centers
- Solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ of the form $\mathbf{x} = \mathbf{e}^{\lambda \mathbf{t}}\mathbf{v}$ (and brief review of 2D matrix theory)
- Real Eigenvalues: Stable or Unstable Nodes, Saddles
- Complex Eigenvalues: Centers, Stable or Unstable Spirals
- Repeated Eigenvalues: Degenerate Nodes, Star nodes

6. Flows in 2D : Nonlinear systems

- Linearization near equilibria
  - Classify linearized system
  - When does linearized system predict nonlinear behaviour?
- Plot Phase Portraits using
  - Nullclines
  - Local behaviour near equilibria
- Conservative systems
  - Energy contours and trajectories.
  - Attracting and repelling fixed points: are they possible? Why not?
  - Effect of a nonconstant conserved quantity
- Pendulum: no damping, underdamped, critically damped, overdamped.

7. Limit Cycles (a nonlinear phenomenon)

- Limit Cycles
  - Significance of limit cycles vs centers
  - Examples using polar coordinates
  - Van der Pool Equation
- Ruling out Closed Orbits
  - Linear systems
  - Conservative systems
  - Gradient systems, Liapunov functions
- Poincare-Bendixson Theorem
  - Determines everything that can possibly happen in 2D!
  - Finding trapping regions

8. Bifurcations in 2D

- Hopf Bifurcations
  - Supercritical, subcritical, degenerate
  - Examples in polar coordinates
- Saddle-node Bifurcations
- Transcritical and Pitchfork Bifurcations

9. Lorenz System

- Properties
  - Volume contraction
  - Bounded trajectories
  - No stable Equilibria if $r > r_H$
  - No stable Limit cycles if $r > r_H$
  - Sensitive dependence on initial conditions
- Definition of chaos
- Definition of an attractor

10. 1-D Maps

- Cobwebs
- Fixed Points and Stability
- Tent map
  - Boundedness and stretching
  - No stable orbits
  - Unstable p-orbits for all p
- Logistic map
  - Fixed points and stability
  - Flip bifurcation: when p-orbit becomes unstable a stable 2p-orbit appears
  - Transverse bifurcation: responsible for periodic windows
  - No stable orbit at finite r
  - Universality: ratio of lengths l_k/l_{k+1}, order of periodic windows

11. Fractals

- Cardinality: countable vs uncountable
- Measure: measure of countable sets
- Dimension: Similarity, Box, Hausdorff, and Correlation dimension
- Ex: Snowflake, Cantor set, randomized Cantor set, Brownian motion, Lorenz attractor

12. Strange Attractors

- Attractor Reconstruction